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MULTIPLE NORMALIZED SOLUTIONS FOR CHOQUARD EQUATIONS INVOLVING KIRCHHOFF TYPE PERTURBATION

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Abstract. In this paper we study the existence of critical points of the C^1 functional

$$E(u)=\frac{a}{2}\int_{\mathbb{R}^N}|\nabla u|^2\,dx+\frac{b}{4}\bigg(\int_{\mathbb{R}^N}|\nabla u|^2\,dx\bigg)^2-\frac{1}{2p}\int_{\mathbb{R}^N}(I_\alpha*|u|^p)|u|^p\,dx$$

under the constraint

$$S_c=\bigg\{u\in H^1(\mathbb{R}^N)\;\bigg|\;\int_{\mathbb{R}^N}|u|^2\,dx=c^2\bigg\},$$

where $a>0$, $b>0$, $N\geq 3$, $\alpha\in(0,N)$, $(N+\alpha)/N<p<(N+\alpha)/(N-2)$ and I_α is the Riesz Potential. When p belongs to different ranges, we obtain the threshold values separating the existence and nonexistence of critical points of E on S_c. We also study the behaviors of the Lagrange multipliers and the energies corresponding to the constrained critical points when $c\to 0$ and $c\to+\infty$, respectively.

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1. Introduction

In this paper we study the following Choquard equation perturbed by a Kirchhoff type nonlocal term

$$-\Big(a+b\int_{\mathbb{R}^N}|\nabla u|^2\,dx\Big)\Delta u-\mu u=(I_\alpha*|u|^p)|u|^{p-2}u,\quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $a>0$, $b>0$, $\mu\in R$, $N\geq 3$, $\alpha\in(0,N)$, $(N+\alpha)/N<p<(N+\alpha)/(N-2)$, Δ is the Laplacian and I_α is the Riesz potential defined as

$$I_\alpha=\frac{\Gamma\Big(\dfrac{N-\alpha}{2}\Big)}{\Gamma\Big(\dfrac{\alpha}{2}\Big)\pi^{N/2}2^\alpha|x|^{N-\alpha}},\quad \text{for all } x\in\mathbb{R}^N\setminus\{0\}.$$

When $a=1$, $b=0$, $\mu=-1$, Problem (1.1) reduces to the nonlinear Choquard equation or Choquard–Pekar equation

$$-\Delta u+u=(I_\alpha*|u|^p)|u|^{p-2}u,\quad \text{in } \mathbb{R}^N, \tag{1.2}$$

which arises in various fields of mathematical physics, such as quantum mechanics, physics of laser beams, the physics of multiple-particle systems. In the case $N=3$, $\alpha=2$, $p=2$, problem (1.2) was introduced in 1954 by S.I. Pekar as a model in quantum theory of a polaron at rest [25]. In 1976, P. Choquard used (1.2) to describe an electron trapped in its own hole, in a certain approximation to Hartree–Fock theory of one component plasma [17]. In 1996, R. Penrose proposed (1.2) as a model of self-gravitating matter, in a programme in which quantum state reduction is understood as a gravitational phenomenon [19]. If u solves (1.2), then the function ψ defined by $\psi(t,x)=e^{it}u(x)$ is a solitary wave of the focussing time dependent Hartree–Fock equation

$$i\psi_t=-\Delta\psi-(I_\alpha*|\psi|^p)|\psi|^{p-2}\psi,\quad \text{in } \mathbb{R}^+\times\mathbb{R}^N.$$

Equation (1.2) has been studied in past decades by variational methods starting with the pioneering works of E.H. Lieb [17]. More recently, some new and improved techniques have been devised to deal with various forms of (1.2). In [18], Ma and Zhao proved, under some assumptions on N, α and p, that every positive solution of (1.2) is radially symmetric and monotone decreasing about some point by the method moving planes in an integral form developed by Chen et al. in [4]. In [20], Moroz and Schaftingen eliminated the restriction given by Ma and Zhao in [18] and established an optimal range of parameters for the existence of a positive ground state solution of (1.2). Moreover, they proved that all positive ground state solutions of (1.2) are radially symmetric and monotone decaying about some point. For more details on this subject, see, for example, the literature [1], [5], [14], [15], [21]–[23], [27], [28], and the survey [24].

Back to (1.1) we should point out that since for all $\varphi \in C_0^\infty(\mathbb{R}^N)$, $I_\alpha * \varphi \to \varphi$ as $\alpha \to 0$, then equation

$$-\bigg(a + b\int_{\mathbb{R}^N} |\nabla u|^2\,dx\bigg)\Delta u - \mu u = |u|^{2p-2}u, \quad \text{in } \mathbb{R}^N, \tag{1.3}$$

can be seen as the limit equation of (1.1) as $\alpha \to 0$. Equation (1.3) is related to the stationary solutions of

$$u_{tt} - \bigg(a + b\int_{\mathbb{R}^N} |\nabla u|^2\,dx\bigg)\Delta u = f(x,u), \tag{1.4}$$

where $f(x,u)$ is a general nonlinearity. Equation (1.4) was proposed by Kirchhoff in [12] and models free vibrations of elastic strings by taking into account the changes in length of the string produced by transverse vibrations. From a mathematical point of view, equation (1.3) is nonlocal, in the sense that, the term $\int_{\mathbb{R}^N} |\nabla u|^2\,dx\Delta u$ depends not only on the pointwise value of Δu, but also on the integral of $|\nabla u|^2$ over the whole space. This new feature brings new mathematical difficulties that make the study of Kirchhoff type equations particularly interesting. Equation (1.3) and its variants have been studied extensively in the literature, see [6]–[9], [13], [16], [26], [32] and the references therein.

In the present paper, motivated by the fact that physicists are often interested in "normalized solutions", we search for solutions in $H^1(\mathbb{R}^N)$ having a prescribed L^2-norm. More precisely, for given $c > 0$,

$$(\mu_c, u_c) \in \mathbb{R} \times H^1(\mathbb{R}^N) \quad \text{solution of (1.1) with } \int_{\mathbb{R}^N} |u|^2\,dx = c^2.$$

Such solutions of Schrödinger–Poisson equations, Kirchhoff type equations, quasilinear Schrödinger equations and Choquard equations have been obtained recently in [2], [3], [5], [11], [14], [15], [28]–[31] and the references therein. Therefore the main aim of this paper is to extend the results of [14], [28]–[31] to the equation (1.1), moreover, we also study the asymptotic behaviors of the normalized solutions as $c \to 0$ and $c \to +\infty$, respectively.

The normalized solutions can be obtained by looking for critical points of the following C^1 functional

$$E(u) = \frac{a}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \frac{b}{4}\bigg(\int_{\mathbb{R}^N} |\nabla u|^2\,dx\bigg)^2 - \frac{1}{2p}\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p\,dx \tag{1.5}$$

constrained on the L^2-spheres in $H^1(\mathbb{R}^N)$:

$$S_c = \bigg\{u \in H^1(\mathbb{R}^N) \;\bigg|\; \int_{\mathbb{R}^N} |u|^2\,dx = c^2\bigg\}.$$

Recalling that in this case the frequency μ is not fixed any longer but appears as a Lagrange multiplier.

For $c > 0$, we set the following constrained minimization problem

$$E_{c^2} = \inf_{u \in S_c} E(u). \tag{1.6}$$

It follows from Lemma 2.1 below that $E(u)$ is bounded from below on S_c if $(N+\alpha)/N < p < \min\{(N+4+\alpha)/(N-2), (N+\alpha)/(N-2)\}$, then we first try to find the minimizers of (1.6) since it is standard that the minimizers of (1.6) are critical points of E on S_c. We note that the main difficulty of obtaining the minimizer of (1.6) is due to the lack of compactness of the minimizing sequence $\{u_n\}$ on S_c. To recover the compactness one can use the concentration-compactness principle and then rule out the cases of vanishing and dichotomy. Inspired by [20], [28], [31], we can deal with this type problem in a simple way by observing the two special nonlocal terms of the functional (1.5), where only technical energy estimates are involved and the concentration-compactness principle is avoided.

To state our main results, we first give some preliminaries. By [20], [28], we know that if $Q_p(x)$ is a positive ground state solution of the equation

$$-\frac{Np - N - \alpha}{2}\,\Delta u + \frac{N + \alpha - (N-2)p}{2}\,u = (I_\alpha * |u|^p)|u|^{p-2}u, \quad \text{in } \mathbb{R}^N, \tag{1.7}$$

then it satisfies the following Pohozaev identity

$$\begin{aligned}\frac{(N-2)(Np - N - \alpha)}{4}\|\nabla Q_p\|_2^2 + \frac{N\big(N + \alpha - (N-2)p\big)}{4}\,\|Q_p\|^2 \\ = \frac{N+\alpha}{2p}\int_{\mathbb{R}^N}\big(I_\alpha * |Q_p|^p\big)|Q_p|^p\,dx,\end{aligned}$$

therefore we can obtain that

$$\|\nabla Q_p\|_2^2 = \|Q_p\|_2^2 = \frac{1}{p}\int_{\mathbb{R}^N}\big(I_\alpha * |Q_p|^p\big)|Q_p|^p\,dx. \tag{1.8}$$

Moreover, $Q_p(x)$ is the optimizer of the following sharp Gagliardo–Nirenberg inequality with best constants

$$\begin{aligned}\int_{\mathbb{R}^N}(I_\alpha * |u|^p)|u|^p\,dx \\ \le \frac{p}{\|Q_p\|_2^{2p-2}}\big(\|\nabla u\|_2^2\big)^{(Np-N-\alpha)/2}\big(\|u\|_2^2\big)^{(N+\alpha-(N-2)p)/2}.\end{aligned} \tag{1.9}$$

for $(N+\alpha)/N < p < (N+\alpha)/(N-2)$.

For $c > 0$, we denote the function $f_c(t)$ and the numbers c_*, c_{**}, $\overline{c}_{**}$ as

$$f_c(t) = \frac{a}{2}\,t + \frac{b}{4}\,t^2 - \frac{c^{N+\alpha-(N-2)p}}{2\|Q_p\|_2^{2p-2}}\,t^{(Np-N-\alpha)/2}, \quad t \ge 0, \tag{1.10}$$

$$
(1.11)\quad c_* = \begin{cases} 0 & \text{if } \dfrac{N+\alpha}{N} < p < \dfrac{N+2+\alpha}{N}, \\ a^{N/(2\alpha+4)}\|Q_{(N+2+\alpha)/N}\|_2 & \\ & \text{if } p = \dfrac{N+2+\alpha}{N}, \\ c_{**} & \text{if } \dfrac{N+2+\alpha}{N} < p < \min\left\{\dfrac{N+4+\alpha}{N}, \dfrac{N+\alpha}{(N-2)}\right\}, \\ \left(\dfrac{b}{2}\right)^{N/2(\alpha+4-N)} \|Q_{(N+\alpha+4)/N}\|_2^{(\alpha+4)/(\alpha+4-N)} & \\ & \text{if } p = \dfrac{N+4+\alpha}{N}, \\ 0 & \text{if } \dfrac{N+4+\alpha}{N} < p < \dfrac{N+\alpha}{(N-2)} \ (N < \alpha+4), \end{cases}
$$

$$
(1.12)\qquad c_{**} = \left(2\|Q_p\|_2^{2p-2}\left(\frac{a}{4-\kappa}\right)^{(4-\kappa)/2}\left(\frac{b}{2(\kappa-2)}\right)^{(\kappa-2)/2}\right)^{1/(2p-\kappa)},
$$

$$
(1.13)\qquad \overline{c}_{**} = \left(\frac{4\|Q_p\|_2^{2p-2}}{\kappa}\left(\frac{a}{4-\kappa}\right)^{(4-\kappa)/2}\left(\frac{b}{\kappa-2}\right)^{(\kappa-2)/2}\right)^{1/(2p-\kappa)}.
$$

where $\kappa := \kappa(N,p,\alpha) = Np - N - \alpha$.

REMARK 1.1. By a simple calculation, we see that $\overline{c}_{**} < c_{**}$ if

$$
\frac{N+2+\alpha}{N} < p < \min\left\{\frac{N+4+\alpha}{N}, \frac{N+\alpha}{N-2}\right\}.
$$

Our main result is the following.

THEOREM 1.2. *Let $a > 0$, $b > 0$, $N \geq 3$, $\alpha \in (0,N)$, $(N+\alpha)/N < p < \min\{(N+4+\alpha)/N, (N+\alpha)/(N-2)\}$ and $Q_p(x)$ is a positive ground state solution of* (1.7), *$\lambda_c = t_c^{1/2}/c$ and $t_c > 0$ is the global minimizer of* (1.10) *for $c \geq c_*$.*

(a) *Assume that $(N+\alpha)/N < p \leq (N+2+\alpha)/N$, then Problem* (1.6) *has a minimizer $u_c \in S_c$ of the form $u_c(x) = c\lambda_c^{N/2}Q_p(\lambda_c x)/\|Q_p\|_2$ with $E(u_c) = E_{c^2}$ if and only if $c > c_*$.*
(b) *Assume that $(N+2+\alpha)/N < p < \min\{(N+4+\alpha)/N, (N+\alpha)/(N-2)\}$, then Problem* (1.6) *has a minimizer $u_c \in S_c$ of the form $u_c(x) = c\lambda_c^{N/2}Q_p(\lambda_c x)/\|Q_p\|_2$ with $E(u_c) = E_{c^2}$ if and only if $c \geq c_*$.*

Then for each minimizer u_c there exists a couple $(\mu_c, u_c) \in \mathbb{R}\times H^1(\mathbb{R}^N)$ solution of (1.1).

From the above theorem we see that problem (1.6) has no global minimizer if $(N+\alpha+2)/N \leq p < \min\{(N+\alpha+4)/N, (N+\alpha)/(N-2)\}$ and $0 < c \leq c_*$ or $p > \min\{(N+4+\alpha)/N, (N+\alpha)/(N-2)\}$ and $c > 0$. We try to find the mountain pass type normalized solutions of (1.1) and local minimizer of E on S_c. Here, we say that a normalized solution $(\mu_c, u_c) \in \mathbb{R} \times H^1(\mathbb{R}^N)$ of (1.1) is a mountain pass type normalized solution if $E(u_c) = m(c)$, where $m(c)$ is defined in the following definition.

DEFINITION 1.3. Given $c > 0$, the functional E is said to have a mountain pass geometry on S_c if there exists $K_c > 0$ and $e \in H^1(\mathbb{R}^N)$ such that

$$m(c) := \inf_{h\in\Gamma(c)} \max_{t\in[0,1]} E(h(t)) > \{E(h(0)), E(h(1))\},$$

where

$$\Gamma(c) = \{h \in C([0,1]; S_c) \mid h(0) \in A_{K(c)}, h(1) = e\},$$

and $A_{K(c)} = \{u \in S_c \mid \|\nabla u\|_2^2 \le K(c)\}$.

Inspired by the results of [2], [10], [11], [31], we investigate the existence of mountain pass type normalized solutions of (1.1) and local minimizers of the functional E on S_c. Our main result is the following.

THEOREM 1.4. *Assume that $a > 0$, $b > 0$, $N \ge 3$, $\alpha \in (0, N)$, $Q_p(x)$ is a positive ground state solution of* (1.7)*, $\lambda_c = (t_c^{1/2})/c$ and $t_c > 0$ is such that $f_c'(t_c) = 0$ for $c > 0$. Let $u_{c,i}$ $(i = 0, 1, 2, 3)$ and $\widetilde{E}_{c^2}$ are given by* (3.3) *and* (3.9)*.*

(a) *Assume that $(N+2+\alpha)/N < p < \min\{(N+4+\alpha)/N, (N+\alpha)/(N-2)\}$.*
   (i) *If $c \in (0, \overline{c}_{**})$, then E does not admit normalized solutions on the constraint S_c.*
   (ii) *If $c = \overline{c}_{**}$, then* (1.1) *has a normalized solution $(\mu_{c,0}, u_{c,0}) \in \mathbb{R} \times H^1(\mathbb{R}^N)$ of the form* (3.3) *with $E(u_{c,0}) > 0$.*
   (iii) *If $c > \overline{c}_{**}$, then* (1.1) *has two normalized solutions $(\mu_{c,1}, u_{c,1})$, $(\mu_{c,2}, u_{c,2}) \in \mathbb{R} \times H^1(\mathbb{R}^N)$ of the form* (3.3) *with $E(u_{c,1}) = m(c) > 0$ and $E(u_{c,1}) = m(c) > E(u_{c,2}) = \widetilde{E}_{c^2}$, which means that $(\mu_{c,1}, u_{c,1})$ is a mountain pass type normalized solution of* (1.1) *and $u_{c,2}$ is a local minimizer of E on S_c when $c \in (\overline{c}_{**}, c_{**})$ and a global minimizer of E on S_c when $c \ge c_{**}$.*

(b) *Assume that $N + 4 + \alpha/N \le p < (N + \alpha)/(N - 2)$ (i.e. $N < \alpha + 4$), then* (1.1) *has a mountain pass type normalized solution $(\mu_{c,3}, u_{c,3}) \in \mathbb{R} \times H^1(\mathbb{R}^N)$ of the form* (3.3) *with $E(u_{c,3}) = m(c) > 0$ if and only if $c > c_*$.*

REMARK 1.5. Theorems 1.2 and 1.4 provide threshold values of $c_* \ge 0$ and $\overline{c}_{**} > 0$ separating the existence and nonexistence of minimizers E on S_c and normalized solutions of (1.1) when p belongs to different ranges. Our results are sharp. Theorems 1.2 and 1.4 extend the results of [14, Theorem 1.1], [28] and [31, Theorems 1.1 and 1.2], where Ye studied (1.1) when $a = 1$ and $b = 0$ in [28], Zeng and Zhang studied (1.1) when $\alpha = 0$ in [31].

When μ is replaced by the external potential $V(x)$, the concentration and symmetry breaking of the minimizers were studied in [5], [15], [28], [30]. In [2], [10], [11], Jeanjean et al. investigated the asymptotic behavior of the Lagrange multiplier μ_c corresponding to the global minimizer u_c as $c \to +\infty$. In [3], Bellazzini et al. show that if the function E_{c^2}/c^2 is monotone decreasing, then they prove that the dichotomy case of the minimizing sequence does not occur, which is crucial to proving the convergence of the minimizing sequences. Motivated by these results and the proofs of our theorems 1.2 and 1.4, we attempt to investigate some properties of normalized solutions of equation (1.1) by introducing some new observations and energy estimates, therefore we study the behaviors of the Lagrange multipliers and the energies corresponding to the constrained critical points of E as $c \to 0$ and $c \to +\infty$, respectively. Our main results are the following.

THEOREM 1.6. *Assume that* $(N+\alpha)/N < p < (N+2+\alpha)/N$ *or* $(N+4+\alpha)/N < p < (N+\alpha)/(N-2)$ $(N < \alpha + 4)$. *Let* (μ_c, u_c) *be the normalized solution obtained by Theorem* 1.2 *and Theorem* 1.4 *with* $u_c(x) = c\lambda_c^{N/2} Q_p(\lambda_c x)/\|Q_p\|_2$.

(a) *If* $(N+\alpha)/N < p < (N+2+\alpha)/N$, *then* λ_c *is monotone increasing for* $c > 0$ *small enough and* E_{c^2}/c^2 *is monotone decreasing for* $c > 0$ *small enough*; *furthermore, as* $c \to 0$,

$$\lambda_c \to 0, \quad \frac{c^{2p-2}}{\lambda_c^{N+\alpha+2-Np}} \to \frac{2\|Q_p\|_2^{2p-2}}{Np-N-\alpha}\, a,$$
$$\frac{E_{c^2}}{c^2} \to 0, \quad \frac{\mu_c}{\lambda_c^2} \to \frac{(N-2)p-N-\alpha}{Np-N-\alpha}\, a.$$

(b) *If* $(N+4+\alpha)/N < p < (N+\alpha)/(N-2)$ (*i.e.* $N < \alpha+4$), *then* λ_c *is monotone decreasing for* $c > 0$ *and as* $c \to 0$,

$$\lambda_c c \to +\infty, \quad \frac{c^{2p-4}}{\lambda_c^{N+4+\alpha-Np}} \to \frac{2\|Q_p\|_2^{2p-2}}{Np-N-\alpha}\, b,$$
$$E(u_c) \to +\infty, \quad \frac{\mu_c}{\lambda_c^4 c^2} \to \frac{(N-2)p-N-\alpha}{Np-N-\alpha}\, b.$$

THEOREM 1.7. *Assume that* $(N+\alpha)/N < p < \min\{(N+\alpha+4)/N, (N+\alpha)/(N-2)\}$. *Let* (μ_c, u_c) *be the normalized solution obtained by Theorem* 1.2 *and Theorem* 1.4 *with* $u_c(x) = c\lambda_c^{N/2} Q_p(\lambda_c x)/\|Q_p\|_2$.

(a) *If* $(N+\alpha)/N < p < \min\{(N+4+\alpha)/N, (N+\alpha)/(N-2)\}$ *and* (μ_c, u_c) *is obtained by Theorem* 1.2, *then, as* $c \to +\infty$,

$$\lambda_c c \to +\infty, \quad \frac{c^{2p-4}}{\lambda_c^{N+4+\alpha-Np}} \to \frac{2\|Q_p\|_2^{2p-2}}{Np-N-\alpha} b,$$
$$E(u_c) \to -\infty, \quad \frac{\mu_c}{c^2\lambda_c^4} \to \frac{(N-2)p-N-\alpha}{Np-N-\alpha} b.$$

Moreover,

(i) *if* $(N+\alpha)/N < p < \min\{2, (N+4+\alpha)/N, (N+\alpha)/(N-2)\}$, *then* $\lambda_c \to 0$ *as* $c \to +\infty$;

(ii) *for* $\alpha + 4 > N$, *if* $2 < p < \min\{(N+4+\alpha)/N, (N+\alpha)/(N-2)\}$, *then* $\lambda_c \to +\infty$ *as* $c \to +\infty$.

(b) *If* (μ_c, u_c) *is obtained by Theorem* 1.4 *with* $E(u_c) > 0$, *then, as* $c \to +\infty$,

$$\lambda_c \to 0, \quad \lambda_c c \to 0, \quad \frac{c^{2p-2}}{\lambda_c^{N+2+\alpha-Np}} \to \frac{2\|Q_p\|_2^{2p-2}}{Np-N-\alpha}\, a,$$
$$E(u_c) \to 0, \quad \frac{\mu_c}{\lambda_c^2} \to \frac{(N-2)p-N-\alpha}{Np-N-\alpha}\, a.$$

Finally, we explicit a relationship between the ground states of (1.1) and the global minimizers of E_{c^2}. Let (μ_c, u_c) be a normalized solution of (1.1). Set

$$\mathcal{A}_{\mu_c} = \{u \mid u \text{ is a solution of (1.1) with } \mu = \mu_c\},$$

then $\mathcal{A}_{\mu_c} \neq \emptyset$ since $u_c \in \mathcal{A}_{\mu_c}$. Here v is called a ground state solution of (1.1) with $\mu = \mu_c$ if $v \in \mathcal{A}_{\mu_c}$ and $I(v) = \inf\{I(u) : u \in \mathcal{A}_{\mu_c}\}$.

Theorem 1.8. *Assume that* $a > 0$, $b > 0$, $N = 3$ *or* $N = 4$, $\alpha \in (0, N)$, $(N+\alpha)/N < p < \min\{(N+4+\alpha)/N, (N+\alpha)/(N-2)\}$. *Let* u_c *be a global minimum of* E *on* S_c *and* $\mu_c < 0$ *be its Lagrange multiplier. Then* u_c *is a ground state solution of* (1.1) *with* $\mu = \mu_c$.

We conclude this introduction by pointing out that whether or not the mountain pass type normalized solution of (1.1) is the ground state solution of (1.1) is an open question.

The paper is organized as follows: Section 2 is devoted to the preliminary and we give some lemmas which are crucial to our proofs. Section 3 contains the proofs of the Theorems 1.2 and 1.4. Section 4 is devoted to the proofs of the Theorems 1.6 and 1.7. In Section 5, we prove Theorem 1.8.

Notations. $L^p(\mathbb{R}^N)$ $(1 < p \leq +\infty)$ is the usual Lebesgue space with the standard norm

$$\|u\|_p = \bigg(\int_{\mathbb{R}^N} |u|^p \, dx\bigg)^{1/p}.$$

$H^1(\mathbb{R}^N)$ is the usual Sobolev spaces with the standard norm

$$\|u\|_{H^1} = \bigg(\int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \int_{\mathbb{R}^N} |u|^2 \, dx\bigg)^{1/2}.$$

2. Preliminaries

For convenience, we set

$$A(u) = \int_{\mathbb{R}^N} |\nabla u|^2 \, dx, \qquad B(u) = \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p \, dx$$

then

$$E(u)=\frac{a}{2}\,A(u)+\frac{b}{4}\,A^2(u)-\frac{1}{2p}\,B(u).$$

It follows from (1.9) that

$$B(u)\le\frac{pc^{N+\alpha-(N-2)p}}{\|Q_p\|_2^{2p-2}}\,(A(u))^{(Np-N-\alpha)/2}. \tag{2.1}$$

Therefore, for any $u\in S_c$, we have

$$\begin{aligned} E(u)&\ge\frac{a}{2}\,A(u)+\frac{b}{4}\,(A(u))^2-\frac{c^{N+\alpha-(N-2)p}}{2\|Q_p\|_2^{2p-2}}\,(A(u))^{(Np-N-\alpha)/2} \\ &=f_c(t)\triangleq\frac{a}{2}\,t+\frac{b}{4}\,t^2-\frac{c^{N+\alpha-(N-2)p}}{2\|Q_p\|_2^{2p-2}}\,t^{(Np-N-\alpha)/2}, \end{aligned} \tag{2.2}$$

where $t=A(u)$.

The following lemma shows that problem (1.6) makes sense if $(N+\alpha)/N<p<\min\{(N+\alpha+4)/N,(N+\alpha)/(N-2)\}$.

Lemma 2.1. *Assume that* $(N+\alpha)/N<p<\min\{(N+\alpha+4)/N,(N+\alpha)/(N-2)\}$*, then* $E_{c^2}\in(-\infty,0)$ *if and only if* $c>c_*$*, where* c_* *is defined by* (1.11).

Proof. For each $c>c_*$ and $u\in S_c$, by (2.2) we know that

$$E(u)\ge\frac{b}{4}(A(u))^2-\frac{c^{N+\alpha-(N-2)p}}{2\|Q_p\|_2^{2p-2}}\,(A(u))^{(Np-N-\alpha)/2}>-\infty$$

since $(N+\alpha)/N<p<\min\{(N+\alpha+4)/N,(N+\alpha)/(N-2)\}$, which means that E is bounded from below on S_c and E_{c^2} is well defined.

For $u\in S_c$ and $\lambda>0$, we have $\lambda^{N/2}u(\lambda x)\in S_c$ and

$$g(\lambda)\triangleq E\big(\lambda^{N/2}u(\lambda x)\big)=\frac{a\lambda^2}{2}\,A(u)+\frac{b\lambda^4}{4}\,(A(u))^2-\frac{\lambda^{Np-N-\alpha}}{2p}\,B(u). \tag{2.3}$$

Hence $E_{c^2}=\inf\limits_{u\in S_c}E(u)\le E(\lambda^{N/2}u(\lambda x))\to 0$ as $\lambda\to 0$, so that $E_{c^2}\le 0$.

If $(N+\alpha)/N<p<(N+\alpha+2)/N$, then $0<Np-N-\alpha<2$, and it follows from (2.3) that $E_{c^2}<0$ for all $c>0$. Note that $c\lambda^{N/2}Q_p(\lambda x)/\|Q_p\|_2\in S_c$, by (1.8), (1.9) and (2.3), we obtain that

$$\begin{aligned} g(\lambda)&=E\bigg(\frac{c\lambda^{N/2}}{\|Q_p\|_2}\,Q_p(\lambda x)\bigg) \\ &=\frac{a}{2}\,c^2\lambda^2+\frac{b}{4}\,(c^2\lambda^2)^2-\frac{c^{N+\alpha-(N-2)p}}{2\|Q_p\|_2^{2p-2}}\,(c\lambda)^{Np-N-\alpha} \\ &=\frac{a}{2}\,c^2\lambda^2+\frac{b}{4}\,(c^2\lambda^2)^2-\frac{c^{2p}}{2\|Q_p\|_2^{2p-2}}\,\lambda^{Np-N-\alpha}. \end{aligned} \tag{2.4}$$

If $Np=N+\alpha+2$, then

$$g(\lambda)=\lambda^2\bigg(\frac{ac^2}{2}-\frac{c^{2p}}{2\|Q_p\|_2^{2p-2}}\bigg)+\frac{b}{4}\,(c^2\lambda^2)^2,$$

there exists $\lambda_0 > 0$ such that $g(\lambda) < 0$ for $\lambda \in (0, \lambda_0)$ and $c > a^{1/(2p-2)}\|Q_p\|_2$, therefore we have $E_{c^2} < 0$.

If $(N+\alpha+2)/N < p < \min\{(N+\alpha+4)/N, (N+\alpha)/(N-2)\}$, that is, $2 < Np - N - \alpha < 4$, by the Young inequality we have

$$\begin{aligned}\frac{a}{2}\,c^2\lambda^2 + \frac{b}{4}\,(c^2\lambda^2)^2 &= \eta\bigg(\frac{a}{2\eta}\,c^2\lambda^2\bigg) + \theta\bigg(\frac{b}{4\theta}\,(c^2\lambda^2)^2\bigg)\\ &\geq \bigg(\frac{a}{2\eta}\bigg)^{\eta}\bigg(\frac{b}{4\theta}\bigg)^{\theta}(c^2\lambda^2)^{(Np-N-\alpha)/2},\end{aligned}$$

where $2\eta = N+\alpha+4-Np$ and $2\theta = Np-N-\alpha-2$, and " $=$ " holds if and only if

$$\frac{ac^2\lambda^2}{2\eta} = \frac{b(c^2\lambda^2)^2}{4\theta}, \quad \text{i.e. } c^2\lambda^2 = \frac{2a\theta}{b\eta} = \frac{2a(Np-N-\alpha-2)}{b(N+\alpha+4-Np)},$$

thus it is deduced from (2.4) that

$$g(\lambda) \geq \frac{c_{**}^{N+\alpha-(N-2)p} - c^{N+\alpha-(N-2)p}}{2\|Q_p\|_2^{2p-2}}\,(c\lambda)^{Np-N-\alpha},$$

where c_{**} is given by (1.12). Therefore, set $\lambda_1^2 = 2a(Np-N-\alpha-2)/(c^2b(N+\alpha+4-Np))$, if $c \in (0, c_{**})$, then $g(\lambda) > 0$ for all $\lambda > 0$; if $c = c_{**}$, then $g(\lambda) > 0$ for all $\lambda \neq \lambda_1$ and $g(\lambda_1) = 0$; if $c > c_{**}$, then we obtain that

$$g(\lambda_1) = \frac{c_{**}^{N+\alpha-(N-2)p} - c^{N+\alpha-(N-2)p}}{2\|Q_p\|_2^{2p-2}}\,(c\lambda_1)^{Np-N-\alpha} < 0$$

so that $E_{c^2} < 0$ for $c > c_*$. □

REMARK 2.2. If $0 < N-\alpha < 4$, then $(N+\alpha+4)/N < (N+\alpha)/(N-2)$. For $p = (N+\alpha+4)/N$, by (2.4), we have

$$\begin{aligned}g(\lambda) &= E\bigg(\frac{c\lambda^{N/2}}{\|Q_p\|_2}\,Q_p(\lambda x)\bigg)\\ &= \frac{a}{2}\,c^2\lambda^2 + \bigg(\frac{b}{4} - \frac{c^{2(\alpha+4-N)/N}}{2\|Q_p\|_2^{2(\alpha+4)/N}}\bigg)(c^2\lambda^2)^2\\ &= \frac{a}{2}\,c^2\lambda^2 + \big(c_*^{2(\alpha+4-N)/N} - c^{2(\alpha+4-N)/N}\big)\frac{(c^2\lambda^2)^2}{2\|Q_p\|_2^{2(\alpha+4)/N}},\end{aligned} \tag{2.5}$$

if $c \in (0, c_*]$, then $g(\lambda) > 0$ for all $\lambda > 0$ and problem (1.6) has no minimizer, if $c > c_*$, then it is deduced from (2.5) that $g(\lambda) \to -\infty$ as $\lambda \to +\infty$, therefore $E_{c^2} = -\infty$ and Problem (1.6) has no minimizer. For $(N+\alpha+4)/N < p < (N+\alpha)/(N-2)$, by (2.3), we know that $g(\lambda) \to -\infty$ as $\lambda \to +\infty$, thus $E_{c^2} = -\infty$ and problem (1.6) has no minimizer.

For $(N+2+\alpha)/N < p < \min\{(N+\alpha+4)/N, (N+\alpha)/(N-2)\}$, our another aim is to find the constrained critical points of E on S_c. By (2.1) and (2.2), we first study the properties of f_c for $t > 0$ in the following lemmas.

LEMMA 2.3. *Assume that* $(N+\alpha+2)/N < p < \min\{(N+\alpha+4)/N, (N+\alpha)/(N-2)\}$. *Then there exists* $\overline{c}_{**} > 0$ *such that*:

(a) *if* $c \in (0, \overline{c}_{**})$ *then* $f_c'(t) > 0$;
(b) *if* $c = \overline{c}_{**}$ *then there is a unique* $t_0 > 0$ *such that* $f_c'(t_0) = 0$;
(c) *if* $c > \overline{c}_{**}$ *then there exist two positive constants* t_1, t_2: $0 < t_1 < t_0 < t_2$ *such that* $f_c(t_1) > 0$, $f_c(t_1) > f_c(t_2)$, $f_c'(t_1) = f_c'(t_2) = 0$, $f_c''(t_1) < 0$, $f_c''(t_2) > 0$, *where* t_1 *is the unique local maximum point of* f_c *and* t_2 *is the unique local minimum point of* f_c.

PROOF. Recall that $2 < Np - N - \alpha < 4$ and

$$f_c(t) = \frac{a}{2}t + \frac{b}{4}t^2 - \frac{c^{N+\alpha-(N-2)p}}{2\|Q_p\|_2^{2p-2}}t^{(Np-N-\alpha)/2}, \quad t > 0$$

then we have

$$f_c'(t) = \frac{a}{2} + \frac{b}{2}t - \frac{(Np-N-\alpha)c^{N+\alpha-(N-2)p}}{4\|Q_p\|_2^{2p-2}}t^{(Np-N-\alpha-2)/2}.$$

Similar to the proof of Lemma 2.1, set $2\eta = N+\alpha+4-Np$ and $2\theta = Np-N-\alpha-2$, then $\eta+\theta = 1$ and by the Young inequality we have

$$\frac{a}{2} + \frac{b}{2}t = \eta\left(\frac{a}{2\eta}\right) + \theta\left(\frac{bt}{2\theta}\right) \geq \left(\frac{a}{2\eta}\right)^\eta\left(\frac{b}{2\theta}\right)^\theta t^\theta,$$

and " = " holds if and only if

$$\frac{a}{2\eta} = \frac{bt}{2\theta}, \quad \text{i.e. } t \triangleq t_0 = \frac{a\theta}{b\eta} = \frac{a(Np-N-\alpha-2)}{b(N+\alpha+4-Np)},$$

we obtain

$$f_c'(t) \geq \frac{(Np-N-\alpha)\big(\overline{c}_{**}^{N+\alpha-(N-2)p} - c^{N+\alpha-(N-2)p}\big)}{4\|Q_p\|_2^{2p-2}}t^{(Np-N-\alpha-2)/2},$$

where $\overline{c}_{**}$ is given by (1.13). If $c \in (0, \overline{c}_{**})$ then $f_c'(t) > 0$ for all $t > 0$; if $c = \overline{c}_{**}$ then $f_c'(t_0) = 0$ and $f_c'(t) > 0$ for all $t > 0$ and $t \neq t_0$; if $c > c_{**}$ then $f_c'(t_0) < 0$ and there exist two positive constants t_1, t_2: $0 < t_1 < t_0 < t_2$ such that $f_c(t_1) > 0$, $f_c(t_1) > f_c(t_2)$, $f_c'(t_1) = f_c'(t_2) = 0$, $f_c''(t_1) < 0$ and $f_c''(t_2) > 0$ since $f_c'(0) = a/2 > 0$ and $f_c'(t) \to +\infty$ as $t \to +\infty$. By standard arguments we can show that the local maximum point t_1 is unique and the local minimum point t_2 is unique. □

REMARK 2.4. By Remark 1.1, we see that $\overline{c}_{**} < c_{**}$. Particularly, if $c > c_{**}$, then by Lemma 2.1, we have $f_c(t_2) = \min_{t>0} f_c(t) < 0$, where t_2 is given in Lemma 2.3.

LEMMA 2.5. *Assume that* $(N+\alpha+4)/N \leq p < (N+\alpha)/(N-2)$ $(0 < N-\alpha < 4)$, *then there is a unique* $t_3 \in (0, +\infty)$ *such that* $f_c'(t_3) = 0$ *and* $f_c''(t_3) < 0$ *if and only if* $c > c_*$, *where* c_* *is given in* (1.11).

PROOF. In the case $Np = N + \alpha + 4$, then

$$f_c(t) = \frac{a}{2}t + \left(\frac{b}{4} - \frac{c^{2(\alpha+4-N)/N}}{2\|Q_{(N+\alpha+4)/N}\|_2^{2(\alpha+4)/N}}\right)t^2,$$

$$f_c'(t) = \frac{a}{2} + \frac{c_*^{2(\alpha+4-N)/N} - c^{2(\alpha+4-N)/N}}{\|Q_{(N+\alpha+4)/N}\|_2^{2(\alpha+4)/N}}t.$$

Therefore, if $c \in (0, c_*]$, then $f_c'(t) > 0$ for all $t > 0$; if $c > c_*$, then there exists a unique $t_3 > 0$ such that $f_c'(t_3) = 0$ and $f_c''(t_3) < 0$.

In the case $(N+\alpha+4)/N < p < (N+\alpha)/(N-2)$ and $c > c_*$, then $f_c(t) > 0$ for $t > 0$ small and $f_c(t) \to -\infty$ as $t \to +\infty$ and by a standard arguments, we can show that there is a unique $t_3 > 0$ such that $f_c'(t_3) = 0$, $f_c(t_3) = \max_{t>0} f_c(t)$ and $f_c''(t_3) < 0$. □

The following lemma reveals the relationship between the critical point of E constrained on S_c and the critical point of f_c, which plays a crucial role in our proofs.

LEMMA 2.6. *Assume that $c > 0$ and $t > 0$ such that $f_c'(t) = 0$, then*

$$Q_{\lambda_t} = \frac{c\lambda_t^{N/2}}{\|Q_p\|_2} Q_p(\lambda_t x) \in S_c, \quad \mu_c = -\frac{\big(N+\alpha-(N-2)\big)c^{2p-2}\lambda_t^{Np-N-\alpha}}{2\|Q_p\|_2^{2p-2}}$$

satisfy (1.1) *and $E(Q_{\lambda_t}) = f_c(t)$, which means that $(\mu_c, Q_{\lambda_t}) \in \mathbb{R} \times H^1(\mathbb{R}^N)$ is a is a normalized solution of* (1.1).

PROOF. Since $f_c'(t) = 0$, we have

$$\frac{a}{2} + \frac{b}{2}t - \frac{(NP-N-\alpha)c^{N+\alpha-(N-2)p}}{4\|Q_p\|_2^{2p-2}} t^{(Np-N-\alpha-2)/2} = 0. \tag{2.6}$$

Set $\lambda_t = t^{1/2}/c$, then $\lambda_t^2 c^2 = t$,

$$A(Q_{\lambda_t}) = \lambda_t^2 c^2, \quad B(Q_{\lambda_t}) = \frac{pc^{N+\alpha-(N-2)p}}{\|Q_p\|_2^{2p-2}} (\lambda_t c)^{Np-N-\alpha} \tag{2.7}$$

and it can be deduced from (2.6) that

$$a + bA(Q_{\lambda_t}) = (a + bc^2\lambda_t^2) = \frac{(Np-N-\alpha)c^{2p-2}}{2\|Q_p\|_2^{2p-2}}\,\lambda_t^{Np-N-\alpha-2},$$

thus we have

$$-\left(a + b\int_{\mathbb{R}^N} |\nabla Q_{\lambda_t}|\,dx\right)\Delta Q_{\lambda_t} = -\frac{(Np-N-\alpha)c^{2p-2}}{2\|Q_p\|_2^{2p-2}}\lambda_t^{Np-N-\alpha-2}\Delta Q_{\lambda_t}$$

$$
\begin{aligned}
&= \frac{c^{2p-1}}{\|Q_p\|_2^{2p-1}} \lambda_t^{Np-(N/2)-\alpha} \\
&\quad \cdot \Bigg(\bigg(\frac{\|Q_p\|_2}{c\lambda_t^{N/2}} \bigg)^{2p-1} \lambda_t^{\alpha} (I_\alpha * |Q_{\lambda_t}(x)|^p) |Q_{\lambda_t}(x)|^{p-2} Q_{\lambda_t}(x) \\
&\quad - \frac{\|Q_p\|_2}{c\lambda_t^{N/2}} \frac{N+\alpha-(N-2)p}{2} Q_{\lambda_t}(x) \Bigg) \\
&= (I_\alpha * |Q_{\lambda_t}(x)|^p) |Q_{\lambda_t}(x)|^{p-2} Q_{\lambda_t}(x) + \mu_c Q_{\lambda_t}(x),
\end{aligned}
$$

where

$$\mu_c = -\frac{c^{2p-2}\lambda_t^{Np-N-\alpha}}{\|Q_p\|_2^{2p-2}} \frac{N+\alpha-(N-2)p}{2},$$

therefore we see that $Q_{\lambda_t}(x)$ satisfies the equation

$$-\bigg(a + b\int_{\mathbb{R}^N} |\nabla u|^2\,dx\bigg)\Delta u - (I_\alpha * |u|^p)|u|^{p-2}u = \mu_c u$$

and $Q_{\lambda_t} \in S_c$, i.e. (μ_c, Q_{λ_t}) is a normalized solution of (1.1). By (1.8), (1.9) and (2.2), we can obtain that $E(Q_{\lambda_t}) = f_c(t)$. □

3. Existence of constrained critical points

Our goal in this section is to prove Theorems 1.2 and 1.4.

PROOF OF THEOREM 1.2. (1) If $(N+\alpha)/N < p < (N+\alpha+2)/N$, then $0 < Np - N - \alpha < 2$, it can be deduced from (2.2) that for each $c > 0$, $f_c(t)$ has a unique global minimizer, denote by t_c, thus by (2.2), we have

$$E_{c^2} = \inf_{u\in S_c} E(u) \geq f_c(t_c). \tag{3.1}$$

Let Q_p be a positive ground state solution of (1.7), we set

$$Q_\lambda(x) = c\lambda^{N/2} Q_p(\lambda x)/\|Q_p\|_2,$$

then $Q_\lambda(x) \in S_c$, by (1.8), (1.9) and (2.7), we obtain

$$E(Q_\lambda) = \frac{a}{2}\, c^2\lambda^2 + \frac{b}{4}\big(c^2\lambda^2\big)^2 - \frac{c^{N+\alpha-(N-2)p}}{2\|Q_p\|_2^{2p-2}}\big(c^2\lambda^2\big)^{(Np-N-\alpha)/2} \triangleq f_c(c^2\lambda^2).$$

We set $\lambda_c = t_c^{1/2}/c$, then $c^2\lambda_c^2 = t_c$ and $E_{c^2} = \inf\limits_{u\in S_c} E(u) \leq E(Q_{\lambda_c}) = f_c(t_c)$. This, together with (3.1), implies that

$$E_{c^2} = \inf_{u\in S_c} E(u) = f_c(t_c) = \min_{t>0} f_c(t),$$

which means that $Q_{\lambda_c}(x)$ is a minimizer of problem (1.6)

If $Np = N + \alpha + 2$, then

$$f_c(t) = \bigg(\frac{a}{2} - \frac{c^{2p-2}}{2\|Q_p\|_2^{2p-2}}\bigg)t + \frac{b}{4}\,t^2.$$

It is easy to see that if $c \le a^{1/(2p-2)}\|Q_p\|_2$, then $f_c(t) > 0$ for all $t > 0$, thus we have $E(u) \ge f_c(A(u)) > 0$ for all $u \in S_c$. It follows from (2.3) that $E_{c^2} \le 0$ for all $c > 0$, therefore (1.6) has no minimizer for $Np = N + \alpha + 2$ and $c \in (0, a^{1/(2p-2)}\|Q_p\|_2]$. If $c > a^{1/(2p-2)}\|Q_p\|_2$, then $f_c(t)$ has a unique minimizer

$$t_c = \frac{1}{b}\left(\frac{c^{2p-2}}{\|Q_p\|_2^{2p-2}} - a\right).$$

Set $\lambda_c = t_c^{1/2}/c$, then we have

$$E(Q_{\lambda_c}) = f_c(t_c) \le E_{c^2} = \inf_{u \in S_c} E(u) \le E(Q_{\lambda_c})$$

and

$$E_{c^2} = E(Q_{\lambda_c}) = f_c(t_c) = -\frac{\left(a\|Q_p\|_2^{2p-2} - c^{2p-2}\right)^2}{4b\|Q_p\|_2^{2p-2}}.$$

(2) For $(N+\alpha+2)/N < p < \min\{(N+\alpha+4)/N, (N+\alpha)/(N-2)\}$, by Lemma 2.1, we have

$$f_c(t) \ge \frac{c_{**}^{N+\alpha-(N-2)p} - c^{N+\alpha-(N-2)p}}{2\|Q_p\|_2^{2p-2}}\, t^{(Np-N-\alpha)/2},$$

and " $=$ " holds if and only if $t_4 = 2a(Np - N - \alpha - 2)/(b(N + \alpha + 4 - Np))$, where c_{**} is defined by (1.12). Clearly, if $c \in (0, c_{**})$ then $f_c(t) > 0$ for all $t > 0$ and problem (1.6) has no minimizer; if $c = c_{**}$ then $f_c(t_4) = 0$ and $f_c(t) > 0$ for all $t > 0$ and $t \ne t_4$, therefore

$$0 = E(Q_{\lambda_{t_4}}) = f_c(t_4) \le E_{c^2} = \inf_{u \in S_c} E(u) \le E(Q_{\lambda_{t_4}}) = f_c(t_4) = 0,$$

which means that $Q_{\lambda_{t_4}}$ is a minimizer of Problem (1.6), where

$$\lambda_{t_4} = \frac{t_4^{1/2}}{c_{**}}, \qquad Q_{\lambda_{t_4}} = \frac{c\lambda_{t_4}^{N/2}}{\|Q_p\|_2} Q_p(\lambda_{t_4} x). \tag{3.2}$$

If $c > c_{**}$, then it follows from Remark 2.4 that

$$E(Q_{\lambda_{t_2}}) = f_c(t_2) = \min_{t>0} f_c(t) \le E_{c^2} = \inf_{u \in S_c} E(u) \le E(Q_{\lambda_{t_2}}) = f_c(t_2),$$

then problem (1.6) has a minimizer $Q_{\lambda_{t_2}}$ of the form (3.2). □

PROOF OF THEOREM 1.4. We denote $u_{c,i}$ and $\mu_{c,i}$ $(i = 0,1,2,3)$ as

$$\begin{aligned} u_{c,i}(x) &= \frac{c\lambda_{c,t_i}^{N/2}}{\|Q_p\|_2} Q_p(\lambda_{c,t_i} x), \\ \mu_{c,i} &= -\frac{c^{2p-2}\lambda_{t_i}^{Np-N-\alpha}}{\|Q_p\|_2^{2p-2}} \frac{N+\alpha-(N-2)p}{2}, \end{aligned} \tag{3.3}$$

then we have

$$\lambda_{c,t_i} = \frac{t_i^{1/2}}{c}, \quad \|u_{c,i}(x)\|_2^2 = c^2, \quad A(u_{c,i}(x)) = c^2\lambda_{c,t_i}^2 = t_i, \tag{3.4}$$

where t_i $(i = 0, 1, 2, 3)$ are given in Lemmas 2.3 and 2.5, respectively.

(1) Assume that $(N+2+\alpha)/N < p < \min\{(N+4+\alpha)/N, (N+\alpha)/(N-2)\}$.

(a) Arguing by contradiction, if $(\mu_c, u_c) \in \mathbb{R} \times S_c$ is a normalized solution of (1.1) for some $c \in (0, \overline{c}_{**})$, then we have

$$aA(u_c) + b\big(A(u_c)\big)^2 - B(u_c) = \mu_c \|u_c\|_2^2 \tag{3.5}$$

and it follows from [6] that the solution of (1.1) satisfies the following Pohozaev identity:

$$\frac{(N-2)}{2}\big(a + bA(u_c)\big)A(u_c) - \frac{N\mu_c}{2}\|u_c\|_2^2 = \frac{N+\alpha}{2p}\, B(u_c), \tag{3.6}$$

thus we get

$$0 = aA(u_c) + b(A(u_c))^2 - \frac{Np-N-\alpha}{2p}\, B(u_c). \tag{3.7}$$

By (2.1), (3.7) and Lemma 2.3, we obtain

$$\begin{aligned} 0 &\geq aA(u_c) + b(A(u_c))^2 - \frac{(Np-N-\alpha)c^{N+\alpha-(N-2)p}}{2\|Q_p\|_2^{2p-2}}(A(u_c))^{(Np-N-\alpha)/2} \\ &\geq \frac{(Np-N-\alpha)\big(\overline{c}_{**}^{N+\alpha-(N-2)p} - c^{N+\alpha-(N-2)p}\big)}{2\|Q_p\|_2^{2p-2}}(A(u_c))^{(Np-N-\alpha)/2} > 0, \end{aligned}$$

this is a contradiction. Therefore, if $c \in (0, \overline{c}_{**})$, then (1.1) does not admit normalized solutions on the constraint S_c.

(b) It follows from Lemmas 2.3 and 2.6 that if $c = \overline{c}_{**}$, then (1.1) has a normalized solution $(\mu_{c,0}, u_{c,0}) \in \mathbb{R} \times H^1(\mathbb{R}^N)$ with $E(u_{c,0}) > 0$,

(c) It follows from Lemmas 2.3 and 2.6 that if $c > \overline{c}_{**}$, then (1.1) has two normalized solutions $(\mu_{c,1}, u_{c,1})$, $(\mu_{c,2}, u_{c,2}) \in \mathbb{R} \times H^1(\mathbb{R}^N)$ with $E(u_{c,1}) > 0$ and $E(u_{c,1}) > E(u_{c,2})$.

Now we first show that $(\mu_{c,1}, u_{c,1}) \in \mathbb{R} \times H^1(\mathbb{R}^N)$ is a mountain pass type normalized solution of (1.1). Set $K(c) = t_1/2$ and $e = u_{c,2}$, then $A(u_{c,2}) = c^2\lambda_{c,t_2}^2 = t_2$ and for any $h \in \Gamma(c)$, by (2.2), we have $E(h(s)) \geq f_c(A(h(s)))$. Since $A(h(0)) \leq K(c)$, we obtain that

$$\max_{s\in[0,1]} E(h(s)) \geq \max_{t\in[t_1/2,t_2]} f_c(t) = f_c(t_1) = E(u_{c,1}).$$

Therefore it holds

$$m(c) \geq f_c(t_1) = E(u_{c,1}), \tag{3.8}$$

where $m(c)$ is defined in the Definition 1.3. On the other hand, set

$$t(s) = \frac{2t_2 - t_1}{2}\, s + \frac{t_1}{2}, \qquad h(s) = \frac{c^{(2-N)/2}}{t}(s)^{N/4}\|Q_p\|_2 Q_p\bigg(\frac{t(s)^{1/2}}{c}x\bigg),$$

then

$$t(0)=\frac{t_1}{2},\quad t(1)=t_2,\quad t\bigg(\frac{t_1}{2t_2-t_1}\bigg)=t_1,\quad \|h(s)\|_2^2=c^2,\quad A(h(s))=t(s),$$

thus we see that $h(s)\in\Gamma(c)$ and

$$m(c)\le\max_{s\in[0,1]}E(h(s))=\max_{s\in[0,1]}f_c(t(s))=f_c(t_1)=E(u_{c,1}).$$

Combining with (3.8), we obtain that $m(c)=f_c(t_1)=E(u_{c,1})$. Therefore $(\mu_{c,1},u_{c,1})\in\mathbb{R}\times H^1(\mathbb{R}^N)$ is a mountain pass type normalized solution of (1.1).

Now we show that $u_{c,2}$ is a local minimizer. We define

$$\widetilde{E}_{c^2}=\inf\big\{E(u)\mid u\in S_c,\ \|\nabla u\|_2^2>t_1\big\}. \tag{3.9}$$

It follows from (2.2), (3.4) and Lemma 2.3 that

$$\begin{aligned}E(u_{c,2})&\ge\widetilde{E}_{c^2}\ge\inf\big\{f_c\big(\|\nabla u\|_2^2\big)\mid u\in S_c,\ \|\nabla u\|_2^2>t_1\big\}\\&\ge\inf\big\{f_c(t)\mid t>t_1\big\}\ge f_c(t_2)=E(u_{c,2}),\end{aligned}$$

hence we obtain $\widetilde{E}_{c^2}=E(u_{c,2})$, which means that $u_{c,2}$ is a local minimizer of E on S_c.

(2) If $(N+4+\alpha)/N\le p<(N+\alpha)/(N-2)$ (i.e. $N<\alpha+4$), then it can be deduced from Lemmas 2.5 and 2.6 that (1.1) has a normalized solution $(\mu_{c,3},u_{c,3})\in\mathbb{R}\times H^1(\mathbb{R}^N)$ with $E(u_{c,3})>0$ if and only if $c>c_*$. We can prove that $(\mu_{c,3},u_{c,3})\in\mathbb{R}\times H^1(\mathbb{R}^N)$ is a mountain pass type normalized solution of (1.1) in the same way as above, here we omit it. □

REMARK 3.1. It follows from Remark 2.4 and the definition of $\widetilde{E}_{c^2}$ that if $c>c_{**}$, then $E_{c^2}=\widetilde{E}_{c^2}$, which means that $u_{c,2}$ is also a global minimizer of E on S_c.

4. Asymptotic behavior

In this section, we consider the asymptotic behaviors of the Lagrange multiplier μ_c and the energy $E(u_c)$ as $c\to0$ and $c\to+\infty$. Let (μ_c,u_c) be the normalized solution of (1.1), where $u_c=c\lambda_c^{N/2}Q_p(\lambda_c x)/\|Q_p\|_2$, μ_c is the corresponding Lagrange multiplier, then we have

$$\begin{aligned}E_{c^2}=E(u_c)=&\frac{a}{2}A(u_c)+\frac{b}{4}(A(u_c))^2-\frac{1}{2p}B(u_c)\\=&\frac{a}{2}c^2\lambda_c^2+\frac{b}{4}\big(c^2\lambda_c^2\big)^2-\frac{c^{N+\alpha-(N-2)p}}{2\|Q_p\|_2^{2p-2}}\big(c^2\lambda_c^2\big)^{(Np-N-\alpha)/2}.\end{aligned}$$

By (3.5), we see that

$$ac^2\lambda_c^2+b\big(c^2\lambda_c^2\big)^2-\frac{pc^{N+\alpha-(N-2)p}}{\|Q_p\|_2^{2p-2}}\big(c^2\lambda_c^2\big)^{(Np-N-\alpha)/2}=\mu_c c^2, \tag{4.1}$$

which, together with (3.6), implies that

$$\begin{aligned}0 &= aA(u_c) + b(A(u_c))^2 - \frac{Np-N-\alpha}{2p} B(u_c) \\ &= ac^2\lambda_c^2 + b\big(c^2\lambda_c^2\big)^2 - \frac{(Np-N-\alpha)c^{2p}}{2\|Q_p\|_2^{2p-2}} \lambda_c^{Np-N-\alpha},\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}E(u_c) &= \frac{a(Np-N-\alpha-2)}{2(Np-N-\alpha)} A(u_c) + \frac{b(Np-N-\alpha-4)}{4(Np-N-\alpha)} (A(u_c))^2 \\ &= \frac{a(Np-N-\alpha-2)}{2(Np-N-\alpha)} c^2\lambda_c^2 + \frac{b(Np-N-\alpha-4)}{4(Np-N-\alpha)} \big(c^2\lambda_c^2\big)^2,\end{aligned} \tag{4.3}$$

therefore we have

$$\frac{E(u_c)}{c^2} = \frac{a(Np-N-\alpha-2)}{2(Np-N-\alpha)} \lambda_c^2 + \frac{b(Np-N-\alpha-4)}{4(Np-N-\alpha)} c^2\lambda_c^4. \tag{4.4}$$

For $c > 0$ and $\lambda > 0$, we set

$$F(\lambda, c) = a\lambda^{N+\alpha+2-Np} + bc^2\lambda^{N+\alpha+4-Np} - \frac{Np-N-\alpha}{2\|Q_p\|_2^{2p-2}} c^{2p-2}, \tag{4.5}$$

by the definition of u_c and (4.2), we obtain that $F(\lambda_c, c)c^2\lambda_c^{Np-N-\alpha} = 0$, which implies that $F(\lambda_c, c) = 0$. Moreover, we have

$$\begin{aligned}F_c(\lambda, c) &:= \frac{\partial F(\lambda, c)}{\partial c} = 2bc\lambda^{N+\alpha+4-Np} - \frac{(p-1)(Np-N-\alpha)}{\|Q_p\|_2^{2p-2}} c^{2p-3}, \\ F_\lambda(\lambda, c) &:= \frac{\partial F(\lambda, c)}{\partial \lambda} = a(N+\alpha+2-Np)\lambda^{N+\alpha+1-Np} \\ &\qquad + b(N+\alpha+4-Np)c^2\lambda^{N+\alpha+3-Np}.\end{aligned}$$

PROOF OF THEOREM 1.6. (1) If $(N+\alpha)/N < p < (N+\alpha+2)/N$, then $E(u_c) = E_{c^2}$ and $F_\lambda(\lambda_c, c) > 0$ for all $c > 0$. It follows from the implicit function theorem that for all $c > 0$, there exists a unique continuous function $\lambda(\,\cdot\,)\colon \mathbb{R} \to \mathbb{R}^+$, hereinafter we denote $\lambda(c)$ by λ_c, such that $F(\lambda_c, c) = 0$ and

$$\begin{aligned}\frac{d\lambda_c}{dc} &= -\frac{F_c(\lambda_c, c)}{F_\lambda(\lambda_c, c)} \\ &= \frac{\dfrac{\lambda_c^{N+\alpha+2-Np}}{c}\bigg(\dfrac{(p-1)(Np-N-\alpha)}{\|Q_p\|_2^{2p-2}} \dfrac{c^{2p-2}}{\lambda_c^{N+\alpha+2-Np}} - 2bc^2\lambda_c^2\bigg)}{a(N+\alpha+2-Np)\lambda_c^{N+\alpha+1-Np} + b(N+\alpha+4-Np)c^2\lambda_c^{N+\alpha+3-Np}}.\end{aligned} \tag{4.6}$$

We first claim that $\lambda_c \to 0$ as $c \to 0$. Otherwise, we assume that there exists $M > 0$ such that $\lambda_c \geq M$ for all $c > 0$ small, then it follows from (4.2) that, as $c \to 0$,

$$a \leq a + bc^2\lambda_c^2 = \frac{Np-N-\alpha}{2\|Q_p\|_2^{2p-2}} \frac{c^{2p-2}}{\lambda_c^{N+\alpha+2-Np}} \leq \frac{Np-N-\alpha}{2\|Q_p\|_2^{2p-2}} \frac{c^{2p-2}}{M^{N+\alpha+2-Np}} \to 0,$$

which contradicts $a > 0$, and our claim holds. By (4.2), (4.5) and $F(\lambda_c, c) = 0$, we obtain that, as $c \to 0$,

$$\frac{c^{2p-2}}{\lambda_c^{N+\alpha+2-Np}} \to \frac{2\|Q_p\|_2^{2p-2}}{Np-N-\alpha} a > 0.$$

This, together with (4.6), implies that there exists $c_0 > 0$ such that for each $c \in (0, c_0)$, $d\lambda_c/dc > 0$, that is, λ_c is monotone increasing for $c \in (0, c_0)$. Moreover, we can deduce from (4.4) that $E_{c^2}/c^2 = E(u_c)/c^2 \to 0$ as $c \to 0$ and E_{c^2}/c^2 is monotone decreasing for $c \in (0, c_0)$.

By (3.5), (4.1) and (4.2), we have

$$\begin{aligned} b\mu_c c^2 &= \left(1 - \frac{2p}{Np-N-\alpha}\right)\left(aA(u_c) + b(A(u_c))^2\right) \\ &= \frac{(N-2)p-N-\alpha}{Np-N-\alpha}\left(ac^2\lambda_c^2 + bc^4\lambda_c^4\right), \end{aligned}$$

which means that $\mu_c < 0$ and as $c \to 0$,

$$\begin{aligned} \mu_c &= \frac{(N-2)p-N-\alpha}{Np-N-\alpha}\,(a\lambda_c^2 + bc^2\lambda_c^4) \to 0, \\ \frac{\mu_c}{\lambda_c^2} &= \frac{(N-2)p-N-\alpha}{Np-N-\alpha}\,(a + bc^2\lambda_c^2) \to \frac{(N-2)p-N-\alpha}{Np-N-\alpha}\,a. \end{aligned}$$

(2) If $(N+\alpha+4)/N < p < (N+\alpha)/(N-2)$ with $N-\alpha < 4$, then $F_\lambda(\lambda_c, c) < 0$. It follows from the implicit function theorem that for all $c > 0$, there exists a unique continuous function $\lambda_c > 0$ such that $F(\lambda_c, c) = 0$ and

$$\begin{aligned} \frac{d\lambda_c}{dc} &= -\frac{F_c(\lambda_c, c)}{F_\lambda(\lambda_c, c)} \\ &= \frac{\left(c\lambda_c^{N+\alpha+4-Np}\right)\left(\dfrac{(p-1)(Np-N-\alpha)}{\|Q_p\|_2^{2p-2}}\dfrac{c^{2p-4}}{\lambda_c^{N+\alpha+4-Np}} - 2b\right)}{a(N+\alpha+2-Np)\lambda^{N+\alpha+1-Np} + b(N+\alpha+4-Np)c^2\lambda_c^{N+\alpha+3-Np}}. \end{aligned} \tag{4.7}$$

We claim that $\lambda_c c \to +\infty$ as $c \to 0$. Otherwise, we assume that there exists $M > 0$ such that $\lambda_c c \in (0, M]$ for all $c > 0$ small, then it follows from (4.2) that, as $c \to 0$,

$$b \le \frac{a}{c^2\lambda_c^2} + b = \frac{Np-N-\alpha}{2\|Q_p\|_2^{2p-2}}\frac{c^{2p-4}}{\lambda_c^{N+\alpha+4-Np}} \le \frac{Np-N-\alpha}{2\|Q_p\|_2^{2p-2}}\frac{c^{N+\alpha-(N-2)p}}{M^{N+\alpha+4-Np}} \to 0$$

since $N+\alpha-(N-2)p > 0$ and $Np-N-\alpha-4 > 0$, this contradicts $b > 0$, and our claim holds, and we also obtain that $\lambda_c \to +\infty$ as $c \to 0$. By (4.2), (4.5) and $F(\lambda_c, c) = 0$, we obtain that, as $c \to 0$,

$$\frac{c^{2p-4}}{\lambda_c^{N+\alpha+4-Np}} \to \frac{2\|Q_p\|_2^{2p-2}}{Np-N-\alpha}\, b > 0.$$

This, together with (4.7), implies that there exists $c_1 > 0$ such that for each $c \in (0, c_1)$, $(d\lambda_c)/dc < 0$, that is, λ_c is monotone decreasing for $c \in (0, c_1)$. Moreover, we can deduce from (4.3) that $E(u_c) \to +\infty$ as $c \to 0$.

As in the above case (1), we have

$$\mu_c c^2 = \frac{(N-2)p - N - \alpha}{Np - N - \alpha}(ac^2\lambda_c^2 + bc^4\lambda_c^4),$$

which means that $\mu_c < 0$ and as $c \to 0$,

$$\frac{\mu_c}{c^2\lambda_c^4} = \frac{(N-2)p - N - \alpha}{Np - N - \alpha}\left(\frac{a}{c^2\lambda_c^2} + b\right) \to \frac{(N-2)p - N - \alpha}{Np - N - \alpha}\, b.$$

This completes the proof of Theorem 1.6. □

To prove Theorem 1.7, we define

$$G(t, c) = a + bt - \frac{(Np - N - \alpha)c^{N+\alpha-(N-2)p}}{2\|Q_p\|_2^{2p-2}}\, t^{(Np-N-\alpha-2)/2}.$$

Set $t_c = c^2\lambda_c^2$, then it follows from (4.2) that $G(t_c, c) = f_c'(t_c) = 0$. Moreover, we have

$$\begin{aligned} G_t(t, c) &:= \frac{\partial G(t, c)}{\partial t} \\ &= b - \frac{(Np - N - \alpha)c^{N+\alpha-(N-2)p}}{2\|Q_p\|_2^{2p-2}}\,\frac{Np - N - \alpha - 2}{2}\, t^{(Np-N-\alpha-4)/2}, \\ G_c(t, c) &:= \frac{\partial G(t, c)}{\partial c} \\ &= -\frac{(Np - N - \alpha)(N + \alpha - (N-2)p)}{2\|Q_p\|_2^{2p-2}}\, c^{N+\alpha-(N-2)p-1} t^{(Np-N-\alpha-2)/2}. \end{aligned}$$

Proof of Theorem 1.7. (1) If $(N+\alpha)/N < p < \min\{(N+4+\alpha)/N, (N+\alpha)/(N-2)\}$ and (u_c, μ_c) is obtained by Theorem 1.2 with $c > c_*$, then it follows from the proof of Theorem 1.2 that $t_c = c^2\lambda_c^2$ is the unique global minimizer of $f_c(t)$, that is, $f_c'(t_c) = 0$ and $f_c''(t_c) = G_t(t_c, c) > 0$. By the implicit function theorem, we obtain that there exists a unique positive function $t(\,\cdot\,)\colon (c_{**}, +\infty) \to \mathbb{R}^+$, hereinafter we denote $t(c)$ by t_c, such that

$$\frac{dt_c}{dc} = -\frac{G_c(t_c, c)}{G_t(t_c, c)} > 0,$$

i.e. t_c is monotone increasing for all $c > c_*$.

We first claim that $t_c \to +\infty$ as $c \to +\infty$. Otherwise, we assume that there exists $M > 0$ such that $t_c \le M$ for all $c > c_*$ and $t_c \to M$ as $c \to +\infty$, then it follows from (3.7) that

$$a + bM \ge a + bt_c = \frac{Np - N - \alpha}{2\|Q_p\|_2^{2p-2}} c^{N+\alpha-(N-2)p} t_c^{(Np-N-\alpha-2)/2} \to +\infty,$$

as $c \to +\infty$. This is a contradiction and our claim holds. Therefore we have $c\lambda_c \to +\infty$ as $c \to +\infty$. Similar to the proof of Theorem 1.6, we obtain that

$$
\begin{aligned}
&E(u_c) \to -\infty, \\
&\frac{c^{2p-4}}{\lambda_c^{N+4+\alpha-Np}} \to \frac{2\|Q_p\|_2^{2p-2}}{Np-N-\alpha}\, b, \\
&\frac{\mu_c^2}{c^2\lambda_c^4} \to \frac{(N-2)p-N-\alpha}{Np-N-\alpha}\, b
\end{aligned} \tag{4.8}
$$

as $c \to +\infty$ since $G(t_c, c) = G(c^2\lambda_c^2, c) = 0$.

If $(N+\alpha)/N < p < \min\{2, (N+2+\alpha)/N, (N+\alpha)/(N-2)\}$, then $2p-4 < 0$ and $N + \alpha + 4 - NP > 0$, and by (4.8), we see that $\lambda_c \to 0$ as $c \to +\infty$.

For $\alpha + 4 > N$, if $2 < p < \min\{(N + 4 + \alpha)/N, (N + \alpha)/(N - 2)\}$, then $2p-4 > 0$ and $N+\alpha+4-Np > 0$, by (4.8), we see that $\lambda_c \to +\infty$ as $c \to +\infty$.

(2) If (u_c, μ_c) is obtained by Theorem 1.4 with $c > c_{**}$ and $E(u_c) > 0$, then it follows from the proof of Theorem 1.4 that $t_c = c^2\lambda_c^2$ is the unique local maximizer of $f_c(t)$, that is, $f_c'(t_c) = 0$ and $f_c''(t_c) = G_t(t_c, c) < 0$. By the implicit function theorem, we obtain that there exists a unique positive function t_c such that

$$\frac{dt_c}{dc} = -\frac{G_c(t_c, c)}{G_t(t_c, c)} < 0,$$

i.e. t_c is monotone decreasing for all $c > c_*$.

We claim that $t_c \to 0$ as $c \to +\infty$. Otherwise, we assume that there exists $M > 0$ such that $t_c \geq M$ for all $c > c_*$ and $t_c \to M$ as $c \to +\infty$, then it follows from (4.2) that,

$$a + bM \leftarrow a + bt_c = \frac{Np-N-\alpha}{2\|Q_p\|_2^{2p-2}}\, c^{N+\alpha-(N-2)p}\, t_c^{(Np-N-\alpha-2)/2} \to +\infty,$$

as $c \to +\infty$. This is a contradiction and our claim holds. Therefore we have $c\lambda_c \to 0$ and $\lambda_c \to 0$ as $c \to +\infty$. Similar to the proof of Theorem 1.6, we obtain that

$$E(u_c) \to 0, \qquad \frac{c^{2p-2}}{\lambda_c^{N+2+\alpha-Np}} \to \frac{2\|Q_p\|_2^{2p-2}}{Np-N-\alpha} a, \qquad \frac{\mu_c}{\lambda_c^2} \to \frac{(N-2)p-N-\alpha}{Np-N-\alpha}\, a$$

as $c \to +\infty$ since $G(t_c, c) = G(c^2\lambda_c^2, c) = 0$. □

5. Relationship between global minimizers on the constraint and ground states

In this section, we describe a relationship between the ground states of (1.1) and the global minimizers of E_{c^2} when $N = 3$ or $N = 4$, $\alpha \in (0, N)$ and $p \in ((N+\alpha)/N, \min\{(N+4+\alpha)/N, (N+\alpha)/(N-2)\})$. It follows from Theorem 1.2 that there exist a global minimizer u_c of E_{c^2} and a Lagrange multiplier $\mu_c < 0$ such that (μ_c, u_c) is a solution of (1.1).

PROOF OF THEOREM 1.8. For each $v \in \mathcal{A}_{\mu_c}$, then v satisfies the Pohozaev identity (3.6). Set

$$\varphi_t(x) = v\left(\frac{x}{t}\right) \quad \text{for } t > 0,$$

then $\|\varphi_t\|_2^2 = t^N \|v\|_2^2$ and, by (3.6),

$$\begin{aligned} I(\varphi_t(x)) &= \frac{at^{N-2}}{2} A(v) + \frac{bt^{2(N-2)}}{4} A^2(v) - \frac{\mu_c t^N}{2} \|v\|_2^2 - \frac{t^{N+\alpha}}{2p} B(v) \\ &= \left(t^{N-2} - \frac{N-2}{N} t^N\right) \frac{a}{2} A(v) \\ &\quad + \left(\frac{t^{2N-4}}{2} - \frac{N-2}{N} t^N\right) \frac{b}{2} A^2(v) + \left(\frac{N+\alpha}{Np} t^N - \frac{t^{N+\alpha}}{p}\right) \frac{1}{2} B(v). \end{aligned}$$

Thus

$$\begin{aligned} \frac{d}{t} I(\varphi_t(x)) &= \big(1 - t^2\big) \frac{a(N-2)t^{N-3}}{2} A(v) \\ &\quad + \big(1 - t^{4-N}\big) \frac{b(N-2)t^{N-5}}{2} A^2(v) + \big(1 - t^{\alpha}\big) \frac{(N+\alpha)t^{N-1}}{2p} B(v). \end{aligned}$$

This implies that for $N = 3$ or $N = 4$, $I(\varphi_t(x)) \leq I(v)$ for all $t > 0$. There exists $t_0 > 0$ such that $\varphi_{t_0} \in S_c$. Therefore we have $I(u_c) \leq I(\varphi_{t_0}(x)) \leq I(v)$, which means that u_c is a ground state of (1.1) with $\mu = \mu_c < 0$. □

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