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A PERIODIC BIFURCATION PROBLEM DEPENDING ON A RANDOM VARIABLE

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Dedicated to the memory of Professor Ioan I. Vrabie

Abstract. We consider an abstract bifurcation equation $P(x) + \varepsilon Q(x,\varepsilon,\omega) = 0$, where P and Q are operators, ε is the bifurcation parameter, $\omega \in \Omega$, is the random variable and $(\Omega, \mathcal{F})$ is a measurable space. The aim of the paper is to provide conditions on P and Q to ensure the existence, for any $\omega \in \Omega$, of a branch of solutions originating from the zeros of the operator P. We show that the considered abstract bifurcation is the model of a random autonomous periodically perturbed differential equation having the property that the unperturbed equation corresponding to $\varepsilon = 0$ has a limit cycle. As a consequence we obtain the existence, for any $\omega \in \Omega$, of a branch of periodic solutions of the perturbed equation emanating from the limit cycle.

1. Introduction

In this paper we consider the bifurcation equation of the form

$$P(x) + \varepsilon Q(x,\varepsilon,\omega) = 0, \tag{1.1}$$

where $P\colon \mathbb{E} \mapsto \mathbb{E}$ and $Q\colon \mathbb{E}\times[0,1]\times\Omega \mapsto \mathbb{E}$ are operators, $\mathbb{E}$ is a separable Banach space, $\varepsilon \geq 0$ is the bifurcation parameter, $\omega \in \Omega$ is the random variable and

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$(\Omega, \mathcal{F})$ is a measurable space. We assume that $P(x) = x - F(x)$ where $F\colon \mathbb{E} \mapsto \mathbb{E}$ is a compact operator and $Q(\,\cdot\,,\,\cdot\,,\omega)$ is a compact operator on $\mathbb{E} \times [0,1]$ for any $\omega \in \Omega$. Assuming the existence of a parametrized smooth curve $\theta \mapsto x_0(\theta)$, $\theta \in [0,T]$, of zeros of the operator P, namely

$$P(x_0(\theta)) = 0,$$

for any $\theta \in [0,T]$, we look for conditions on P and Q ensuring the existence of a measurable function $\omega \mapsto \theta_\omega \in [0,T], \omega \in \Omega$, such that for each $\omega \in \Omega$ there exists a family of solutions $x(\varepsilon,\omega), \varepsilon \geq 0$ small, of (1.1) originating from the point $x_0(\theta_\omega) \in \{x_0(\theta), \theta \in [0,T]\}$. The abstract bifurcation equation (1.1) can be aimed to deal with the existence of branches of T-periodic solutions of a random autonomous periodically perturbed differential equation of the form

$$x'(t) = f(x(t)) + \varepsilon\varphi(t, x(t), \varepsilon, \omega) \tag{1.2}$$

where $\varepsilon \in [0,1]$, $f\colon \mathbb{R}^n \mapsto \mathbb{R}^n$ and $\varphi\colon \mathbb{R} \times \mathbb{R}^n \times [0,1] \times \Omega \mapsto \mathbb{R}^n$ is T-periodic. Indeed, for any $\omega \in \Omega$, the zeros $x(\varepsilon,\omega), \varepsilon \geq 0$ small, of (1.1) represent the fixed points v of the Poincaré operator $\mathcal{P}_\varepsilon(v,\omega) = x(T,v,\varepsilon,\omega), \varepsilon \geq 0$, associated to (1.2), where $x(T,v,\varepsilon,\omega)$ is the evaluation at $t = T$ of the solution $x(t,v,\varepsilon,\omega)$ of equation (1.2) such that $x(0,v,\varepsilon,\omega) = v$. To see this, define

$$P(v) = \mathcal{P}_0(v) - v \quad \text{and} \quad Q(v,\varepsilon,\omega) = \frac{\mathcal{P}_\varepsilon(v,\omega) - \mathcal{P}_0(v)}{\varepsilon},$$

hence, we can write $\mathcal{P}_\varepsilon(v,\omega) - v = P(v) + \varepsilon Q(v,\varepsilon,\omega)$ with singular $P'(x_0(\theta))$, for any $\theta \in [0,T]$, where x_0 is the limit cycle of the unperturbed autonomous equation $x'(t) = f(x(t))$. Thus, the existence, for any $\omega \in \Omega$, of a branch of solutions of (1.1) emanating from $x_0(\theta_\omega)$ is equivalent to the existence of a family of T-periodic solutions of (1.2) originating at $x_0(\theta_\omega)$.

The mathematical models, as (1.2), which describe processes of the real world are usually derived from experimental data, hence they describe real phenomena only approximately. In particular, this concerns real processes which can be described by means of ordinary differential equations. These processes are usually affected by a large number of small external fluctuation whose resulting action is natural to consider random. This explains the presence of the random variable ω in (1.2) which models the imprecise knowledge of the perturbation φ of the autonomous equation, which, instead, is assumed to be deterministically known. In this paper, in spite of the presence of the random variable ω, we aim at keeping the same qualitative behavior of (1.2), that is we want to guarantee the bifurcation of periodic solutions from the limit cycle of the unperturbed equation for any $\omega \in \Omega$.

For a deterministic differential equation of the form (1.2), namely

$$x'(t) = f(x(t)) + \varepsilon\varphi(t, x(t), \varepsilon) \tag{1.3}$$

existence, uniqueness and asymptotic stability of branches of periodic solutions originating from a limit cycle of the autonomous equation when it is periodically perturbed are very classical problems, see e.g. [3], [18] and [21].

The main tool employed in these papers is the so-called Malkin bifurcation function associated to (1.3)

$$M(\theta) = \int_0^T \langle z_0(\tau), \varphi(\tau - \theta, x_0(\tau), 0)\rangle \, d\tau, \tag{1.4}$$

where $\langle \cdot\,, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^n$ and z_0 is a T-periodic solution of

$$z' = -(f'(x_0(t)))^* z, \tag{1.5}$$

the adjoint equation of the linearized equation

$$y' = (f'(x_0(t)))y, \tag{1.6}$$

where $x_0(t), t \in [0,T]$, is the limit cycle of the unperturbed equation. Clearly, the linearized equation (1.6) has at least a characteristic multiplier with absolute value 1.

Since the pioneering papers [18] and [21] a relevant bibliography has been devoted to this subject. We mention here some of the papers more related to the approach employed in this paper, namely papers based on the abstract bifurcation equation

$$P(x) + \varepsilon Q(x, \varepsilon) = 0. \tag{1.7}$$

In [13] by a convenient scaling of the variable x in (1.7), we introduce an equivalent equation $\Psi(w,\varepsilon) = 0$. Then, assuming the existence of a simple zero θ_0 of the Malkin bifurcation function $M(\theta)$ the classical Implicit Function Theorem ensures the existence of a branch of zeros of Ψ originating from $x_0(\theta_0)$. The same approach has been used in [14] to deal with the same problem, in the case when the operators P and Q satisfy regularity conditions only along certain directions. This approach has been extended to infinite dimensional bifurcation problem in [5] and [15] with the aim of studying the bifurcation of periodic solutions respectively for a functional differential equation of neutral type and for a class of parabolic problems. In all the previous papers it is assumed that the characteristic multiplier with absolute value 1 of (1.6) is simple. In [6] this assumption has been removed. In [13] and [14] in order to employ a suitable version of the Implicit Function Theorem we assume that f and φ in (1.3) are sufficiently smooth, i.e. $f \in C^2$ and $\varphi \in C^1$. Papers are also devoted to the case when the functions f, φ in (1.3) are less regular. Roughly speaking, in this case, instead of an Implicit Function Theorem a suitable topological degree theory is used. Indeed, it can be shown that the existence of a simple zero of the Malkin bifurcation function implies that the topological degree is different from zero, see e.g. [11], [12], [19] and [20]. Moreover, random versions of suitable topological tools, such as fixed point theorems, have been employed to show existence and stability results for perturbed random differential equations, see e.g. [7], [22] and [24]. For a comprehensive presentation of this topic both for ordinary and partial differential equations we refer e.g. to [8] and [16].

Coming back to the present paper, following the lines of [13], in Section 3 we provide conditions on the operators P and Q of the bifurcation equation (1.1) which guarantee the existence of a measurable function $\omega \mapsto \theta_\omega \in [0,T]$ with the property that for any $\omega \in \Omega$ there exists a measurable family of solutions $x(\varepsilon,\omega), \varepsilon \geq 0$ small, of (1.1) originating from $x_0(\theta_\omega) \in \{x_0(\theta), \theta \in [0,T]\}$. This represents the main result of the paper, i.e. Theorem 3.4. More precisely, to prove this result we introduce the Malkin bifurcation function for (1.1), namely

$$M(\theta,\omega) = \langle Q(x(\theta),0,\omega), z_0(\theta)\rangle, \tag{1.8}$$

here $\langle \cdot\, , \cdot \rangle$ is the duality pairing between E and E^*, the dual space of E, $z_0(\theta)$ is an eigenvector of the operator $(P'(x(\theta)))^*$ corresponding to the eigenvalue zero such that $\|z_0(\theta))\| = 1$ and $\langle x_0'(\theta), z_0(\theta)\rangle > 0$. It can be seen, compare with [13], that the classical Malkin bifurcation function (1.4) takes the form (1.8) when we consider the abstract bifurcation equation (1.1) instead of (1.2).

Lemma 3.5 provides a necessary and sufficient condition on P and Q to ensure that a zero $\theta_\omega \in [0,T], \omega \in \Omega$, of the Malkin function is simple, namely $M(\theta_\omega,\omega) = 0$ and $M'(\theta_\omega,\omega) \neq 0$.

Under this condition and (H1)–(H3) precised in Section 3, the proof of Theorem 3.4 consists in showing that the random Implicit Function Theorem (Lemma 2.4) applies to an equation $\Psi(w,\varepsilon,\omega) = 0$, which is equivalent to (1.1). This allows to deduce the existence of a branch of measurable solutions $x(\varepsilon,\omega), \varepsilon \geq 0$ small, of (1.1) originating from $x_0(\theta_\omega) \in \{x_0(\theta), \theta \in [0,T]\}$, where the function $\omega \to \theta_\omega$ is mesurable by Lemma 3.2.

Indeed, one of the main task of the paper is to establish the measurability of the function $\omega \mapsto \theta_\omega$ and that of the function $(\varepsilon,\omega) \mapsto x(\varepsilon,\omega)$ via the random Implicit Function Theorem. To this end, Section 2 is devoted to the needed measurability results. These results are either well known or easy, in the latter case, e.g. for the random Implicit Function Theorem, we give the proof since we do not know any reference where it can be found. Finally, in Section 4 we apply the abstract bifurcation result to show the existence of branches of T-periodic solutions of (1.2) originating from the limit cycle x_0 of the unperturbed equation at any point $x_0(\theta_\omega)$, $\omega \in \Omega$.

2. Some measurability lemmas

In the sequel, $\mathbb{E}$, $\mathbb{F}$, $\mathbb{G}$ denote separable Banach spaces, whereas $\mathbb{X}$ denotes a non-necessarily separable Banach space.

The open (respectively, closed) ball with center a and radius ρ is denoted by $\mathrm{B}(a,\rho)$ (respectively, $\overline{\mathrm{B}}(a,\rho)$).

A mapping $\xi\colon \Omega \mapsto \mathbb{X}$ is said to be *Borel measurable*, or simply *measurable*, if $\xi^{-1}(B) \in \mathcal{F}$ for every Borel subset B of $\mathbb{X}$. We say that ξ is *strongly measurable*, or *Bochner measurable*, if there exists a sequence $\{\xi_n\}_{n\in\mathbb{N}}$ of simple Borel measurable functions converging pointwise to ξ. The mapping ξ is strongly measurable if, and only if, ξ is Borel measurable and separably valued, see e.g. [2, Theorem 3.5].

An example of a non-separable space (if $\mathbb{E}$ and $\mathbb{F}$ are infinite dimensional) is the space $\mathrm{L_c}(\mathbb{E},\mathbb{F})$ of bounded linear operators from $\mathbb{E}$ to $\mathbb{F}$, endowed with the operator norm. A simple characterization of strong measurability for random operators can be found in [2, Theorem 4.1]. A mapping $\Phi\colon \Omega \mapsto \mathrm{L_c}(\mathbb{E},\mathbb{F})$ is strongly measurable if, and only if, Φ is separably valued and, for every $x \in \mathbb{E}$ and every y^* in the topological dual $\mathbb{F}^*$ of $\mathbb{F}$, the mapping $\omega \mapsto \langle y^*, \Phi(\omega)x\rangle$ is measurable.

Let $U \subset \mathbb{E}$, $U \neq \emptyset$. We say that a function $\varphi\colon U \times \Omega \mapsto \mathbb{X}$ is *Carathéodory* if $\varphi(\,\cdot\,,\omega)$ is continuous for each $\omega \in \Omega$ and $\varphi(u,\,\cdot\,)$ is strongly measurable for each $u \in U$. It is well known that any Carathéodory function is globally strongly measurable, see [4, Lemma III.14] (the separability of U is crucial in this result). The proof of [4, Lemma III.14] can be easily adapted to the case of a random domain of definition.

Lemma 2.1. *Let $a\colon \Omega \mapsto \mathbb{E}$ and $\rho\colon \Omega \mapsto\,]0,+\infty[$ be measurable, and, for each $\omega \in \Omega$, let $U_\omega = \mathrm{B}(a(\omega),\rho(\omega))$. Let $U = \{(u,\omega) \in \mathbb{E}\times\Omega;\ u \in U_\omega\}$ and $\varphi\colon U \mapsto \mathbb{X}$ such that $\varphi(\,\cdot\,,\omega)$ is continuous on U_ω for each $\omega \in \Omega$ and $\varphi(u,\,\cdot\,)$ is strongly measurable on $\Omega_u := \{\omega \in \Omega;\ u \in U_\omega\}$ for each $u \in \mathbb{E}$ (in the case when $\Omega_u = \emptyset$, we set $\varphi(u,\,\cdot\,)$ strongly measurable). Then φ is globally strongly measurable.*

Proof. Let $\{e_n\}_{n\in\mathbb{N}}$ be a dense sequence in $\mathbb{E}$. For each integer $p \geq 1$, and for $(u,\omega) \in U$, let $\varphi_p(u,\omega) = \varphi(e_n,\omega)$, where n is the smallest integer such that $u \in \mathrm{B}(e_n,1/p)$ (set arbitrarily $\varphi(e_n,\omega) = 0$ if $e_n \notin U_\omega$). Then $\varphi_p(u,\omega) \to \varphi(u,\omega)$ for all $(u,\omega) \in U$ when $p \to \infty$, and each φ_p is strongly measurable since φ_p coincides with $\varphi(e_n,\,\cdot\,)$ on $\Big(\mathrm{B}(e_n,1/p) \setminus \bigcup_{m<n} \mathrm{B}(e_m,1/p)\Big) \times \Omega$. □

Lemma 2.2. *Let $I \subset \mathbb{R}$ be a compact interval, and let us denote by $C(I,\mathbb{E})$ the space of continuous mappings from I to $\mathbb{E}$, endowed with the topology of uniform convergence on I. Let $\varphi\colon I \times \Omega \mapsto \mathbb{E}$ be a mapping such that $\varphi(t,\,\cdot\,)$ is measurable for each $t \in I$ and $\varphi(\,\cdot\,,\omega)$ is continuous for each $\omega \in \Omega$. Then the mapping*

$$\begin{cases} \Omega \mapsto C(I,\mathbb{E}), \\ \omega \mapsto \varphi(\,\cdot\,,\omega), \end{cases}$$

is measurable.

Proof. Since I is compact and $\mathbb{E}$ is separable, $C(I,\mathbb{E})$ is a separable Banach space, and thus it is a Suslin space. Therefore all separated topologies on $C(I,\mathbb{E})$ which are comparable with the topology of uniform convergence share the same Borel subsets, see [23, Lemma 17 p. 108]. In particular, the Borel σ-algebra on $C(I,\mathbb{E})$ generated by the topology of pointwise convergence coincides with the Borel σ-algebra generated by the topology of uniform convergence. □

Lemma 2.3. *Let $I=[0,T]\subset\mathbb{R}$ be an interval, and let $\varphi\colon I\times\mathbb{E}\times\Omega\mapsto\mathbb{F}$ be a Carathéodory function (measurable with respect to $\omega\in\Omega$ and continuous with respect to $(t,x)\in I\times\mathbb{E}$) such that, for each $\omega\in\Omega$, $\varphi(t,\,\cdot\,,\omega)$ is Lipschitz, uniformly with respect to $t\in I$. Let $a\colon\Omega\mapsto\mathbb{E}$ be measurable and, for each $\omega\in\Omega$, let $t\mapsto x(a(\omega),t,\omega)$ be the unique solution of the Cauchy problem*

$$x'=\varphi(t,x,\omega),\quad x(0)=a(\omega).$$

Then the mapping

$$\begin{cases}\Omega\mapsto C(I,\mathbb{E}),\\ \omega\mapsto x(a(\omega),\,\cdot\,,\omega),\end{cases}$$

is measurable.

Proof. For each ω, since $\varphi(\,\cdot\,,\,\cdot\,,\omega)$ is Lipschitz with respect to the second variable, uniformly with respect to the first one, the existence of the solution $x(a(\omega),\,\cdot\,,\omega)$ is well known. Furthermore, $x_\omega:=x(a(\omega),\,\cdot\,,\omega)$ can be obtained by Euler's forward method. Let $t_k^{(n)}=kT/n$, $n\geq 1$, $0\leq k\leq n$, and, for $t_k^{(n)}\leq s\leq t_{k+1}^{(n)}$, set

$$x_{n,\omega}(s)=x_{n,\omega}\Big(t_k^{(n)}\Big)+\varphi'_{(1)}\Big(t_k^{(n)},x_{n,\omega}(t_k^{(n)}),\omega\Big),\quad x_{n,\omega}(0)=a(\omega),$$

where $\varphi'_{(1)}$ denotes the derivative with respect to the first variable. Then each $x_{n,\omega}(s)$ is measurable, and the sequence $\{x_{n,\omega}(s)\}_{n\in\mathbb{N}}$ converges to $x_\omega(s)$, which proves that $\omega\mapsto x_\omega(s)$ is measurable. We conclude by using Lemma 2.2. □

We shall also need the following random Implicit Function Theorem. The proof of this result closely follows the classical constructive proof which makes use of Banach's fixed point theorem. We provide it for the reader convenience. In the sequel $\mathrm{L_c}(\mathbb{E},\mathbb{G})$ denotes the space of bounded linear operators from $\mathbb{E}$ to $\mathbb{G}$ endowed with the uniform operator norm $\|\cdot\|$ and $\mathrm{Isom_c}(\mathbb{E},\mathbb{G})$ the space of continuous isomorphisms. In the sequel, when no confusion may arise, the same notation $\|\cdot\|$ will denote the norm for all the normed spaces.

Lemma 2.4. *Let $a\colon\Omega\mapsto\mathbb{E}$ and $b\colon\Omega\mapsto\mathbb{F}$ be measurable mappings. Let $\rho\colon\Omega\mapsto\,]0,+\infty[$ be measurable, and, for each $\omega\in\Omega$, let $U_\omega=\mathrm{B}(a(\omega),\rho(\omega))$ and $V_\omega=\mathrm{B}(b(\omega),\rho(\omega))$. Let $\psi\colon\mathbb{E}\times\mathbb{F}\times\Omega\mapsto\mathbb{G}$ be a measurable map. Assume that ψ is differentiable with respect to the first variable on $U_\omega\times V_\omega$, for each $\omega\in\Omega$.*

Let $\psi'_{(1)}(u,v,\omega) \in \mathrm{L_c}(\mathbb{E},\mathbb{G})$, with $(u,v) \in U_\omega \times V_\omega$ and $\omega \in \Omega$, be the differential. Assume that $\psi'_{(1)}(\,\cdot\,,\,\cdot\,,\omega)$ is continuous on $U_\omega \times V_\omega$ for each $\omega \in \Omega$, and that $\psi'_{(1)}(u,v,\,\cdot\,)$ is strongly measurable on $\Omega_{(u,v)} := \{\omega \in \Omega; (u,v) \in U_\omega \times V_\omega\}$. Finally, assume that, for every $\omega \in \Omega$,

$$\psi'_{(1)}(a(\omega),b(\omega),\omega) \text{ is an isomorphism from } \mathbb{E} \text{ to } \mathbb{G},$$
$$\psi(a(\omega),b(\omega),\omega) = 0.$$

Then, there exist random neighbourhoods U'_ω and V'_ω of a and b respectively, of the form

$$\omega \mapsto U'_\omega := \mathrm{B}(a(\omega),\rho_1(\omega)) \subset U_\omega,$$
$$\omega \mapsto V'_\omega := \mathrm{B}(b(\omega),\rho_2(\omega)) \subset V_\omega,$$

where ρ_1 and ρ_2 are measurable functions with positive values and there exists a measurable mapping $\varphi\colon \mathbb{F} \times \Omega \mapsto \mathbb{E}$ defined on $V' := \{(y,\omega) \in \mathbb{F} \times \Omega;\ y \in V'_\omega\}$, such that, for every $\omega \in \Omega$, $\varphi(\,\cdot\,,\omega)$ is continuous, and, for every $(u_\omega,v_\omega) \in U'_\omega \times V'_\omega, \omega \in \Omega$, the following equivalence holds true:

$$\begin{aligned} &((u_\omega,v_\omega) \in U'_\omega \times V'_\omega \text{ and } \psi(u_\omega,v_\omega,\omega) = 0) \\ &\qquad\qquad \Leftrightarrow (v_\omega \in V'_\omega \text{ and } u_\omega = \varphi(v_\omega,\omega)). \end{aligned} \tag{2.1}$$

Proof. For every $\omega \in \Omega$ put $A_\omega = \psi'_{(1)}(a(\omega),b(\omega),\omega) \in \mathrm{Isom_c}(\mathbb{E},\mathbb{G})$ and define the mapping $\Phi\colon U_\omega \times V_\omega \times \{\omega\} \to \mathbb{E}$ as follows

$$\Phi(u,v,\omega) = u - A_\omega^{-1}\psi(u,v,\omega)$$

By Lemma 2.1, $\psi'_{(1)}$ is strongly measurable. Since the mapping $A \mapsto A^{-1}$ is continuous from $\mathrm{Isom_c}(\mathbb{E},\mathbb{G})$ to $\mathrm{Isom_c}(\mathbb{G},\mathbb{E})$, we deduce that Φ is measurable. Furthermore, the differential $\Phi'_{(1)}$ of Φ with respect to u is well defined and strongly measurable, with $\Phi'_{(1)}(a(\omega),b(\omega),\omega) = 0$ in $\mathrm{L_c}(\mathbb{E},\mathbb{E})$. The strong measurability of $\Phi'_{(1)}$ also yields the measurability of $\|\Phi'_{(1)}\|$. Thus, since $\Phi'_{(1)}$ is continuous with respect to the first two variables, we can find a random variable ρ_1 such that $0 < \rho_1 \le \rho$ and

$$(\|u - a(\omega)\| \le \rho_1(\omega) \text{ and } \|v - b(\omega)\| \le \rho_1(\omega)) \Rightarrow \|\Phi'_{(1)}(u,v,\omega)\| \le \frac{1}{2}.$$

Then, by the mean value inequality, $\Phi(\,\cdot\,,v,\omega)$ is $1/2$-Lipschitz on $\overline{\mathrm{B}}(a(\omega),\rho_1(\omega)) \times \overline{\mathrm{B}}(b(\omega),\rho_1(\omega))$.

Furthermore, we can find a random variable ρ_2, with $0 < \rho_2 \le \rho_1$, such that

$$\|\Phi(a(\omega),v,\omega)\| \le \frac{\rho_1(\omega)}{2}$$

for all $\omega \in \Omega$ and $v \in \overline{\mathrm{B}}(b(\omega), \rho_2(\omega))$. Then we have, for $\|u - a(\omega)\| \le \rho_1(\omega)$ and $\|v - b(\omega)\| \le \rho_2(\omega)$,

$$\begin{aligned}\|\Phi(u, v, \omega)\| &\le \|\Phi(a(\omega), v, \omega)\| + \|\Phi(u, v, \omega) - \Phi(a(\omega), v, \omega)\| \\ &\le \frac{\rho_1(\omega)}{2} + \frac{1}{2}\|u - a(\omega)\| \le \rho_1(\omega).\end{aligned}$$

Thus, for $\|v - b(\omega)\| \le \rho_2(\omega)$, $\Phi(\,\cdot\,, v, \omega)$ maps the complete metric space $\overline{\mathrm{B}}(a(\omega), \rho_1(\omega))$ into itself. We can now apply the fixed point theorem for random contractive operators: for each $v \in \overline{\mathrm{B}}(v(\omega), \rho_2(\omega))$, the map $\Phi(\,\cdot\,, v, \omega)$ has a unique fixed point $\varphi(v, \omega) \in \overline{\mathrm{B}}(v(\omega), \rho_2(\omega))$, and the map φ is measurable since

$$\varphi(v, \omega) = \lim_{n \to \infty} \varphi_n(v, \omega),$$

where $\varphi_0(v, \omega) = b(\omega)$ and $\varphi_{n+1}(v, \omega) = \Phi(\varphi_n, v, \omega)$, $n \ge 0$.

Observe that the random fixed point theorem can be also obtained by a trivial adaptation of [1, Theorem 7]. Clearly, φ satisfies (2.1). There remains to show that $\varphi(\,\cdot\,, \omega)$ is continuous for each $\omega \in \Omega$. Let $v, v' \in \overline{\mathrm{B}}(v(\omega), \rho_2(\omega))$. We have

$$\begin{aligned}&\|\varphi(v', \omega) - \varphi(v, \omega)\| = \|\Phi(\varphi(v', \omega), v', \omega) - \Phi(\varphi(v, \omega), v, \omega)\| \\ &\le \|\Phi(\varphi(v', \omega), v', \omega) - \Phi(\varphi(v, \omega), v', \omega)\| + \|\Phi(\varphi(v, \omega), v', \omega) - \Phi(\varphi(v, \omega), v, \omega)\| \\ &\le \frac{1}{2}\|\varphi(v', \omega) - \varphi(v, \omega)\| + \|\Phi(\varphi(v, \omega), v', \omega) - \Phi(\varphi(v, \omega), v, \omega)\|,\end{aligned}$$

thus

$$\|\varphi(v', \omega) - \varphi(v, \omega)\| \le 2\|\Phi(\varphi(v, \omega), v', \omega) - \Phi(\varphi(v, \omega), v, \omega)\|. \tag{2.2}$$

By continuity of $\Phi(\varphi(v, \omega), \,\cdot\,, \omega)$, we can find a neighbourhood W_ω of v such that the last term in (2.2) is arbitrary small when $v' \in W_\omega$, which yields the continuity of $\varphi(\,\cdot\,, \omega)$ at v. □

3. The abstract bifurcation result

This section is devoted to the formulation and the proof of the abstract bifurcation result for (1.1), that is, Theorem 3.4. To this end we precise in the following the assumptions on the operators $P\colon \mathbb{E} \mapsto \mathbb{E}$ and $Q\colon \mathbb{E} \times [0, 1] \times \Omega \mapsto \mathbb{E}$, where $\varepsilon \in [0, 1]$ is the bifurcation parameter. In this Section $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the duality pairing between E and E^*, the dual space of E.

(H1) There exists a curve $\theta \mapsto x_0(\theta)$, $\theta \in [0, T]$, $x_0 \in C^1([0, T], \mathbb{E})$ satisfying

$$P(x_0(\theta)) = 0 \quad \text{and} \quad x_0'(\theta) \ne 0, \quad \text{for any } \theta \in [0, T].$$

(H2) The derivative $P''(x)$ is continuous at every $x \in U$, where U is a neighbourhood of the curve $\{x_0(\theta) : \theta \in [0, T]\}$, and $Q'_{(1)}(x, \varepsilon, \omega)$ is continuous for each $\omega \in \Omega$ at any $(x, \varepsilon) \in U \times [0, 1]$, and $Q'_{(1)}$ is strongly measurable on $U \times [0, 1] \times \Omega$. Moreover, P has the form $P(x) = x - F(x)$, where the operator $F\colon \mathbb{E} \mapsto \mathbb{E}$ is compact. The operator Q is also compact with respect to (x,ε) for any $\omega \in \Omega$.

By deriving $P(x_0(\theta)) = 0$ with respect to the parameter $\theta \in [0,T]$ we obtain that $P'(x_0(\theta))x_0'(\theta) = 0$, hence $0 \in \mathfrak{S}(P'(x(\theta)))$, i.e. 0 belongs to the spectrum of $P'(x_0(\theta))$ or equivalently $1 \in \mathfrak{S}(F'(x(\theta)))$. Hence 1 is an eigenvalue of $F'(x_0(\theta))$ of finite multiplicity, since F' is a compact operator. From now on we assume:

(H3) The eigenvalue 1 of $F'(x_0(\theta))$ is simple, for all $\theta \in [0,T]$.

Observe that the unique eigenvector $z_0(\theta)$ corresponding to the eigenvalue zero of the operator $P'(x(\theta))$, satisfying $\|z_0(\theta)\| = 1$ and $\langle x_0'(\theta), z_0(\theta)\rangle > 0$, is such that the mapping $\theta \mapsto z_0(\theta)$ is continuous.

Define the Riesz projector $\pi(\theta)$ associated with the operator $P'(x_0(\theta))$ corresponding to the simple eigenvalue 0 by means of the well-known formula, see [9],

$$\pi(\theta) = \frac{1}{2\pi i}\int_\gamma (\lambda I - P'(x_0(\theta)))^{-1}\,d\lambda, \tag{3.1}$$

where γ is a circumference centered at the origin containing in the closure of its interior only the zero eigenvalue of $P'(x_0(\theta))$.

Remark 3.1. We can easily check that

$$\pi(\theta)h = \frac{\langle h, z_0(\theta)\rangle}{\langle x_0'(\theta), z_0(\theta)\rangle}\, x_0'(\theta),$$

and that $\theta \mapsto \pi(\theta)$ is continuous.

Consider now the Malkin bifurcation function

$$M(\theta,\omega) = \langle Q(x_0(\theta),0,\omega), z_0(\theta)\rangle. \tag{3.2}$$

Lemma 3.2. *Assume that, for each $\omega \in \Omega$, there exists $\theta \in [0,T]$ such that $M(\theta,\omega) = 0$ and $M'_{(1)}(\theta,\omega) \neq 0$. Then there exists a measurable function*

$$\begin{cases} \Omega \mapsto [0,T], \\ \omega \mapsto \theta_\omega, \end{cases}$$

such that, for each $\omega \in \Omega$, $M(\theta_\omega,\omega) = 0$ and $M'_{(1)}(\theta_\omega,\omega) \neq 0$.

Proof. Put $N = M'_{(1)}$. Since the functions M and N, from $[0,T]\times\Omega$ to $\mathbb{R}$, are measurable, we deduce that the mappings $\omega \mapsto M(\,\cdot\,,\omega)$ and $\omega \mapsto N(\,\cdot\,,\omega)$, with values in $C([0,T];\mathbb{R})$, are measurable. Indeed, let $\{\theta_n\}_{n\in\mathbb{N}}$ be a dense sequence in $[0,T]$. For every $\xi \in C([0,T];\mathbb{R})$, and for every $\delta > 0$, we have

$$\{\omega \in \Omega;\ \|M(\,\cdot\,,\omega) - \xi\| \le \delta\} = \bigcap_{n\in\mathbb{N}} \{\omega \in \Omega;\ |M(\theta_n,\omega) - \xi(\theta)| \le \delta\} \in \mathcal{F},$$

and similarly $\{\omega \in \Omega;\ \|N(\,\cdot\,,\omega) - \xi\| \le \delta\} \in \mathcal{F}$.

Let $\{\xi_n\}_{n\in\mathbb{N}}$ be a sequence in $C^1([0,T];\mathbb{R})$ such that $\{\xi_n\}_{n\in\mathbb{N}}$ and $\{\xi'_n\}_{n\in\mathbb{N}}$ are both dense in $C([0,T];\mathbb{R})$. Let us define, for all integers $n\geq 0$ and $k\geq 1$, the possibly empty compact set

$$Z_{n,k}=\biggl\{\theta\in[0,T];\ |\xi_n(\theta)|\leq\frac{2}{k}\text{ and }|\xi'_n(\theta)|\geq\frac{2}{k}\biggr\}.$$

For every $\omega\in\Omega$ and each $k\geq 1$, set

$$n_k(\omega)=\min\{n\in\mathbb{N};\ \|M(\,\cdot\,,\omega)-\xi_n\|\leq 1/k\text{ and }\|N(\,\cdot\,,\omega)-\xi'_n\|\leq 1/k\},$$
$$H_k(\omega)=Z_{n_k(\omega),k}.$$

The measurability of the mapping n_k follows from the measurability of the mappings $\omega\mapsto M(\,\cdot\,,\omega)$ and $\omega\mapsto N(\,\cdot\,,\omega)$. Furthermore, for any open subset U of $[0,T]$, we have

$$\begin{aligned}H_k^-(U)&:=\{\omega\in\Omega;\ U\cap Z_{n_k(\omega),k}\neq\emptyset\}\\&=\bigcup_{j\in\mathbb{N}}\{\omega\in\Omega;\ n_k(\omega)=j\text{ and }Z_{j,k}\cap U\neq\emptyset\}\in\mathcal{F},\end{aligned}$$

which proves that the multifunction $H_k\colon\Omega\mapsto[0,T]$ is measurable.

Now, for all $(\theta,\omega)\in[0,T]\times\Omega$, we have that

$$\begin{aligned}&\bigl(M(\theta,\omega)=0\text{ and }N(\theta,\omega)\neq 0\bigr)\\&\quad\Rightarrow\biggl(\forall\,k\geq 1\ |M(\theta,\omega)|\leq\frac{1}{k}\text{ and }\exists\,m\geq 1\ \forall\,k\geq m\ |N(\theta,\omega)|\geq\frac{3}{k}\biggr)\\&\quad\Rightarrow\biggl(\forall\,k\geq 1,\ |\xi_{n_k(\omega)}(\theta)|\leq\frac{2}{k}\text{ and }\exists\,m\geq 1\ \forall\,k\geq m\ |\xi'_{n_k(\omega)}(\theta)|\geq\frac{2}{k}\biggr)\\&\quad\Rightarrow\bigl(\exists\,m\geq 1\ \forall\,k\geq m\ \theta\in H_k(\omega)\bigr).\end{aligned}$$

Conversely, we have

$$\begin{aligned}&\bigl(\exists\,m\geq 1\ \forall\,k\geq m\ \theta\in H_k(\omega)\bigr)\\&\quad\Rightarrow\biggl(\exists\,m\geq 1\ \forall\,k\geq m\ |\xi_{n_k(\omega)}(\theta)|\leq\frac{2}{k}\text{ and }|\xi'_{n_k(\omega)}(\theta)|\geq\frac{2}{k}\biggr)\\&\quad\Rightarrow\biggl(\exists\,m\geq 1\ \forall\,k\geq\ m\ |M(\theta,\omega)|\leq\frac{1}{k}\text{ and }|N(\theta,\omega)|\geq\frac{1}{k}\biggr)\\&\quad\Rightarrow\bigl(M(\theta,\omega)=0\text{ and }N(\theta,\omega)\neq 0\bigr).\end{aligned}$$

Thus

$$\bigl(M(\theta,\omega)=0\text{ and }N(\theta,\omega)\neq 0\bigr)\Leftrightarrow\biggl(\theta\in\bigcup_{m\geq 1}\bigcap_{k\geq m}H_k(\omega)\biggr).$$

From our hypothesis, we deduce that, for every $\omega \in \Omega$, there exists $m \geq 1$ such that $\bigcap_{k \geq m} H_k(\omega) \neq \emptyset$. Let

$$k_0(\omega) = \min\left\{m \geq 1;\ \bigcap_{k\geq m} H_k(\omega) \neq \emptyset\right\} < \infty.$$

By [4, Proposition III.4], for each $m \geq 1$, the multifunction $G_m = \bigcap_{k\geq m} H_k$ is measurable. Thus k_0 is measurable. Let $\mathcal{M} \subset \mathbb{N}$ be the set of values taken by k_0, i.e. $\mathcal{M} = \{m \in \mathbb{N};\ k_0^{-1}(m) \neq \emptyset\}$. Let $\Omega_m = \{\omega \in \Omega;\ k_0(\omega) = m\}$ for each $m \in \mathcal{M}$. On each Ω_m, by [4, Theorem III.6], the compact valued multifunction G_m admits a measurable selection $\omega \mapsto \theta_\omega^{(m)}$. To conclude we only need to set $\theta_\omega = \theta_\omega^{(m)}$ for $\omega \in \Omega_m$, $m \in \mathcal{M}$. □

REMARK 3.3. Let us endow $(\Omega, \mathcal{F})$ with a probability measure $\mathbf{P}$. Then, for every $\delta > 0$, there exists a measurable subset Ω_δ of Ω and a number $\eta > 0$ such that $\mathbf{P}(\Omega_\delta) > 1 - \delta$ and, for each $\omega \in \Omega_\delta$, $M(\theta_\omega, \omega) = 0$ and $M'_{(1)}(\theta_\omega, \omega) \geq \eta$. Indeed, since the random variable $\omega \mapsto M'_{(1)}(\theta_\omega, \omega)$ takes its values in the Polish space $\mathbb{R} \setminus \{0\}$, its distribution is tight, that is, for each $\delta > 0$, we can find a compact subset K_δ of $\mathbb{R}\setminus\{0\}$ such that $\mathbf{P}(M'_{(1)}(\theta_\omega, \cdot)) \in K_\delta > 1 - \delta$. Then we can take $\Omega_\delta = \{\omega \in \Omega;\ M'_{(1)}(\theta_\omega, \omega) \in K_\delta\}$ and $\eta = \min\{|k|;\ k \in K_\delta\}$.

Now, let

$$\pi(\theta)Q(x_0(\theta), 0, \omega) = \frac{M(\theta, \omega)}{\langle x_0'(\theta), z_0(\theta)\rangle}\, x_0'(\theta). \tag{3.3}$$

As we will see, Lemma 3.5 in what follows states that $M'_{(1)}(\theta_\omega, \omega) \neq 0$ is equivalent to

$$\frac{d}{d\theta}\big[\pi(\theta)Q(x_0(\theta), 0, \omega)\big]\bigg|_{\theta=\theta_\omega} \neq 0.$$

Let $x_0(\theta_\omega) = v_\omega$, $\pi(\theta_\omega) = \pi_\omega$, $x_0'(\theta_\omega) = e_\omega \neq 0$ and

$$y_\omega = -\big(P'(v_\omega)|_{(I-\pi_\omega)\mathbb{E}}\big)^{-1} Q(v_\omega, 0, \omega). \tag{3.4}$$

We are now in the position to state the abstract bifurcation result.

THEOREM 3.4. *Assume* (H1)–(H3). *Moreover assume that for any $\omega \in \Omega$ we have that*

$$\pi_\omega\big[P''(v_\omega)y_\omega e_\omega + Q'_{(1)}(v_\omega, 0, \omega)e_\omega\big] \neq 0. \tag{3.5}$$

Then (1.1) *has a measurable solution $x(\varepsilon, \omega)$, for $\varepsilon \geq 0$ small, of the form*

$$x(\varepsilon, \omega) = v_\omega + \varepsilon w_\omega + h(\varepsilon, \omega), \tag{3.6}$$

where $h(\varepsilon, \omega)/\varepsilon \to 0$ as $\varepsilon \to 0$, $w_\omega = x_\omega + y_\omega$, $x_\omega = \alpha_\omega e_\omega$ with α_ω uniquely determined by the equation

$$\pi_\omega\big[Q'_{(1)}(v_\omega, 0, \omega)x_\omega + \pi_\omega P'(v_\omega)y_\omega\, x_\omega\big] = -\frac{1}{2}\big[P''(v_\omega)y_\omega\, y_\omega + \pi_\omega Q'_{(1)}(v_\omega, 0, \omega)y_\omega\big].$$

Let us first prove the following result concerning the Malkin bifurcation function $M(\theta,\omega)$ introduced in (3.2).

Lemma 3.5. *Let* $\theta_\omega \in [0,T]$ *be such that* $M(\theta_\omega,\omega)=0$. *Then* $M'_{(1)}(\theta_\omega,\omega)\neq 0$ *if, and only if,* (3.5) *holds true.*

Proof. Consider

$$\pi(\theta)Q(x_0(\theta),0,\omega)=\frac{M(\theta,\omega)}{\langle x_0'(\theta),z_0(\theta)\rangle}\,x_0'(\theta). \tag{3.7}$$

By deriving (3.7) with respect to θ we obtain

$$\begin{aligned}\pi'(\theta)Q(x_0(\theta),0,\omega)&+\pi(\theta)Q'_{(1)}(x_0(\theta),0,\omega)x_0'(\theta)\\ &=\frac{1}{\langle x_0'(\theta),z_0(\theta)\rangle}\,[M'_{(1)}(\theta,\omega)x_0'(\theta)+M(\theta,\omega)x_0''(\theta)]\\ &\qquad\qquad+M(\theta,\omega)x_0'(\theta)\,\frac{d}{d\theta}\bigg(\frac{1}{\langle x_0'(\theta),z_0(\theta)\rangle}\bigg).\end{aligned}$$

Let $\theta=\theta_\omega$, since $M(\theta_\omega,\omega)=0$ we have that

$$\pi'(\theta_\omega)Q(v_\omega,0,\omega)+\pi(\theta_\omega)Q'_{(1)}(v_\omega,0,\omega)e_\omega=\frac{1}{\langle e_\omega,z_0(\theta_\omega)\rangle}\,M'_{(1)}(\theta_\omega,\omega)e_\omega, \tag{3.8}$$

and $\pi(\theta_\omega)Q(v_\omega,0,\omega)=0$. From the integral representation of the Riesz projector (3.1) we obtain

$$\pi'(\theta_\omega)y=\frac{1}{2\pi i}\int_\gamma(\lambda I-P'(v_\omega))^{-1}P''(v_\omega)e_\omega(\lambda I-P'(v_\omega))^{-1}y\,d\lambda.$$

For notational convenience we let

$$\widehat{\mathbb{E}}_\omega=(I-\pi(\theta_\omega))\mathbb{E},\quad Q_\omega=Q(v_\omega,0,\omega)\quad\text{and}\quad(\lambda I-P'(v_\omega))^{-1}=R_\omega(\lambda).$$

Since $Q_\omega\in\widehat{\mathbb{E}}_\omega$ we have

$$\begin{aligned}\pi'(\theta_\omega)\,Q_\omega&=\frac{1}{2\pi i}\int_\gamma R_\omega(\lambda)P''(v_\omega)e_\omega R_\omega(\lambda)|_{\widehat{\mathbb{E}}_\omega}Q_\omega\,d\lambda\\ &=\frac{1}{2\pi i}\int_\gamma R_\omega(\lambda)\pi(\theta_\omega)P''(v_\omega)e_\omega R_\omega(\lambda)|_{\widehat{\mathbb{E}}_\omega}Q_\omega\,d\lambda\\ &\quad+\frac{1}{2\pi i}\int_\gamma R_\omega(\lambda)(I-\pi(\theta_\omega)P''(v_\omega)e_\omega R_\omega(\lambda)|_{\widehat{\mathbb{E}}_\omega}Q_\omega\,d\lambda.\end{aligned}$$

The second integral is zero, since the integrand is an analytic function of λ in $\overline{\mathrm{int}(\gamma)}$. For the first integral we consider the Taylor series of the function $\lambda\mapsto R_\omega(\lambda)|_{\widehat{\mathbb{E}}_\omega}$ in $\mathrm{int}(\gamma)$ and the Laurent series for $\lambda\mapsto R_\omega(\lambda)|_{\pi(\theta_\omega)\mathbb{E}}$ which has a pole $\lambda=0$ of first order in $\mathrm{int}(\gamma)$. We have

$$\pi'(\theta_\omega)Q_\omega=\frac{1}{2\pi i}\int_\gamma\frac{\pi(\theta_\omega)P''(v_\omega)e_\omega y_\omega}{\lambda}\,d\lambda=\pi(\theta_\omega)P''(v_\omega)e_\omega y_\omega,$$

where y_ω is given in (3.4). In conclusion from (3.8) we obtain

$$\pi(\theta_\omega)[P''(v_\omega)e_\omega y_\omega + Q'_{(1)}(v_\omega, 0, \omega)e_\omega] = \frac{1}{\langle e_\omega, z_0(\theta_\omega)\rangle} M'_{(1)}(\theta_\omega, \omega)e_\omega.$$

Hence (3.5) is equivalent to $M'_{(1)}(\theta_\omega, \omega)) \neq 0$. □

Proof of Theorem 3.4. Since P is twice continuously differentiable we have that

$$\|P'(v_1) - P'(v_2)\| \leq L\|v_1 - v_2\| \tag{3.9}$$

for some $L > 0$, v_1, v_2 belonging to a neighbourhood U of the set $\{x_0(\theta), \theta \in [0, T]\}$ and $\varepsilon \in [0, 1]$. Taking into account that $P(v_\omega) = 0$ we have

$$\begin{aligned} P(v) + \varepsilon Q(v, \varepsilon, \omega) &= P(v) - P(v_\omega) + \varepsilon Q(v, \varepsilon, \omega) \\ &= P'(v_\omega)(v - v_\omega) + \varepsilon Q(v, \varepsilon, \omega) + \gamma_\omega(v), \end{aligned}$$

where

$$\begin{aligned} \gamma_\omega(v) &= P(v) - P(v_\omega) - P'(v_\omega)(v - v_\omega) \\ &= \int_0^1 [P'(v_\omega + \tau(v - v_\omega)) - P'(v_\omega)](v - v_\omega)\, d\tau. \end{aligned}$$

From (3.9) we get

$$\begin{aligned} &\|\gamma_\omega(v_1) - \gamma_\omega(v_2)\| \\ &\quad = \bigg\| \int_0^1 \{[P'(v_\omega + \tau(v_1 - v_\omega)) - P'(v_\omega)](v_1 - v_\omega) \\ &\qquad - [P'(v_\omega + \tau(v_2 - v_\omega)) - P'(v_\omega)](v_2 - v_\omega)\}\, d\tau \bigg\| \\ &\quad \leq \int_0^1 \|P'(v_\omega + \tau(v_1 - v_\omega)) - P'(v_\omega + \tau(v_2 - v_\omega))\| \|v_1 - v_\omega\|\, d\tau \\ &\qquad + \int_0^1 \|P'(v_\omega + \tau(v_2 - v_\omega)) - P'(v_\omega)\| \|v_2 - v_1\|\, d\tau \\ &\quad \leq \int_0^1 \tau L(\|v_2 - v_1\| \|v_1 - v_\omega\| + \|v_2 - v_\omega\| \|v_2 - v_1\|)\, d\tau \\ &\quad \leq L \max(\|v_1 - v_\omega\|, \|v_2 - v_\omega\|) \|v_2 - v_1\|. \end{aligned}$$

Hence,

$$\|\gamma_\omega(v_1) - \gamma_\omega(v_2)\| \leq L \max(\|v_1 - v_\omega\|, \|v_2 - v_\omega\|) \|v_2 - v_1\|. \tag{3.10}$$

Since $\gamma_\omega(v_\omega) = 0$ for any $\varepsilon \in [0, 1]$, equation (1.1) is equivalent to

$$\Phi(v, \varepsilon, \omega) = 0, \tag{3.11}$$

where $\Phi(v, \varepsilon, \omega) = \mathbb{P}(v) + \varepsilon \mathbb{Q}(v, \varepsilon, \omega) + \gamma_\omega(v)$, $\mathbb{P}(v) = P'(v_\omega)(v - v_\omega)$ and $\mathbb{Q}(v, \varepsilon, \omega) = Q(v, \varepsilon, \omega)$. Note that $\Phi(v_\omega, 0, \omega) = \mathbb{P}(v_\omega) = 0$ and $\mathbb{P}'(v_\omega) = P'(v_\omega)$.

Let $v = v_\omega + \varepsilon w$, $\pi_\omega = \pi(\theta_\omega)$ and observe that for $\varepsilon > 0$ equation (3.11) is equivalent to

$$\Psi(w, \varepsilon, \omega) = 0, \tag{3.12}$$

where

$$\Psi(w, \varepsilon, \omega) = \frac{1}{\varepsilon}\bigg(\Phi(v_\omega + \varepsilon w, \varepsilon, \omega) - \pi_\omega \Phi(v_\omega + \varepsilon w, \varepsilon, \omega) + \frac{1}{\varepsilon}\,\pi_\omega \Phi(v_\omega + \varepsilon w, \varepsilon, \omega)\bigg). \tag{3.13}$$

We can rewrite (3.13) as follows

$$\Psi(w, \varepsilon, \omega) = I_1(w, \varepsilon, \omega) + \widetilde{I}_1(w, \varepsilon, \omega) + I_2(w, \varepsilon, \omega) + I_3(w, \varepsilon, \omega) + \widetilde{I}_3(w, \varepsilon, \omega),$$

where

$$\begin{aligned}
I_1(w, \varepsilon, \omega) &= \frac{1}{\varepsilon}\,(\mathbb{P}(v_\omega + \varepsilon w) + \varepsilon\,\mathbb{Q}(v_\omega + \varepsilon w, \varepsilon, \omega)),\\
\widetilde{I}_1(w, \varepsilon, \omega) &= \frac{1}{\varepsilon}\,\gamma_\omega(v_\omega + \varepsilon w),\\
I_2(w, \varepsilon, \omega) &= -\,\frac{1}{\varepsilon}\,\pi_\omega \Phi(v_\omega + \varepsilon w, \varepsilon, \omega),\\
I_3(w, \varepsilon, \omega) &= \frac{1}{\varepsilon^2}\,\pi_\omega(\mathbb{P}(v_\omega + \varepsilon w) + \varepsilon\,\mathbb{Q}(v_\omega + \varepsilon w, \varepsilon, \omega)),\\
\widetilde{I}_3(w, \varepsilon, \omega) &= \frac{1}{\varepsilon^2}\,\pi_\omega \gamma_\omega(v_\omega + \varepsilon w).
\end{aligned}$$

By the differentiability of P we have

$$\begin{aligned}
I_1(w, \varepsilon, \omega) &= P'(v_\omega)w + Q(v_\omega + \varepsilon w, \varepsilon, \omega)\\
&= P'(v_\omega)w + Q(v_\omega, 0, \omega) + O(w, \varepsilon, \omega),
\end{aligned}$$

where

$$O(w, \varepsilon, \omega) \to 0 \quad \text{as } \varepsilon \to 0 \tag{3.14}$$

uniformly with respect to $w \in \mathrm{B}(0, r)$, $r > 0$. By (3.10) we get

$$\|\widetilde{I}_1(w, \varepsilon, \omega)\| \le L\|\varepsilon w\|\|w\|,$$

hence $\widetilde{I}_1(w, \varepsilon, \omega) = O(w, \varepsilon, \omega)$. Moreover, we have

$$I_2(w, \varepsilon, \omega) = -\pi_\omega P'(v_\omega)w - \pi_\omega Q(v_\omega, 0, \omega) + O(w, \varepsilon, \omega).$$

By (3.3) we obtain

$$I_3(w, \varepsilon, \omega) = \frac{1}{\varepsilon}\,\pi_\omega P'(v_\omega)w + \pi_\omega Q'_{(1)}(v_\omega, 0, \omega)w + O(w, \varepsilon, \omega).$$

Since $\pi_\omega P'(v_\omega) = 0$ we obtain

$$I_3(w, \varepsilon, \omega) = \pi_\omega Q'_{(1)}(v_\omega, 0, \omega)w + O(w, \varepsilon, \omega).$$

Finally,

$$\begin{aligned}\widetilde{I}_3(w,\varepsilon,\omega) &= \frac{1}{\varepsilon}\,\pi_\omega \int_0^1 [P'(v_\omega+\tau\varepsilon w)-P'(v_\omega)]w\,d\tau \\ &= \pi_\omega \int_0^1 \tau P''(v_\omega)ww\,d\tau + O(w,\varepsilon,\omega) = \frac{1}{2}\,\pi_\omega P''(v_\omega)ww + O(w,\varepsilon,\omega).\end{aligned}$$

Letting

$$\begin{aligned}\Psi(w,0,\omega) = (I-\pi_\omega)[P'(v_\omega)w + Q(v_\omega,0,\omega)]& \\ + \pi_\omega Q'_{(1)}(v_\omega,0,\omega)w + \frac{1}{2}\pi_\omega P''(v_\omega)ww&\end{aligned}$$

it results that Ψ is continuous with respect to $(w,\varepsilon)\in B(0,1)\times[0,1]$.

We now prove the existence of $w_\omega\in\mathbb{E}$ such that $\Psi(w_\omega,0,\omega)=0$, namely, the existence of $w_\omega\in\mathbb{E}$ for which

$$\begin{aligned}(3.15)\quad (I-\pi_\omega)[P'(v_\omega)w_\omega + Q(v_\omega,0,\omega)]& \\ + \pi_\omega\Big[Q'_{(1)}(v_\omega,0,\omega)w_\omega + \frac{1}{2}\,P''(v_\omega)w_\omega\, w_\omega\Big] &= 0.\end{aligned}$$

Let $x_\omega=\pi_\omega w_\omega$ and $y_\omega=(I-\pi_\omega)w_\omega$. Applying $(I-\pi_\omega)$ to (3.15) and taking into account that $P'(v_\omega)|_{(I-\pi_\omega)\mathbb{E}}$ is invertible we get (3.4) for y_ω. Note that $\pi_\omega P''(v_\omega)\,\pi_\omega r\,\pi_\omega s=0$ for any $r,s\in\mathbb{E}$. Indeed, avoiding the dependence on ω, by deriving two times $P(x(\theta))$ with respect to θ we obtain $P''(x(\theta))x'(\theta)\,x'(\theta)+P'(x(\theta))x''(\theta)=0$, on the other hand $\pi x'(\theta)=x'(\theta)$ and $\pi P'(x(\theta))=P'(x(\theta))\pi$, thus $\pi P''(x(\theta))\pi x'(\theta)\,\pi x'(\theta)=-\pi P'(x(\theta))\pi x''(\theta)$, but $\pi P'(x(\theta))\pi=0$ and so $\pi P''(x(\theta))\pi x'(\theta)\,\pi x'(\theta)=0$. If we apply π_ω to (3.15), we obtain the following equation for x_ω:

$$\pi_\omega Q'_{(1)}(v_\omega,0,\omega)x_\omega + \pi_\omega P''(v_\omega)y_\omega\, x_\omega = -\frac{1}{2}\big[P''(v_\omega)y_\omega\, y_\omega + \pi_\omega Q'_{(1)}(v_\omega,0,\omega)y_\omega\big].$$

Since $x_\omega=\alpha_\omega e_\omega$ for some $\alpha_\omega\in\mathbb{R}$, condition (3.5) allows to uniquely determine α_ω. Moreover, the function $\omega\mapsto\ \alpha_\omega$ is measurable. In conclusion, w_ω is given by $w_\omega=x_\omega+y_\omega$.

To complete the proof of Theorem 3.4 we must show that w_ω is a simple zero of $\Psi(w,0,\omega)$. In fact, the application of the random Implicit Function Theorem (Lemma 2.4) to $\Psi(w,\varepsilon,\omega)$ at $(w_\omega,0)$ ensures, by the equivalence of (1.1) and (3.12), the existence of a measurable branch of solution of (1.1) of the form (3.6). For this, evaluate $\Psi'_{(1)}(w_\omega,\varepsilon,\omega)h$, $h\in B(0,1)$. By our assumptions on P and Q we obtain

$$\begin{aligned}\Psi'_{(1)}(w_\omega,\varepsilon,\omega)h =& P'(v_\omega)h + \varepsilon Q'_{(1)}(v_\omega+\varepsilon w_\omega,\varepsilon,\omega)h + \varepsilon P''(v_\omega)hw_\omega \\ &- \pi_\omega P'(v_\omega)h - \pi_\omega\varepsilon Q'_{(1)}(v_\omega+\varepsilon w_\omega,\varepsilon,\omega)h - \varepsilon\pi_\omega P''(v_\omega)hw_\omega\end{aligned}$$

$$+ \frac{1}{\varepsilon}\,\pi_\omega P'(v_\omega)h + \pi_\omega Q'_{(1)}(v_\omega + \varepsilon w_\omega, \varepsilon, \omega)h$$
$$+ \pi_\omega P''(v_\omega)hw_\omega + \frac{1}{\varepsilon}\,r(\varepsilon w_\omega, \omega)h,$$

where $r(w,\omega)/\|w\| \to 0$ as $w \to 0$. Therefore, we have that $\Psi'_{(1)}(w_\omega, \varepsilon, \omega)h$ has a limit when $\varepsilon \to 0$ uniformly with respect to $h \in B(0,1)$, that is,

$$\lim_{\varepsilon\to 0} \Psi'_{(1)}(w_\omega, \varepsilon, \omega)h = (I - \pi_\omega)P'(v_\omega)h + \pi_\omega Q'_{(1)}(v_\omega, 0, \omega)h + \pi_\omega P''(v_\omega)hw_\omega.$$

Furthermore, the map $(\varepsilon, \omega) \mapsto \Psi'_{(1)}(w_\omega, \varepsilon, \omega)$ is strongly measurable. It remains to show that the operator

$$(I - \pi_\omega)P'(v_\omega) + \pi_\omega Q'_{(1)}(v_\omega, 0, \omega) + \pi_\omega P''(v_\omega)w_\omega \tag{3.16}$$

is invertible. We have $P'(v_\omega) = I - F'(v_\omega)$, where the operator $F'(v_\omega)$ is compact and the operator

$$-\pi_\omega I + \pi_\omega F'(v_\omega) + \pi_\omega Q'_{(1)}(v_\omega, 0, \omega) + \pi_\omega P''(v_\omega)w_\omega$$

is also compact, since it takes value in $\mathrm{span}(e_\omega)$, thus the operator given in (3.16) is invertible if we prove that its kernel is trivial. For this, consider

$$(I - \pi_\omega)P'(v_\omega)h + \pi_\omega Q'_{(1)}(v_\omega, 0, \omega)h + \pi_\omega P''(v_\omega)w_\omega h = 0. \tag{3.17}$$

Applying to (3.17) the projector $I - \pi_\omega$ we obtain

$$(I - \pi_\omega)P'(v_\omega)h = 0, \tag{3.18}$$

hence $(I - \pi_\omega)h = 0$.

Finally, if we apply to (3.17) the projector π_ω, taking into account (3.18) and the fact that $\pi_\omega P''(v_\omega)\pi_\omega w_\omega \pi_\omega h = 0$, we get

$$\pi_\omega Q'_{(1)}(v_\omega, 0, \omega)\pi_\omega h + \pi_\omega P''(v_\omega)y_\omega \pi_\omega h = 0.$$

From (3.5) we have $\pi_\omega h = 0$. In conclusion, we have $h = 0$. □

4. Application to a random differential equation

We consider now the random periodically perturbed autonomous differential equation

$$x'(t) = f(x(t)) + \varepsilon\varphi(t, x(t), \varepsilon, \omega) \tag{4.1}$$

where $\varepsilon \in [0,1]$ and $f\colon \mathbb{R}^n \mapsto \mathbb{R}^n$, $\varphi\colon \mathbb{R} \times \mathbb{R}^n \times [0,1] \times \Omega \mapsto \mathbb{R}^n$ are measurable functions, precise conditions on f and φ are given in what follows. We assume that the equation

$$x'(t) = f(x(t)) \tag{4.2}$$

has a T-periodic solution x_0. Then every function of the form $x_\theta(t) = x_0(t+\theta)$, for $\theta \in [0,T]$, is also a T-periodic solution of (4.2).

Let U be a bounded neighbourhood of the set

$$\{x_\theta(t);\, t, \theta \in [0, T]\} \subset \mathbb{R}^n. \tag{4.3}$$

We assume that

- ($\widehat{\mathrm{H}}1$) $f \in \mathcal{C}^2(\overline{U})$,
- ($\widehat{\mathrm{H}}2$) φ is T-periodic with respect to t, $\varphi(t, \cdot, \varepsilon, \omega) \in \mathcal{C}^1(\overline{U})$, and the derivative $\varphi'_{(2)}$ of φ with respect to the second variable is well defined, continuous with respect to (t, x, ε) for each $\omega \in \Omega$.

The global measurability of $\varphi'_{(2)}$ is easily obtained from the fact that the continuous first order partial derivatives of φ with respect to the components of $x \in \mathbb{R}^n$ are limits of measurable difference quotients.

Note that $x_\theta(t)$ is twice differentiable with respect to t and θ, since

$$x''_\theta(t) = x''_0(t+\theta) = f'(x_0(t+\theta))x'_0(t+\theta).$$

In the following, we assume that $x'_0(t) \neq 0$ for any $t \in [0, t]$. Let us denote $a_\theta(t) = f'(x_\theta(t))$. Then x'_θ is a nontrivial solution of the linear equation

$$y' = a_\theta(t)y. \tag{4.4}$$

For each (ε, ω) the Poincaré operator for the equation (4.1) is well defined in a neighbourhood of the set (4.3), that is, for each v in the neighbourhood of (4.3), the solution $x(t, v, \varepsilon, \omega)$ of (4.1) with the initial condition

$$x(0) = v \tag{4.5}$$

exists and is unique on the interval $[0, T]$. We denote the Poincaré operator by $\mathcal{P}_\varepsilon(v, \omega)$, that is,

$$\mathcal{P}_\varepsilon(v, \omega) = x(T, v, \varepsilon, \omega).$$

Thus, for any $\omega \in \Omega$ the fixed points v of $\mathcal{P}_\varepsilon(v, \omega)$ are T-periodic solutions of (4.1).

Note that the map $((\varepsilon, v), \omega) \mapsto \mathcal{P}_\varepsilon(v, \omega)$ is Carathéodory, and thus globally measurable. Indeed, by classical results on continuous dependence of the solution with respect to the initial condition (e.g. [10, Chapter V]), $\mathcal{P}_\varepsilon(v, \omega)$ is continuous with respect to (ε, v) for any ω. The conclusion follows from Lemma 2.3.

Since the operator $\mathcal{P}_0(w)$ does not depend on ω, we denote it simply by $\mathcal{P}_0$. By [10, Theorem 4.1], $\mathcal{P}_0$ is of class $\mathcal{C}^2$ in a neighbourhood of the set (4.3). The points $u_\theta = x_\theta(0), \theta \in [0, T]$, describes a curve of fixed points of $\mathcal{P}_0$. Then $\mathcal{P}'_0(u_\theta)$ is the translation operator from 0 to T along the trajectories of (4.4) (see [17, Theorem 2.1]).

Since, for any $\theta \in [0, T]$, $\mathcal{P}_0(u_\theta) = u_\theta$, we have $\mathcal{P}'_0(u_\theta)x'_\theta(0) = x'_\theta(0)$. Thus $1 \in \mathfrak{S}(\mathcal{P}'_0(u_\theta))$ for any $\theta \in [0, T]$. We assume that

- ($\widehat{\mathrm{H}}3$) 1 is a simple eigenvalue of $\mathcal{P}'_0(u_0)$.

Then, 1 is a simple eigenvalue of all operators $\mathcal{P}_0'(u_\theta)$, $\theta \in [0,T]$. Indeed, (4.4) can be written

$$y' = a_0(t+\theta)y. \tag{4.6}$$

Let us denote by $\mathcal{U}_\theta^\tau$ the operator of translation (or Poincaré operator) from 0 to τ along the trajectories of (4.6). Then $\mathcal{P}_0'(u_\theta) = \mathcal{U}_\theta^T$. If y is a solution of (4.6) with the initial condition $y(0) = y_0$, then the function $z(t) = y(t-\theta)$ is the solution of the equation

$$z'(t) = a_0(t)z(t), \tag{4.7}$$

with the initial condition $z(0) = y(-\theta) = \mathcal{U}_\theta^{-\theta} y_0$. Then $z(T) = \mathcal{U}_0^T \mathcal{U}_\theta^{-\theta} y_0$. But $z(T) = y(T-\theta) = \mathcal{U}_\theta^{-\theta}\mathcal{U}_\theta^T y_0$. Since $\mathcal{U}_\theta^{-\theta}\mathcal{U}_\theta^\theta = \mathcal{I}$ (the identity operator), we get

$$\mathcal{U}_\theta^T = \mathcal{U}_\theta^\theta \mathcal{U}_0^T \mathcal{U}_\theta^{-\theta}.$$

Thus, if 1 is a simple root of the polynomial $\det(\lambda\mathcal{I} - \mathcal{U}_0^T)$, it is also a simple root of $\det(\lambda\mathcal{I} - \mathcal{U}_\theta^T)$.

Let us now investigate the dependence with respect to θ of the eigenvectors of the operators $\mathcal{P}_0'(u_\theta)$ and $\big(\mathcal{P}_0'(u_\theta)\big)^*$ corresponding to the eigenvalue 1. By ($\widehat{\text{H}}$3), we have

$$\mathbb{R}^n = \operatorname{Im}\big(\mathcal{I} - \mathcal{P}_0'(u_0)\big) \oplus \operatorname{Ker}\big(\mathcal{I} - \mathcal{P}_0'(u_0)\big).$$

To see this, since $n = \dim \operatorname{Im}\big(\mathcal{I} - \mathcal{P}_0'(u_0)\big) + 1$, it is sufficient to prove that $\operatorname{Im}\big(\mathcal{I} - \mathcal{P}_0'(u_0)\big) \cap \operatorname{Ker}\big(\mathcal{I} - \mathcal{P}_0'(u_0)\big) = \{0\}$. Arguing by contradiction, let $e \neq 0$, $e \in \operatorname{Ker}\big(\mathcal{I} - \mathcal{P}_0'(u_0)\big)$, and $q \in \mathbb{R}^n$ such that $\big(\mathcal{I} - \mathcal{P}_0'(u_0)\big)q = e$. Then

$$\mathcal{P}_0'(u_0)(-q) = (-q) + e,$$

that is, $-q$ is an adjoint vector to e corresponding to a Jordan block associated with the eigenvalue 1 of $\mathcal{P}_0'(u_0)$, which contradicts ($\widehat{\text{H}}$3).

Now, if $e \in \operatorname{Ker}\big(\mathcal{I} - \mathcal{P}_0'(u_0)\big)$, $e \neq 0$, and $g^* \in \operatorname{Ker}\big(\mathcal{I} - \mathcal{P}_0'(u_0)^*\big)$, $g^* \neq 0$, we have $\langle e, g^*\rangle \neq 0$. Indeed, for any $x \in \mathbb{R}^n$, we have

$$\big\langle \big(\mathcal{I} - \mathcal{P}_0'(u_0)\big)x, g^*\big\rangle = \big\langle x, \big(\mathcal{I} - \mathcal{P}_0'(u_0)^*\big)g^*\big\rangle = 0.$$

If we had $\langle e, g^*\rangle = 0$, this would imply $g^*(\mathbb{R}^n) = 0$, which is a contradiction.

Let $g^* \in \operatorname{Ker}\big(\mathcal{I} - \mathcal{P}_0'(u_0)^*\big)$, $g^* \neq 0$. Then g^* is an eigenvector of $\mathcal{P}_0'(u_0)^*$ corresponding to the eigenvalue 1, that is, $\mathcal{P}_0'(u_0)^* g^* = g^*$. Since $\mathcal{P}_0'(u_0)^*$ is invertible on $\operatorname{span}(g^*)$, we have $g^* = \big(\mathcal{P}_0'(u_0)^*\big)^{-1} g^*$. Using the fact that $\mathcal{P}_0'(u_0)^*$ is the translation operator for the adjoint equation to (4.7)

$$z'(t) = -(a_0(t))^* z(t), \tag{4.8}$$

and taking the solution z which satisfies the initial condition $z(0) = g^*$, we obtain a T-periodic solution z_0 to (4.8) and, by Perron's theorem,

$$\langle x_0'(t), z_0(t)\rangle \equiv \text{const.} \neq 0.$$

Since $z_\theta(t) := z_0(t+\theta)$ is a T periodic solution of the equation

$$z'(t) = -(a_\theta(t))^* z(t),$$

we obtain a branch $z_\theta(0)$ of eigenvectors of $\mathcal{P}'_\theta(u_0)^*$ corresponding to the eigenvalue 1, which depends continuously on θ. Furthermore, we have

$$\langle x'_\theta(0), z_\theta(0)\rangle \equiv \text{const.} \neq 0.$$

We can now state the main result. Let

$$M(\theta,\omega) = \int_0^T \langle \varphi(t, x_\theta(t), 0, \omega), z_\theta(t)\rangle\, dt. \tag{4.9}$$

the Malkin bifurcation function associated to (4.1)

Theorem 4.1. *Assume* ($\widehat{\mathrm{H}}$1)–($\widehat{\mathrm{H}}$3). *Assume furthermore that for any* $\omega \in \Omega$ *there exists* $\theta \in [0,T]$ *such that* $M(\theta,\omega) = 0$ *and* $M'_{(1)}(\theta,\omega) \neq 0$. *Then there exists a measurable function* $\omega \mapsto \theta_\omega$ *from* Ω *to* $[0,T]$ *such that* $M(\theta_\omega,\omega) = 0$ *and* $M'_{(1)}(\theta_\omega,\omega) \neq 0$ *for each* $\omega \in \Omega$, *and there exists a measurable function* $\omega \mapsto w_\omega$ *such that, for* $\varepsilon > 0$ *small enough, equation* (4.1) *has a periodic solution* $x(\varepsilon,\omega)$ *of the form*

$$x(\varepsilon,\omega) = x_{\theta_\omega} + \varepsilon w_\omega + o(\varepsilon,\omega)$$

where $\lim\limits_{\varepsilon\to 0} \|o(\varepsilon,\omega)\|/\varepsilon = 0$, *for any* $\omega \in \Omega$.

Proof. Take

$$P(v) = \mathcal{P}_0(v) - v \quad \text{and} \quad Q(v,\varepsilon,\omega) = \frac{\mathcal{P}_\varepsilon(v,\omega) - \mathcal{P}_0(v)}{\varepsilon}.$$

It is easy to verify that all conditions of the abstract Theorem 3.4 are satisfied. We only need to check that $M(\theta,\omega)$ coincides with the Malkin bifurcation function defined by (3.2). But $Q(v,0,\omega)$ is the value at time T of the solution of the Cauchy problem

$$\begin{cases} w' = f'(x_\theta(t))w + \varphi(t, x_\theta(t), 0, \omega), \\ w(0) = 0. \end{cases} \tag{4.10}$$

Multiplying the first equation of (4.10) by $z_\theta(t)$ and integrating on $[0,T]$, we get

$$\langle w(T), z_\theta(T)\rangle = \int_0^T \langle \varphi(t, x_\theta(t), 0, \omega), z_\theta(t)\rangle\, dt. \qquad \square$$

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