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NONLINEAR VECTOR DUFFING INCLUSIONS WITH NO GROWTH RESTRICTION ON THE ORIENTOR FIELD

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Abstract. We consider nonlinear multivalued Dirichlet Duffing systems. We do not impose any growth condition on the multivalued perturbation. Using tools from the theory of nonlinear operators of monotone type, we prove existence theorems for the convex and the nonconvex problems. Also we show the existence of extremal trajectories and show that such solutions are $C^1_0(T,\mathbb{R}^N)$-dense in the solution set of the convex problem (strong relaxation theorem).

1. Introduction

In this paper, we continue our work on multivalued nonlinear Duffing systems initiated in Papageorgiou–Vetro–Vetro [14]. So, the system under consideration is the following:

$$
(1.1)\qquad \begin{cases} -a(u'(t))' - r(t)|u'(t)|^{p-2}u'(t) \in F(t,u(t)) & \text{for a.a. } t \in T=[0,b], \\ u(0)=u(b)=0, \quad 1<p<+\infty. \end{cases}
$$

Here $a\colon \mathbb{R}^N \to \mathbb{R}^N$ is a monotone homeomorphism and incorporates as special cases many differential operators of interest. In [14] we proved existence theorems for both the convex and nonconvex problems (that is, F is convex valued and respectively nonconvex valued). Also, we proved a relaxation theorem showing

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that the solutions of the nonconvex problem are $C_0^1(T,\mathbb{R}^N)$-dense in the solution set of the convex problem. In [14] the hypotheses on the multivalued perturbation $F(t,x)$ (the orientor field), dictated a sublinear growth for $F(t,\,\cdot\,)$. In contrast here we do not impose any global growth condition on $F(t,\,\cdot\,)$. Instead, we employ a Hartman-type condition on $F(t,\,\cdot\,)$. As in [14], we prove existence theorems for the convex and nonconvex problems. We also show the existence of "extremal solutions", that is, solutions of (1.1) when $F(t,x)$ is replaced by $\operatorname{ext}F(t,x)$ (= the extreme points of $F(t,x)$). In the context of control systems such solutions correspond to states generated by "bang-bang controls". Finally we prove a "strong relaxation theorem" showing that the extremal solutions are $C_0^1(T,\mathbb{R}^N)$-dense in the solution set of the convex problem. The last two results were mentioned as open problems in [14].

The presence of the drift term $r(\,\cdot\,)|u'(\,\cdot\,)|^{p-2}u'(\,\cdot\,)$, characterizes problem (1.1) as nonvariational. So, our method of proof is topological but it is different from the one in [14]. There, the main tool was fixed point theory. Here our arguments are based on the theory of nonlinear operators of monotone type.

We mention that the starting point for the work in [14], was the recent paper of Kalita–Kowalski [10], where the authors studied scalar semilinear Duffing inclusions (the convex problem only). Earlier results for single-valued such equations, can be found in Galewski [2], Kowalski [11], Tomiczek [15].

2. Mathematical background – hypotheses

Let X be a reflexive Banach space and X^* its topological dual. By $\langle\,\cdot\,,\,\cdot\,\rangle$ we denote the duality brackets for the pair (X^*,X). A multivalued map $A\colon D\subseteq X\to 2^{X^*}$ is said to be *monotone*, if

$$\langle u^*-x^*,u-x\rangle\geq 0 \quad \text{for all } (u,u^*),(x,x^*)\in \operatorname{Gr}A.$$

The map $A(\,\cdot\,)$ is *strictly monotone*, if it is monotone and

$$\langle u^*-x^*,u-x\rangle = 0 \;\Rightarrow\; u=x.$$

We say that $A(\,\cdot\,)$ is *maximal monotone*, if it is monotone and

$$\langle u^*-x^*,u-x\rangle\geq 0 \quad \text{for all } (u,u^*)\in \operatorname{Gr}A \;\Rightarrow\; (x,x^*)\in \operatorname{Gr}A.$$

This condition is equivalent to saying that $\operatorname{Gr}A$ is maximal with respect to inclusion among the graphs of all monotone maps.

A nonlinear operator $K\colon X\to X^*$ is said to be of type $(\mathrm{S})_+$, if the following property holds

$$u_n\xrightarrow{w}u \quad\text{in } X \quad\text{and}\quad \limsup_{n\to+\infty}\langle K(u_n),u_n-u\rangle\leq 0 \;\Rightarrow\; u_n\to u \text{ in } X.$$

A multivalued map $V\colon X\to 2^{X^*}$ is said to be *pseudomonotone*, if

(a) for every $x\in X$, $V(x)\subseteq X^*$ is nonempty, convex and w-compact;

(b) for any sequences $\{u_n\}_{n\geq 1} \subseteq X$, $\{u_n^*\}_{n\geq 1} \subseteq X^*$ such that

$$u_n \xrightarrow{w} u \quad \text{in } X, \quad u_n^* \xrightarrow{w} u^* \quad \text{in } X^*, \quad u_n^* \in V(u_n) \quad \text{for all } n \in \mathbb{N},$$
$$\limsup_{n\to+\infty} \langle u_n^*, u_n - u\rangle \leq 0,$$

we have $u^* \in V(u)$ and $\langle u_n^*, u_n\rangle \to \langle u^*, u\rangle$.

Pseudomonotone maps exhibit remarkable surjectivity properties. Recall that a multivalued map $V\colon X \to 2^{X^*} \setminus \{\emptyset\}$ is said to be *coercive*, if

$$\frac{\inf[\langle u^*, u\rangle : u^* \in V(u)]}{\|u\|} \to +\infty \quad \text{as } \|u\| \to +\infty.$$

We have the following surjectivity result for pseudomonotone maps (see Gasiński–Papageorgiou [3, Theorem 3.2.52, p. 336]), see also Francŭ [1].

THEOREM 2.1. *If $V\colon X \to 2^{X^*}$ is pseudomonotone and coercive, then $V(\,\cdot\,)$ is surjective.*

Let Y be a separable Banach space. We introduce the following families of subsets of Y:

$$P_{f(c)}(Y) = \{A \subseteq Y : A \text{ is nonempty, closed (and convex)}\},$$
$$P_{(w)k(c)}(Y) = \{A \subseteq Y : A \text{ is nonempty, (w-) compact (and convex)}\}.$$

Let (Ω, Σ) be a measurable space and $F\colon \Omega \to 2^Y \setminus \{\emptyset\}$. We say that F is *graph measurable* if $\operatorname{Gr} F = \{(\omega, y) \in \Omega \times Y : y \in F(\omega)\} \in \Sigma \otimes B(Y)$ with $B(Y)$ being the Borel σ-field of Y. Suppose that $\mu(\,\cdot\,)$ is a finite measure defined on Σ. By the Yankov–von Neumann–Aumann selection theorem (see Hu and Papageorgiou [8, Theorem 2.14, p. 158]), a graph measurable multifunction $F\colon \Omega \to 2^Y \setminus \{\emptyset\}$ admits a measurable selection, that is, there exists a function $f\colon \Omega \to Y$ which is $(\Sigma, B(Y))$-measurable and $f(\omega) \in F(\omega)$ μ-almost everywhere. Given a graph measurable multifunction $F\colon \Omega \to 2^Y \setminus \{\emptyset\}$ and $1 \leq p \leq +\infty$ we set

$$S_F^p = \{f \in L^p(\Omega, Y) : f(\omega) \in F(\omega) \ \mu\text{-a.e. on } \Omega\}.$$

It is easy to check that

$$S_F^p \neq \emptyset \text{ if and only if } \inf[\|y\| : y \in F(\omega)] \leq \psi(\omega) \ \mu\text{-a.e., with } \psi \in L^p(\Omega).$$

The set S_F^p is *decomposable*, that is,

$$(A, f_1, f_2) \in \Sigma \times S_F^p \times S_F^p \ \Rightarrow\ \chi_A f_1 + \chi_{\Omega\setminus A} f_2 \in S_F^p.$$

Recall that for $C \subseteq \Omega$, χ_C is the characteristic function of C defined by

$$\chi_C(\omega) = \begin{cases} 1 & \text{if } \omega \in C, \\ 0 & \text{if } \omega \notin C. \end{cases}$$

On $P_f(Y)$ we can define an extended metric, known as the *Hausdorff metric*, by setting

$$h(C,K) = \max\left\{\max_{c\in C} d(c,K), \max_{k\in K} d(k,C)\right\} \quad \text{for all } C,K \in P_f(Y).$$

We know that $(P_f(Y),h)$ is a complete metric space. We say that a multifunction $F\colon Y \to P_f(Y)$ is *h-continuous*, if it is continuous from Y into $(P_f(Y),h)$.

Suppose that E, Z are Hausdorff topological spaces and $G\colon E \to 2^Z \setminus \{\emptyset\}$ a multifunction. We say that $G(\,\cdot\,)$ is *upper semicontinuous* (u.s.c.) (resp. *lower semicontinuous* (l.s.c.)), if for all $C \subseteq Z$ closed, $G^-(C) = \{e \in E : G(e)\cap C \neq \emptyset\}$ (resp. $G^+(C) = \{e \in E : G(e) \subseteq C\}$) is closed in E.

For any $A \subseteq \mathbb{R}^N$, we set $|A| = \sup[|a| : a \in A]$ (hereafter $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^N$). Let $T = [0,b]$. By $L^1_w(T,\mathbb{R}^N)$ we denote the Lebesgue space $L^1(T,\mathbb{R}^N)$ equipped with the weak norm $\|\cdot\|_w$ defined by

$$\|u\|_w = \sup\left[\left|\int_s^t u(\tau)\,d\tau\right| : 0 \le s \le t \le b\right], \quad u \in L^1(T,\mathbb{R}^N).$$

Equivalently, we can define the weak norm of u by

$$\|u\|_w = \sup\left[\left|\int_0^t u(\tau)\,d\tau\right| : 0 \le t \le b\right].$$

On the other hand by $L^\theta(T,\mathbb{R}^N)_w$ $(1 \le \theta < +\infty)$, we denote the space $L^\theta(T,\mathbb{R}^N)$ furnished with the weak topology. The following simple fact can be found in Hu and Papageorgiou [9, Lemma 2.8, p. 24].

Proposition 2.2. *If $\{u_n,u\}_{n\ge1} \subseteq L^p(T,\mathbb{R}^N)$, $1 < p < +\infty$, $u_n \xrightarrow{\|\cdot\|_w} u$ and $\sup_{n\ge1}\|u_n\|_p < +\infty$, then $u_n \xrightarrow{w} u$ in $L^p(T,\mathbb{R}^N)$.*

Consider the following nonlinear vector eigenvalue problem:

$$-(|u'(t)|^{p-2}u'(t))' = \widehat{\lambda}|u(t)|^{p-2}u(t) \quad \text{a.e. on } T, u(0) = u(b) = 0.$$

This problem has a smallest eigenvalue $\widehat{\lambda}_1 > 0$, which admits the following variational characterization

$$\widehat{\lambda}_1 = \inf\left[\frac{\|u'\|_p^p}{\|u\|_p^p} : u \in W_0^{1,p}((0,b),\mathbb{R}^N),\ u \neq 0\right] \tag{2.1}$$

(see Gasiński and Papageorgiou [3, p. 768]).

Now we introduce the hypotheses on the map $a\colon \mathbb{R}^N \to \mathbb{R}^N$ and on the drift coefficient $r(\,\cdot\,)$.

H(a) $a\colon \mathbb{R}^N \to \mathbb{R}^N$ is continuous, monotone, $a(0) = 0$ and

$$a(y) = \widehat{c}(|y|)y \quad \text{for all } y \in \mathbb{R}^N,$$

with $\widehat{c}\colon (0,+\infty) \to (0,+\infty)$ continuous, $c_0 t^p \le \widehat{c}(t)t^2$ for all $t > 0$, some $c_0 > 0$ and

$$|a(y)| \le c_1\big[1 + |y|^{p-1}\big] \quad \text{for some } c_1 > 0 \text{ and all } y \in \mathbb{R}^N.$$

Remark 2.3. Evidently $a\colon \mathbb{R}^N \to \mathbb{R}^N$ is maximal monotone and

$$c_0|y|^p \le (a(y), y)_{\mathbb{R}^N} \quad \text{for all } y \in \mathbb{R}^N.$$

Hypotheses $\mathrm{H}(a)$ are more restrictive than those in [14], where no growth restriction was imposed on $a(\,\cdot\,)$. Nevertheless, as the examples which follow illustrate, conditions $\mathrm{H}(a)$ are still very general and incorporate many differential operators of interest.

Example 2.4. The following maps $a\colon \mathbb{R}^N \to \mathbb{R}^N$ satisfy hypotheses $\mathrm{H}(a)$:

$$a(y) = |y|^{p-2}y \quad \text{for all } y \in \mathbb{R}^N,\ 1 < p < +\infty$$

(this map corresponds to the vector p-Laplacian),

$$a(y) = |y|^{p-2}y + |y|^{q-2}y \quad \text{for all } y \in \mathbb{R}^N,\ 1 < q < p < +\infty$$

(this map corresponds to the vector (p,q)-Laplacian),

$$a(y) = (1 + |y|^2)^{(p-2)/2}y \quad \text{for all } y \in \mathbb{R}^N,\ 1 < p < +\infty.$$

$\mathrm{H}(r)$ $r \in L^\infty(T)$ $(T = [0,b])$, $r(t) \ge 0$ for almost all $t \in T$ and $\|r\|_\infty < c_0\widehat{\lambda}_1^{1/p}$.

Finally, in what follows by $\|\cdot\|$, we denote the norm of the Sobolev space $W_0^{1,p}((0,b),\mathbb{R}^N)$. On account of the Poincaré inequality, we have

$$\|u\| = \|u'\|_p \quad \text{for all } u \in W_0^{1,p}((0,b),\mathbb{R}^N).$$

In what follows to simplify our notation, we write

$$W_0^{1,p} = W_0^{1,p}\big((0,b),\mathbb{R}^N\big).$$

Also we write

$$C_0(T,\mathbb{R}^N) = \{u \in C(T,\mathbb{R}^N) : u(0) = u(b) = 0\},$$
$$C_0^1(T,\mathbb{R}^N) = C^1(T,\mathbb{R}^N) \cap C_0(T,\mathbb{R}^N).$$

Finally, if $1 \le p < +\infty$, then $1 < p' \le +\infty$ is defined by $1/p + 1/p' = 1$.

3. The convex problem

In this section we prove an existence theorem for the convex problem (that is, F is convex valued). The precise hypotheses on $F(t,x)$ are the following:

$\mathrm{H}(F)_1$ $F\colon T \times \mathbb{R}^N \to P_{kc}(\mathbb{R}^N)$ is a multifunction such that

(i) for all $x \in \mathbb{R}^N$, the multifunction $t \to F(t,x)$ admits a measurable selection;

(ii) for almost all $t \in T$, $\mathrm{Gr}F(t,\,\cdot\,) \subseteq \mathbb{R}^N \times \mathbb{R}^N$ is closed;

(iii) for every $\eta > 0$, there exists $a_\eta \in L^{p'}(T)_+$ such that

$$|F(t,x)| \le a_\eta(t) \quad \text{for almost all } t \in T, \text{ all } |x| \le \eta$$

and there exists $M > 0$ such that

$$(h,x)_{\mathbb{R}^N} < 0 \quad \text{for almost all } t \in T, \text{ all } |x| \ge M, \text{ all } h \in F(t,x).$$

Remark 3.1. Hypothesis $\mathrm{H}(F)_1$ (i) is satisfied if for all $x \in \mathbb{R}^N$ the multifunction $t \to F(t,x)$ is graph measurable. Hypotheses $\mathrm{H}(F)_1$ (ii), (iii) imply that for almost all $t \in T$, $F(t,\,\cdot\,)$ is usc (see Hu and Papageorgiou [8, Proposition 2.23, p. 43]). The second part of hypothesis $\mathrm{H}(F)_1$ (iii) is a Hartman-type condition on $F(t,\,\cdot\,)$ (see Hartman [6], [7]). In the context of scalar, single-valued Duffing equations, this hypothesis is in fact a sign condition saying that $F(t,x) < 0$ for $x \ge M$ and $F(t,x) > 0$ for $x \le -M$. We stress that no global growth condition is imposed on $F(t,\,\cdot\,)$. This distinguishes our work here from that in [14].

Consider the nonlinear operator $A\colon W_0^{1,p} \to W^{-1,p'} = (W_0^{1,p})^*$ defined by

$$\langle A(u), y\rangle = \int_0^b (a(u'), y')_{\mathbb{R}^N}\, dt \quad \text{for all } u, y \in W_0^{1,p}.$$

From Gasiński and Papageorgiou [5, Problem 2.192, p. 279], we have:

Proposition 3.2. *If hypotheses* $\mathrm{H}(a)$ *hold, then* $A(\,\cdot\,)$ *is continuous, monotone* (*hence maximal monotone too*) *and of type* $(\mathrm{S})_+$.

Then let $V\colon W_0^{1,p} \to W^{-1,p'}$ be defined by

$$\langle V(u), y\rangle = \langle A(u), y\rangle - \int_0^b r(t)|u'|^{p-2}(u', y)_{\mathbb{R}^N}\, dt \quad \text{for all } u, y \in W_0^{1,p},$$

that is,

$$V(u) = A(u) - r(\,\cdot\,)|u'(\,\cdot\,)|^{p-2}u'(\,\cdot\,) \quad \text{for all } u \in W_0^{1,p}. \tag{3.1}$$

Proposition 3.3. *If hypotheses* $\mathrm{H}(a)$, $\mathrm{H}(r)$ *hold, then* $V\colon W_0^{1,p} \to W^{-1,p'}$ *is pseudomonotone and coercive.*

Proof. Suppose that

$$\begin{gathered} u_n \xrightarrow{w} u \quad \text{in } W_0^{1,p}, \qquad V(u_n) \xrightarrow{w} u^* \quad \text{in } W^{-1,p'}, \\ \limsup_{n\to+\infty} \langle V(u_n), u_n - u\rangle \le 0. \end{gathered} \tag{3.2}$$

From (3.2) and since $W_0^{1,p}$ is embedded compactly in $C_0(T,\mathbb{R}^N)$, we have

$$\begin{aligned} u_n \to u \quad &\text{in } C_0(T,\mathbb{R}^N),\\ &\Rightarrow \int_0^b r(t)|u_n'|^{p-2}(u_n', u_n-u)_{\mathbb{R}^N}\,dt \to 0,\\ &\Rightarrow \limsup_{n\to+\infty}\langle A(u_n), u_n-u\rangle \le 0 \quad \text{(see (3.1), (3.2))},\\ &\Rightarrow u_n \to u \text{ in } W_0^{1,p} \quad \text{(see Proposition 3.2)}. \end{aligned} \tag{3.3}$$

From (3.3) it follows that $u_n' \to u'$ in $L^p(T,\mathbb{R}^N)$. So, by passing to a subsequence if necessary, we may assume that

$$u_n'(t) \to u'(t), \qquad |u_n'(t)|, |u'(t)| \le \xi(t) \tag{3.4}$$

for almost all $t \in T$ and all $n \in \mathbb{N}$, with $\xi \in L^p(T)_+$. Then, from (3.4) and the dominated convergence theorem, we have

$$\int_0^b \big||u_n'|^{p-2}u_n' - |u'|^{p-2}u'\big|^{p'}\,dt \to 0 \tag{3.5}$$
$$\Rightarrow\ r|u_n'|^{p-2}u_n' \to r|u'|^{p-2}u' \quad \text{in } L^{p'}(T,\mathbb{R}^N)$$

(see H(r)). On account of (3.3) and Proposition 3.2, we have

$$A(u_n) \to A(u) \quad \text{in } W^{-1,p'}. \tag{3.6}$$

From (3.5) and (3.6) it follows that for the original sequence we have

$$\langle V(u_n), u_n\rangle \to \langle V(u), u\rangle,\ V(u)=u^* \ \Rightarrow\ V(\,\cdot\,) \text{ is pseudomonotone.}$$

Also, for all $u \in W_0^{1,p}$, we have

$$\begin{aligned} \langle V(u), u\rangle &= \int_0^b (a(u'), u')_{\mathbb{R}^N}\,dt - \int_0^b r(t)|u'|^{p-2}(u', u)_{\mathbb{R}^N}\,dt\\ &\ge c_0\|u'\|_p^p - \|r\|_\infty \|u'\|_p^{p-1}\|u\|_p \end{aligned} \tag{3.7}$$

(see hypothesis H(a) and use Hölder's inequality)

$$\begin{aligned} &\ge \bigg[c_0 - \frac{\|r\|_\infty}{\widehat{\lambda}_1^{1/p}}\bigg]\|u'\|_p^p \quad \text{(see (2.1))},\\ &\Rightarrow\ V(\,\cdot\,) \quad \text{is coercive (see H}(r)). \end{aligned}$$

□

Let $N\colon W_0^{1,p} \to 2^{L^{p'}(T,\mathbb{R}^N)}$ be the multifunction defined by

$$N(u) = S^{p'}_{F(\,\cdot\,,u(\,\cdot\,))} \quad \text{for all } u \in W_0^{1,p}.$$

From [14, Proposition 3], we have:

PROPOSITION 3.4. *If hypotheses* H(r), H$(F)_1$ *hold, then the multifunction* $N(\,\cdot\,)$ *has values in* $P_{wkc}(L^{p'}(T,\mathbb{R}^N))$ *and it is usc from* $W_0^{1,p}$ *into* $L^{p'}(T,\mathbb{R}^N)_w$.

Recall that $W_0^{1,p} \hookrightarrow L^p(T,\mathbb{R}^N)$ compactly. Hence

$$L^{p'}(T,\mathbb{R}^N) \hookrightarrow W^{-1,p'} \text{ compactly}$$

(see Gasiński and Papageorgiou [3, Lemma 2.2.27, p. 141])

$$\Rightarrow\ N\colon W_0^{1,p} \to P_{kc}(W^{-1,p'}) \text{ is compact}$$

(that is, $N(\,\cdot\,)$ maps bounded sets to relatively compact ones)

$$\Rightarrow\ u \to V(u) - N(u) \text{ is pseudomonotone} \tag{3.8}$$

(see Gasiński and Papageorgiou [3, Proposition 3.2.51, p. 334]). Also for all $u \in W_0^{1,p}$ and all $f \in S^{p'}_{F(\,\cdot\,,u(\,\cdot\,))}$ we have

$$\begin{aligned}&\langle V(u),u\rangle - \int_0^b (f,u)_{\mathbb{R}^N}\,dt\\ &\geq \bigg[c_0 - \frac{\|r\|_\infty}{\widehat{\lambda}_1^{1/p}}\bigg]\|u'\|_p^p - \int_{\{|u|\geq M\}} (f,u)_{\mathbb{R}^N}\,dt - \int_{\{|u|<M\}} (f,u)_{\mathbb{R}^N}\,dt\end{aligned}$$

(see (3.7))

$$\geq \bigg[c_0 - \frac{\|r\|_\infty}{\widehat{\lambda}_1^{1/p}}\bigg]\|u'\|_p^p - c_2 \quad \text{for some } c_2 > 0$$

(see hypothesis $\mathrm{H}(F)_1$ (iii))

$$\Rightarrow\ u \to V(u) - N(u) \text{ is coercive.} \tag{3.9}$$

Theorem 3.5. *If hypotheses* $\mathrm{H}(a)$*,* $\mathrm{H}(r)$*,* $\mathrm{H}(F)_1$ *hold, then problem* (1.1) *has a solution* $u_0 \in C_0^1(T,\mathbb{R}^N)$*.*

Proof. On account of (3.8), (3.9) and Theorem 2.1, we can find $u_0 \in W_0^{1,p}$ such that

$$\begin{aligned}&0 \in V(u_0) - N(u_0),\\ &\Rightarrow\ V(u_0) = f \quad \text{for some } f \in N(u_0) = S^{p'}_{F(\,\cdot\,,u_0(\,\cdot\,))}\\ &\Rightarrow\ \langle A(u_0),y\rangle - \int_0^b r(t)|u_0'|^{p-2}(u_0',y)_{\mathbb{R}^N}\,dt\\ &\qquad = \int_0^b (f,y)_{\mathbb{R}^N}\,dt \quad \text{for all } y \in W_0^{1,p}\end{aligned}$$

$$\Rightarrow\ -a(u_0'(t))' - r(t)|u_0'(t)|^{p-2}u_0'(t) = f(t) \tag{3.10}$$

for almost all $t \in T$, $u_0(0) = u_0(b) = 0$. From (3.10) as in the proof of [14, Proposition 2], we conclude that $u_0 \in C_0^1(T,\mathbb{R}^N)$. $\square$

REMARK 3.6. An interesting consequence of the above proof, is that if by $\widehat{S}_c \subseteq C_0^1(T, \mathbb{R}^N)$ we denote the solution set of the convex problem, then $\widehat{S}_c \in P_k(C_0^1(T, \mathbb{R}^N))$ (see also [14]).

4. The nonconvex problem

In this section we prove an existence theorem for the nonconvex problem (that is, F has nonconvex values). The precise hypotheses on the orientor field $F(t,x)$ are the following:

$\mathrm{H}(F)_2$ $F\colon T \times \mathbb{R}^N \to P_f(\mathbb{R}^N)$ is a multifunction such that
- (i) $(t,x) \to F(t,x)$ is graph measurable;
- (ii) for almost all $t \in T$, $x \to F(t,x)$ is l.s.c.
- (iii) the same as hypothesis $\mathrm{H}(F)_1$ (iii).

THEOREM 4.1. *If hypotheses* $\mathrm{H}(a)$, $\mathrm{H}(r)$, $\mathrm{H}(F)_2$ *hold, then problem* (1.1) *has a solution* $u_0 \in C_0^1(T, \mathbb{R}^N)$.

PROOF. Consider the multifunction $N\colon W_0^{1,p} \to P_f(L^{p'}(T, \mathbb{R}^N))$ defined by

$$N(u) = S^{p'}_{F(\,\cdot\,,u(\,\cdot\,))} \quad \text{for all } u \in W_0^{1,p}.$$

According to Theorem 7.27 of Hu and Papageorgiou [8, p. 237], $N(\,\cdot\,)$ is l.s.c. and of course it has decomposable values. So, we can apply Theorem 8.7 of Hu and Papageorgiou [8, p. 245] and produce a continuous map $g\colon W_0^{1,p} \to L^{p'}(T, \mathbb{R}^N)$ such that

$$g(u) \in N(u) \quad \text{for all } u \in W_0^{1,p}.$$

We consider the following nonlinear Duffing system

$$\begin{cases} -a(u'(t))' - r(t)|u'(t)|^{p-2}u'(t) = g(u)(t) & \text{for a.a. } t \in T, \\ u(0) = u(b) = 0. \end{cases}$$

Then this problem has a solution $u_0 \in C_0^1(T, \mathbb{R}^N)$ (see Theorem 3.5). Evidently $u_0 \in C_0^1(T, \mathbb{R}^N)$ is a solution of (1.1).

5. Extremal solutions

Let $\mathrm{ext}F(t,x)$ denote the set of extreme points of $F(t,x)$. In this section we deal with the following nonlinear multivalued Duffing system:

$$\begin{cases} -a(u'(t))' - r(t)|u'(t)|^{p-2}u'(t) \in \mathrm{ext}F(t,u(t)) & \text{for almost all } t \in T, \\ u(0) = u(b) = 0. \end{cases} \tag{5.1}$$

The solutions of (5.1) are of course solutions of (1.1) and are known as "extremal solutions". To produce extremal solutions, first we prove an a priori pointwise bound for the solutions of (1.1), using hypothesis $\mathrm{H}(F)_1$ (iii).

Proposition 5.1. *If* H(a), H(r), H$(F)_1$ (iii) *hold and* $u \in C_0^1(T, \mathbb{R}^N)$ *is a solution of problem* (1.1), *then* $\|u\|_\infty \le M$.

Proof. Let $\theta(t) = |u(t)|^2/2$ for all $t \in T$. Suppose that the proposition is not true. Then

$$\theta(t_0) = \max[\theta(t) : t \in T] > \frac{1}{2} M^2.$$

Since $\theta(0) = \theta(b) = 0$, we see that we may assume that $t_0 \in (0, b)$. Then we can find $\delta > 0$ small such that

$$|u(t)| \ge M \quad \text{and} \quad \frac{d}{dt}|u(t)|^2 \le 0 \quad \text{for all } t \in [t_0, t_0 + \delta] \subseteq T. \tag{5.2}$$

We have $-a(u'(t))' - r(t)|u'(t)|^{p-2}u'(t) = f(t)$ for almost all $t \in T$, $u(0) = u(b) = 0$, with $f \in S^{p'}_{F(\,\cdot\,, u(\,\cdot\,))}$.

We take inner product with $u(t)$, integrate over $[t_0, t]$ with $t \le t_0 + \delta$ and perform integration by parts. We obtain

$$\begin{aligned} &(a(u'(t)), u(t))_{\mathbb{R}^N} - (a(u'(t_0)), u(t_0))_{\mathbb{R}^N} \\ &\quad - \int_{t_0}^t (a(u'), u')_{\mathbb{R}^N}\, ds + \int_{t_0}^t r(s)|u'|^{p-2}(u', u)_{\mathbb{R}^N}\, ds = \int_{t_0}^t (-f, u)_{\mathbb{R}^N}\, ds. \end{aligned} \tag{5.3}$$

Note that

$$\frac{d}{dt}\theta(t)\bigg|_{t=t_0} = 0 \;\Rightarrow\; \frac{1}{2}\frac{d}{dt}|u(t)|^2\bigg|_{t=t_0} = 0 \;\Rightarrow\; (u'(t_0), u(t_0))_{\mathbb{R}^N} = 0.$$

Therefore we have

$$(a(u'(t_0)), u(t_0))_{\mathbb{R}^N} = \widehat{c}(|u'(t_0)|)(u'(t_0), u(t_0))_{\mathbb{R}^N} = 0 \tag{5.4}$$

(see hypothesis H(a)). Also we have

$$\int_{t_0}^t r(s)|u'|^{p-2}(u', u)_{\mathbb{R}^N}\, ds = \frac{1}{2}\int_{t_0}^t r(s)|u'|^{p-2}\frac{d}{ds}|u|^2\, ds \le 0 \tag{5.5}$$

(see hypothesis H(r) and (5.2)). Finally, on account of (5.2) and hypothesis H$(F)_1$ (iii), we have

$$\int_{t_0}^t (-f, u)_{\mathbb{R}^N}\, ds > 0. \tag{5.6}$$

Returning to (5.3) and using (5.4)–(5.6), we obtain

$$\begin{aligned} &(a(u'(t)), u(t))_{\mathbb{R}^N} - \int_{t_0}^t (a(u'), u')_{\mathbb{R}^N}\, dt > 0 \\ &\Rightarrow\; (a(u'(t)), u(t))_{\mathbb{R}^N} > 0 \qquad\qquad \text{for all } t \in (t_0, t_0 + \delta] \end{aligned}$$

(see hypothesis H(a))

$$\Rightarrow\ \widehat{c}(|u'(t)|)(u'(t),u(t))_{\mathbb{R}^N} > 0 \qquad \text{for all } t\in(t_0,t_0+\delta],$$
$$\Rightarrow\ (u'(t),u(t))_{\mathbb{R}^N} = \frac{1}{2}\frac{d}{dt}|u(t)|^2 > 0 \quad \text{for all } t\in(t_0,t_0+\delta],$$

a contradiction (see (5.2)). Therefore

$$\theta(t_0)=\frac{1}{2}|u(t_0)|^2 \le \frac{1}{2}M^2 \ \Rightarrow\ |u(t)|\le M \quad \text{for all } t\in T. \qquad \square$$

Let $p_M\colon \mathbb{R}^N\to\mathbb{R}^N$ be the M-radial retraction defined by

$$p_M(x)=\begin{cases} x & \text{if } |x|\le M,\\ \dfrac{Mx}{|x|} & \text{if } M<|x|,\end{cases} \qquad \text{for all } x\in\mathbb{R}^N.$$

Evidently $p_M(\,\cdot\,)$ is nonexpansive, that is,

$$|p_M(x)-p_M(v)|\le|x-v| \quad \text{for all } x,v\in\mathbb{R}^N.$$

On account of Proposition 5.1, in (5.1) we may replace F by F_0 defined by

$$F_0(t,x)=F(t,p_M(x)) \quad \text{for all } t\in T \text{ and all } x\in\mathbb{R}^N.$$

Note that $F_0(t,x)$ satisfies the same conditions as $F(t,x)$ and in addition

$$|F_0(t,x)|\le a_M(t) \quad \text{for a.a. } t\in T,\ \text{all } x\in\mathbb{R}^N,\ \text{with } a_M\in L^{p'}(T)_+.$$

Therefore without any loss of generality we may assume that

$$|F(t,x)|\le a_M(t) \quad \text{for a.a. } t\in T \text{ and all } x\in\mathbb{R}^N.$$

Let $\mathcal{D}=\{h\in L^{p'}(T,\mathbb{R}^N) : |h(t)|\le a_M(t) \text{ for almost all } t\in T\}$. We consider the following Duffing system

$$-a(u'(t))' - r(t)|u'(t)|^{p-2}u'(t) = h(t) \tag{5.7}$$

for almost all $t\in T$, $u(0)=u(b)=0$. Let

$$C_0=\{u\in C_0^1(T,\mathbb{R}^N) : u \text{ is a solution of (5.7) with } h\in\mathcal{D}\}.$$

PROPOSITION 5.2. *If hypotheses* H(a), H(r) *hold, then* $C_0\subseteq C_0^1(T,\mathbb{R}^N)$ *is compact.*

PROOF. Let $\{u_n\}_{n\ge1}\subseteq C_0$. We have

$$-a(u_n')' - r(t)|u_n'|^{p-2}u_n' = h_n, \quad u_n(0)=u_n(b)=0 \quad \text{for all } n\in\mathbb{N}. \tag{5.8}$$

We act with u_n in (5.8) and after integration by parts, we obtain

$$c_0\|u_n'\|_p^p - \|r\|_\infty \|u_n'\|_p^{p-1}\|u_n\|_p \le \int_0^b (h_n,u_n)_{\mathbb{R}^N}\,dt \le c_3\|u_n'\|_p \quad \text{for some } c_3>0, \tag{5.9}$$

$$\Rightarrow \bigg[c_0 - \frac{\|r\|_\infty}{\widehat{\lambda}_1^{1/p}}\bigg]\|u_n'\|_p^{p-1} \le c_3 \quad \text{for all } n\in\mathbb{N},$$

$$\Rightarrow \{u_n\}_{n\ge1}\subseteq W_0^{1,p} \quad \text{is bounded (see hypothesis H}(r)),$$

$$\Rightarrow \{u_n\}_{n\ge1}\subseteq C_0(T,\mathbb{R}^N) \quad \text{is relatively compact.}$$

From (5.8) we have

$$a(u_n'(t)) = a(u_n'(0)) - \int_0^t r(s)|u_n'|^{p-2}u_n'\,ds - \int_0^t h_n\,ds \tag{5.10}$$

for all $t\in T$ and all $n\in\mathbb{N}$,

$$\Rightarrow\ u_n'(t) = a^{-1}\bigg[a(u_n'(0)) - \int_0^t \big[r(s)|u_n'|^{p-2}u_n' + h_n\big]\,ds\bigg] \tag{5.11}$$

for all $t\in T$ and all $n\in\mathbb{N}$.

Note that $\int_0^b u_n'(t)\,dt = 0$. So, from (5.11) and Proposition 3.1 (ii) of Manásevich and Mawhin [12], we have that

$$\{a(u_n'(0))\}_{n\ge1}\subseteq \mathbb{R}^{\mathbb{N}} \quad \text{is bounded.} \tag{5.12}$$

Then, from (5.10), (5.12) and Arzelà–Ascoli theorem, it follows that

$$\{a(u_n'(\,\cdot\,))\}_{n\ge1}\subseteq C(T,\mathbb{R}^{\mathbb{N}}) \quad \text{is relatively compact.} \tag{5.13}$$

Consider the map $\widehat{a}^{-1}\colon C(T,\mathbb{R}^{\mathbb{N}})\to C(T,\mathbb{R}^{\mathbb{N}})$ defined by

$$\widehat{a}^{-1}(u)(\,\cdot\,) = a^{-1}(u(\,\cdot\,)) \quad \text{for all } u\in C(T,\mathbb{R}^{\mathbb{N}}).$$

Evidently this map is continuous and bounded (that is, maps bounded sets to bounded sets). So, from (5.13) it follows that

$$\{u_n'\}_{n\ge1}\subseteq C(T,\mathbb{R}^{\mathbb{N}}) \quad \text{is relatively compact.} \tag{5.14}$$

From (5.9) and (5.14), we have that

$$\{u_n\}_{n\ge1}\subseteq C_0^1(T,\mathbb{R}^{\mathbb{N}}) \quad \text{is relatively compact.}$$

So, we may assume that $u_n\to u$ in $C_0^1(T,\mathbb{R}^{\mathbb{N}})$. Evidently $\mathcal{D}\subseteq L^{p'}(T,\mathbb{R}^N)$ is w-compact and so we may assume that $h_n \xrightarrow{w} h$ in $L^{p'}(T,\mathbb{R}^N)$, for $h\in\mathcal{D}$.

We have

$$\int_0^b (a(u_n'),y')_{\mathbb{R}^N}\,dt - \int_0^b r(t)|u_n'|^{p-2}(u_n',y)_{\mathbb{R}^N}\,dt = \int_0^b (h_n,y)_{\mathbb{R}^N}\,dt$$

for all $y\in W_0^{1,p}$ and all $n\in\mathbb{N}$.

Passing to the limit as $n \to +\infty$ and using (5.12) and (5.13), we obtain

$$\int_0^b (a(u'), y')_{\mathbb{R}^N}\, dt - \int_0^b r(t)|u'|^{p-2}(u', y)_{\mathbb{R}^N}\, dt = \int_0^b (h, y)_{\mathbb{R}^N}\, dt$$

for all $y \in W_0^{1,p}$, then $u \in C_0$. Therefore $C_0 \subseteq C_0^1(T, \mathbb{R}^N)$ is compact. □

Now let $C = \overline{\text{conv}} C_0 \in P_{kc}(C_0^1(T, \mathbb{R}^N))$ (see Gasiński and Papageorgiou [4, Theorem 5.86, p. 852]). If by $\widehat{S}_c$ we denote the solution set of the convex problem, then $\widehat{S}_c \subseteq C$.

To produce extremal solutions (that is, solutions of (5.1)), we introduce the following conditions on the orientor field $F(t,x)$.

$\text{H}(F)_3$ $F\colon T \times \mathbb{R}^N \to P_{kc}(\mathbb{R}^N)$ is a multifunction such that
- (i) for all $x \in \mathbb{R}^N$, $t \to F(t,x)$ is graph measurable;
- (ii) for almost all $t \in T$, $x \to F(t,x)$ is h-continuous;
- (iii) the same as hypothesis $\text{H}(F)_1$ (iii).

Remark 5.3. Hypotheses $\text{H}(F)_3$ (i), (ii) imply that $(t,x) \to F(t,x)$ is graph measurable (see Hu and Papageorgiou [8, Proposition 7.9, p. 229]). As we already mentioned, by replacing F with F_0 if necessary (see Proposition 5.1), without any loss of generality we may assume that

$$|F(t,x)| \le a_M(t) \tag{5.15}$$

for almost all $t \in T$, all $x \in \mathbb{R}^N$, with $a_M \in L^{p'}(T)_+$.

Theorem 5.4. *If hypotheses* $\text{H}(a)$, $\text{H}(r)$, $\text{H}(F)_3$ *hold, then problem* (5.1) *has a solution* $u_0 \in C_0^1(T, \mathbb{R}^N)$.

Proof. Recall that $C = \overline{\text{conv}} C_0 \in P_{kc}(C_0^1(T, \mathbb{R}^N))$ (see Proposition 5.2). We consider the multifunction $G\colon C \to P_{wkc}(L^{p'}(T, \mathbb{R}^N))$ defined by

$$G(u) = S^{p'}_{F(\,\cdot\,, u(\,\cdot\,))} \quad \text{for all } u \in C.$$

Using Theorem 8.31 of Hu and Papageorgiou [8, p. 260], we can find a continuous map $g\colon C \to L^1_w(T, \mathbb{R}^N)$ such that

$$g(u) \in \text{ext}\, G(u) = \text{ext} S^{p'}_{F(\,\cdot\,, u(\,\cdot\,))} = S^{p'}_{\text{ext} F(\,\cdot\,, u(\,\cdot\,))} \quad \text{for all } u \in C \tag{5.16}$$

(see Hu and Papageorgiou [8, Theorem 4.6, p. 192]). On account of (5.15) and Proposition 2.2, we have that $g\colon C \to L^{p'}(T, \mathbb{R}^N)_w$ is continuous. We consider the following Duffing system:

$$\begin{cases} -a(u'(t))' - r(t)|u'(t)|^{p-2}u'(t) = g(u)(t) & \text{for a.a. } t \in T, \\ u(0) = u(b) = 0. \end{cases}$$

By Theorem 3.5, this problem has a solution $u_0 \in C_0^1(T, \mathbb{R}^N)$. Evidently (5.16) implies that u_0 is an extremal solution (that is, solves problem (5.1)). □

6. Strong relaxation theorem

In this section, under stronger conditions on $a(\,\cdot\,)$ and $F(\,\cdot\,,\,\cdot\,)$ we prove a strong relaxation theorem. Recall that if $E \subseteq \mathbb{R}^N$ is compact, then $\operatorname{conv} E \in P_{kc}(\mathbb{R}^N)$ (see Gasiński–Papageorgiou [4, Problem 5.93, p. 889]). We consider the following multivalued Duffing systems:

$$-a(u'(t))' - r(t)u(t) \in \operatorname{ext}\operatorname{conv} F(t, u(t)) \tag{6.1}$$

for almost all $t \in T$, $u(0) = u(b) = 0$,

$$-a(u'(t))' - r(t)u(t) \in \operatorname{conv} F(t, u(t)) \tag{6.2}$$

for almost all $t \in T$, $u(0) = u(b) = 0$.

Let $\widehat{S}_e \subseteq C_0^1(T, \mathbb{R}^N)$ be the solution set of (6.1) and $\widehat{S}_c \subseteq C_0^1(T, \mathbb{R}^N)$ the solution set of (6.2). We know that $\widehat{S}_c \in P_k(C_0^1(T, \mathbb{R}^N))$. Our aim is to show that

$$\overline{\widehat{S}_e}^{C_0^1(T,\mathbb{R}^N)} = \widehat{S}_c.$$

Such a density result is known as *strong relaxation theorem*. It is important in control theory in connection with the "bang-bang principle".

To have a strong relaxation theorem, we need stronger conditions on $a(\,\cdot\,)$ and $F(\,\cdot\,,\,\cdot\,)$.

H$(a)'$ $a\colon \mathbb{R}^N \to \mathbb{R}^N$ is continuous, monotone, $a(0) = 0$, $a(y) = \widehat{c}(|y|)y$ for all $y \in \mathbb{R}^N$, with $\widehat{c}\colon (0, +\infty) \to (0, +\infty)$ continuous, $c_0 t^2 \le \widehat{c}(t)t^2$ for all $t > 0$, some $c_0 > 0$, for every $\eta > 0$, there exists $\widetilde{c}_\eta > 0$ such that

$$\widetilde{c}_\eta |y - v|^2 \le (a(y) - a(v), y - v)_{\mathbb{R}^N} \quad \text{for all } |y|, |v| \le \eta$$

and $|a(y)| \le c_1(1 + |y|)$ for some $c_1 > 0$ and all $y \in \mathbb{R}^N$.

Remark 6.1. Evidently $a(\,\cdot\,)$ is strictly monotone and maximal monotone. Also $a(\,\cdot\,)$ is a homeomorphism onto $\mathbb{R}^N$ and $|a^{-1}(y)| \to +\infty$ as $|y| \to +\infty$.

Example 6.2. The following maps satisfy hypotheses H$(a)'$:

$$a(y) = y \qquad \text{for all } y \in \mathbb{R}^N,$$

$$a(y) = |y|^{q-2}y + y \qquad \text{for all } y \in \mathbb{R}^N, \text{ with } 1 < q < 2,$$

$$a(y) = \begin{cases} 2|y|^{q-2}y & \text{if } |y| \le 1 \\ 2y & \text{if } 1 < |y| \end{cases} \qquad \text{for all } y \in \mathbb{R}^N, \text{ with } 1 < q < 2,$$

$$a(y) = (1 + |y|^q)^{(2-q)/q}|y|^{q-2}y \qquad \text{for all } y \in \mathbb{R}^N, \text{ with } 1 < q \le 2.$$

H$(F)_4$ $F\colon T \times \mathbb{R}^N \to P_k(\mathbb{R}^N)$ is a multifunction such that

(i) for all $x \in \mathbb{R}^N$, $t \to F(t, x)$ is graph measurable;

(ii) for every $\eta > 0$, there exists $k_\eta \in L^1(T)_+$ such that

$$\widetilde{c}_\eta - \frac{\|r\|_\infty}{\widehat{\lambda}_1^{1/p}} - \|k_\eta\|_\infty b > 0,$$

and $h(\operatorname{conv} F(t,x), \operatorname{conv} F(t,v)) \le k_\eta(t)|x-v|$ for almost all $t \in T$, all $|x|, |v| \le \eta$;

(iii) the same as hypothesis $\mathrm{H}(F)_1$ (iii).

Remark 6.3. As before on account of Proposition 5.1 and by replacing F with F_0, without any loss of generality, we may assume that

(6.3) $|F(t,x)| \le a_M(t)$ for almost all $t \in T$, all $x \in \mathbb{R}^N$, with $a_M \in L^{p'}(T)_+$.

Hypotheses $\mathrm{H}(F)_4$ (i), (ii) imply that $(t,x) \to F(t,x)$ is graph measurable and then so is $(t,x) \to \operatorname{conv} F(t,x)$. Finally, note that

$$\operatorname{ext} \operatorname{conv} F(t,x) \subseteq F(t,x) \quad \text{for all } (t,x) \in T \times \mathbb{R}^N.$$

Theorem 6.4. *If hypotheses* $\mathrm{H}(a)'$, $\mathrm{H}(r)$, $H(F)_4$ *hold, then* $\overline{\widehat{S}_e}^{C_0^1(T,\mathbb{R}^N)} = \widehat{S}_c$.

Proof. As before $C = \overline{\operatorname{conv}}\, C_0 \in P_{kc}(C_0^1(T,\mathbb{R}^N))$ (see Proposition 5.2). Let $u \in \widehat{S}_c$. We have

$$-a(u')' - r(t)|u'|^{p-2}u' = f \quad \text{with } f \in S^{p'}_{\operatorname{conv} F(\,\cdot\,, u(\,\cdot\,))}.$$

For $v \in C$ and $\varepsilon > 0$, let $R^v_\varepsilon \colon T \to 2^{\mathbb{R}^N} \setminus \{\emptyset\}$ be defined by

$$R^v_\varepsilon(t) = \{h \in \mathbb{R}^N : |f(t) - h| < \varepsilon + d(f(t), \operatorname{conv} F(t, v(t))),\ h \in \operatorname{conv} F(t, v(t))\}.$$

Clearly $R^v_\varepsilon(\,\cdot\,)$ is graph measurable. So, we can use the Yankov–von Neumann–Aumann selection theorem (see Hu and Papageorgiou [8, Theorem 2.14, p. 158]) and infer that $R^v_\varepsilon(\,\cdot\,)$ admits a measurable selection. Such a selection belongs in $L^{p'}(T,\mathbb{R}^N)$ (see (6.3)).

Next let $L_\varepsilon \colon C \to 2^{L^{p'}(T,\mathbb{R}^N)}$ be defined by

$$L_\varepsilon(v) = S^{p'}_{R^v_\varepsilon} \quad \text{for all } v \in C.$$

From the previous argument, we see that $L_\varepsilon(v) \neq \emptyset$ for all $v \in C$. Also $v \to L_\varepsilon(v)$ is l.s.c. (see Hu and Papageorgiou [8, Lemma 8.3, p. 239]). It follows that $v \to \overline{L_\varepsilon(v)}$ is l.s.c. and has decomposable values. So, we can find $l_\varepsilon \colon C \to L^{p'}(T,\mathbb{R}^N)$ continuous such that

$$l_\varepsilon(v) \in \overline{L_\varepsilon(v)} \quad \text{for all } v \in C$$

(see Hu and Papageorgiou [8, Theorem 8.7, p. 245]). Then, on account of [8, Theorem 8.31, p. 260], we can find a continuous map $\theta_\varepsilon \colon C \to L^{p'}_w(T,\mathbb{R}^N)$ such that

$$\theta_\varepsilon(v) \in S^{p'}_{\operatorname{ext} \operatorname{conv} F(\,\cdot\,, v(\,\cdot\,))}, \quad \|\theta_\varepsilon(v) - l_\varepsilon(v)\|_w < \varepsilon \quad \text{for all } v \in C. \tag{6.4}$$

Now, let $\varepsilon_n \to 0^+$ and set $l_n = l_{\varepsilon_n}$, $\theta_n = \theta_{\varepsilon_n}$ for all $n \in \mathbb{N}$. We consider the following Duffing system

$$-a(u'(t))' - r(t)u'(t) = \theta_n(u)(t) \quad \text{for a.a. } t \in T,\ u(0) = u(b) = 0,\ n \in \mathbb{N}.$$

This problem has a solution $u_n \in C_0^1(T, \mathbb{R}^N)$ (see Theorem 3.5). Note that $u_n \in C$ for all $n \in \mathbb{N}$ (see (6.3)) and $C \in P_{kc}(C_0^1(T, \mathbb{R}^N))$. So, we may assume that

$$u_n \to \widehat{u} \quad \text{in } C_0^1(T, \mathbb{R}^N). \tag{6.5}$$

We have

$$-a(u_n')' + a(u')' - r(t)[u_n' - u'] = \theta_n(u_n) - f \quad \text{for all } n \in \mathbb{N}.$$

We act with $u_n - u$ and after integration by parts, we have

$$\begin{aligned}\int_0^b (a(u_n') - a(u'), u_n' - u')_{\mathbb{R}^N}\, dt - \int_0^b r(t)(u_n' - u', u_n - u)_{\mathbb{R}^N}\, dt \\ = \int_0^b (\theta_n(u_n) - f, u_n - u)_{\mathbb{R}^N}\, dt.\end{aligned} \tag{6.6}$$

Let $\eta = \max\Big\{\sup_{n\ge 1} \|u_n\|_{C_0^1(T,\mathbb{R}^N)}, \|u\|_{C_0^1(T,\mathbb{R}^N)}\Big\} < +\infty$ (see (6.5)). Hypothesis H$(a)'$ implies that

$$\widetilde{c}_\eta \|u_n' - u'\|_2^2 \le \int_0^b (a(u_n') - a(u'), u_n' - u')_{\mathbb{R}^N}\, dt. \tag{6.7}$$

Also, we have

$$\begin{aligned}\int_0^b r(t)(u_n' - u', u_n - u)_{\mathbb{R}^N}\, dt \\ \le \|r\|_\infty \|u_n' - u'\|_2 \|u_n - u\|_2 \le \frac{\|r\|_\infty}{\widehat{\lambda}_1^{1/2}} \|u_n' - u'\|_2^2.\end{aligned} \tag{6.8}$$

Finally, note that for all $n \in \mathbb{N}$

$$\begin{aligned}&\int_0^b (\theta_n(u_n) - f, u_n - u)_{\mathbb{R}^N}\, dt \\ &= \int_0^b (\theta_n(u_n) - l_n(u_n), u_n - u)_{\mathbb{R}^N} dt + \int_0^b (l_n(u_n) - f, u_n - u)_{\mathbb{R}^N}\, dt \\ &\le \int_0^b (\theta_n(u_n) - l_n(u_n), u_n - u)_{\mathbb{R}^N}\, dt + \varepsilon_n b + \int_0^b k_\eta(t)|u_n - u|^2\, dt.\end{aligned} \tag{6.9}$$

From (6.4) and Proposition 2.2, we have

$$\int_0^b (\theta_n(u_n) - l_n(u_n), u_n - u)_{\mathbb{R}^N}\, dt \to 0 \quad \text{as } n \to +\infty. \tag{6.10}$$

We return to (6.6), use (6.7)–(6.9), pass to the limit as $n \to +\infty$ and finally use (6.5), (6.10) and Jensen's inequality. Then

$$\bigg[\widetilde{c}_\eta - \frac{\|r\|_\infty}{\widehat{\lambda}_1^{1/2}} - \|k_\eta\|_\infty b\bigg]\|\widehat{u}' - u'\|_2^2 \leq 0 \ \Rightarrow\ \widehat{u} = u$$

(see hypothesis $\mathrm{H}(F)_4$ (ii)). Since $u_n \in \widehat{S}_e$ and $u_n \to u$ in $C_0^1(T,\mathbb{R}^N)$ (see (6.5)), we conclude that $\overline{\widehat{S}_e}^{C_0^1(T,\mathbb{R}^N)} = \widehat{S}_c$. $\square$

Remark 6.5. A careful inspection of the proofs, reveals that the positivity of the drift coefficient $r(\,\cdot\,)$ (see hypothesis $\mathrm{H}(r)$) was first used in the proof of Proposition 5.1. So, Theorems 3.5 and 4.1 are valid without the assumption that $r(t) \geq 0$ for almost all $t \in T$. It will be interesting to know if we can remove this restriction also in Theorems 5.4 and 6.4. Theorems 5.4 and 6.4 provide answers to questions raised at the end of [14]. Extremal trajectories and a strong relaxation theorem, were proved for a different class of multivalued second order systems, in Papageorgiou, Vetro and Vetro [13].

We conclude with an example of a control system.

Example 6.6. Consider the following control system:

$$\begin{cases} -a(u'(t))' - r(t)|u'(t)|^{p-2}u'(t) = f(t,u(t))v(t) & \text{a.e. on } T, \\ u(0) = u(b) = 0, \quad v(t) \in V(t) & \text{for a.a. } t \in T. \end{cases} \tag{6.11}$$

In system (6.11), $f\colon T \times \mathbb{R}^N \to \mathcal{L}(\mathbb{R}^m,\mathbb{R}^N)$ is a Carathéodory function and $V\colon T \to P_k(\mathbb{R}^m)$ is graph measurable. We assume that

$$\begin{aligned} &|f(t,x)| \leq a_\eta(t) && \text{for a.a. } t \in T, \text{ all } |x| \leq \eta \text{ with } a_\eta \in L^{p'}(T), \\ &|U(t)| \leq \theta && \text{for a.a. } t \in T, \text{ some } \theta > 0, \end{aligned}$$

and there exists M such that

$$(f(t,x)u,x)_{\mathbb{R}^N} < 0 \quad \text{for a.a. } t \in T, \text{ all } |x| \geq M \text{ and all } u \in U(t).$$

Then the control system (6.11) has admissible state-control pairs and if $f(t,\,\cdot\,)$ is locally Lipschitz with local Lipschitz constant $\widehat{k}_\eta(\,\cdot\,) \in L^1(T)$, then the states of the nonconvex problem are $C_0^1(T,\mathbb{R}^N)$-dense in those of the convexified system (control constraint set $\operatorname{conv} U(t)$ for all $t \in T$, see Theorem 6.4).

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