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A WEIGHTED TRUDINGER–MOSER TYPE INEQUALITY AND ITS APPLICATIONS TO QUASILINEAR ELLIPTIC PROBLEMS WITH CRITICAL GROWTH IN THE WHOLE EUCLIDEAN SPACE

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Abstract. We establish a version of the Trudinger–Moser inequality involving unbounded or decaying radial weights in weighted Sobolev spaces. In the light of this inequality and using a minimax procedure we also study existence of solutions for a class of quasilinear elliptic problems involving exponential critical growth.

1. Introduction and main results

We recall that if Ω is a bounded domain in $\mathbb{R}^n$ $(n \geq 2)$, the classical Trudinger–Moser inequality (cf. [31], [38]) asserts that $e^{\alpha|u|^{n'}} \in L^1(\Omega)$, for all $u \in W_0^{1,n}(\Omega)$ and $\alpha > 0$ and there exists a constant $C(n) > 0$ such that

$$\sup_{\|u\|_n \leq 1} \int_\Omega e^{\alpha|u|^{n'}}\, dx \leq C(n)|\Omega|, \quad \text{if } \alpha \leq \alpha_n, \tag{1.1}$$

where $n' = n/(n-1)$, $\alpha_n = n\omega_{n-1}^{1/(n-1)}$, $\|u\|_n := \big(\int_\Omega |\nabla u|^n\, dx\big)^{1/n}$ and ω_{n-1} is the surface area of the unit sphere in $\mathbb{R}^n$. Moreover, the inequality (1.1)

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is sharp in the sense that if $\alpha > \alpha_n$ the correspondent supremum is $+\infty$ and clearly as the Lebesgue's measure $|\Omega| \to +\infty$ no uniform bound can be retained in (1.1). In recent years, related inequalities for unbounded domains have been proposed by D. Cao [18] and B. Ruf [33] in two dimensions and by S. Adachi and K. Tanaka [1], J.M. do Ó [23] and Y. Li and B. Ruf [30] in high dimensions. See also R. Adams [2]. On the other hand, Adimurthi and K. Sandeep in [4] extended the Trudinger–Moser inequality (1.1) for singular weights. More precisely, they have proved that if Ω is a smooth bounded domain in $\mathbb{R}^n$ containing the origin, $u \in W_0^{1,n}(\Omega)$ and $\beta \in [0,n)$, then there exists a constant $C(n,\beta) > 0$ such that

$$\sup_{\|\nabla u\|_n \le 1} \int_\Omega \frac{e^{\alpha|u|^{n'}}}{|x|^\beta}\,dx < C(n,\beta)|\Omega| \quad \text{if and only if} \quad 0 < \alpha \le \alpha_n\biggl(1 - \frac{\beta}{n}\biggr). \tag{1.2}$$

Note that the supremums in inequalities (1.1) and (1.2) become infinite for domains Ω which do not have finite measure, and therefore the Trudinger–Moser inequality is not available for this class of domains in the classical formulation.

In [23], the author has proved that if $\|\nabla u\|_n \le 1$, $\|u\|_n \le M < \infty$ and $0 < \alpha < \alpha_n$, then there exists a constant $C(n,M,\alpha) > 0$ such that

$$\int_{\mathbb{R}^n} \Phi_\alpha(u)\,dx \le C(n,M,\alpha), \quad \text{where} \quad \Phi_\alpha(s) := e^{\alpha|s|^{n'}} - \sum_{j=0}^{n-2} \frac{\alpha^j |s|^{jn'}}{j!}.$$

A further result in this direction was obtained by S. Adachi and K. Tanaka in [1] who proved that for any $0 < \alpha < \alpha_n$, there exists a constant $C(n,\alpha) > 0$ such that

$$\int_{\mathbb{R}^n} \Phi_\alpha\biggl(\frac{|u|}{\|\nabla u\|}\biggr)\,dx \le C(n,\alpha)\frac{\|u\|_n^n}{\|\nabla u\|_n^n}, \quad \text{for all } u \in W_0^{1,n}(\mathbb{R}^n).$$

In [30], the authors have proved that if the Dirichlet norm $\|\nabla u\|_n$ is replaced by the standard Sobolev norm

$$\|u\|_{1,n} := \biggl[\int_\Omega (|\nabla u|^n + |u|^n)\,dx\biggr]^{1/n},$$

then there exists a constant $d_n(\alpha) > 0$ (independent of Ω) such that for any domain $\Omega \subset \mathbb{R}^n$,

$$\sup_{\|u\|_{1,n} \le 1} \int_\Omega \Phi_\alpha(u)\,dx \le d_n(\alpha), \tag{1.3}$$

where

$$d_n(\alpha) = \begin{cases} < \infty & \text{if } \alpha \le \alpha_n, \\ = +\infty & \text{if } \alpha > \alpha_n. \end{cases}$$

Moreover, this inequality (1.3) is sharp in the sense that the correspondent supremum becomes infinite as $\alpha > \alpha_n$. Finally, in [5], the inequality (1.2) has been generalized to the whole space $\mathbb{R}^n$, $n \geq 2$, as follows

$$\sup_{\|u\|_{1,\tau}\leq 1} \int_{\mathbb{R}^n} \frac{\Phi_\alpha(u)}{|x|^\beta}\, dx < \infty \quad \text{if and only if} \quad 0 < \alpha \leq \alpha_n\left(1 - \frac{\beta}{n}\right),$$

where $\tau > 0$ and

$$\|u\|_{1,\tau} = \left(\int_{\mathbb{R}^n} \left(|\nabla u|^n + \tau |u|^n\right) dx\right)^{1/n}, \quad u \in W^{1,n}(\mathbb{R}^n).$$

Throughout this work, we consider some weight functions $V(|x|)$ and $Q(|x|)$ satisfying the following assumptions:

(V) $V \in C(0,\infty)$, $V(r) > 0$ and there exist $a, a_0, a_1 > -n$ such that

$$\liminf_{r\to+\infty} \frac{V(r)}{r^a} > 0, \quad \liminf_{r\to 0^+} \frac{V(r)}{r^{a_0}} > 0 \quad \text{and} \quad \limsup_{r\to 0^+} \frac{V(r)}{r^{a_1}} < \infty.$$

(Q) $Q \in C(0,\infty)$, $Q(r) > 0$ and there exist $b < a$, $b_0 > -n$ such that

$$\limsup_{r\to 0^+} \frac{Q(r)}{r^{b_0}} < \infty \quad \text{and} \quad \limsup_{r\to+\infty} \frac{Q(r)}{r^b} < \infty.$$

Example 1.1. (a) The standard models of potentials V and Q satisfying (V) and (Q), respectively, are of the form

$$V(x) = \begin{cases} |x|^{\gamma_1} & \text{if } |x| \leq 1, \\ |x|^{\gamma_2} & \text{if } |x| \geq 1, \end{cases} \quad \text{and} \quad Q(x) = |x|^{\gamma_3},$$

with $\gamma_1 > 0$, $\gamma_2 > -n$ and $0 < \gamma_3 < \gamma_2$. Indeed, take $a = \gamma_2$, $a_0 = a_1 = \gamma_1$, $b_0 = \gamma_2$ and $b = \gamma_3$.

(b) The potentials introduced by Ambrosetti, Felli and Malchiodi in [11] in the frame of nonlinear Schrödinger equations,

$$\frac{A_1}{1+|x|^{\gamma_1}} \leq V(|x|) \leq A_2 \quad \text{and} \quad 0 < Q(|x|) \leq \frac{A_3}{1+|x|^{\gamma_2}},$$

for positive constants A_1, A_2, A_3, with $\gamma_1 \in (0,n)$ and $\gamma_2 \geq 0$ also satisfy (V) and (Q), respectively.

In order to state our results, we need to introduce some notations. If $1 \leq p < \infty$ we define the weighted Lebesgue spaces

$$L^p(\mathbb{R}^n;Q) := \left\{u\colon \mathbb{R}^n \to \mathbb{R} : u \text{ is measurable and } \int_{\mathbb{R}^n} Q(|x|)|u|^p\, dx < \infty\right\},$$

$$L^p(\mathbb{R}^n;V) := \left\{u\colon \mathbb{R}^n \to \mathbb{R} : u \text{ is measurable and } \int_{\mathbb{R}^n} V(|x|)|u|^p\, dx < \infty\right\},$$

endowed, respectively, with the norms

$$\|u\|_{L^p(\mathbb{R}^n;Q)} = \left(\int_{\mathbb{R}^n} Q(|x|)|u|^p\, dx\right)^{1/p}$$

and

$$\|u\|_{L^p(\mathbb{R}^n;V)} = \left(\int_{\mathbb{R}^n} V(|x|)|u|^p\,dx\right)^{1/p}.$$

Let $C_0^\infty(\mathbb{R}^n)$ be the set of smooth functions with compact support and

$$C_{0,\mathrm{rad}}^\infty(\mathbb{R}^n) = \{u \in C_0^\infty(\mathbb{R}^n) : u \text{ is radial}\}.$$

Here, we define the energy space $W^{1,n}_{\mathrm{rad}}(\mathbb{R}^n;V)$ as the subspace of radially symmetric functions in the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|u\| = \left[\int_{\mathbb{R}^n} \big(|\nabla u|^n + V(|x|)|u|^n\big)\,dx\right]^{1/n}.$$

Equivalently, $W^{1,n}_{\mathrm{rad}}(\mathbb{R}^n;V)$ can be considered as the Sobolev space modeled on the Lebesgue space $L^n_{\mathrm{rad}}(\mathbb{R}^n;V) = \{u \in L^n(\mathbb{R}^n;V) : u \text{ is radial}\}$ defined by

$$W^{1,n}_{\mathrm{rad}}(\mathbb{R}^n;V) := \big\{u \in L^n_{\mathrm{rad}}(\mathbb{R}^n;V) : |\nabla u| \in L^n_{\mathrm{rad}}(\mathbb{R}^n)\big\},$$

where the derivative above is understood in the sense of distributions. We use the notation $E = W^{1,n}_{\mathrm{rad}}(\mathbb{R}^n;V)$.

Remark 1.2. Under the hypothesis (V) we can show that

$$\|u\|^n = \int_{\mathbb{R}^n} \big(|\nabla u|^n + V(|x|)|u|^n\big)dx$$

is a norm in E. In fact, if $\|u\| = 0$, then

$$\int_{\mathbb{R}^n} |\nabla u|^n\,dx = 0$$

and so u is a constant. But from behaviour of V at infinity, we should have $u = 0$.

With the aid of inequalities (1.1), (1.2) and inspired by similar arguments developed in [18], [23], [33], we establish in this work a Trudinger–Moser inequality in the functional space E. More precisely, one has:

Theorem 1.3. *Assume that* (V) *and* (Q) *hold. Then, for any $u \in E$ and $\alpha > 0$, we have that $\Phi_\alpha(u) \in L^1(\mathbb{R}^n;Q)$. Furthermore, if*

$$\alpha < \lambda := \min\{\alpha_n, \alpha_n(1 + b_0/n)\},$$

there holds

$$\sup_{u\in E:\|u\|\le 1} \int_{\mathbb{R}^n} Q(|x|)\Phi_\alpha(u)\,dx < \infty. \tag{1.4}$$

Moreover, if the function Q is nonincreasing in $|x|$, then

$$\sup_{u\in E:\|u\|\le 1} \int_{\mathbb{R}^n} Q(|x|)\Phi_\lambda(u)\,dx < \infty.$$

Furthermore, if we also assume that $-n < b_0 \le 0$ *and* $\liminf_{r\to 0^+} Q(r)/r^{b_0} > 0$, *then the value* λ *is optimal, that is*

$$\sup_{u\in E:\|u\|\le 1} \int_{\mathbb{R}^n} Q(|x|)\Phi_\alpha(u)\,dx = +\infty, \quad \textit{for all } \alpha > \lambda.$$

Remark 1.4. Here, we point out that in general, due to technical reasons, the inequality (1.4) holds for $\alpha < \lambda$ but not necessarily for $\alpha = \lambda$. In fact, the space E consists of radial functions that are not necessarily nonincreasing in $|x|$, $x \in \mathbb{R}^n$. Thus, in general (i.e. without the additional condition that the radial weight Q is nonincreasing in $|x|$) the "radial lemma" (see inequality (2.16) below), which also has been used in [5] and [33], is not true.

We quote that inequality (1.4) is a natural generalization for high dimensions to the one obtained in [8, Theorem 1.1] when $n = 2$. See also [6, Theorem 2.1] for the sharpness and the existence of extremal function for this inequality. Here, we have to highlight the fact that, in contrast with [8] and [6] (where it was assumed that $b < (a-2)/2$), we only assume that $b < a$. Thus, even in the case where $n = 2$, there is some novelty in the result proved in the present paper. Also, in two dimensions, do Ó, Sani and Zhang in [28] have obtained a similar inequality in the nonradial case but considering only the potentials introduced by Ambrosetti, Felli and Malchiodi in [11] and in the subcritical case, that is, in the sense that the range of the exponent is the open interval $(0, \lambda)$.

As an application of the previous theorem and using a minimax procedure, we will study the existence of a nontrivial solution for the following quasilinear elliptic problem:

$$\begin{cases} -\operatorname{div}(|\nabla u|^{n-2}\nabla u) + V(|x|)|u|^{n-2}u = Q(|x|)f(u) & \text{in } \mathbb{R}^n, \\ u(x) \to 0 & \text{as } |x| \to \infty, \end{cases} \tag{1.5}$$

$n \ge 2$, when the nonlinear term $f(s)$ is allowed to enjoy an exponential critical growth. When $n = 2$, problem (1.5) has been treated in [8, 36]. In [36] the authors considered the nonlinearity $f(s) = |s|^{p-2}s$, with $2 < p < 2^\star = 2n/(n-2)$ for $n \ge 3$ and $2 < p < \infty$ if $n = 2$, and in [8] the authors have studied the critical case suggested by the classical Trudinger–Moser inequality (1.1).

Here, we are interested in the case where the nonlinearity $f(s)$ has maximal growth on s which allows us to treat the problem (1.5) variationally. Explicitly, in view of the classical Trudinger–Moser inequality (1.1) and [19], we say that $f(s)$ has α_0-*exponential critical growth* at $+\infty$ if there exists $\alpha_0 > 0$ such that

$$\begin{aligned} &\lim_{s\to+\infty} f(s)e^{-\alpha|s|^{n'}} = 0, && \text{for all } \alpha > \alpha_0, \\ &\lim_{s\to+\infty} f(s)e^{-\alpha|s|^{n'}} = +\infty, && \text{for all } \alpha < \alpha_0. \end{aligned} \tag{1.6}$$

Similarly, we define α_0-exponential critical growth at $-\infty$.

We would like to mention that problems involving exponential critical growth have received a special attention in last years, see for example, [7]–[10], [13], [16], [18]–[20], [27] for semilinear elliptic equations, and [3], [14], [15], [17], [21]–[23], [25], [39], [40] for quasilinear equations.

In order to perform the minimax approach to the problem (1.5), we also need to make some suitable assumptions on the behaviour of $f(s)$. More precisely, we shall assume the following conditions:

(f_1) $f\colon [0,+\infty) \to \mathbb{R}$ is continuous and $f(s)/|s|^{n-1} \to 0$ as $s \to 0^+$;

(f_2) there exists $\theta > n$ such that

$$0 < \theta F(s) := \theta \int_0^s f(t)\,dt \leq s f(s), \quad \text{for all } s > 0;$$

(f_3) there exist $\theta_0 > n$ and $\mu > 0$ such that

$$F(s) \geq \frac{\mu}{\theta_0} s^{\theta_0}, \quad \text{for all } s \geq 0.$$

We observe that the hypotheses (f_1)–(f_3) have been used in many papers to find a solution using the classical Mountain Pass Theorem introduced by Ambrosetti and Rabinowitz in the celebrated paper [12], see for instance [24], [26] and their references.

In this work, we say that a function $u : \mathbb{R}^n \to \mathbb{R}$ is a weak solution of (1.5) if $u \in E$ and it holds the identity

$$\int_{\mathbb{R}^n} \big(|\nabla u|^{n-2}\nabla u \nabla \varphi + V(|x|)|u|^{n-2}u\varphi\big)\,dx = \int_{\mathbb{R}^n} Q(|x|)f(u)\varphi\,dx,$$

for all $\varphi \in E$. We point out that from (f_1) the identically zero function is the trivial solution of (1.5). So, our goal is to show the existence of nontrivial solution for (1.5).

Next, we state our existence result.

Theorem 1.5. *Suppose that* (V) *and* (Q) *hold. If f has α_0-exponential critical growth at $+\infty$ and* (f_1)–(f_3) *are satisfied, then there exists $\mu_0 > 0$ such that problem* (1.5) *has a nontrivial nonnegative weak solution u in E for all $\mu > \mu_0$.*

Example 1.6. Let $q > n$ and $\alpha_0 > 0$ be constants. The hypotheses of Theorem 1.5 are satisfied by the nonlinearity

$$f(s) = \bigg(qs^{q-1} + \frac{\alpha_0 n}{n-1}\, s^{q+1/(n-1)}\bigg)e^{\alpha_0 s^{n/(n-1)}}, \quad s \geq 0.$$

For this example, we have: $F(s) = s^q e^{\alpha_0 s^{n/(n-1)}}$, $s \geq 0$.

Remark 1.7. Our existence result complements the study made in [8], [36] in the sense that, in this paper, we study a class of problems involving the n-Laplacian ($n \geq 2$) with exponential critical growth and in [36] only the Sobolev subcritical growth was considered. Moreover, the conditions taken on the behaviour of the weight Q and the nonlinearity f are less restrictive. In fact, in contrast with [8], we only assume that $b < a$. Furthermore, we no longer assume the existence of a positive constant M_0 such that $F(s) \leq M_0|f(s)|$ for s sufficiently large.

Remark 1.8. If the nonlinearity f has exponential subcritical growth at $+\infty$ (or $-\infty$), that is, if f satisfies the condition

$$\lim_{s\to+\infty} f(s)e^{-\alpha|s|^{n'}} = 0$$

for all $\alpha > 0$, it is standard to check that the problem (1.5) has a Mountain Pass type solution.

Remark 1.9. The difficulties in treating this class of problems (1.5) are the possible lack of compactness due to the unboundedness of the domain besides the fact that the nonlinear term $f(s)$ is allowed to enjoy the exponential critical growth and the behaviour of the nonlinear operator $\operatorname{div}(|\nabla u|^{n-2}\nabla u)$.

The outline of the paper is as follows: Section 2 contains some embedding results and the proof of our Trudinger–Moser inequality; Theorem 1.3. In Section 3, we set up the setting which will allow us to follow a variational approach. Specifically, we check the geometric conditions of the associated functional, we treat with the Palais–Smale sequences and we get a more precise information about the minimax level obtained by the Mountain Pass Theorem. Finally, in Section 4, we prove our existence result; Theorem 1.5.

2. Embedding results and proof of Theorem 1.3

In order to prove the Theorem 1.3, we need to establish some embeddings from E into the Lebesgue weighted spaces $L^p(\mathbb{R}^n; Q)$ beginning by introducing a radial lemma in the spirit of that due to W. Strauss [35] (see also [34]). In the following, we denote by $B(x,R) \subset \mathbb{R}^n$ the open ball centered at $x \in \mathbb{R}^n$ with radius $R > 0$ and, to simplify notations, we set $B_R := B(0,R)$, $B_R^c := \mathbb{R}^n \setminus B_R$ and $B_R \setminus B_r$ denotes the annulus with interior radius r and exterior radius R. We use $C, C_0, C_1, C_2, \dots$ to denote (possibly different) positive constants.

Lemma 2.1. *Suppose that* (V) *holds. Then there exist* $C > 0$ *and* $R > 1$ *such that, for all* $u \in E$*, we have*

$$|u(x)| \leq C\|u\||x|^{-(n-1)(a+n)/n^2}, \quad \textit{for } |x| \geq R.$$

The proof is very much similar to the case of $n = 2$ which is given in [8], and is omitted.

Next, we recall some basic embeddings (see J. Su *et al.* [37]). Let $A \subset \mathbb{R}^n$ and define

$$W^{1,n}_{\mathrm{rad}}(A;V) = \{u_{|_A} : u \in E\}.$$

Lemma 2.2. *Assume* (V) *and* (Q). *For any fixed* $0 < r < R < \infty$, *the embeddings*

$$W^{1,n}_{\mathrm{rad}}(B_R \setminus B_r;V) \hookrightarrow L^p(B_R \setminus B_r;Q), \quad 1 \le p \le \infty$$

are compact.

Remark 2.3. For $R \gg 1$, the embedding $W^{1,n}_{\mathrm{rad}}(B_R;V) \hookrightarrow W^{1,n}(B_R)$ is continuous. That last result can be obtained by proceeding exactly as in [36, Lemma 4].

Lemma 2.4. *Assume that* (V) *and* (Q) *hold. Then the embeddings* $E \hookrightarrow L^p(\mathbb{R}^n;Q)$ *are compact for all* $n \le p < \infty$.

Proof. We first show the continuity of the embeddings. That is to show

$$S_p(V;Q) := \inf_{u \in E\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^n} \big(|\nabla u|^n + V(|x|)|u|^n\big)\,dx}{\bigg(\displaystyle\int_{\mathbb{R}^n} Q(|x|)|u|^p\,dx\bigg)^{n/p}} > 0.$$

Suppose by contradiction that $S_p(V;Q) = 0$. Thus, there exists $(u_k) \subset E$ such that

$$\begin{aligned} &\int_{\mathbb{R}^n} Q(|x|)|u_k|^p\,dx = 1, \quad \text{for all } k \in \mathbb{N}, \\ &\lim_{k\to\infty} \int_{\mathbb{R}^n} \big(|\nabla u_k|^n + V(|x|)|u_k|^n\big)\,dx = 0. \end{aligned} \tag{2.1}$$

By (V) and (Q), there exist constants $R_0 > r_0 > 0$ and $C_0 > 0$ such that

$$\begin{aligned} V(|x|) &\ge C_0|x|^a, & Q(|x|) &\le C_0|x|^b, \quad \text{for all } |x| \ge R_0, \\ V(|x|) &\ge C_0|x|^{a_0}, & Q(|x|) &\le C_0|x|^{b_0}, \quad \text{for all } 0 < |x| \le r_0. \end{aligned}$$

Now for $R > R_0$, by Lemma 2.1, we have

$$\begin{aligned} \int_{B_R^c} Q(|x|)|u_k|^p\,dx &\le \int_{B_R^c} C_0|x|^b|u_k|^p\,dx \\ &= \int_{B_R^c} |x|^{b-a}|u_k|^{p-n}C_0|x|^a|u_k|^n\,dx \\ &\le C_1\|u_k\|^{p-n} \int_{B_R^c} |x|^{b-a-(n-1)(p-n)(a+n)/n^2}V(|x|)|u_k|^n\,dx. \end{aligned}$$

Since $a > -n$, $b < a$ and $p \geq n$, we have that $b-a-(n-1)(p-n)(a+n)/n^2 < 0$. Thus, we obtain

$$
\begin{aligned}
(2.2) \qquad \int_{B_R^c} Q(|x|)|u_k|^p\,dx &\leq C_1 R^{b-a-(n-1)(p-n)(a+n)/n^2}\|u_k\|^p \\
&= C_1 R^{b-a-(n-1)(p-n)(a+n)/n^2} o_k(1), \quad \text{as } k \to \infty.
\end{aligned}
$$

Now, we estimate the integral on ball B_r, for $0 < r < \min\{r_0, 1/2\}$. To reach this aim, we introduce a cutoff function $\varphi \in C^\infty_{0,\mathrm{rad}}(B_1)$ such that

$$
0 \leq \varphi \leq 1 \text{ in } B_1, \quad \varphi \equiv 1 \text{ in } B_{1/2}, \quad \varphi \equiv 0 \text{ in } B_1 \setminus B_{3/4} \quad \text{and} \quad |\nabla\varphi| \leq C \text{ in } B_1.
$$

Then $\varphi u_k \in W_0^{1,n}(B_1)$. Let $\sigma > 1$ be such that $b_0\sigma > -n$. So, by Hölder's inequality, we have

$$
\begin{aligned}
(2.3) \qquad \int_{B_r} Q(|x|)|u_k|^p\,dx &\leq C_0 \int_{B_r} |x|^{b_0}|u_k|^p\,dx = C_0 \int_{B_r} |x|^{b_0}|\varphi u_k|^p\,dx \\
&\leq C_0 \bigg(\int_{B_r} |x|^{b_0\sigma}\,dx\bigg)^{1/\sigma} \bigg(\int_{B_r} |\varphi u_k|^{p\sigma/(\sigma-1)}\,dx\bigg)^{(\sigma-1)/\sigma} \\
&\leq C_2\, r^{b_0\sigma+n} \bigg(\int_{B_1} |\varphi u_k|^{p\sigma/(\sigma-1)}\,dx\bigg)^{(\sigma-1)/\sigma} \\
&\leq C_3\, r^{b_0\sigma+n} \bigg(\int_{B_1} |\nabla(\varphi u_k)|^n\,dx\bigg)^{p/n} \\
&\leq C_4\, r^{b_0\sigma+n} \bigg(\int_{B_1} |u_k\nabla\varphi|^n\,dx + \int_{B_1} |\varphi\nabla u_k)|^n\,dx\bigg)^{p/n} \\
&\leq C_5\, r^{b_0\sigma+n} \bigg(\int_{B_{3/4}\setminus B_{1/2}} |u_k|^n\,dx + \int_{B_1} |\nabla u_k|^n\,dx\bigg)^{p/n} \\
&\leq C_6\, r^{b_0\sigma+n} \bigg(\int_{B_{3/4}\setminus B_{1/2}} V(|x|)|u_k|^n\,dx + \int_{B_1} |\nabla u_k|^n\,dx\bigg)^{p/n} \\
&\leq C_6\, r^{b_0\sigma+n}\|u_k\|^p = C_6\, r^{b_0\sigma+n} o_k(1),
\end{aligned}
$$

as $k \to \infty$, where we have used the fact that $\min\limits_{B_{3/4}\setminus B_{1/2}} V(|x|) > 0$. Now, writing

$$
\begin{aligned}
\int_{\mathbb{R}^n} Q(|x|)|u_k|^p\,dx &= \int_{B_r} Q(|x|)|u_k|^p\,dx \\
&\quad + \int_{B_R\setminus B_r} Q(|x|)|u_k|^p\,dx + \int_{B_R^c} Q(|x|)|u_k|^p\,dx,
\end{aligned}
$$

using (2.2), (2.3) and Lemma 2.2 we get

$$
\lim_{k\to\infty} \int_{\mathbb{R}^n} Q(|x|)|u_k|^p\,dx = 0,
$$

which contradicts the fact that $\int_{\mathbb{R}^n} Q(|x|)|u_k|^p\,dx = 1$. This proves the continuity of the embedding. For the compactness, let (u_k) be a sequence in E such that

$\|u_k\| \le C$. Without loss of generality, we may assume that $u_k \rightharpoonup 0$ weakly in E. We need to prove that, up to a subsequence, $u_k \to 0$ strongly in $L^p(\mathbb{R}^n;Q)$ for all $n \le p < \infty$. As in (2.2), we infer

$$\begin{aligned}\int_{B_R^c} Q(|x|)|u_k|^p\,dx &\le C_7 R^{b-a-(n-1)(p-n)(a+n)/n^2}\|u_k\|^p \\ &\le C_8 R^{b-a-(n-1)(p-n)(a+n)/n^2}.\end{aligned}$$

Since $b-a-(n-1)(p-n)(a+n)/n^2 < 0$, given $\varepsilon > 0$, for $R > 0$ sufficiently large we deduce

$$\int_{B_R^c} Q(|x|)|u_k|^p\,dx \le C_8 R^{b-a-(n-1)(p-n)(a+n)/n^2} < \frac{\varepsilon}{3}. \tag{2.4}$$

On the other hand, as in (2.3) and by choosing r small enough, it yields

$$\int_{B_r} Q(|x|)|u_k|^p\,dx \le C_9 r^{b_0\sigma+n} < \frac{\varepsilon}{3}. \tag{2.5}$$

Now, from Lemma 2.2, $u_k \to 0$ strongly in $L^p(B_R \setminus B_r;Q)$ for all $1 \le p < \infty$. Thus, for $k \in \mathbb{N}$ large enough,

$$\int_{B_R\setminus B_r} Q(|x|)|u_k|^p\,dx < \frac{\varepsilon}{3}. \tag{2.6}$$

Combining (2.4), (2.5) and (2.6), we get

$$\lim_{k\to\infty}\|u_k\|^p_{L^p(\mathbb{R}^n;Q)} = \lim_{k\to\infty}\int_{\mathbb{R}^n} Q(|x|)|u_k|^p\,dx = 0,$$

and this ends the proof of Lemma 2.4. $\square$

Proof of Theorem 1.3. Let $\alpha < \lambda$. Recall that by the hypothesis (Q), we have

$$\begin{aligned} Q(|x|) &\le C_0|x|^b &&\text{for } |x| \ge R_0,\\ Q(|x|) &\le C_0|x|^{b_0} &&\text{for } 0 < |x| \le r_0.\end{aligned} \tag{2.7}$$

Let $R > 0$ to be chosen later during the proof independently of u. We write

$$\int_{\mathbb{R}^n} Q(|x|)\Phi_\alpha(u)\,dx = I_1(\alpha,u) + I_2(\alpha,u),$$

where

$$I_1(\alpha,u) = \int_{B_R} Q(|x|)\Phi_\alpha(u)\,dx, \qquad I_2(\alpha,u) = \int_{B_R^c} Q(|x|)\Phi_\alpha(u)\,dx.$$

We shall estimate $I_1(\alpha,u)$ and $I_2(\alpha,u)$. For the integral $I_1(\alpha,u)$, we have two cases to analyze.

Case 1. $b_0 \ge 0$. Since Q is a positive continuous function, then there exists $C > 0$ such that

$$I_1(\alpha,u) \le C\int_{B_R} e^{\alpha|u|^{n'}}\,dx. \tag{2.8}$$

Let us recall two elementary inequalities. There exists a positive constant $A = A(n)$ such that

$$(s_1 + s_2)^{n'} \le s_1^{n'} + A s_1^{1/(n-1)} s_2 + s_2^{n'}, \quad \text{for all } s_1, s_2 \ge 0. \tag{2.9}$$

If γ and γ' are positive real numbers such that $\gamma + \gamma' = 1$, then for all $\varepsilon > 0$, we have

$$s_1^{\gamma} s_2^{\gamma'} \le \varepsilon s_1 + \varepsilon^{-\gamma/\gamma'} s_2, \quad \text{for all } s_1, s_2 \ge 0. \tag{2.10}$$

Let $v(x) = u(x) - u(R)$ for $x \in B_R$. Note that by Remark 2.3, $v \in W_0^{1,n}(B_R)$. Then by (2.9) and (2.10), for each $\varepsilon > 0$ given, we have

$$|u|^{n'} \le (1+\varepsilon)|v|^{n'} + C_n \left(\frac{|u(R)|^n}{\varepsilon} \right)^{1/(n-1)}. \tag{2.11}$$

Thus, by Lemma 2.1, there exists a positive constant $C_{\varepsilon,n}$ such that

$$|u|^{n'} \le (1+\varepsilon)|v|^{n'} + C_{\varepsilon,n} R^{-(a+n)/n} \|u\|^{n'}.$$

Then, fixing $R \gg \max\left\{1, R_0, (C_{\varepsilon,n})^{n/(a+n)}\right\}$, we get $|u|^{n'} \le (1+\varepsilon)|v|^{n'} + \|u\|^{n'}$. Hence,

$$\int_{B_R} e^{\alpha |u|^{n'}} dx \le e^{\alpha \|u\|^{n'}} \int_{B_R} e^{\alpha(1+\varepsilon)|v|^{n'}} dx < \infty, \tag{2.12}$$

by the classical Trudinger–Moser inequality (1.1). Furthermore, taking $\varepsilon > 0$ such that $\alpha(1+\varepsilon) \le \lambda \le \alpha_n$ there exists $C = C(n,R) > 0$ such that

$$\sup_{v \in W_0^{1,n}(B_R) : \|v\|_{W_0^{1,n}(B_R)} \le 1} \int_{B_R} e^{\alpha(1+\varepsilon)|v|^{n'}} dx \le C.$$

From this, (2.8) and (2.12) we obtain

$$\sup_{u \in E : \|u\| \le 1} I_1(\alpha, u) \le C.$$

Case 2. $-n < b_0 < 0$. We write

$$\begin{aligned}
\int_{B_R} Q(|x|) e^{\alpha |u|^{n'}} dx &= \int_{B_{r_0}} Q(|x|) e^{\alpha |u|^{n'}} dx + \int_{B_R \setminus B_{r_0}} Q(|x|) e^{\alpha |u|^{n'}} dx \\
&\le C_0 \int_{B_{r_0}} |x|^{b_0} e^{\alpha |u|^{n'}} dx + C \int_{B_R \setminus B_{r_0}} e^{\alpha |u|^{n'}} dx \\
&\le C_0 \int_{B_R} |x|^{b_0} e^{\alpha |u|^{n'}} dx + C \int_{B_R} e^{\alpha |u|^{n'}} dx.
\end{aligned}$$

By similar computations done above, we obtain

$$\int_{B_R} |x|^{b_0} e^{\alpha |u|^{n'}} dx \le e^{\alpha \|u\|^{n'}} \int_{B_R} \frac{e^{\alpha(1+\varepsilon)|v|^{n'}}}{|x|^{-b_0}} dx. \tag{2.13}$$

Having in mind that $\alpha < \alpha_n(1+b_0/n)$, then we can take $\varepsilon > 0$ such that $\alpha(1+\varepsilon) \le \alpha_n(1+b_0/n)$. Thus, since $v \in W_0^{1,n}(B_R)$,

$$\|v\|_{W_0^{1,n}(B_R)} = \|\nabla v\|_{L^n(B_R)} \le \|u\| \le 1$$

and $-b_0 \in (0,n)$, thanks to (1.2)

$$\sup_{v \in W_0^{1,n}(B_R) : \|v\|_{W_0^{1,n}(B_R)} \le 1} \int_{B_R} \frac{e^{\alpha(1+\varepsilon)|v|^{n'}}}{|x|^{-b_0}}\, dx \le C.$$

Using that inequality together with (2.13), we obtain

$$\sup_{u \in E : \|u\| \le 1} \int_{B_R} |x|^{b_0} e^{\alpha |u|^{n'}}\, dx \le C.$$

Therefore, in both cases we have

$$\sup_{u \in E : \|u\| \le 1} \int_{B_R} Q(|x|) e^{\alpha |u|^{n'}}\, dx < \infty,$$

and consequently

$$\sup_{u \in E : \|u\| \le 1} I_1(\alpha, u) < \infty. \tag{2.14}$$

Next, we will estimate the integral $I_2(\alpha,u)$. It follows from the first inequality in (2.7), Lemma 2.4 and the Monotone Convergence Theorem that, for any $u \in E$,

$$\begin{aligned} I_2(\alpha,u) &= \int_{B_R^c} Q(|x|) \sum_{j=n-1}^{\infty} \frac{\alpha^j |u|^{jn'}}{j!}\, dx \\ &= \int_{B_R^c} Q(|x|) \sum_{j=n}^{\infty} \frac{\alpha^j |u|^{jn'}}{j!}\, dx + \frac{\alpha^{n-1}}{(n-1)!} \int_{B_R^c} Q(|x|)|u|^n\, dx \\ &\le C_0 \sum_{j=n}^{\infty} \frac{\alpha^j}{j!} \int_{B_R^c} |x|^b |u|^{jn'}\, dx + C_1 \|u\|^n. \end{aligned}$$

Using Lemma 2.1, we can estimate the last integral above as follows

$$\begin{aligned} \int_{B_R^c} |x|^b |u|^{jn'}\, dx &\le (C\|u\|)^{n'j} \int_{B_R^c} |x|^{b - j(a+n)/n}\, dx \\ &= \omega_{n-1} (C\|u\|)^{n'j} \int_R^{\infty} t^{b - j(a+n)/n + n - 1}\, dt \le \frac{\omega_{n-1}}{(a-b)R^{a-b}} (C\|u\|)^{n'j}, \end{aligned}$$

where we have used that $a > -n$, $b - j(a+n)/n + n - 1 < b - a$, for all $j \ge n$

and $R > 1$. Thus,

$$
\begin{aligned}
I_2(\alpha, u) &\leq \frac{\omega_{n-1} C_0}{(a-b)R^{a-b}} \sum_{j=n}^{\infty} \frac{\big(\alpha C^{n'} \|u\|^{n'}\big)^j}{j!} + C_1 \|u\|^n \\
&= \frac{\omega_{n-1} C_0}{(a-b)R^{a-b}} \left(\Phi_{\alpha C^{n'}}(u) - \frac{\alpha^{n-1} C^n \|u\|^n}{(n-1)!} \right) + C_1 \|u\|^n < \infty,
\end{aligned}
$$

for all $u \in E$. Hence,

$$
\sup_{u \in E:\ \|u\| \leq 1} I_2(\alpha, u) < \infty. \tag{2.15}
$$

Thereby, from (2.14) and (2.15) we conclude that

$$
\sup_{u \in E: \|u\| \leq 1} \int_{\mathbb{R}^n} Q(|x|) \Phi_\alpha(u)\, dx < \infty,
$$

and this ends the proof of the first part of Theorem 1.3. Now, we will try to prove that, if we add the condition that the function Q is nonincreasing in $|x|$, then the value $\alpha = \lambda$ is permitted. A similar results have been obtained in [6] (for the case $n = 2$) and [5] (for the case V constant and $Q(|x|) = 1/|x|^\beta$, $0 < \beta < n$). In order to reach our claimed result, we need to replace the function $u \in E$ by its Schwarz rearrangement to profit of its nonincreaseness in $|x|$, $x \in \mathbb{R}^n$. Since Q is assumed to be nonincreasing, then it is immediate that $Q^*(|x|) = Q(|x|)$, for all $x \in \mathbb{R}^n$, where Q^* denotes the Schwarz rearrangement of Q. On the other hand, taking $u \in E$ and denoting u^* its Schwarz rearrangement, by the well known Hardy–Littlewood inequality, it yields

$$
\int_{\mathbb{R}^n} Q(|x|) \Phi_\lambda(u)\, dx \leq \int_{\mathbb{R}^n} Q(|x|) \Phi_\lambda(u^*)\, dx.
$$

For basic properties on rearrangements, we refer to [29], [32]. Therefore, we can restrict our analysis to the functions $u \in E$ which are nonincreasing in $|x|$. Now, we recall that $V(|x|) \geq C_0 |x|^a$, for all $|x| \geq R_0$. Hence, for $|x| > 2R_0$, we infer

$$
\begin{aligned}
\|u\|^n_{L^n_{\mathrm{rad}}(\mathbb{R}^n; V)} &= \int_{\mathbb{R}^n} V(|x|)|u|^n\, dx \geq \int_{|x|/2}^{|x|} V(t) |\varphi_u(t)| t^{n-1}\, dt \\
&\geq C_0 |\varphi_u(|x|)|^n |x|^{a+n} \left(1 - \left(\frac{1}{2}\right)^{a+n}\right) \\
&= C_0 |u(x)|^n |x|^{a+n} \left(1 - \left(\frac{1}{2}\right)^{a+n}\right),
\end{aligned}
$$

where $\varphi_u(t) = u(x)$, $t = |x|$. It follows,

$$
|u(x)| \leq C |x|^{-(a+n)/n} \|u\|_{L^n_{\mathrm{rad}}(\mathbb{R}^n; V)}, \quad \text{for all } |x| > 2R_0. \tag{2.16}
$$

Observe that, since $1/(n-1) \leq 1$, then

$$
\left(\frac{1}{1-x}\right)^{1/(n-1)} \geq 1 + \frac{x}{n-1}, \quad \text{for all } 0 \leq x < 1.
$$

That last inequality implies

$$\frac{1}{\|\nabla u\|_n^{n'}} = \left(\frac{1}{\|\nabla u\|_n^n}\right)^{1/(n-1)} \geq \left(\frac{1}{1-\|u\|^n_{L^n_{\mathrm{rad}}(\mathbb{R}^n;V)}}\right)^{1/(n-1)} \geq 1 + \frac{\|u\|^n_{L^n_{\mathrm{rad}}(\mathbb{R}^n;V)}}{n-1}, \tag{2.17}$$

for all $u \in E$ such that $\|u\| \leq 1$. Choosing ε in (2.11) such as $1+\varepsilon = 1/\|\nabla u\|_n^{n'}$ and using (2.17), we obtain

$$\varepsilon \geq \frac{\|u\|^n_{L^n_{\mathrm{rad}}(\mathbb{R}^n;V)}}{n-1}.$$

Taking R large enough such that $R > 2R_0$, by (2.16) we deduce the existence of a positive constant C such that

$$C_n\left(\frac{|u(R)|^n}{\varepsilon}\right)^{1/(n-1)} \leq C.$$

Now, by (2.11), we infer

$$|u|^{n'} \leq \left(\frac{|v|}{\|\nabla u\|_n}\right)^{n'} + C, \quad \text{for all } u \in E,\ u \text{ is nonincreasing in } |x|.$$

Thus, for R large enough, we have

$$\int_{B_R} Q(|x|)e^{\lambda|u|^{n'}}\,dx \leq C_1 \int_{B_R} Q(|x|)e^{\lambda(|v|/\|\nabla u\|_n)^{n'}}\,dx.$$

Taking into account that $\nabla v(x) = \nabla u(x)$, $x \in B_R$, one can continue as previously (i.e. considering the two cases $b_0 \geq 0$ and $-n < b_0 < 0$, and using (1.1) or (1.2)) and we can deduce that $\sup\limits_{u\in E:\|u\|\leq 1} I_1(\lambda,u) < \infty$. The fact that $\sup\limits_{u\in E:\|u\|\leq 1} I_2(\lambda,u) < \infty$ can be reached using exactly the same arguments as for the case $\alpha < \lambda$ treated previously.

Finally, assume that $-n < b_0 \leq 0$ and $\liminf\limits_{r\to 0^+} Q(r)/r^{b_0} > 0$. We claim that

$$\sup_{u\in E:\|u\|\leq 1} \int_{\mathbb{R}^n} Q(|x|)\Phi_\alpha(u)\,dx = \infty, \quad \text{for all } \alpha > \lambda = \alpha_n(1+b_0/n). \tag{2.18}$$

For that aim, we recall the Moser sequence,

$$M_k(x,r) = \omega_{n-1}^{-1/n}\begin{cases} (\log k)^{(n-1)/n} & \text{if } |x| \leq r/k, \\ \dfrac{\log(r/|x|)}{(\log k)^{1/n}} & \text{if } r/k \leq |x| \leq r, \\ 0 & \text{if } |x| \geq r, \end{cases}$$

for $x \in \mathbb{R}^n$, $r > 0$. Notice that

$$\int_{\mathbb{R}^n} |\nabla M_k(x,r)|^n dx = 1, \quad \text{for all } r > 0, \text{ for all } k \geq 1. \tag{2.19}$$

Recall that by (V), there exist $C_1 > 0$ and $r_1 > 0$ such that

$$V(|x|) \leq C_1 |x|^{a_1}, \quad \text{for all } 0 < |x| \leq r_1.$$

Thus, for $r < r_1$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} V(|x|)|M_k(x,r)t|^n dx &= \omega_{n-1}(\log k)^{n-1} \int_0^{r/k} V(t)t^{n-1}\, dt \\ &\quad + \omega_{n-1}(\log k)^{-1} \int_{r/k}^{r} V(t)(\log(r/t))^n t^{n-1}\, dt \\ &\leq C_1(\log k)^{n-1} \frac{(r/k)^{a_1+n}}{a_1+n} + C_1\omega_{n-1}(\log k)^{-1} \int_{r/k}^{r} (\log(r/t))^n t^{a_1+n-1}\, dt.\end{aligned}$$

Since, $a_1 + n > 0$, then $(\log k)^{n-1}/k^{a_1+n} \to 0$ as $k \to +\infty$, and the function $t \mapsto (\log(r/t))^n\, t^{a_1+n-1}$ belongs to $L^1(]0,r[)$. It follows,

$$\int_{\mathbb{R}^n} V(|x|)|M_k(x,r)|^n dx = O\left(\frac{1}{\log k}\right), \quad \text{as } k \to +\infty. \tag{2.20}$$

Let $\widetilde{M_k} = M_k/\|M_k\|$. Combining (2.19) and (2.20), we can easily see that

$$\widetilde{M_k}^{n'}(x,r) = \omega_{n-1}^{-1/(n-1)} \log k + d_k, \quad \text{for all } |x| \leq r/k, \tag{2.21}$$

where d_k is a bounded sequence of nonnegative numbers. On the other hand, there exist $C_2 > 0$ and $r_2 > 0$ such that

$$Q(|x|) \geq C_2 |x|^{b_0}, \quad \text{for all } 0 < |x| \leq r_2. \tag{2.22}$$

Using (2.21) and (2.22), and choosing $r < r_2$, we obtain

$$\begin{aligned}\sup_{u \in E:, \|u\| \leq 1} \int_{\mathbb{R}^n} Q(|x|)\Phi_\alpha(u)\, dx &\geq \int_{\mathbb{R}^n} Q(|x|)\Phi_\alpha\big(\widetilde{M_k}\big)\, dx \\ &\geq \int_{|x| \leq r/k} Q(|x|)\Phi_\alpha\big(\widetilde{M_k}\big)\, dx \\ &\geq \Big(e^{\alpha\omega_{n-1}^{-1/(n-1)} \log k + O(1)} + O\big((\log k)^{n-2}\big)\Big) \int_0^{r/k} Q(t)t^{n-1}\, dt \\ &\geq \frac{C_2 r^{b_0+n}}{b_0+n}\, k^{-(b_0+n)}\big(k^{n\alpha/\alpha_n} e^{O(1)} + O\big((\log k)^{n-2}\big)\big).\end{aligned} \tag{2.23}$$

If $\alpha > \alpha_n(1 + b_0/n)$, then

$$\frac{C_2 r^{b_0+n}}{b_0+n} k^{-(b_0+n)}\big(k^{n\alpha/\alpha_n} e^{O(1)} + O\big((\log k)^{n-2}\big)\big) \to +\infty,$$

as $k \to +\infty$ and (2.23) leads to the claimed result (2.18). □

An immediate consequence of Theorem 1.3, which will be very practical and useful in our application, is the following:

Corollary 2.5. *Under the assumptions of Theorem* 1.3, *if* $u \in E$ *is such that* $\|u\| \le M < (\lambda/\alpha)^{1/n'}$, *then there exists a constant* $C = C(n, M, \alpha) > 0$ *independent of* u *such that*

$$\int_{\mathbb{R}^n} Q(|x|)\Phi_\alpha(u)\,dx \le C.$$

3. An application to a quasilinear elliptic problem

In this section, we consider the quasilinear elliptic problem (1.5) taken under the hypotheses of Theorem 1.5. Initially, we introduce the functional setting for a variational approach to the problem (1.5).

3.1. The variational formulation. Using assumption (f_1), we can see that $f(0) = 0$ and since we are interested in nontrivial positive solutions, we will assume, without loss of generality, that $f(s) = 0$ for all $s \le 0$. Let $\alpha > \alpha_0$ and $q \ge n$. From (1.6) and (f_1), for any given $\varepsilon > 0$, there exist $b_1, b_2 > 0$ such that

$$|f(s)| \le \varepsilon |s|^{n-1} + b_1 |s|^{q-1}\Phi_\alpha(s), \quad \text{for all } s \in \mathbb{R}, \tag{3.1}$$

$$|F(s)| \le \frac{\varepsilon}{n}|s|^n + b_2|s|^q\Phi_\alpha(s), \qquad \text{for all } s \in \mathbb{R}. \tag{3.2}$$

Given $u \in E$, by (3.2) it yields

$$\int_{\mathbb{R}^n} Q(|x|)F(u)\,dx \le \frac{\varepsilon}{n}\int_{\mathbb{R}^n} Q(|x|)|u|^n\,dx + b_2\int_{\mathbb{R}^n} Q(|x|)|u|^q\Phi_\alpha(u)\,dx.$$

Now, let $r_1, r_2 > 1$ be such that $1/r_1 + 1/r_2 = 1$. Hölder's inequality, Lemma 2.4 and the first part of Theorem 1.3 imply that

$$\begin{aligned}\int_{\mathbb{R}^n} & Q(|x|)|u|^q\Phi_\alpha(u)\,dx \\ &\le \bigg(\int_{\mathbb{R}^n} Q(|x|)|u|^{qr_1}\,dx\bigg)^{1/r_1}\bigg(\int_{\mathbb{R}^n} Q(|x|)\Phi_{r_2\alpha}(u)\,dx\bigg)^{1/r_2} < \infty,\end{aligned}$$

where we have used the elementary inequality

$$\bigg(e^s - \sum_{j=0}^{n-2}\frac{s^j}{j!}\bigg)^r \le e^{rs} - \sum_{j=0}^{n-2}\frac{(rs)^j}{j!}, \tag{3.3}$$

for all $r \ge 1$, $s \ge 0$. Therefore, the energy functional I associated to problem (1.5) and defined by

$$I(u) := \frac{1}{n}\|u\|^n - \int_{\mathbb{R}^n} Q(|x|)F(u)\,dx, \quad u \in E,$$

is well defined. Using standard arguments, one can easily show that $I \in C^1(E, \mathbb{R})$ with derivative given by

$$I'(u)v = \int_{\mathbb{R}^n} \big(|\nabla u|^{n-2}\nabla u \nabla v + V(|x|)|u|^{n-2}uv\big)dx - \int_{\mathbb{R}^n} Q(|x|)f(u)v\,dx,$$

for all $u, v \in E$. Thus, critical points of I correspond to weak solutions of the problem (1.5) and reciprocally. In the next lemma, we prove that the functional I has the geometric structure required by the Mountain Pass Theorem.

LEMMA 3.1. *Suppose that* (V) *and* (Q), (1.6) *and* (f_1)–(f_2) *hold. Then*

(a) *there exist $\tau, \rho > 0$ such that $I(u) \geq \tau$ for any $u \in E$ with $\|u\| = \rho$,*

(b) *for any $u \in E \setminus \{0\}$ with compact support and $u \geq 0$, we have*

$$I(tu) \to -\infty \quad \text{as } t \to \infty.$$

PROOF. From (3.2), we get

$$\int_{\mathbb{R}^n} Q(|x|)F(u)\,dx \leq \frac{\varepsilon}{n}\int_{\mathbb{R}^n} Q(|x|)|u|^n\,dx + b_2\int_{\mathbb{R}^n} Q(|x|)|u|^q\Phi_\alpha(u)\,dx.$$

Let $r_1, r_2 > 1$ be such that $1/r_1 + 1/r_2 = 1$. By Hölder's inequality and (3.3), we infer

$$\int_{\mathbb{R}^n} Q(|x|)|u|^q\Phi_\alpha(u)\,dx \\ \leq \bigg(\int_{\mathbb{R}^n} Q(|x|)|u|^{qr_1}\,dx\bigg)^{1/r_1}\bigg(\int_{\mathbb{R}^n} Q(|x|)\Phi_{r_2\alpha}(u)\,dx\bigg)^{1/r_2}.$$

Choosing $r_2 > 1$ sufficiently close to 1 and $0 < M < (\lambda/(r_2\alpha))^{n'}$, then for $\|u\| \leq M$, it follows from Corollary 2.5 that

$$\int_{\mathbb{R}^n} Q(|x|)\Phi_{r_2\alpha}(u)\,dx \leq C.$$

From Lemma 2.4, we deduce that

$$\int_{\mathbb{R}^n} Q(|x|)F(u)\,dx \leq \frac{C\varepsilon}{n}\,\|u\|^n + C_2\|u\|^q.$$

Therefore,

$$I(u) \geq \bigg(\frac{1}{n} - \frac{C\varepsilon}{n}\bigg)\|u\|^n - C_2\|u\|^q = \bigg(\frac{1}{n} - \frac{C\varepsilon}{n}\bigg)\rho^n - C_2\rho^q$$

and, choosing $\varepsilon > 0$ sufficiently small such that $C_1 := 1/n - C\varepsilon/n$ is positive,

$$I(u) \geq C_1\rho^n - C_2\rho^q.$$

Since $q > n$, for $\rho > 0$ small enough, there exists $\tau > 0$ such that

$$I(u) \geq \tau \quad \text{for any } u \in E \text{ with } \|u\| = \rho.$$

In order to verify (b), let $u \in E \setminus \{0\}$ with compact support and $u \geq 0$. We notice that from the Ambrosetti–Rabinowitz condition (f_2), there exist $A, B > 0$ such that $F(s) \geq A|s|^\theta - Bs^2$, for all $s \in \mathbb{R}$. Thus,

$$I(tu) \leq \frac{t^n}{n}\|u\|^n - At^\theta\int_{\operatorname{supp}(u)} Q(|x|)|u|^\theta\,dx + Bt^2\int_{\operatorname{supp}(u)} Q(|x|)|u|^2\,dx,$$

which implies (b), since $\theta > n \geq 2$. This completes the proof of the lemma. $\square$

3.2. Palais–Smale sequences. First, we recall that $(u_k) \subset E$ is a Palais–Smale (for short (PS)) sequence at a level $c \in \mathbb{R}$ for the functional I if

$$I(u_k) \to c \quad \text{and} \quad I'(u_k) \to 0, \quad \text{as } k \to \infty,$$

where the second convergence occurs in the dual space E'. We say that I satisfies the (PS) condition if any (PS) sequence has a convergent subsequence.

Lemma 3.2. *Any Palais–Smale sequence for I is bounded.*

Proof. Let $(u_k) \subset E$ be a (PS) sequence at a level $c \in \mathbb{R}$ for the functional I. Thus

$$\frac{1}{n}\|u_k\|^n - \int_{\mathbb{R}^n} Q(|x|)F(u_k)\,dx \to c \quad \text{as } k \to \infty \tag{3.4}$$

and

$$\bigg| \int_{\mathbb{R}^n} \big(|\nabla u_k|^{n-2}\nabla u_k \nabla v + V(|x|)|u_k|^{n-2}u_k v\big)\,dx - \int_{\mathbb{R}^n} Q(|x|)f(u_k)v\,dx \bigg| \le \varepsilon_k \|v\|, \tag{3.5}$$

for all $v \in E$, where $\varepsilon_k \to 0$ as $k \to \infty$. From (3.4), (3.5) and the Ambrosetti–Rabinowitz assumption (f_2), we deduce that

$$c+\varepsilon_k\|u_k\| \ge \bigg(\frac{\theta}{n}-1\bigg)\|u_k\|^n - \int_{\mathbb{R}^n} Q(|x|)(\theta F(u_k)-f(u_k)u_k)\,dx \ge \bigg(\frac{\theta}{n}-1\bigg)\|u_k\|^n,$$

which implies that (u_k) is bounded in E, concluding the proof of the lemma. $\square$

In view of Lemma 3.1, the minimax level

$$c_\mu = \inf_{g\in\Gamma} \max_{0\le t\le 1} I(g(t))$$

is positive, where $\Gamma = \{g \in C([0,1],E) : g(0) = 0 \text{ and } I(g(1)) < 0\}$. Furthermore, we have the following estimate for c_μ:

Lemma 3.3. *There exists $\mu_0 > 0$ such that*

$$c_\mu < \bigg(\frac{1}{n}-\frac{1}{\theta}\bigg)\bigg(\frac{\lambda}{2\alpha_0 n'}\bigg)^{n-1} \quad \textit{for all } \mu > \mu_0.$$

Proof. Since $\theta_0 > n$, then E is compactly immersed in $L^{\theta_0}(\mathbb{R}^n;Q)$ (see Lemma 2.4). Thus, there exists a nonnegative function $u_{\theta_0} \in E$ such that

$$S_{\theta_0} := \int_{\mathbb{R}^n} \big(|\nabla u_{\theta_0}|^n + V(|x|)|u_{\theta_0}|^n\big)\,dx \quad \text{and} \quad \int_{\mathbb{R}^n} Q(|x|)|u_{\theta_0}|^{\theta_0}\,dx = 1.$$

By the definition of c_μ and (f_3), we deduce that

$$c_\mu \le \max_{t\ge 0}\bigg[\frac{t^n}{n}S_{\theta_0} - t^{\theta_0}\frac{\mu}{\theta_0}\bigg] = \frac{\theta_0-n}{n\theta_0}\frac{S_{\theta_0}^{\theta_0/(\theta_0-n)}}{\mu^{n/(\theta_0-n)}} \to 0, \quad \text{as } \mu \to +\infty.$$

Choosing μ_0 sufficiently large, we immediately reach our desired result. $\square$

4. Proof of Theorem 1.5

In view of Lemma 3.1, we may apply the Mountain Pass Theorem without Palais–Smale condition to obtain a (PS) sequence (u_k) in E which, from Lemma 3.2, is bounded. Then, there exists $u \in E$ such that, up to a subsequence, $u_k \rightharpoonup u$ weakly in E. We will prove that, up to a subsequence, $u_k \to u$ strongly in E. Set

$$I_k := \int_{\mathbb{R}^n} Q(|x|) f(u_k)(u_k - u)\, dx.$$

We claim that $I_k \to 0$ as $k \to +\infty$. In fact, by Hölder's inequality

$$|I_k| \leq \|f(u_k)\|_{L^{n'}(\mathbb{R}^n;Q)} \|(u_k - u)\|_{L^n(\mathbb{R}^n;Q)}.$$

By Lemma 2.4, the embedding $E \hookrightarrow L^n(\mathbb{R}^n; Q)$ is compact which implies that $u_k \to u$ strongly in $L^n(\mathbb{R}^n; Q)$. Consequently,

$$\|(u_k - u)\|_{L^n(\mathbb{R}^n;Q)} \to 0, \quad k \to +\infty.$$

At this point, it suffices to prove that

$$\sup_{k\geq 1} \|f(u_k)\|_{L^{n'}(\mathbb{R}^n;Q)} < +\infty.$$

By (f_1) and the α_0-exponential critical growth of f at infinity, and using the fact that $2\alpha_0 > \alpha_0$, there exists a positive constant C such that

$$|f(u_k)|^{n'} \leq C(|u_k|^n + \Phi_{2\alpha_0 n'}(u_k)).$$

On the other hand, we have

$$I(u_k) = I(u_k) - \frac{1}{\theta} I'(u_k)u_k + o_k(1) = \left(\frac{1}{n} - \frac{1}{\theta}\right)\|u_k\|^n + o_k(1) \to c_\mu.$$

By Lemma 3.3, for any $\mu > \mu_0$, we have

$$\left(\frac{c_\mu}{1/n - 1/\theta}\right)^{1/n} < \left(\frac{\lambda}{2\alpha_0 n'}\right)^{1/n'}.$$

Hence, we deduce that

$$\limsup_{k\to+\infty} \|u_k\| < \left(\frac{\lambda}{2\alpha_0 n'}\right)^{1/n'},$$

and in view of Corollary 2.5 we conclude that

$$\sup_{k\geq 1} \int_{\mathbb{R}^n} Q(|x|)\Phi_{2\alpha_0 n'}(u_k)\, dx < +\infty.$$

Finally, using the fact that (u_k) is bounded in $L^{n'}(\mathbb{R}^n; Q)$, our claim immediately follows. Thus, since $\lim_{k\to\infty} I'(u_k)(u_k - u) = 0$, we obtain

$$\int_{\mathbb{R}^n} \big(|\nabla u_k|^{n-2}\nabla u_k \nabla(u_k - u) + V(|x\|)u_k(u_k - u)\big)\, dx = o_k(1).$$

Now, as an immediate consequence of the weak convergence $u_k \rightharpoonup u$ in E, we have

$$\int_{\mathbb{R}^n} \big(|\nabla u|^{n-2}\nabla u \nabla(u_k - u) + V(|x|)u(u_k - u)\big)\,dx = o_k(1).$$

Combining that last identities, we conclude that $u_k \to u$ strongly in E. Since I and I' are continuous, then $I'(u_k) = o_k(1) \to I'(u) = 0$ and $I(u_k) \to I(u) = c_\mu > 0$, proving that u is a nontrivial critical point of the functional I. To finish the proof, it remains to check that u is nonnegative. But, it just suffices to observe that $I'(u)(u^-) = 0$ which leads to $\|u^-\|^n = 0$ and therefore $u = u^+ \geq 0$, where $u^+ := \max\{u, 0\}$ and $u^- := \min\{u, 0\}$ denote the positive and negative part of u, respectively. □

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References

[1] S. Adachi and K. Tanaka, *Trudinger type inequalities in $\mathbb{R}^N$ and their best exponents*, Proc. Amer. Math. Soc. **128** (2000), 2051–2057.

[2] R. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.

[3] Adimurthi, *Existence of Positive solutions of the semilinear Dirichlet problem with critical growth for the N-Laplacian*, Ann. Sci. Norm. Super. Pisa **17** (1990), 393–413.

[4] Adimurthi and K. Sandeep, *A singular Moser–Trudinger embedding and its applications*, NoDEA Nonlinear Differential Equations Appl. **13** (2007), 585–603.

[5] Adimurthi, and Y. Yang, *An interpolation of Hardy inequality and Trudinger–Moser inequality in $\mathbb{R}^N$ and its applications*, Int. Math. Res. Not. IMRN, vol. 2010, no. 13, 2394–2426.

[6] F.S.B. Albuquerque, *Sharp constant and extremal function for weighted Trudinge-r-Moser type inequalities in $\mathbb{R}^2$*, J. Math. Anal. Appl. **421** (2015), 963–970.

[7] F.S.B. Albuquerque, *Standing wave solutions for a class of nonhomogeneous systems in dimension two*, Complex Var. Elliptic Equ. **61** (2016), 1157–1175.

[8] F.S.B. Albuquerque, C.O. Alves and E.S. Medeiros, *Nonlinear Schrödinger equation with unbounded or decaying radial potentials involving exponential critical growth in $\mathbb{R}^2$*, J. Math. Anal. Appl. **409** (2014), 1021–1031.

[9] C.O. Alves, *Multiplicity of solutions for a class of elliptic problem in $\mathbb{R}^2$ with Neumann conditions*, J. Differential Equations **219** (2005), 20–39.

[10] C.O. Alves, J.M.B. do Ó and O.H. Miyagaki, *On nonlinear perturbations of a periodic elliptic problem in $\mathbb{R}^2$ involving critical growth*, Nonlinear Anal. **56** (2004), 781–791.

[11] A. Ambrosetti, V. Felli and A. Malchiodi, *Ground states of nonlinear Schrödinger equations with potentials vanishing at infinity*, J. Eur. Math. Soc. (JEMS) **7** (2005), 117–144.

[12] A. Ambrosetti and P. H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funct. Anal. **14** (1973), 349–381.

[13] S. Aouaoui, *On some semilinear elliptic equation involving exponential growth*, Appl. Math. Lett. **33** (2014), 23–28.

[14] S. Aouaoui, *Multiplicity result for some degenerate elliptic problem involving critical exponential growth in* $\mathbb{R}^N$, Differential Integral Equations **28** (2015), no. 3–4, 255–270.

[15] S. Aouaoui, *Three nontrivial solutions for some elliptic equation involving the* N*-Laplacian*, Electron. J. Qual. Theory Differ. Equ. **2** (2015), 1–12.

[16] S. Aouaoui, *On some local-nonlocal elliptic equation involving nonlinear terms with exponential growth*, Comm. Pure Appl. Anal. **16** (2017), 1767–1784.

[17] S. Aouaoui, *On some quasilinear equation with critical exponential growth at infinity and a singular behavior at the origin*, J. Elliptic Parabol. Equ. **4** (2018), 27–50.

[18] D.M. Cao, *Nontrivial solution of semilinear elliptic equation with critical exponent in* $\mathbb{R}^2$, Comm. Partial Differential Equations **17** (1992), 407–435.

[19] D.G. de Figueiredo, O.H. Miyagaki and B. Ruf, *Elliptic equations in* $\mathbb{R}^2$ *with nonlinearities in the critical growth range*, Calc. Var. Partial Differential Equations **3** (1995), 139–153.

[20] D.G. de Figueiredo, J.M. do Ó and B. Ruf, *On an inequality by N. Trudinger and J. Moser and related elliptic equations*, Comm. Pure Appl. Math. **55** (2002), 1–18.

[21] J.M.B. do Ó, *Quasilinear elliptic equations with exponential nonlinearities*, Comm. Appl. Nonlin. Anal. **2** (1995), 63–72.

[22] J.M.B. do Ó, *Semilinear Dirichlet problems for the* n*-Laplacian in* $\mathbb{R}^n$ *with nonlinearities in critical growth range*, Differential Integral Equations **9** (1996), 967–979.

[23] J.M.B. do Ó, N*-Laplacian equations in* $\mathbb{R}^N$ *with critical growth*, Abstr. Appl. Anal. **2** (1997), 301–315.

[24] J.M.B. do Ó and M. de Souza, *On a class of singular Trudinger–Moser inequalities*, Math. Nachr. **284** (2011), 1754–1776.

[25] J.M.B. do Ó, M. de Souza, E. de Medeiros and U. Severo, *An improvement for the Trudinger–Moser inequality and applications*, J. Differential Equations **256** (2014), 1317–1349.

[26] J.M.B. do Ó, E. Medeiros and U. Severo, *A nonhomogeneous elliptic problem involving critical growth in dimension two*, J. Math. Anal. Appl. **345** (2008), 286–304.

[27] J.M.B. do Ó and B. Ruf, *On a Schrödinger equation with periodic potential and critical growth in* $\mathbb{R}^2$, NoDEA Nonlinear Differential Equations Appl. **13** (2006), 167–192.

[28] J.M.B do Ó, F. Sani and J. Zhang, *Stationary nonlinear Schrödinger equations in* $\mathbb{R}^2$ *with potential vanishing at infinity*, Ann. Mat. Pura Appl. **196** (2017), 363–393.

[29] S. Kesavan, *Symmetrization and Applications*, Series in Analysis, vol. 3, World Scientific, 2006.

[30] Y. Li and B. Ruf, *A sharp Trudinger–Moser type inequality for unbounded domains in* $\mathbb{R}^n$, Indiana Univ. Math. J. **57** (2008), 451–480.

[31] J. Moser, *A sharp form of an inequality by N. Trudinger*, Indiana Univ. Math. J. **20** (1971), 1077–1092.

[32] G. Polya and G. Szegö, *Isoperimetric Inequalities in Mathematical Physics*, Princeton University Press, 1951.

[33] B. Ruf, *A sharp Trudinger–Moser type inequality for unbounded domains in* $\mathbb{R}^2$, J. Funct. Anal. **219** (2005), no. 2, 340–367.

[34] P. Sintzoff and M. Willem, *A semilinear elliptic equation on* $\mathbb{R}^N$ *with unbounded coefficients*, Variational and Topological Methods in the Study of Nonlinear Phenomena (Pisa, 2000), Progr. Nonlinear Differential Equations Appl., vol. 49, Birkhäuser Boston, Boston, MA, 2002, 105–113.

[35] W.A. Strauss, *Existence of solitary waves in higher dimensions*, Comm. Math. Phys. **55** (1977), 149–162.

[36] J. Su, Z.-Q. Wang and M. Willem, *Nonlinear Schrödinger equations with unbounded and decaying radial potentials*, Comm. Contemp. Math. **9** (2007), 571–583.

[37] J. Su, Z.-Q. Wang and M. Willem, *Weighted Sobolev embedding with unbounded and decaying radial potentials*, J. Differential Equations **238** (2007), 20-1-219.

[38] N.S. Trudinger, *On the embedding into Orlicz spaces and some applications*, J. Math. Mech. **17** (1967), 473–484.

[39] Y. Yang, *Existence of positive solutions to quasilinear elliptic equations with exponential growth in the whole Euclidean space*, J. Funct. Anal. **262** (2012), 1679–1704.

[40] Y. Yang, *Trudinger–Moser inequalities on complete noncompact Riemannian manifolds*, J. Funct. Anal. **263** (2012), 1894–1938.

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