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CONLEY INDEX CONTINUATION FOR A SINGULARLY PERTURBED PERIODIC BOUNDARY VALUE PROBLEM

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Abstract. We establish spectral convergence and Conley index continuation results for a class of singularly perturbed periodic boundary value problems.

1. Introduction

This paper is a sequel to our previous articles [2] and [3]. In the paper [3] we considered, with $\varepsilon > 0$ small, a family

$$(\mathrm{E}_\varepsilon, \mathrm{S}_\varepsilon) \qquad \begin{cases} u_t = (a_\varepsilon u_x)_x + g_\varepsilon(x,u), & 0 < x < 1,\ t > 0, \\ \rho u - (1-\rho) a_\varepsilon u_x = 0, & x = 0,\ t > 0, \\ \sigma u + (1-\sigma) a_\varepsilon u_x = 0, & x = 1,\ t > 0 \end{cases}$$

of semilinear boundary value problems.

Here, $0 \leq \rho, \sigma \leq 1$ and $g_\varepsilon(x,u)$ is a nonlinearity satisfying certain (mild) regularity assumptions. The diffusion coefficient a_ε is large except in some small neighbourhood of each of the $n+1$ subdivision points of $[0,1]$ in which a_ε, divided by the length of the neighbourhood, is small as $\varepsilon \to 0$. Moreover, there is some transitory behavior between such neighbourhoods.

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The precise conditions on a_ε are presented in [3, Assumption 2.1], which generalizes an earlier condition introduced in [5], [4].

Let A_ε be the set of all pairs (u,w) with $u \in H^1(0,1)$ and $w \in L^2(0,1)$ such that $a_\varepsilon u \in H^1(0,1)$, $\rho u(0) - (1-\rho)a_\varepsilon(0)u'(0) = \sigma u(1) + (1-\sigma)a_\varepsilon(1)u'(1) = 0$ and $w = -(a_\varepsilon u')'$.

It is known that A_ε is (the graph of) a densely defined nonnegative self-adjoint linear operator in $L^2(0,1)$. If f_ε is the Nemytskĭ operator defined by the function $g_\varepsilon(x,u)$, then problem $(\mathrm{E}_\varepsilon, \mathrm{S}_\varepsilon)$ can be written as the abstract parabolic equation

$$\dot{u} + A_\varepsilon u = f_\varepsilon(u)$$

which generates a local semiflow on π_ε on $H^1(0,1)$.

Now, by results in [3], there is $n \times n$ matrix A_0 which is symmetric with respect to some scalar product on $\mathbb{R}^n$ and such that the first n eigenvalues $\lambda_{l,\varepsilon}$, $l \in [1\,..\,n]$, of A_ε converge, as $\varepsilon \to 0^+$, to the corresponding eigenvalues $\lambda_{l,0}$, $l \in [1\,..\,n]$, of A_0, while $\lambda_{l,\varepsilon} \to \infty$ for $l > n$. One can also choose corresponding eigenfunctions $\widehat{\varphi}_{l,\varepsilon}$ of A_ε converging, in some sense, to a corresponding eigenfunction $\widehat{\varphi}_{l,0}$ of A_0, $l \in [1\,..\,n]$. This is the contents of the spectral convergence result [3, Theorem 2.6], which extends the corresponding spectral convergence result from [4].

If there is a limit $g_0(x,u)$ for the family $g_\varepsilon(x,u)$, then we may consider the limit ordinary differential equation

$$\dot{z} + A_0 z = f_0(z)$$

generating a local (semi)-flow on $\mathbb{R}^n$. Here, f_0 is obtained by properly averaging g_0 on $[0,1]$.

We can now define the linear ε-dependent embedding

$$J_\varepsilon \colon \mathbb{R}^n \to H^1(0,1) \quad \text{by} \quad \varphi_{l,0} \mapsto \varphi_{l,\varepsilon}, \quad l \in [1\,..\,n]. \tag{1.1}$$

It turns out that, with this embedding, the abstract Conley index continuation principles established in [2, Theorems 2.4 and 2.5] are applicable in this situation and yield singular continuation results for the concrete family π_ε, $\varepsilon \geq 0$, as $\varepsilon \to 0$, see [3, Theorem 5.3], cf also [2, Theorem 4.5].

In the present paper we extend the results from [3] to the technically more difficult case with periodic boundary conditions, i.e. to the family of equations

$$(\mathrm{E}_\varepsilon, \mathrm{P}) \qquad \begin{cases} u_t = (a_\varepsilon u_x)_x + g_\varepsilon(x,u), & 0 < x < 1,\ t > 0 \\ u(t,0) = u(t,1), & t > 0. \end{cases}$$

To our knowledge, the periodic case was not considered in this context before.

Our hypotheses on the diffusion coefficients $(a_\varepsilon)_{\varepsilon \in]0,\varepsilon_0[}$ are similar to [3, Assumption 2.1], see Assumption 2.1 below. The hypotheses on the nonlinearity $g_\varepsilon(x,u)$ are as in [2], [3], see Assumption 4.3 below.

The map $u \mapsto -(a_\varepsilon u_x)_x$ with periodic boundary conditions again generates a linear operator A_ε in $L^2(0,1)$. The spectrum of A_ε consists of a sequence of eigenvalues which, however, do not have to be simple, cf. Proposition 2.7.

As we shall describe now, it is the very lack of simplicity of eigenvalues which, compared to the situation in [3], leads to a more restrictive statement of the spectral convergence result and more involved proofs of the Conley index continuation results.

Let $(\lambda_{l,\varepsilon})_l$ be the repeated sequence of eigenvalues of A_ε, i.e. the nondecreasing sequence of eigenvalues of A_ε in which each eigenvalue is repeated according to its multiplicity. Choose an L^2-orthonormal sequence $(\varphi_{l,\varepsilon})_l$ such that $\varphi_{l,\varepsilon}$ is an eigenfunction of A_ε corresponding to $\lambda_{l,\varepsilon}$, $l \in \mathbb{N}$. We prove, in Theorem 2.5 below, that there exists a linear operator A_0 on $\mathbb{R}^n$, symmetric with respect to some scalar product $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathbb{L}}$, with repeated sequence of eigenvalues $(\lambda_{l,0})_{l\in[1..n]}$ (some of which may be double) and such that, for $\varepsilon \to 0$, $\lambda_{l,\varepsilon} \to \lambda_{l,0}$ for $l \in [1..n]$ and $\lambda_{l,\varepsilon} \to \infty$ for $l > n$.

Moreover, we prove that for each null sequence $(\varepsilon_m)_m$ in $]0,\varepsilon_0[$ there is a subsequence $(\varepsilon^1_m)_m$ of $(\varepsilon_m)_m$ and an $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathbb{L}}$-orthonormal sequence $(z_l)_{l\in[1..n]}$ such that, for each $l \in [1..n]$, z_l is an eigenvector of A_0 corresponding to $\lambda_{l,0}$ and such that in some sense, $\varphi_{l,\varepsilon^1_m} \to z_l$ as $m \to \infty$.

We may then consider a limit problem

$$\text{(E}_0\text{)} \qquad \dot z + A_0 z = g_0(z).$$

Here, similarly as in [2], [3], the function $g_0(z)$ is obtained from $g_0(x,u)$ by an averaging procedure.

In order to compare problem (E$_0$) to problem (E$_\varepsilon$, P), we again have to find an appropriate embedding. Unfortunately, in the present case an embedding cannot be defined as in (1.1), since the families $(\varphi_{l,\varepsilon})_\varepsilon$ do not necessarily have a unique limit for $\varepsilon \to 0^+$. Fortunately, we are able to construct an appropriate embedding J_ε, see the beginning of Section 4, but the procedure is more involved.

The boundary value problem (E$_\varepsilon$, P) generates a local semiflow π_ε on the space $H^1_{\text{per}}(0,1)$ of 1-periodic H^1-functions. Moreover, the ODE system (E$_0$) generates a local (semi)flow π_0 on $\mathbb{R}^n$.

Using the constructed family J_ε of embeddings we now proceed as in [2], [3] to establish Conley index and homology index braid continuation results for the family π_ε, $\varepsilon \geq 0$ small, showing in particular that isolated invariant sets S_0 of π_0 continue, for small $\varepsilon > 0$, to isolated invariant sets S_ε of π_ε with S_ε 'close' to $J_\varepsilon(S_0)$ and such that S_0 and S_ε have the same Conley index, see Theorems 4.5 and 4.8. In particular, some aspects of the dynamics of the simpler flow π_0 can be found in the more complicated semiflow π_ε.

The above 'closeness' is with respect to certain ε-dependent Hilbert norms $\|\cdot\|_\varepsilon$ on the space $H^1_{\text{per}}(0,1)$ and it implies $C([0,1])$-closeness. On the other hand, the embedding J_ε, though necessary for the applicability of Conley index theory, is not very explicit. A more natural, ε-independent and explicit embedding Θ is obtained by interpreting each element of $\mathbb{R}^n$ as a step function relative to the decomposition of $[0,1]$ given in Assumption 2.1 below.

It turns out that, as a consequence of $\|\cdot\|_\varepsilon$-closeness and our construction of J_ε, the set S_ε actually is $L^r(0,1)$-close to $\Theta(S_0)$ for any $r\in[1,\infty[$, see Proposition 4.7.

Our proof methods are kept at an abstract level and permit applications to other types of singularly perturbed infinite dimensional dynamical systems with a finite dimensional limit. This will be treated in a subsequent publication.

In this paper, all linear spaces are defined over the real numbers field.

2. The spectral convergence result

In this section we will state one of the main results of this paper, the spectral convergence theorem. Throughout this paper, $\mathbf{m}$ is the one-dimensional Lebesgue measure.

We begin by stating our linear hypothesis:

Assumption 2.1.

(1a) $n\in\mathbb{N}$, $\varepsilon_0\in\,]0,\infty]$ and x_0, $x_{n+1}\in\mathbb{R}$ with $x_0<x_{n+1}$;

(1b) $(a_\varepsilon)_{\varepsilon\in]0,\varepsilon_0[}$ is a family of continuous positive functions on $[x_0,x_{n+1}]$;

(1c) $(x_j)_{j\in[1..n]}$ is a strictly increasing sequence in $]x_0,x_{n+1}[$, $(\tau_j)_{j\in[1..n]}$ is a sequence in $]0,\infty[$ and $\xi'_{j,\varepsilon}$, $\xi_{j,\varepsilon}$, $\zeta_{j,\varepsilon}$, $\zeta'_{j,\varepsilon}$ are families in $]x_0,x_{n+1}[$ with $\xi'_{j,\varepsilon}<\xi_{j,\varepsilon}<x_j<\zeta_{j,\varepsilon}<\zeta'_{j,\varepsilon}$, $j\in[1\,.\,.\,n]$, $\varepsilon\in\,]0,\varepsilon_0[$.

Furthermore, $\zeta'_{j,\varepsilon}<\xi'_{j+1,\varepsilon}$ if $j\le n-1$. For each $j\in[1\,.\,.\,n]$, $\mathbf{m}([\xi'_{j,\varepsilon},\zeta'_{j,\varepsilon}])\to 0$ as $\varepsilon\to 0$.

(2a) If $(\Gamma_\varepsilon)_{\varepsilon\in]0,\varepsilon_0[}$ is any of the following families:

$$([x_0,\xi'_{1,\varepsilon}])_{\varepsilon\in]0,\varepsilon_0[},\qquad ([\zeta'_{j,\varepsilon},\xi'_{j+1,\varepsilon}])_{\varepsilon\in]0,\varepsilon_0[}\quad\text{or}\quad ([\zeta'_{n,\varepsilon},x_{n+1}])_{\varepsilon\in]0,\varepsilon_0[},$$

for $j\in[1\,.\,.\,n-1]$, or else any of the families

$$([\xi'_{j,\varepsilon},\xi_{j,\varepsilon}])_{\varepsilon\in]0,\varepsilon_0[},\qquad ([\zeta_{j,\varepsilon},\zeta'_{j,\varepsilon}])_{\varepsilon\in]0,\varepsilon_0[},$$

for $j\in[1\,.\,.\,n]$, then

$$\frac{\inf_{\Gamma_\varepsilon} a_\varepsilon}{\mathbf{m}(\Gamma_\varepsilon)}\to\infty\quad\text{as }\varepsilon\to 0.$$

(2b) For each $j\in[1\,.\,.\,n]$ and $\varepsilon\in\,]0,\varepsilon_0[$, set $\Gamma_\varepsilon=[\xi_{j,\varepsilon},\zeta_{j,\varepsilon}]$. Then

$$\frac{\inf_{\Gamma_\varepsilon} a_\varepsilon}{\mathbf{m}(\Gamma_\varepsilon)}\to\tau_j\quad\text{and}\quad\frac{\sup_{\Gamma_\varepsilon} a_\varepsilon}{\mathbf{m}(\Gamma_\varepsilon)}\to\tau_j\quad\text{as }\varepsilon\to 0.$$

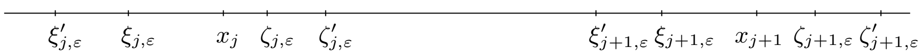
Notation. In the sequel, we write

$$K_{j,\varepsilon} = [\zeta_{j,\varepsilon}, \xi_{j+1,\varepsilon}]\,, \quad K'_{j,\varepsilon} = [\zeta'_{j,\varepsilon}, \xi'_{j+1,\varepsilon}]\,, \quad K_j = [x_j, x_{j+1}]\,,$$

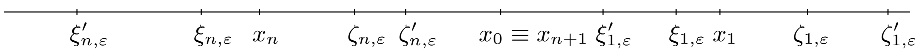
for $j \in [1\,..\,n-1]$, and

$$K_{n,\varepsilon} = [\zeta_{n,\varepsilon}, x_{n+1}] \cup [x_0, \xi_{1,\varepsilon}]\,, \qquad K'_{n,\varepsilon} = [\zeta'_{n,\varepsilon}, x_{n+1}] \cup [x_0, \xi'_{1,\varepsilon}]\,,$$
$$K_n = [x_n, x_{n+1}] \cup [x_0, x_1]\,, \qquad L_j = \mathbf{m}(K_j), \quad j \in [1\,..\,n]\,.$$

Remark 2.2. These notations and Assumption 2.1 are best understood by viewing the above number families in the interval $]x_0, x_{n+1}[$ as number families in the one-sphere $S = [x_0, x_{n+1}]\,/\{x_0, x_{n+1}\}$. Then, for $j \in [1\,..\,n-1]$ we have the following picture:



while, for $j = n$ with the identification of x_0 with x_{n+1} we have the following picture:



In particular, the set $K_{n,\varepsilon}$ (resp. $K'_{n,\varepsilon}$, resp. K_n) is the interval in S from $\zeta_{n,\varepsilon}$ to $\xi_{1,\varepsilon}$ (resp. from $\zeta'_{n,\varepsilon}$ to $\xi'_{1,\varepsilon}$, resp. from x_n to x_1).

Let $j \in [1\,..\,n]$ be arbitrary. Since $\mathbf{m}(K'_{j,\varepsilon}) \to L_j > 0$ as $\varepsilon \to 0$, part (2a) of Assumption 2.1 implies that $a_\varepsilon \to \infty$ for $\varepsilon \to 0$, uniformly in $K'_{j,\varepsilon}$. Moreover, by part (2b), on the small intervals $[\xi_{j,\varepsilon}, \zeta_{j,\varepsilon}]$ around x_j, a_ε is of the same order as the measure of these intervals so $a_\varepsilon \to 0$ for $\varepsilon \to 0$, uniformly in $[\xi_{j,\varepsilon}, \zeta_{j,\varepsilon}]$. Finally, there is some transitional behavior on the remaining small intervals $[\xi'_{j,\varepsilon}, \xi_{j,\varepsilon}]$ and $[\zeta_{j,\varepsilon}, \zeta'_{j,\varepsilon}]$ around x_j, as a_ε is of lower order than the measure of these intervals.

The following result further clarifies the above hypothesis.

Proposition 2.3. *If Assumption* 2.1 *holds, then*

$$\frac{\mathbf{m}([\xi'_{j,\varepsilon}, \xi_{j,\varepsilon}]) + \mathbf{m}([\zeta_{j,\varepsilon}, \zeta'_{j,\varepsilon}])}{\mathbf{m}([\xi_{j,\varepsilon}, \zeta_{j,\varepsilon}])} \to 0 \quad \text{as } \varepsilon \to 0,\ j \in [1\,..\,n]. \tag{2.1}$$

Conversely, if parts (1a), (1c) *of Assumption* 2.1 *together with estimate* (2.1) *hold, then there is a family* $(a_\varepsilon)_{\varepsilon \in]0,\varepsilon_0[}$, *such that parts* (1b), (2a) *and* (2b) *of that assumption are also satisfied. In addition, we may assume that each function* a_ε *can be extended to a* $(x_{n+1} - x_0)$*-periodic* C^∞*-function defined on all of* $\mathbb{R}$.

Proof. If Assumption 2.1 holds, then, for each $j \in [1\,..\,n]$ by (2a),

$$\frac{a_\varepsilon(\zeta_{j,\varepsilon})}{\mathbf{m}([\zeta_{j,\varepsilon}, \zeta'_{j,\varepsilon}])} \to \infty \quad \text{and} \quad \frac{a_\varepsilon(\xi_{j,\varepsilon})}{\mathbf{m}([\xi'_{j,\varepsilon}, \xi_{j,\varepsilon}])} \to \infty \quad \text{as } \varepsilon \to 0$$

while, by (2b),

$$\frac{a_\varepsilon(\zeta_{j,\varepsilon})}{\mathbf{m}([\xi_{j,\varepsilon},\zeta_{j,\varepsilon}])} \to \tau_j \quad \text{and} \quad \frac{a_\varepsilon(\xi_{j,\varepsilon})}{\mathbf{m}([\xi_{j,\varepsilon},\zeta_{j,\varepsilon}])} \to \tau_j \quad \text{as } \varepsilon \to 0.$$

These estimates imply estimate (2.1).

Conversely, if parts (1a), (1c) of Assumption 2.1 together with estimate (2.1) hold, then define, for each $\varepsilon \in \,]0,\varepsilon_0[$ the uniquely determined continuous function $a_\varepsilon\colon [x_0,x_{n+1}] \to \mathbb{R}$ such that, for each $j \in [1\,.\,.\,n]$,

$$\begin{aligned} a_\varepsilon(x) &= \varepsilon^{-1} && \text{on } K'_{j,\varepsilon}, \\ a_\varepsilon(x) &= \tau_j \cdot \mathbf{m}([\xi_{j,\varepsilon},\zeta_{j,\varepsilon}]) && \text{on } [\xi_{j,\varepsilon},\zeta_{j,\varepsilon}] \end{aligned}$$

and a_ε is affine on $[\xi'_{j,\varepsilon},\xi_{j,\varepsilon}]$ and on $[\zeta_{j,\varepsilon},\zeta'_{j,\varepsilon}]$. With this choice of $(a_\varepsilon)_{\varepsilon\in]0,\varepsilon_0[}$ and estimate (2.1) it is easily proved that parts (1b), (2a) and (2b) of Assumption 2.1 also hold. Each function a_ε is constant on $K_{n,\varepsilon}$ so it can be extended to a continuous $(x_{n+1}-x_0)$-periodic function defined on all of $\mathbb{R}$. Applying to the latter function the usual smoothing procedure via mollifiers, we obtain, for every $b_\varepsilon \in \,]0,\infty[$ a smooth $(x_{n+1}-x_0)$-periodic function $\widetilde{a}_\varepsilon$ on $\mathbb{R}$, which differs from a_ε by at most b_ε on $[x_0,x_{n+1}]$. Choosing $(b_\varepsilon)_{\varepsilon\in]0,\varepsilon_0[}$ so small that $b_\varepsilon < \inf_{[x_0,x_{n+1}]} a_\varepsilon$ and $b_\varepsilon/\mathbf{m}(K_\varepsilon) \to 0$, where $(K_\varepsilon)_{\varepsilon\in]0,\varepsilon_0[}$ is any family occurring in Assumption 2.1, we see that with the choice of the family $(\widetilde{a}_\varepsilon)_{\varepsilon\in]0,\varepsilon_0[}$, parts (1b), (2a) and (2b) of Assumption 2.1 also hold. □

Define $H^1_{\mathrm{per}} = H^1_{\mathrm{per}}(x_0,x_{n+1})$ (see the Appendix). For each $\varepsilon \in \,]0,\varepsilon_0[$ define the bilinear form $b_\varepsilon := b_{a_\varepsilon}$. Let

$$\langle\,\cdot\,,\,\cdot\,\rangle_{L^2} = \langle\,\cdot\,,\,\cdot\,\rangle_{L^2(x_0,x_{n+1})}$$

be the standard scalar product on $L^2 = L^2(x_0,x_{n+1})$ and

(2.2) let $A_\varepsilon\colon D_\varepsilon := D(A_\varepsilon) \subset H^1_{\mathrm{per}} \to L^2$ be the linear operator defined by the pair $(b_\varepsilon, \langle\,\cdot\,,\,\cdot\,\rangle_{L^2})$.

In this paper we will consider the following norm in H^1_{per}:

$$\|u\|^2_\varepsilon := b_\varepsilon(u,u) + \|u\|^2_{L^2}, \quad u \in H^1_{\mathrm{per}}. \tag{2.3}$$

Let $(\lambda_{l,\varepsilon})_l$ be the repeated sequence of eigenvalues of A_ε, i.e. the nondecreasing sequence of eigenvalues of A_ε in which each eigenvalue is repeated according to its multiplicity. Choose an L^2-orthonormal sequence $(\varphi_{l,\varepsilon})_l$ such that $\varphi_{l,\varepsilon}$ is an eigenfunction of A_ε corresponding to $\lambda_{l,\varepsilon}$, $l \in \mathbb{N}$.

Now define the 'limit' bilinear form $b_0\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by

$$b_0(y,z) = \tau_1(y_1-y_n)(z_1-z_n) + \sum_{j=2}^n \tau_j(y_j-y_{j-1})(z_j-z_{j-1})$$

and the scalar product $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathbb{L}}$ on $\mathbb{R}^n$ by

$$\langle y, z\rangle_{\mathbb{L}} = \sum_{j=1}^{n} L_j y_j z_j, \quad y = (y_j)_{j\in[1..n]},\ z = (z_j)_{j\in[1..n]} \in \mathbb{R}^n.$$

(2.4) Let $A_0\colon \mathbb{R}^n \to \mathbb{R}^n$ be the linear map defined by the pair $(b_0, \langle\,\cdot\,,\,\cdot\,\rangle_{\mathbb{L}})$.

The map A_0 is $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathbb{L}}$-symmetric.

Remark 2.4. Note that, unlike in the boundary value case considered in [5] and [4], the operator A_0 may have a double eigenvalue as the following example shows: Let $L_j = 1/3$ and $\tau_j = 1$ for $j \in [1\,.\,.\,3]$. Then the matrix of A_0 relative to the canonical basis of $\mathbb{R}^3$ takes the form

$$\begin{pmatrix} 6 & -3 & -3 \\ -3 & 6 & -3 \\ -3 & -3 & 6 \end{pmatrix}.$$

Thus $\lambda = 0$ is an eigenvalue of A_0 with geometric multiplicity 1 and $\lambda = 9$ is an eigenvalue of A_0 with geometric multiplicity 2.

Now let $(\lambda_{l,0})_{l\in[1..n]}$ be the repeated sequence of eigenvalues of A_0. Define also the following norm on $\mathbb{R}^n$:

$$\|z\|_0^2 := b_0(z, z) + \|z\|_{\mathbb{L}}^2, \quad z \in \mathbb{R}^n. \tag{2.5}$$

We can now state our spectral convergence result.

Theorem 2.5. *With the notation introduced above the following assertions hold:*

(a) $\lambda_{n+1,\varepsilon} \to \infty$ *as* $\varepsilon \to 0$.
(b) *For each* $l \in [1\,.\,.\,n]$, $\lambda_{l,\varepsilon} \to \lambda_{l,0}$ *as* $\varepsilon \to 0$.
(c) *For each null sequence* $(\varepsilon_m)_m$ *in* $]0, \varepsilon_0[$ *there is a subsequence* $(\varepsilon_m^1)_m$ *of* $(\varepsilon_m)_m$ *and an* $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathbb{L}}$*-orthonormal sequence* $(z_l)_{l\in[1..n]}$ *such that, for each* $l \in [1\,.\,.\,n]$, z_l *is an eigenvector of* A_0 *corresponding to* $\lambda_{l,0}$ *and such that for each* $j \in [1\,.\,.\,n]$

$$\sup_{x\in K_{j,\varepsilon_m^1}} |\varphi_{l,\varepsilon_m^1}(x) - z_{l,j}| \to 0, \quad \text{as } m \to \infty,$$

where $z_{l,j}$ *is the* j*-th component of the vector* z_l.

Theorem 2.5 extends the main part of [3, Theorem 2.6] to the periodic case. We will give a proof of Theorem 2.5 in the next section.

In the remaining part of this section we will show that, if Assumption 2.1 is satisfied and some additional periodicity hypotheses hold, then the corresponding operators A_ε have double eigenvalues which remain bounded as $\varepsilon \to 0^+$.

For the rest of this section suppose n, ε_0, $x_0 = 0$, $x_{n+1} = (1/3)$, a_ε, x_j, τ_j, $\xi'_{j,\varepsilon}$, $\xi_{j,\varepsilon}$, $\zeta_{j,\varepsilon}$ and $\zeta'_{j,\varepsilon}$, $\varepsilon \in\ ]0, \varepsilon_0[$, $j \in [1\,.\,.\,n]$, satisfy Assumption 2.1.

Additionally assume that $a_\varepsilon(0) = a_\varepsilon(1/3)$ for each $\varepsilon \in \,]0,\varepsilon_0[$ (cf. Proposition 2.3) and let $\widetilde{a}_\varepsilon\colon [0,1] \to \mathbb{R}$ be the (1/3)-periodic extension of a_ε. Let $A_\varepsilon\colon D(A_\varepsilon) \subset H^1_{\mathrm{per}}(0,1/3) \to L^2(0,1/3)$ be the linear operator defined by a_ε with (1/3)-periodic condition.

Let $\widetilde{n} = 3n$. Let $\varepsilon \in \,]0,\varepsilon_0[$ be arbitrary. Whenever α_j, $j \in [1\,.\,.\,n]$, is any of the sequences x_j, $\xi'_{j,\varepsilon}$, $\xi_{j,\varepsilon}$, $\zeta_{j,\varepsilon}$ and $\zeta'_{j,\varepsilon}$, $j \in [1\,.\,.\,n]$, define the sequence $\widetilde{\alpha}_j$, $j \in [1\,.\,.\,\widetilde{n}]$, in $]0,1[$ by $\widetilde{\alpha}_j = \alpha_j$, $\widetilde{\alpha}_{n+j} = \alpha_j + (1/3)$ and $\widetilde{\alpha}_{2n+j} = \alpha_j + (2/3)$, $j \in [1\,.\,.\,n]$. Moreover, define the sequence $\widetilde{\tau}_j$, $j \in [1\,.\,.\,\widetilde{n}]$, in $\mathbb{R}$ by $\widetilde{\tau}_j = \tau_j$, $\widetilde{\tau}_{n+j} = \tau_j$ and $\widetilde{\tau}_{2n+j} = \tau_j$, $j \in [1\,.\,.\,n]$.

It is easy to show that $\widetilde{n}$, ε_0, $\widetilde{x}_0 = 0$, $\widetilde{x}_{\widetilde{n}+1} = 1$, $\widetilde{a}_\varepsilon$, $\widetilde{x}_j$, $\widetilde{\tau}_j$, $\widetilde{\xi}'_{j,\varepsilon}$, $\widetilde{\xi}_{j,\varepsilon}$, $\widetilde{\zeta}_{j,\varepsilon}$ and $\widetilde{\zeta}'_{j,\varepsilon}$, $\varepsilon \in \,]0,\varepsilon_0[$, $j \in [1\,.\,.\,n]$, satisfy Assumption 2.1.

Let $\widetilde{A}_\varepsilon\colon D(\widetilde{A}_\varepsilon) \subset H^1_{\mathrm{per}}(0,1) \to L^2(0,1)$ be the linear operator defined by $\widetilde{a}_\varepsilon$ with 1-periodic condition.

Let $(\lambda_{l,\varepsilon})_l$ be the repeated sequence of eigenvalues of A_ε in which each eigenvalue is repeated according to its multiplicity. Choose an $L^2(0,1/3)$-orthonormal sequence $(\varphi_{l,\varepsilon})_l$ such that $\varphi_{l,\varepsilon}$ is defined and continuous on $[0,1/3]$ and is an eigenfunction of A_ε corresponding to $\lambda_{l,\varepsilon}$, $l \in \mathbb{N}$.

Let $(\widetilde{\lambda}_{l,\varepsilon})_l$ be the repeated sequence of eigenvalues of $\widetilde{A}_\varepsilon$ in which each eigenvalue is repeated according to its multiplicity. Choose an $L^2(0,1)$-orthonormal sequence $(\widetilde{\varphi}_{l,\varepsilon})_l$ such that $\widetilde{\varphi}_{l,\varepsilon}$ is defined and continuous on $[0,1]$ and is an eigenfunction of $\widetilde{A}_\varepsilon$ corresponding to $\widetilde{\lambda}_{l,\varepsilon}$, $l \in \mathbb{N}$.

For each $p \in \mathbb{N}$ and $\varepsilon \in \,]0,\varepsilon_0[$, let $U_{p,\varepsilon}$ be the span of the eigenfunctions $\varphi_{l,\varepsilon}$, for $l \in [1\,.\,.\,p]$ and let $\widetilde{U}_{p,\varepsilon}$ be the span of the eigenfunctions $\widetilde{\varphi}_{l,\varepsilon}$, for $l \in [1\,.\,.\,p]$.

Theorem 2.5 implies that there exist an $\widehat{\varepsilon} \in \,]0,\varepsilon_0[$ and an $M \in \,]0,\infty[$ such that

$$\widetilde{\lambda}_{\widetilde{n},\varepsilon} \le M < \lambda_{n+1,\varepsilon}, \quad \text{for all } \varepsilon \in \,]0,\widehat{\varepsilon}]. \tag{2.6}$$

Lemma 2.6. *With the notation introduced above, for each $\varepsilon \in \,]0,\widehat{\varepsilon}]$ there exists a $l_\varepsilon \in [1\,.\,.\,\widetilde{n}]$ such that $\widetilde{\varphi}_{l_\varepsilon,\varepsilon}$ is not (1/3)-periodic or $\widetilde{\varphi}'_{l_\varepsilon,\varepsilon}$ is not (1/3)-periodic.*

Proof. Suppose that there exists an $\varepsilon \in \,]0,\widehat{\varepsilon}]$ such that for all $l \in [1\,.\,.\,\widetilde{n}]$, $\widetilde{\varphi}_{l,\varepsilon}$ and $\widetilde{\varphi}'_{l,\varepsilon}$ are (1/3)-periodic.

Since $\widetilde{\varphi}_{l,\varepsilon}$ is a (1/3)-periodic continuous function for all $l \in [1\,.\,.\,\widetilde{n}]$, it follows that φ is (1/3)-periodic and continuous (on $[0,1]$) for all $\varphi \in \widetilde{U}_{\widetilde{n},\varepsilon}$. Define $\Gamma\colon \widetilde{U}_{\widetilde{n},\varepsilon} \to C([0,1/3])$ by $\Gamma\varphi = \varphi|_{[0,1/3]}$, $\varphi \in \widetilde{U}_{\widetilde{n},\varepsilon}$. It is clear that Γ is a linear map. Moreover, Γ is injective. Indeed, let $\varphi \in \widetilde{U}_{\widetilde{n},\varepsilon}$ be such that $\Gamma\varphi = 0$. Then $\varphi(x) = 0$ for all $x \in [0,1/3]$. Since φ is (1/3)-periodic, it follows that $\varphi = 0$.

Fix $l \in [1\,.\,.\,\widetilde{n}]$ and set $\varphi = \Gamma\widetilde{\varphi}_{l,\varepsilon}$. Since $\widetilde{\varphi}_{l,\varepsilon} \ne 0$, it follows that $\varphi \ne 0$. Moreover, we claim that $A_\varepsilon\varphi$ is defined and $A_\varepsilon\varphi = \widetilde{\lambda}_{l,\varepsilon}\varphi$.

Indeed, since $\widetilde{\varphi}_{l,\varepsilon} \in D(\widetilde{A}_\varepsilon)$ and $\widetilde{\varphi}_{l,\varepsilon}$ is an eigenfunction of $\widetilde{A}_\varepsilon$ and is continuous on $[0,1]$, it follows that $\widetilde{A}_\varepsilon\widetilde{\varphi}_{l,\varepsilon}$ is continuous and so the result in the Appendix implies that $\widetilde{\varphi}_{l,\varepsilon}$ is of class C^1 on $[0,1]$, both $\widetilde{\varphi}_{l,\varepsilon}$ and $\widetilde{\varphi}'_{l,\varepsilon}$ (in the classical sense) are 1-periodic functions and $\widetilde{a}_\varepsilon\widetilde{\varphi}'_{l,\varepsilon} \in C^1([0,1],\mathbb{R})$ with $(\widetilde{a}_\varepsilon\widetilde{\varphi}'_{l,\varepsilon})'(x) = -\widetilde{\lambda}_{l,\varepsilon}\widetilde{\varphi}_{l,\varepsilon}(x)$ for all $x \in [0,1]$.

Since $\varphi = \widetilde{\varphi}_{l,\varepsilon}|_{[0,1/3]}$, it follows that φ is of class C^1 on $[0,1/3]$ and $a_\varepsilon\varphi' = (\widetilde{a}_\varepsilon\widetilde{\varphi}'_{l,\varepsilon})|_{[0,1/3]} \in C^1([0,1/3])$. Since $\widetilde{\varphi}_{l,\varepsilon}$ and $\widetilde{\varphi}'_{l,\varepsilon}$ are (1/3)-periodic, it follows that $\widetilde{\varphi}_{l,\varepsilon}(0) = \widetilde{\varphi}_{l,\varepsilon}(1/3)$ and $\widetilde{\varphi}'_{l,\varepsilon}(0) = \widetilde{\varphi}'_{l,\varepsilon}(1/3)$ and so $\varphi(0) = \varphi(1/3)$ and $\varphi'(0) = \varphi'(1/3)$. Therefore, φ and φ' are (1/3)-periodic. Moreover,

$$(a_\varepsilon\varphi')'(x) = (\widetilde{a}_\varepsilon\widetilde{\varphi}'_{l,\varepsilon})'(x) = -\widetilde{\lambda}_{l,\varepsilon}\widetilde{\varphi}_{l,\varepsilon}(x) = -\widetilde{\lambda}_{l,\varepsilon}\varphi(x), \quad \text{for all } x \in [0,1/3].$$

Hence, the result in the Appendix implies that $\varphi \in D(A_\varepsilon)$ and $A_\varepsilon\varphi = \widetilde{\lambda}_{l,\varepsilon}\varphi$. The claim is proved. Therefore, φ is an eigenfunction of A_ε.

Formula (2.6) implies that $\widetilde{\lambda}_{l,\varepsilon} \le M$ and that there exists a $j \in [1\,.\,.\,n]$ such that $\widetilde{\lambda}_{l,\varepsilon} = \lambda_{j,\varepsilon}$. Therefore, $\varphi \in U_{n,\varepsilon}$. This implies that $\Gamma(\widetilde{U}_{\widetilde{n},\varepsilon}) \subset U_{n,\varepsilon}$. Since Γ is injective and $\dim U_{n,\varepsilon} = n < \widetilde{n} = \dim \widetilde{U}_{\widetilde{n},\varepsilon}$, we obtain a contradiction. □

PROPOSITION 2.7. *With the notation introduced above, for each* $\varepsilon \in \,]0,\widehat{\varepsilon}]$ *let* $l_\varepsilon \in [1\,.\,.\,\widetilde{n}]$ *be as in Lemma* 2.6. *Then* $\lambda = \widetilde{\lambda}_{l_\varepsilon,\varepsilon}$ *is a double eigenvalue of* $\widetilde{A}_\varepsilon$.

PROOF. Let $\varepsilon \in \,]0,\widehat{\varepsilon}]$ be fixed and let $l_\varepsilon \in [1\,.\,.\,\widetilde{n}]$ be as in Lemma 2.6. Set $\varphi = \widetilde{\varphi}_{l_\varepsilon,\varepsilon}$. The result in the Appendix implies that $\varphi \in C^1([0,1])$, $\varphi(0) = \varphi(1)$, $\varphi'(0) = \varphi'(1)$, $v := \widetilde{a}_\varepsilon\varphi' \in C^1([0,1])$ and $v'(x) = -\lambda\varphi(x)$ for all $x \in [0,1]$. Let $\check{a}_\varepsilon\colon \mathbb{R} \to \mathbb{R}$ be the (1/3)-periodic extension of a_ε, which is also the 1-periodic extension of $\widetilde{a}_\varepsilon$. Let $\check{\varphi}\colon \mathbb{R} \to \mathbb{R}$ be the 1-periodic extension of φ. It follows that $\check{a}_\varepsilon\check{\varphi}'$ is the 1-periodic extension of v and so $(\check{a}_\varepsilon\check{\varphi}')'(x) = -\lambda\check{\varphi}(x)$ for all $x \in \mathbb{R}$. Let $\gamma\colon \mathbb{R} \to \mathbb{R}$ be the (1/3)-translate of $\check{\varphi}$, $\gamma(x) = \check{\varphi}(x + (1/3))$, $x \in \mathbb{R}$. Thus $\gamma \in C^1(\mathbb{R})$, and since $\check{a}_\varepsilon$ is (1/3)-periodic, we also have that $(\check{a}_\varepsilon\gamma')'(x) = -\lambda\gamma(x)$ for all $x \in \mathbb{R}$. Since $\check{\varphi}$ is 1-periodic, so is γ and therefore γ' is 1-periodic as well. Therefore, we have proved that $\psi = \gamma_{|[0,1]}$ is an eigenfunction of $\widetilde{A}_\varepsilon$ corresponding to the eigenvalue λ.

We claim now that φ and ψ are linearly independent. Suppose this does not hold. Since $\varphi \ne 0$ and $\psi \ne 0$, there exists a $\rho \in \mathbb{R}$ such that $\psi = \rho\varphi$. Since $\|\varphi\|_{L^2(0,1)} = 1$ and $\check{\varphi}$ is the 1-periodic extension of φ to $\mathbb{R}$, it follows that $\|\psi\|_{L^2(0,1)} = 1$. Hence $\rho = 1$ or $\rho = -1$. Suppose first that $\rho = 1$ so $\varphi = \psi$ on $[0,1]$. In particular,

$$\varphi(0) = \psi(0) = \varphi(1/3) \quad \text{and} \quad \varphi'(0) = \psi'(0) = \varphi'(1/3),$$

which contradicts the choice of φ.

Thus $\rho = -1$ and so

$$\check{\varphi}(x) = -\check{\varphi}(x + 1/3), \quad \text{for all } x \in [0,1].$$

Let $x \in [0,1]$. We have

$$\breve{\varphi}(x+2/3) = -\breve{\varphi}(x+1/3) = -(-\breve{\varphi}(x)) = \breve{\varphi}(x)$$

which implies that

$$\breve{\varphi}(x) = \breve{\varphi}(x+1) = -\breve{\varphi}(x+2/3) = -\breve{\varphi}(x).$$

We have proved that $\breve{\varphi}(x) = 0$ for all $x \in [0,1]$ which is a contradiction. Therefore, the claim is proved and this concludes the proof of the proposition. □

3. Proof of Theorem 2.5

This section is devoted to the proof of Theorem 2.5. We will consider the case $x_0 = 0$ and $x_{n+1} = 1$. The general case follows by a simple change of coordinates.

We follow the proof of the spectral convergence result from [3] but, for brevity, omit those steps in the proof which are very similar to the ones given in [3]. The following lemma was proved in [3, Lemma 3.1].

Lemma 3.1. *If $M \in \,]0,\infty[$, $I \subset [0,1]$ is a compact interval, $a\colon I \to \mathbb{R}$ is a continuous positive function and $\varphi \in H^1(0,1)$ is such that $\int_I a \cdot (\varphi')^2\,dx \le M$, then*

$$|\varphi(x) - \varphi(y)|^2 \le M \frac{\mathbf{m}(I)}{\inf\limits_I a}, \quad \text{for } x, y \in I.$$

For each $\varepsilon \in \,]0,\varepsilon_0[$ and $j \in [1\,.\,.\,n]$ define $\psi_{j,\varepsilon}\colon [0,1] \to \mathbb{R}$ as the uniquely determined continuous function such that

(1) if $j \in [1\,.\,.\,n-1]$, then $\psi_{j,\varepsilon}(x) = 1$ for $x \in [\zeta_{j,\varepsilon}, \xi_{j+1,\varepsilon}]$, $\psi_{j,\varepsilon}(x) = 0$ for $x \notin [\xi_{j,\varepsilon}, \zeta_{j+1,\varepsilon}]$ and $\psi_{j,\varepsilon}$ is affine on each of the intervals $[\xi_{j,\varepsilon}, \zeta_{j,\varepsilon}]$ and $[\xi_{j+1,\varepsilon}, \zeta_{j+1,\varepsilon}]$.

(2) $\psi_{n,\varepsilon}(x) = 1$ for $x \in [0, \xi_{1,\varepsilon}] \cup [\zeta_{n,\varepsilon}, 1]$, $\psi_{n,\varepsilon}(x) = 0$ for $x \notin [0, \zeta_{1,\varepsilon}] \cup [\xi_{n,\varepsilon}, 1]$ and $\psi_{n,\varepsilon}$ is affine on each of the intervals $[\xi_{1,\varepsilon}, \zeta_{1,\varepsilon}]$ and $[\xi_{n,\varepsilon}, \zeta_{n,\varepsilon}]$.

For each $\varepsilon \in \,]0,\varepsilon_0[$, let W_ε be the span of the functions $\psi_{j,\varepsilon}$, $j \in [1\,.\,.\,n]$, i.e. the n-dimensional subspace of $H^1_{\text{per}}(0,1)$ given by

$$W_\varepsilon = \left\{ \sum_{j=1}^n u_j \psi_{j,\varepsilon} \;\middle|\; u_j \in \mathbb{R},\ \text{for } j \in [1\,.\,.\,n] \right\}.$$

Lemma 3.2. *There exists a $C_1' \in \,]0,\infty[$ and an $\varepsilon_1' \in \,]0,\varepsilon_0[$ such that*

$$\frac{b_\varepsilon(u,u)}{\|u\|^2_{L^2}} \le C_1' \quad \textit{for all } \varepsilon \in \,]0,\varepsilon_1'] \textit{ and all } u \in W_\varepsilon \textit{ with } u \ne 0.$$

Proof. There is a $c \in \,]0,\infty[$ and an $\varepsilon_1 \in \,]0,\varepsilon_0[$ such that

$$c \le \min\Big(\min_{j \in [1..n-1]} (\xi_{j+1,\varepsilon} - \zeta_{j,\varepsilon}), \xi_{1,\varepsilon} + 1 - \zeta_{n,\varepsilon} \Big)$$

for all $\varepsilon \in]0, \varepsilon_1]$. Let $\varepsilon \in]0, \varepsilon_1]$ and let $u \in W_\varepsilon$ be arbitrary with $\|u\|_{L^2}^2 = 1$. Hence $u = \sum_{j=1}^{n} u_j \psi_{j,\varepsilon}$ with $u_j \in \mathbb{R}$, for $j \in [1..n]$. Thus

$$1 = \|u\|_{L^2}^2 \geq \int_0^{\xi_{1,\varepsilon}} u^2\,dx + \sum_{j=1}^{n-1} \int_{\zeta_{j,\varepsilon}}^{\xi_{j+1,\varepsilon}} u^2\,dx + \int_{\zeta_{n,\varepsilon}}^{1} u^2\,dx$$

$$= u_n^2\,\xi_{1,\varepsilon} + \sum_{j=1}^{n-1} u_j^2(\xi_{j+1,\varepsilon} - \zeta_{j,\varepsilon}) + u_n^2(1 - \zeta_{n,\varepsilon}) \geq c \sum_{j=1}^{n} u_j^2,$$

so $|u_j| \leq c^{-1/2}$ for all $j \in [1..n]$. Notice that $u'(x) = 0$ for $x \in [0, \xi_{1,\varepsilon}] \cup \bigcup_{j=1}^{n-1} [\zeta_{j,\varepsilon}, \xi_{j+1,\varepsilon}] \cup [\zeta_{n,\varepsilon}, 1]$. Moreover, for $j \in [1..n]$ and $x \in [\xi_{j,\varepsilon}, \zeta_{j,\varepsilon}]$, $u(x) = u_{j-1}\psi_{j-1,\varepsilon}(x) + u_j\psi_{j,\varepsilon}(x)$ with $|\psi'_{j-1,\varepsilon}(x)| = |\psi'_{j,\varepsilon}(x)| = (1/(\zeta_{j,\varepsilon} - \xi_{j,\varepsilon}))$. Here, we set $u_0 = u_n$ and $\psi_{0,\varepsilon} = \psi_{n,\varepsilon}$. It follows that

$$b_\varepsilon(u,u) = \int_0^1 a_\varepsilon \cdot (u')^2\,dx = \sum_{j=0}^{n} \int_{x_j}^{x_{j+1}} a_\varepsilon \cdot (u')^2\,dx = \sum_{j=1}^{n} \int_{\xi_{j,\varepsilon}}^{\zeta_{j,\varepsilon}} a_\varepsilon \cdot (u')^2\,dx$$

$$= \sum_{j=1}^{n} \int_{\xi_{j,\varepsilon}}^{\zeta_{j,\varepsilon}} a_\varepsilon \cdot (u_{j-1}\psi'_{j-1,\varepsilon} + u_j\psi'_{j,\varepsilon})^2\,dx$$

$$\leq \sum_{j=1}^{n} \int_{\xi_{j,\varepsilon}}^{\zeta_{j,\varepsilon}} \Big(\sup_{[\xi_{j,\varepsilon},\zeta_{j,\varepsilon}]} a_\varepsilon\Big) \frac{4}{c(\zeta_{j,\varepsilon} - \xi_{j,\varepsilon})^2}\,dx$$

$$= \sum_{j=1}^{n} \Big(\sup_{[\xi_{j,\varepsilon},\zeta_{j,\varepsilon}]} a_\varepsilon\Big) \frac{4}{c(\zeta_{j,\varepsilon} - \xi_{j,\varepsilon})} \to \frac{4}{c} \sum_{j=1}^{n} \tau_j \in]0, \infty[\,,$$

so

$$\sum_{j=1}^{n} \Big(\sup_{[\xi_{j,\varepsilon},\zeta_{j,\varepsilon}]} a_\varepsilon\Big) \frac{4}{c(\zeta_{j,\varepsilon} - \xi_{j,\varepsilon})} \leq C'_1$$

for some $C'_1 \in]0, \infty[$, some $\varepsilon'_1 \in]0, \varepsilon_1]$ and all $\varepsilon \in]0, \varepsilon'_1]$. □

Notation. For $C \in]0, \infty[$ and $\varepsilon \in]0, \varepsilon_0[$ let $\widetilde{B}_{\varepsilon,C}$ be the closed ball in H^1_{per} with center in zero and radius C with respect to the norm $\|\cdot\|_\varepsilon$.

Lemma 3.3. *The following two assertions hold:*

(a) *There exist an* $\varepsilon'_2 \in]0, \varepsilon_0[$ *and a* $C'_2 \in]0, \infty[$ *such that, for every* $v \in H^1_{\mathrm{per}}$ *and every* $\varepsilon \in]0, \varepsilon'_2]$,

$$\sup_{x,y \in [0,1]} |v(x) - v(y)| \leq C'_2 b_\varepsilon(v,v)^{1/2}, \tag{3.1}$$

$$\sup_{x \in [0,1]} |v(x)| \leq C'_2 \|v\|_\varepsilon. \tag{3.2}$$

(b) *Let* $M \in]0, \infty[$ *be arbitrary. For each* $j \in [1..n]$ *we have*

$$\sup_{v \in \widetilde{B}_{\varepsilon,M}} \sup_{x,y \in K_{j,\varepsilon}} |v(x) - v(y)| \to 0, \quad \text{as } \varepsilon \to 0. \tag{3.3}$$

Proof. By our assumptions there are an $\varepsilon_1 \in]0,\varepsilon_0[$ and a $C_1 \in]0,\infty[$ such that, for $\varepsilon \in]0,\varepsilon_1]$,

$$\frac{\mathbf{m}(\Gamma_\varepsilon)}{\inf_{\Gamma_\varepsilon} a_\varepsilon} \le C_1,$$

where Γ_ε is any of the $\ell = 4n+1$ intervals $[0,\xi'_{1,\varepsilon}]$, $[\zeta'_{j,\varepsilon},\xi'_{j+1,\varepsilon}]$, $[\zeta'_{n,\varepsilon},1]$, $j \in [1\,.\,.\,n-1]$ or else any of the intervals $[\xi'_{j,\varepsilon},\xi_{j,\varepsilon}]$, $[\xi_{j,\varepsilon},\zeta_{j,\varepsilon}]$, $[\zeta_{j,\varepsilon},\zeta'_{j,\varepsilon}]$, $j \in [1\,.\,.\,n]$. Thus, whenever $\varepsilon \in]0,\varepsilon_1]$ and $v \in H^1_{\mathrm{per}}$, it follows from Lemma 3.1 that

$$\operatorname{diam} v(\Gamma_\varepsilon) \le (C_1 b_\varepsilon(v,v))^{1/2}.$$

The above ℓ intervals can be ordered to form a sequence $(I_j)_{j\in[1..\ell]}$ such that for $j \in [1\,.\,.\,\ell-1]$ the endpoint of I_j is the initial point of I_{j+1}. Consequently,

$$\operatorname{diam} v([0,1]) \le (4n+1)(C_1 b_\varepsilon(v,v))^{1/2},$$

so

$$|v(x)| \le |v(y)| + (4n+1)(C_1 b_\varepsilon(v,v))^{1/2}, \quad x,y \in [0,1]$$

which implies

$$|v(x)| \le C_2\|v\|_\varepsilon, \quad x \in [0,1],$$

where $C_2 = 1 + (4n+1)C_1^{1/2}$. These estimates prove part (a) of the lemma. Now, let $M \in]0,\infty[$ be arbitrary and for each $\varepsilon \in]0,\varepsilon_0[$ let β_ε be the maximum of all the values

$$M\biggl(\frac{\mathbf{m}(\Gamma_\varepsilon)}{\inf_{\Gamma_\varepsilon} a_\varepsilon}\biggr)^{1/2},$$

where Γ_ε is any of the intervals $[0,\xi'_{1,\varepsilon}]$, $[\zeta'_{j,\varepsilon},\xi'_{j+1,\varepsilon}]$, $[\zeta'_{n,\varepsilon},1]$, $j \in [1\,.\,.\,n-1]$ or else any of the intervals $[\xi'_{j,\varepsilon},\xi_{j,\varepsilon}]$, $[\zeta_{j,\varepsilon},\zeta'_{j,\varepsilon}]$, $j \in [1\,.\,.\,n]$. For $j \in [1\,.\,.\,n-1]$ it follows from Lemma 3.1 that

$$\sup_{v\in\widetilde{B}_{\varepsilon,M}} \sup_{x,y\in K_{j,\varepsilon}} |v(x)-v(y)| \le 3\beta_\varepsilon.$$

If $\Gamma_\varepsilon = [0,\xi_{1,\varepsilon}]$ or $\Gamma_\varepsilon = [\zeta_{n,\varepsilon},1]$ we have

$$\sup_{v\in\widetilde{B}_{\varepsilon,M}} \sup_{x,y\in\Gamma_\varepsilon} |v(x)-v(y)| \le 2\beta_\varepsilon.$$

Finally, since $v(0)=v(1)$ for each $v \in H^1_{\mathrm{per}}$, it follows that

$$\sup_{v\in\widetilde{B}_{\varepsilon,M}} \sup_{x,y\in K_{n,\varepsilon}} |v(x)-v(y)| \le 4\beta_\varepsilon.$$

Now Assumption 2.1 implies that $\beta_\varepsilon \to 0$ as $\varepsilon \to 0$. This proves part (b) of the lemma. □

Lemma 3.4. *Let $(\varepsilon_m)_m$ be a null sequence in $]0,\varepsilon_0[$. Let $(u_m)_m$, $(v_m)_m$ be sequences in H^1_{per} such that $u_m \in \widetilde{B}_{\varepsilon_m,M}$ and $v_m \in \widetilde{B}_{\varepsilon_m,M'}$ for some M,*

$M' \in \,]0,\infty[$ and all $m \in \mathbb{N}$. Let $(\gamma_{j,m})_{j\in[1..n],m\in\mathbb{N}}$ be such that $\gamma_{j,m} \in K_{j,\varepsilon_m}$ for $m \in \mathbb{N}$ and $j \in [1..n]$. Then

$$\langle u_m, v_m\rangle_{L^2} - \sum_{j=1}^{n} L_j u_m(\gamma_{j,m}) v_m(\gamma_{j,m}) \to 0, \quad \text{as } m \to \infty.$$

Proof. For each $m \in \mathbb{N}$ we have

$$\begin{aligned}\int_0^1 u_m v_m\,dx &= \sum_{j=1}^{n} \mathbf{m}(K_{j,\varepsilon_m}) u_m(\gamma_{j,m}) v_m(\gamma_{j,m}) + \sum_{j=1}^{n} \int_{[\xi_{j,\varepsilon_m},\zeta_{j,\varepsilon_m}]} u_m v_m\,dx \\ &\quad + \sum_{j=1}^{n} \int_{K_{j,\varepsilon_m}} (u_m v_m - u_m(\gamma_{j,m}) v_m(\gamma_{j,m}))\,dx \\ &=: \sum_{j=1}^{n} \mathbf{m}(K_{j,\varepsilon_m}) u_m(\gamma_{j,m}) v_m(\gamma_{j,m}) + T_{1,m} + T_{2,m}.\end{aligned}$$

It follows from Lemma 3.3 that all functions u_m and v_m are uniformly bounded by the same constant C. Thus

$$\left|T_{1,m}\right| \le C^2 \sum_{j=1}^{n} (\zeta_{j,\varepsilon_m} - \xi_{j,\varepsilon_m})$$

and Assumption 2.1 implies that

$$T_{1,m} \to 0, \quad \text{as } m \to \infty. \tag{3.4}$$

For $x \in K_{j,\varepsilon_m}$ we have

$$\begin{aligned}&|u_m(x)v_m(x) - u_m(\gamma_{j,m})v_m(\gamma_{j,m})| \\ &\quad \le |u_m(x) - u_m(\gamma_{j,m})| \cdot |v_m(x)| + |v_m(x) - v_m(\gamma_{j,m})| \cdot |u_m(\gamma_{j,m})| \\ &\quad\quad \le C(|u_m(x) - u_m(\gamma_{j,m})| + |v_m(x) - v_m(\gamma_{j,m})|).\end{aligned}$$

Therefore

$$|T_{2,m}| \le C \sum_{j=1}^{n} \mathbf{m}(K_{j,\varepsilon_m}) \sup_{x\in K_{j,\varepsilon_m}} \left(|u_m(x) - u_m(\gamma_{j,m})| + |v_m(x) - v_m(\gamma_{j,m})|\right).$$

Again, Lemma 3.3 implies that

$$T_{2,m} \to 0, \quad \text{as } m \to \infty. \tag{3.5}$$

Moreover, it follows from Assumption 2.1 that $\mathbf{m}(K_{j,\varepsilon_m}) - L_j \to 0$, as $m \to \infty$, for each $j \in [1..n]$. This together with (3.4) and (3.5) implies the assertion of the lemma. □

Corollary 3.5. *Let $M' \in \,]0,\infty[$ and $(\varepsilon_m)_m$ be a null sequence in $]0,\varepsilon_0[$. Let $(v_m)_m$ and $(\gamma_{j,m})_m$, $j \in [1..n]$, be sequences such that $v_m \in \widetilde{B}_{\varepsilon_m,M'}$ and $\gamma_{j,m} \in K_{j,\varepsilon_m}$ for $m \in \mathbb{N}$ and $j \in [1..n]$. Then for each $j \in [1..n]$,*

$$\langle \psi_{j,\varepsilon_m}, v_m\rangle_{L^2} - L_j v_m(\gamma_{j,m}) \to 0, \quad \text{as } m \to \infty.$$

Proof. Lemma 3.2 and the fact that the functions $u_m = \psi_{j,\varepsilon_m}$, $j \in [1\,.\,.\,n]$, $m \in \mathbb{N}$, are nonnegative and bounded by 1 imply that $u_m \in \widetilde{B}_{\varepsilon_m,M}$ for some constant $M \in \,]0,\infty[$ and for all $m \in \mathbb{N}$. Hence the assumptions of Lemma 3.4 are satisfied. Now that lemma implies that, for each $j \in [1\,.\,.\,n]$,

$$\int_0^1 \psi_{j,\varepsilon_m} v_m\,dx - \sum_{l=1}^n L_l \psi_{j,\varepsilon_m}(\gamma_{l,m}) v_m(\gamma_{l,m}) \to 0, \quad \text{as } m \to \infty.$$

The definition of the map ψ_{j,ε_m}, $m \in \mathbb{N}$, implies that $\psi_{j,\varepsilon_m}(\gamma_{l,m}) = 1$ if $j = l$ and $\psi_{j,\varepsilon_m}(\gamma_{l,m}) = 0$ otherwise and so

$$\sum_{l=1}^n L_l \psi_{j,\varepsilon_m}(\gamma_{l,m}) v_m(\gamma_{l,m}) = L_j v_m(\gamma_{j,m}).$$

Passing to the limit as $m \to \infty$ we complete the proof. □

Lemma 3.6. *Let $\varepsilon_2' \in \,]0,\varepsilon_0[$ be as in Lemma 3.3. Then, for every $M \in \,]0,\infty[$, there is an $\varepsilon_3' = \varepsilon_3'(M) \in \,]0,\varepsilon_2']$ such that $v \notin W_\varepsilon^\perp$ for all $v \in \widetilde{B}_{\varepsilon,M}$ with $\|v\|_2 = 1$ and $\varepsilon \in \,]0,\varepsilon_3']$. (Here, the orthogonal complement is taken with respect to the L^2-scalar product.)*

Proof. Suppose the conclusion of the lemma does not hold. Then, for some $M \in \,]0,\infty[$, there exists a null sequence $(\varepsilon_m)_m$ in $]0,\varepsilon_2']$ such that for each $m \in \mathbb{N}$ there exists a $v_m \in \widetilde{B}_{\varepsilon_m,M} \cap W_{\varepsilon_m}^\perp$ with with $\|v_m\|_2 = 1$. Let $m \in \mathbb{N}$. Hence $\langle v_m, \psi_{j,\varepsilon_m}\rangle_{L^2} = 0$ for all $j \in [1\,.\,.\,n]$.

For each $j \in [1\,.\,.\,n-1]$ choose $\gamma_j \in \,]x_j, x_{j+1}[$ and choose $\gamma_n \in \,]0,x_1[\cup]x_n,1[$ independently of $m \in \mathbb{N}$. Then there exists an $m_0 \in \mathbb{N}$ such that $\gamma_j \in K_{j,\varepsilon_m}$ for all $j \in [1\,.\,.\,n]$ and $m \geq m_0$. Now Corollary 3.5 implies that, for each $j \in [1\,.\,.\,n]$,

$$v_m(\gamma_j) \to 0, \quad \text{as } m \to \infty$$

and so Lemma 3.3 implies that $v_m(x) \to 0$ as $m \to \infty$ for each $x \in \,]0,1[\setminus \bigcup_{j=1}^n \{x_j\}$. Moreover, it follows from Lemma 3.3 that there exists an $m_1 \in \mathbb{N}$ such that the functions v_m, for all $m \geq m_0$, are pointwise bounded by the same constant. This implies that

$$\int_0^1 v_m^2\,dx \to 0 \quad \text{as } m \to \infty.$$

However, this is a contradiction as

$$\int_0^1 v_m^2\,dx = 1 \quad \text{for all } m \in \mathbb{N}.$$

□

Lemma 3.7. *The following statements hold*:

(a) $\lambda_{n+1,\varepsilon} \to \infty$ *as* $\varepsilon \to 0$.

(b) *There exists an $\varepsilon_4' \in \,]0,\varepsilon_0[$ and a $C_3' \in \,]0,\infty[$ such that*

$$\lambda_{n,\varepsilon} \leq C_3' \quad \textit{for all } \varepsilon \in \,]0,\varepsilon_4'].$$

Proof. For each positive integer p and $\varepsilon \in]0, \varepsilon_0[$ let $U_{p,\varepsilon}$ be the span of the eigenfunctions $\varphi_{l,\varepsilon}$, for $l \in [1\,..\,p]$. Moreover, let $U_{0,\varepsilon} = \{0\} \subset L^2$. If assertion (a) is not true, then there is a null sequence $(\varepsilon_m)_m$ in $]0, \varepsilon_0[$ such that $(\lambda_{n+1,\varepsilon_m})_m$ is bounded by some $C \in]0, \infty[$.

We claim that $U_{n+1,\varepsilon_m} \cap W_{\varepsilon_m}^{\perp} = \{0\}$ for all $m \in \mathbb{N}$ large enough. If this is not true, then there is a subsequence $(\varepsilon_m^1)_m$ of $(\varepsilon_m)_m$ such that for each $m \in \mathbb{N}$ there is a v_m in $U_{n+1,\varepsilon_m^1} \cap W_{\varepsilon_m^1}^{\perp}$ with $\|v_m\|_{L^2} = 1$. It easily follows that $b_{\varepsilon_m^1}(v_m, v_m) \leq C$ so $v_m \in \widetilde{B}_{\varepsilon_m^1,K}$ for all $m \in \mathbb{N}$, where $K^2 = C + 1$. However, this contradicts Lemma 3.6 and the claim is proved.

The claim implies that $n + 1 \leq n$, a contradiction which implies the first assertion. Let D be the set of all nonnegative integers ℓ_1 such that, for some $\widehat{\varepsilon} \in]0, \varepsilon_0[$ the eigenvalue family $(\lambda_{\ell_1,\varepsilon})_{\varepsilon \in]0,\widehat{\varepsilon}]}$ is bounded by some $C_1 \in]0, \infty[$. Let ℓ be the supremum of D if D is nonempty and $\ell = 0$ otherwise. From what we have proved so far, we have $\ell \leq n$. If $\ell < n$, then $U_{\ell,\varepsilon}^{\perp} \cap W_\varepsilon \neq \{0\}$ and so, for each $\varepsilon \in]0, \varepsilon_0[$ there is a $w_\varepsilon \neq 0$ lying in $U_{\ell,\varepsilon}^{\perp} \cap W_\varepsilon$. It follows that

$$\lambda_{\ell+1,\varepsilon} = \inf_{w \in H^1_{\mathrm{per}} \setminus \{0\},\, w \in U_{\ell,\varepsilon}^{\perp}} \frac{b_\varepsilon(w, w)}{\|w\|_{L^2}} \leq \frac{b_\varepsilon(w_\varepsilon, w_\varepsilon)}{\|w_\varepsilon\|_{L^2}} \leq C_1'$$

for all $\varepsilon \in]0, \varepsilon_1']$, where $C_1' \in [0, \infty[$ and $\varepsilon_1' \in]0, \varepsilon_0[$ are as in Lemma 3.2. This shows in particular, that D is nonempty. Moreover, this also shows that $\ell + 1 \in D$, a contradiction proving that $\ell = n$. Since D is nonempty and finite, we have $\ell \in D$. This proves assertion (b). $\square$

In the sequel

(3.6) for each $\varepsilon \in]0, \varepsilon_0[$ fix an arbitrary L^2-orthonormal sequence $(\varphi_{l,\varepsilon})_l$ such that $\varphi_{l,\varepsilon}$ is an eigenfunction of A_ε corresponding to $\lambda_{l,\varepsilon}$, $l \in \mathbb{N}$.

Lemma 3.8. *Let $(\varepsilon_m)_m$ be a null sequence in $]0, \varepsilon_0[$ and $(\gamma_{j,m})_m$ be a (double) sequence with $\gamma_{j,m} \in K_{j,\varepsilon_m}$, for $m \in \mathbb{N}$ and $j \in [1\,..\,n]$. For each i, $j \in [1\,..\,n]$, we then have*

(a) $\langle \psi_{j,\varepsilon_m}, \varphi_{i,\varepsilon_m} \rangle_{L^2} - L_j \varphi_{i,\varepsilon_m}(\gamma_{j,m}) \to 0$ *as* $m \to \infty$.

(b) $\sum_{j=1}^{n} L_j \varphi_{i,\varepsilon_m}(\gamma_{j,m}) \varphi_{k,\varepsilon_m}(\gamma_{j,m}) \to \delta_{i,k}$ *as* $m \to \infty$.

Proof. This follows from Lemma 3.7, Corollary 3.5 and Lemma 3.4. $\square$

Notation. For each $\varepsilon \in]0, \varepsilon_0[$, define $\Psi_\varepsilon \colon W_\varepsilon \to \mathbb{R}^n$ by

$$\Psi_\varepsilon(u) := \widehat{u} := (u_j)_{j \in [1..n]}, \quad \text{for } u = \sum_{j=1}^{n} u_j \psi_{j,\varepsilon} \in W_\varepsilon.$$

Consider the $n \times n$ matrix $B_\varepsilon = (b_{i,j,\varepsilon})_{i,j=1}^{n}$ given by

$$b_{i,j,\varepsilon} = \langle \psi_{i,\varepsilon}, \psi_{j,\varepsilon} \rangle_{L^2}, \quad \text{for } i,\, j \in [1\,..\,n].$$

Assume that

(3.7) $(\alpha_{j,\varepsilon})_{(j,\varepsilon)\in[1..n]\times]0,\varepsilon_0[}$ is an arbitrary family such that $\alpha_{j,\varepsilon}\in K_{j,\varepsilon}$, for $(j,\varepsilon)\in[1..n]\times]0,\varepsilon_0[$.

Let $\|\cdot\|_{\mathbb{L}}$ be the norm on $\mathbb{R}^n$ induced by the scalar product $\langle\cdot,\cdot\rangle_{\mathbb{L}}$. In what follows $\langle\cdot,\cdot\rangle$ (respectively, $\|\cdot\|$) denotes the canonical inner product (respectively, the induced norm) on $\mathbb{R}^n$. Let a, $b\in]0,\infty[$ such that

$$a\|z\|_{\mathbb{L}}\le\|z\|\le b\|z\|_{\mathbb{L}},\quad \text{for all } z\in\mathbb{R}^n.$$

Lemma 3.9. *Let $\varepsilon_4'\in]0,\varepsilon_0[$ be as in Lemma 3.7. There is an $\varepsilon_5'\in]0,\varepsilon_4']$ such that for each $\varepsilon\in]0,\varepsilon_5']$, there are constants c_ε, $C_\varepsilon\in]0,\infty[$ such that*

$$c_\varepsilon\|\Psi_\varepsilon(u)\|_{\mathbb{L}}\le\|u\|_{L^2}\le C_\varepsilon\|\Psi_\varepsilon(u)\|_{\mathbb{L}},\quad u\in W_\varepsilon.$$

Moreover, $c_\varepsilon\to 1$, $C_\varepsilon\to 1$ as $\varepsilon\to 0$.

The proof is identical to the proof of [3, Lemma 3.9].

Notation. Define the $n\times n$ matrix $G_\varepsilon=(g_{i,j,\varepsilon})_{i,j=1}^n$ by $g_{i,j,\varepsilon}=\langle\varphi_{i,\varepsilon},\psi_{j,\varepsilon}\rangle_{L^2}$ for i, $j\in[1..n]$ and $\varepsilon\in]0,\varepsilon_0[$. Clearly

$$G_\varepsilon\Psi_\varepsilon(u)=(\langle u,\varphi_{i,\varepsilon}\rangle_{L^2})_{i\in[1..n]},\quad \varepsilon\in]0,\varepsilon_0[,\ u\in W_\varepsilon. \tag{3.8}$$

Lemma 3.10. *There exists an $\varepsilon_6'\in]0,\varepsilon_5']$ and for each $k\in[1..n]$ there exists a family $(v_{k,\varepsilon})_{\varepsilon\in]0,\varepsilon_6']}$ such that $v_{k,\varepsilon}\in W_\varepsilon$, $\|v_{k,\varepsilon}\|_{L^2}=1$ for $\varepsilon\in]0,\varepsilon_6']$ and*

$$\langle v_{k,\varepsilon},\varphi_{i,\varepsilon}\rangle=0\quad \textit{for } i\ne k.$$

Moreover, if (3.7) holds, then $v_{k,\varepsilon}(\alpha_{j,\varepsilon})-\varphi_{k,\varepsilon}(\alpha_{j,\varepsilon})\to 0$ as $\varepsilon\to 0$.

The proof is identical to the proof of [3, Lemma 3.10].

Lemma 3.11. *Let $\varepsilon_4'\in]0,\varepsilon_0[$ be as in Lemma 3.7 and let $(u_\varepsilon)_{\varepsilon\in]0,\varepsilon_4']}$ be such that $u_\varepsilon\in W_\varepsilon$ and $\|u_\varepsilon\|_{L^2}=1$ for each $\varepsilon\in]0,\varepsilon_4']$. Then*

$$b_\varepsilon(u_\varepsilon,u_\varepsilon)-b_0(\Psi_\varepsilon(u_\varepsilon),\Psi_\varepsilon(u_\varepsilon))\to 0,\quad \textit{as } \varepsilon\to 0.$$

Proof. Set $\widehat{u}_\varepsilon=\Psi_\varepsilon(u_\varepsilon)$, where $u_\varepsilon=\sum_{j=1}^n\widehat{u}_{\varepsilon,j}\psi_{j,\varepsilon}\in W_\varepsilon$. Thus, $\widehat{u}_\varepsilon=(\widehat{u}_{\varepsilon,j})_{j\in[1..n]}$. We also set $\widehat{u}_{\varepsilon,0}=\widehat{u}_{\varepsilon,n}$ and $\psi_{0,\varepsilon}=\psi_{n,\varepsilon}$. We then have

$$\begin{aligned}b_\varepsilon(u_\varepsilon,u_\varepsilon)&=\sum_{j=1}^n\int_{[\xi_{j,\varepsilon},\zeta_{j,\varepsilon}]}a_\varepsilon\cdot(u_\varepsilon')^2\,dx\le\sum_{j=1}^n\sup_{[\xi_{j,\varepsilon},\zeta_{j,\varepsilon}]}a_\varepsilon\int_{[\xi_{j,\varepsilon},\zeta_{j,\varepsilon}]}(u_\varepsilon')^2\,dx\\&=\sum_{j=1}^n\sup_{[\xi_{j,\varepsilon},\zeta_{j,\varepsilon}]}a_\varepsilon\int_{[\xi_{j,\varepsilon},\zeta_{j,\varepsilon}]}(\widehat{u}_{\varepsilon,j-1}\psi_{j-1,\varepsilon}'+\widehat{u}_{\varepsilon,j}\psi_{j,\varepsilon}')^2\,dx.\end{aligned}$$

Notice that

$$\psi_{j-1,\varepsilon}'(x)=-\frac{1}{\zeta_{j,\varepsilon}-\xi_{j,\varepsilon}}\quad\text{and}\quad\psi_{j,\varepsilon}'(x)=\frac{1}{\zeta_{j,\varepsilon}-\xi_{j,\varepsilon}}\quad\text{for } x\in[\xi_{j,\varepsilon},\zeta_{j,\varepsilon}]$$

and so

$$
\begin{aligned}
b_\varepsilon(u_\varepsilon, u_\varepsilon) &\le \sum_{j=1}^{n} \frac{\sup\limits_{[\xi_{j,\varepsilon},\zeta_{j,\varepsilon}]} a_\varepsilon}{\zeta_{j,\varepsilon} - \xi_{j,\varepsilon}} (\widehat{u}_{\varepsilon,j} - \widehat{u}_{\varepsilon,j-1})^2 \\
&= \sum_{j=1}^{n} (\tau_j + h_{1,j,\varepsilon})(\widehat{u}_{\varepsilon,j} - \widehat{u}_{\varepsilon,j-1})^2 = \sum_{j=1}^{n} \tau_j (\widehat{u}_{\varepsilon,j} - \widehat{u}_{\varepsilon,j-1})^2 + h_{2,\varepsilon},
\end{aligned}
$$

with $h_{1,j,\varepsilon} \to 0$, $j \in [1\,.\,.\,n]$, and $h_{2,\varepsilon} \to 0$ as $\varepsilon \to 0$. This follows from Assumption 2.1, the assumption that $\|u_\varepsilon\|_{L^2} = 1$, for $\varepsilon \in \,]0, \varepsilon_4']$, and Lemma 3.9. Similarly, working with 'inf' instead of 'sup', we show that

$$
b_\varepsilon(u_\varepsilon, u_\varepsilon) \ge \sum_{j=1}^{n} \tau_j (\widehat{u}_{\varepsilon,j} - \widehat{u}_{\varepsilon,j-1})^2 + h_{3,\varepsilon},
$$

with $h_{3,\varepsilon} \to 0$ as $\varepsilon \to 0$. Therefore

$$
b_\varepsilon(u_\varepsilon, u_\varepsilon) - \sum_{j=1}^{n} \tau_j (\widehat{u}_{\varepsilon,j} - \widehat{u}_{\varepsilon,j-1})^2 \to 0, \quad \text{as } \varepsilon \to 0. \tag{3.9}
$$

Now estimate (3.9) and the definition of b_0 and $\widehat{u}_\varepsilon$ imply the assertion. □

COROLLARY 3.12. *Let $\varepsilon_6' \in \,]0, \varepsilon_0[$ be as in Lemma 3.10 and $k \in [1\,.\,.\,n]$ be arbitrary. Then*

$$
\begin{gathered}
\{b_\varepsilon(u,u) \mid u \in W_\varepsilon,\ \|u\|_{L^2} = 1,\ u \in U_{k-1,\varepsilon}^\perp\} \ne \emptyset, \\
\{b_0(\Psi_\varepsilon(u), \Psi_\varepsilon(u)) \mid u \in W_\varepsilon,\ \|u\|_{L^2} = 1, u \in U_{k-1,\varepsilon}^\perp\} \ne \emptyset
\end{gathered}
$$

for all $\varepsilon \in \,]0, \varepsilon_6']$. Moreover, as $\varepsilon \to 0$, the following holds:

$$
\begin{aligned}
&\inf\{b_\varepsilon(u,u) \mid u \in W_\varepsilon,\ \|u\|_{L^2} = 1,\ u \in U_{k-1,\varepsilon}^\perp\} \\
&\qquad - \inf\{ b_0(\Psi_\varepsilon(u), \Psi_\varepsilon(u)) \mid u \in W_\varepsilon,\ \|u\|_{L^2} = 1,\ u \in U_{k-1,\varepsilon}^\perp \} \to 0.
\end{aligned}
$$

LEMMA 3.13. *Let $\varepsilon_6' \in \,]0, \varepsilon_0[$ be as in Lemma 3.10 and, for each $k \in [1\,.\,.\,n]$, let the family $(v_{k,\varepsilon})_{\varepsilon \in]0,\varepsilon_6']}$ be also as in Lemma 3.10. Then*

$$
\begin{aligned}
\lambda_{k,\varepsilon} - \inf\{b_0(\Psi_\varepsilon(u), \Psi_\varepsilon(u)) \mid u \in W_\varepsilon,\ \|u\|_{L^2} = 1,\ u \in U_{k-1,\varepsilon}^\perp\} &\to 0, \quad \textit{as } \varepsilon \to 0, \\
\lambda_{k,\varepsilon} - b_\varepsilon(v_{k,\varepsilon}, v_{k,\varepsilon}) &\to 0, \quad \textit{as } \varepsilon \to 0.
\end{aligned}
$$

PROOF. Lemma 3.10 implies that $\{b_0(\Psi_\varepsilon(u), \Psi_\varepsilon(u)) \mid u \in W_\varepsilon,\ \|u\|_{L^2} = 1,$ $u \in U_{k-1,\varepsilon}^\perp\} \ne \emptyset$ for all $\varepsilon \in \,]0, \varepsilon_6']$. It follows from Lemma 3.10, choosing first $\alpha_{j,\varepsilon} = \xi_{j,\varepsilon}$ for $(j, \varepsilon) \in [1\,.\,.\,n] \times]0, \varepsilon_0[$ and then $\alpha_{j,\varepsilon} = \zeta_{j,\varepsilon}$ for $(j, \varepsilon) \in [1\,.\,.\,n] \times]0, \varepsilon_0[$, that

$$
\begin{aligned}
v_{k,\varepsilon}(\xi_{j,\varepsilon}) - \varphi_{k,\varepsilon}(\xi_{j,\varepsilon}) &\to 0 \quad \text{as } \varepsilon \to 0, \\
v_{k,\varepsilon}(\zeta_{j,\varepsilon}) - \varphi_{k,\varepsilon}(\zeta_{j,\varepsilon}) &\to 0 \quad \text{as } \varepsilon \to 0.
\end{aligned} \tag{3.10}
$$

Thus

$$\begin{aligned} b_\varepsilon(\varphi_{k,\varepsilon},\varphi_{k,\varepsilon}) &\geq \sum_{j=1}^{n} \int_{[\xi_{j,\varepsilon},\zeta_{j,\varepsilon}]} a_\varepsilon \cdot (\varphi'_{k,\varepsilon})^2\,dx \\ &\geq \sum_{j=1}^{n} \inf_{[\xi_{j,\varepsilon},\zeta_{j,\varepsilon}]} a_\varepsilon \int_{[\xi_{j,\varepsilon},\zeta_{j,\varepsilon}]} (\varphi'_{k,\varepsilon})^2\,dx \\ &\geq \sum_{j=1}^{n} \frac{\inf_{[\xi_{j,\varepsilon},\zeta_{j,\varepsilon}]} a_\varepsilon}{\zeta_{j,\varepsilon}-\xi_{j,\varepsilon}} \bigg(\int_{[\xi_{j,\varepsilon},\zeta_{j,\varepsilon}]} \varphi'_{k,\varepsilon}\,dx\bigg)^2 \\ &= \sum_{j=1}^{n} \frac{\inf_{[\xi_{j,\varepsilon},\zeta_{j,\varepsilon}]} a_\varepsilon}{\zeta_{j,\varepsilon}-\xi_{j,\varepsilon}} (\varphi_{k,\varepsilon}(\zeta_{j,\varepsilon})-\varphi_{k,\varepsilon}(\xi_{j,\varepsilon}))^2. \end{aligned}$$

Define $h_{1,j,\varepsilon}$, $h_{2,j,\varepsilon}$ and $h_{3,j,\varepsilon}$, $j \in [1\,.\,.\,n]$, such that

$$\varphi_{k,\varepsilon}(\xi_{j,\varepsilon}) = v_{k,\varepsilon}(\xi_{j,\varepsilon}) + h_{1,j,\varepsilon}, \qquad \varphi_{k,\varepsilon}(\zeta_{j,\varepsilon}) = v_{k,\varepsilon}(\zeta_{j,\varepsilon}) + h_{2,j,\varepsilon},$$
$$\frac{\inf_{[\xi_{j,\varepsilon},\zeta_{j,\varepsilon}]} a_\varepsilon}{\zeta_{j,\varepsilon}-\xi_{j,\varepsilon}} = \tau_j + h_{3,j,\varepsilon}.$$

Assumption 2.1 and (3.10) imply that $h_{1,j,\varepsilon} \to 0$, $h_{2,j,\varepsilon} \to 0$ and $h_{3,j,\varepsilon} \to 0$ as $\varepsilon \to 0$. Therefore

$$\begin{aligned} b_\varepsilon(\varphi_{k,\varepsilon},\varphi_{k,\varepsilon}) &\geq \sum_{j=1}^{n} (\tau_j + h_{3,j,\varepsilon})(v_{k,\varepsilon}(\zeta_{j,\varepsilon}) + h_{2,j,\varepsilon} - v_{k,\varepsilon}(\xi_{j,\varepsilon}) - h_{1,j,\varepsilon})^2 \\ &= \sum_{j=1}^{n} \tau_j (v_{k,\varepsilon}(\zeta_{j,\varepsilon}) - v_{k,\varepsilon}(\xi_{j,\varepsilon}))^2 + h_{4,\varepsilon} \\ &= \sum_{j=1}^{n} \tau_j (\widehat{v}_{k,\varepsilon,j} - \widehat{v}_{k,\varepsilon,j-1})^2 + h_{4,\varepsilon}, \end{aligned}$$

where $h_{4,\varepsilon} \to 0$ as $\varepsilon \to 0$. Here, we write $\widehat{v}_{k,\varepsilon} = \Psi_\varepsilon(v_{k,\varepsilon})$ and $\widehat{v}_{k,\varepsilon,l}$ is the l-th component of $\widehat{v}_{k,\varepsilon} \in \mathbb{R}^n$. We also set $\widehat{v}_{k,\varepsilon,0} = \widehat{v}_{k,\varepsilon,n}$. By Lemma 3.11,

$$\sum_{j=1}^{n} \tau_j (\widehat{v}_{k,\varepsilon,j} - \widehat{v}_{k,\varepsilon,j-1})^2 = b_0(\Psi_\varepsilon(v_{k,\varepsilon}), \Psi_\varepsilon(v_{k,\varepsilon})) = b_\varepsilon(v_{k,\varepsilon}, v_{k,\varepsilon}) + h_{5,\varepsilon}$$

with $h_{5,\varepsilon} \to 0$ as $\varepsilon \to 0$. Thus,

$$b_\varepsilon(\varphi_{k,\varepsilon},\varphi_{k,\varepsilon}) - h_{6,\varepsilon} \geq b_\varepsilon(v_{k,\varepsilon}, v_{k,\varepsilon}) \tag{3.11}$$

with $h_{6,\varepsilon} \to 0$ as $\varepsilon \to 0$. For $\varepsilon \in \,]0, \varepsilon_0[$ small enough and for all $k \in [1\,.\,.\,n]$ we have

$$\begin{aligned} b_\varepsilon(\varphi_{k,\varepsilon},\varphi_{k,\varepsilon}) &= \inf\{b_\varepsilon(\varphi,\varphi) \mid \varphi \in H^1_{\mathrm{per}},\ \|\varphi\|_{L^2} = 1,\ \varphi \in U^\perp_{k-1,\varepsilon}\} \\ &\leq \inf\{\, b_\varepsilon(u,u) \mid u \in W_\varepsilon,\ \|u\|_{L^2} = 1,\ u \in U^\perp_{k-1,\varepsilon}\}. \end{aligned}$$

It follows from (3.11) that

$$\begin{aligned} b_\varepsilon(\varphi_{k,\varepsilon},\varphi_{k,\varepsilon}) - h_{6,\varepsilon} &\geq b_\varepsilon(v_{k,\varepsilon},v_{k,\varepsilon}) \\ &\geq \inf\{b_\varepsilon(u,u) \mid u \in W_\varepsilon,\ \|u\|_{L^2}=1,\ u \in U^\perp_{k-1,\varepsilon}\}. \end{aligned}$$

Since $b_\varepsilon(\varphi_{k,\varepsilon},\varphi_{k,\varepsilon}) = \lambda_{k,\varepsilon}$, we have

$$\begin{gathered} \lambda_{k,\varepsilon} - \inf\{b_\varepsilon(u,u) \mid u \in W_\varepsilon,\ \|u\|_{L^2}=1,\ u \in U^\perp_{k-1,\varepsilon}\} \to 0, \quad \text{as } \varepsilon \to 0, \\ \lambda_{k,\varepsilon} - b_\varepsilon(v_{k,\varepsilon},v_{k,\varepsilon}) \to 0, \quad \text{as } \varepsilon \to 0. \end{gathered}$$

Now Corollary 3.12 completes the proof. □

Lemma 3.14. *Let $\varepsilon'_6 \in]0,\varepsilon_0[$ be as in Lemma 3.10. Let $(\varepsilon_m)_m$ be a null sequence in $]0,\varepsilon'_6]$ and suppose that there exists a sequence $(z_l)_{l\in[1..n]}$ in $\mathbb{R}^n$ such that for each $l \in [1..n]$ and $j \in [1..n]$,*

$$\sup_{x\in K_{j,\varepsilon_m}} |\varphi_{l,\varepsilon_m}(x) - z_{l,j}| \to 0, \qquad \text{as } m \to \infty.$$

Here $z_l = (z_{l,j})_{j\in[1..n]} \in \mathbb{R}^n$. Then $(z_l)_{l\in[1..n]}$ is an $\langle\cdot\,,\cdot\rangle_{\mathbb{L}}$-orthonormal sequence. Define $Y_0 = \{0\} \subset \mathbb{R}^n$ and for each $p \in [1..n]$, let Y_p be the span of the vectors z_l, for $l \in [1..p]$. Moreover, let $Y_p^\perp$, $p \in [0..n]$, be the $\langle\cdot\,,\cdot\rangle_{\mathbb{L}}$-orthogonal complement of Y_p. Then, for each $k \in [1..n]$,

$$\begin{aligned} \text{(3.12)}\quad &\inf\{b_0(\Psi_{\varepsilon_m}(u),\Psi_{\varepsilon_m}(u)) \mid u \in W_{\varepsilon_m},\ \|u\|_{L^2}=1,\ u \in U^\perp_{k-1,\varepsilon_m}\} \\ &\qquad - \inf\{b_0(y,y) \mid y \in \mathbb{R}^n,\ \|y\|_{\mathbb{L}}=1 \text{ and } y \in Y^\perp_{k-1}\} \to 0, \end{aligned}$$

as $m \to \infty$. Moreover, $\lambda_{k,\varepsilon_m} \to b_0(z_k,z_k)$, as $m \to \infty$.

The proof is identical to the proof of [3, Lemma 3.14].

Lemma 3.15. *Let $(\varepsilon_m)_m$ be a null sequence in $]0,\varepsilon_0[$ and suppose that there exists a sequence $(z_l)_{l\in[1..n]}$ in $\mathbb{R}^n$ such that for all $l \in [1..n]$ and $j \in [1..n]$,*

$$\sup_{x\in K_{j,\varepsilon_m}} |\varphi_{l,\varepsilon_m}(x) - z_{l,j}| \to 0, \quad \text{as } m \to \infty.$$

For each $k \in [1..n]$ consider the following statement (P_k):

(P_k) *For each $l \in [1..k]$, z_l is an eigenvector corresponding to $\lambda_{l,0}$.*

Then (P_k) holds for each $k \in [1..n]$. Moreover, for each $k \in [1..n]$,

$$\lambda_{k,\varepsilon_m} \to \lambda_{k,0}, \quad \text{as } m \to \infty.$$

The proof is identical to the proof of [3, Lemma 3.15].

Lemma 3.16. *For every null sequence $(\varepsilon_m)_m$ in $]0,\varepsilon_0[$ there are a subsequence $(\varepsilon^1_m)_m$ of $(\varepsilon_m)_m$ and a sequence $(z_l)_{l\in[1..n]}$ in $\mathbb{R}^n$ such that for each $l \in [1..n]$ and $j \in [1..n]$,*

$$\sup_{x\in K_{j,\varepsilon^1_m}} \left|\varphi_{l,\varepsilon^1_m}(x) - z_{l,j}\right| \to 0, \quad \text{as } m \to \infty.$$

The proof is identical to the proof of [3, Lemma 3.16].

Proof of Theorem 2.5. Part (a) of the theorem was established in Lemma 3.7. Now Lemmas 3.16, 3.14 and statement (P_n) from Lemma 3.15 shows part (c) of the theorem. The arbitrariness of the sequence $(\varepsilon_m)_m$ in part (c) and Lemma 3.14 imply part (b) of the theorem. □

4. Conley index continuation for scalar reaction-diffusion equations with periodic boundary conditions

In this section we will extend the Conley index continuation results from [2] and [3] to the present more general case. We assume the reader's familiarity with the papers [2], [3]. Moreover, for the rest of this section, assume Assumption 2.1 for $x_0 = 0$ and $x_{n+1} = 1$, with the ensuing definitions and notation of Section 2.

Let $\varepsilon_0 \in \,]0, \infty]$ be as in Assumption 2.1. For each $\varepsilon \in \,]0, \varepsilon_0[$ define $H^\varepsilon = L^2$, $\langle \cdot\,, \cdot \rangle_{H^\varepsilon} = \langle \cdot\,, \cdot \rangle_{L^2}$ and A_ε as in (2.2). Define also $H^0 = \mathbb{R}^n$, $\langle \cdot\,, \cdot \rangle_{H^0} = \langle \cdot\,, \cdot \rangle_{\mathbb{L}}$ and A_0 as in (2.4). Notice that for each $\varepsilon \in \,]0, \varepsilon_0[$ it follows that $H^\varepsilon_1 = H^1_{\mathrm{per}}$ and $|\cdot|_{H^\varepsilon_1} = \|\cdot\|_\varepsilon$. Moreover, $H^0_1 = \mathbb{R}^n$ and $|\cdot|_{H^0_1} = \|\cdot\|_0$.

To prove the existence of an embedding family $(J_\varepsilon)_{\varepsilon \in]0, \widetilde{\varepsilon}]}$, for some $\widetilde{\varepsilon} \in \,]0, \varepsilon_0[$, let us introduce some notation and establish some preliminary estimates.

Define $\mathcal{B}$ to be the set of all $\langle \cdot\,, \cdot \rangle_{\mathbb{L}}$-orthonormal sequences $Z = (z_l)_{l \in [1..n]}$ such that $A_0 z_l = \lambda_{l,0} z_l$, $l \in [1\,..\,n]$. For each $Z = (z_l)_{l \in [1..n]} \in \mathcal{B}$ and $\varepsilon \in \,]0, \varepsilon_0[$ define $I_{\varepsilon,Z} \colon \mathbb{R}^n \to H^\varepsilon_1 = H^1_{\mathrm{per}}$ by

$$I_{\varepsilon,Z}(u) = \sum_{p=1}^{n} \langle u, z_p \rangle_{\mathbb{L}} \cdot \varphi_{p,\varepsilon}, \quad u \in \mathbb{R}^n.$$

It follows that $I_{\varepsilon,Z}$ is $\mathbb{R}$-linear. Suppose that $I_{\varepsilon,Z}(u) = 0$. Since $\varphi_{p,\varepsilon}$, $p \in [1\,..\,n]$ is linearly independent, we have $\langle u, z_p \rangle_{\mathbb{L}} = 0$ for all $p \in [1\,..\,n]$. Recall that Z is an $\langle \cdot, \cdot \rangle_{\mathbb{L}}$-orthonormal basis of $\mathbb{R}^n$. Therefore, $u = \sum_{p=1}^{n} \langle u, z_p \rangle_{\mathbb{L}} z_p = 0$. Thus $I_{\varepsilon,Z}$ is injective.

Let $u \in \mathbb{R}^n$ and $v = I_{\varepsilon,Z}(u) \in H^\varepsilon_1$. We have

$$\|v\|_\varepsilon^2 = \sum_{l=1}^{\infty} (\lambda_{l,\varepsilon} + 1) |\langle v, \varphi_{l,\varepsilon} \rangle_{L^2}|^2$$

so a quick calculation shows that

$$\|v\|_\varepsilon^2 = \sum_{l=1}^{n} (\lambda_{l,\varepsilon} + 1) |\langle u, z_l \rangle_{\mathbb{L}}|^2.$$

Moreover,

$$\|u\|_0^2 = \sum_{l=1}^{n} (\lambda_{l,0} + 1) |\langle u, z_l \rangle_{\mathbb{L}}|^2.$$

Now it follows from Lemma 3.7 and Theorem 2.5 that there are a constant $C \in]1,\infty[$ and an $\varepsilon_7' \in]0,\varepsilon_6']$ such that $0 \le \lambda_{l,\varepsilon}+1 \le C^2$ and $0 \le \lambda_{l,0}+1 \le C^2$, $\lambda_{l,\varepsilon}+1 \le C^2(\lambda_{l,0}+1)$ and $\lambda_{l,0}+1 \le C^2(\lambda_{l,\varepsilon}+1)$ for $l \in [1\,.\,.\,n]$ and $\varepsilon \in]0,\varepsilon_7']$. Therefore

$$\|u\|_0^2 \le C^2 \|I_{\varepsilon,Z}(u)\|_\varepsilon^2 \quad \text{and} \quad \|I_{\varepsilon,Z}(u)\|_\varepsilon^2 \le C^2 \|u\|_0^2 \tag{4.1}$$

for all $u \in \mathbb{R}^n$, $Z \in \mathcal{B}$ and $\varepsilon \in]0,\varepsilon_7']$. For each $Z = (z_l)_{l\in[1..n]} \in \mathcal{B}$ and $\varepsilon \in]0,\varepsilon_0[$ define

$$T_{\varepsilon,Z} := \sup_{l,j\in[1..n]} \sup_{x\in K_{j,\varepsilon}} |\varphi_{l,\varepsilon}(x) - z_{l,j}|. \tag{4.2}$$

Note that

$$\sup_{l,j\in[1..n]} |z_{l,j} - z'_{l,j}| \le T_{\varepsilon,Z} + T_{\varepsilon,Z'}, \quad Z = (z_l)_{l\in[1..n]}, \quad Z' = (z'_l)_{l\in[1..n]} \in \mathcal{B}. \tag{4.3}$$

The set $\mathcal{B}$ is compact in $(\mathbb{R}^n)^n$ and, for each $\varepsilon \in]0,\varepsilon_0[$, the map $T_\varepsilon \colon \mathcal{B} \to \mathbb{R}$, $Z \mapsto T_{\varepsilon,Z}$ is continuous, so there is a

$$Z(\varepsilon) = (z(\varepsilon)_l)_{l\in[1..n]} \in \mathcal{B} \tag{4.4}$$

such that

$$T_{\varepsilon,Z(\varepsilon)} = \inf_{Z\in\mathcal{B}} T_{\varepsilon,Z}. \tag{4.5}$$

Set

$$J_\varepsilon = I_{\varepsilon,Z(\varepsilon)}, \quad \varepsilon \in]0,\varepsilon_0[\,. \tag{4.6}$$

Lemma 4.1. *For every $k \in \mathbb{N}$ there exists an $\varepsilon'' = \varepsilon''(k) \in]0,\varepsilon_0[$ such that, for all $\varepsilon \in]0,\varepsilon'']$, there exists a $Z \in \mathcal{B}$ with $T_{\varepsilon,Z} \le 1/2^k$.*

Proof. Suppose the conclusion does not hold. Then there exists a $k_0 \in \mathbb{N}$ such that for all $\varepsilon'' \in]0,\varepsilon_0[$ there exists an $\varepsilon \in]0,\varepsilon'']$ such that $T_{\varepsilon,Z} > 1/2^{k_0}$, for all $Z \in \mathcal{B}$. Thus, there exists a sequence $(\varepsilon_m)_m$ in $]0,\varepsilon_0[$ with $\varepsilon_m \to 0$ as $m \to 0$ such that for all $Z \in \mathcal{B}$

$$T_{\varepsilon_m,Z} > \frac{1}{2^{k_0}}, \quad \text{for all } m \in \mathbb{N}. \tag{4.7}$$

Now, Lemma 3.16 and Lemma 3.15 imply that there are a subsequence $(\varepsilon_m^1)_m$ of $(\varepsilon_m)_m$ and a $Z \in \mathcal{B}$ such that $T_{\varepsilon_m^1,Z} \to 0$, as $m \to \infty$. This contradicts (4.7). □

Lemma 4.1 and formula (4.5) imply that

$$T_{\varepsilon,Z(\varepsilon)} \to 0 \quad \text{as } \varepsilon \to 0. \tag{4.8}$$

In the next result we will establish, for the present case, the validity of condition (FSpec) introduced in [2].

Theorem 4.2. *The family $(H^\varepsilon, \langle\,\cdot\,,\,\cdot\,\rangle_{H^\varepsilon}, A_\varepsilon, J_\varepsilon)_{\varepsilon\in[0,\varepsilon_7']}$ satisfies* (FSpec).

Proof. It is clear that (1) and (2) of condition (FSpec) hold. Inequalities (4.1) imply (3) and (4) of condition (FSpec).

For every $\varepsilon \in]0, \varepsilon'_7]$, let $(\lambda_{l,\varepsilon})_l$ be the repeated sequence of eigenvalues of A_ε and $(\varphi_{l,\varepsilon})_l$ be a corresponding H^ε-orthonormal sequence of eigenfunctions. Furthermore, let $(\lambda_{l,0})_{l\in[1..n]}$ be the repeated sequence of eigenvalues of A_0. Let $(\varepsilon_m)_m$ be an arbitrary null sequence in $\varepsilon \in]0, \varepsilon'_7]$.

It follows from Theorem 2.5 that (5)(a) and (5)(b) of condition (FSpec) hold. To complete the proof we need to show that (5)(c) and (5)(d) of condition (FSpec) also hold. Lemmas 3.15 and 3.16 imply that there are a subsequence $(\varepsilon^1_m)_m$ of $(\varepsilon_m)_m$ and a $\widetilde{Z} = (z_l)_{l\in[1..n]} \in \mathcal{B}$ such that

$$T_{\varepsilon^1_m,\widetilde{Z}} \to 0, \quad \text{as } m \to \infty. \tag{4.9}$$

Formulas (4.3), (4.9) and (4.8) imply that

$$\sup_{l,j\in[1..n]} |z(\varepsilon^1_m)_{l,j} - z_{l,j}| \to 0, \quad \text{as } m \to \infty. \tag{4.10}$$

Let $l \in [1..n]$ be arbitrary. We have

$$\begin{aligned}
\varphi_{l,\varepsilon^1_m} - J_{\varepsilon^1_m}(z_l) &= \varphi_{l,\varepsilon^1_m} - I_{\varepsilon^1_m,Z(\varepsilon^1_m)}(z_l) \\
&= \sum_{p=1}^n \delta_{l,p}\varphi_{p,\varepsilon^1_m} - \sum_{p=1}^n \langle z_l, z(\varepsilon^1_m)_p\rangle_{\mathbb{L}}\varphi_{p,\varepsilon^1_m} \\
&= \sum_{p=1}^n (\delta_{l,p} - \langle z_l, z(\varepsilon^1_m)_p\rangle_{\mathbb{L}})\varphi_{p,\varepsilon^1_m}.
\end{aligned}$$

Thus

$$\begin{aligned}
\|\varphi_{l,\varepsilon^1_m} - J_{\varepsilon^1_m}(z_l)\|_{\varepsilon^1_m} &\le \sum_{p=1}^n |\delta_{l,p} - \langle z_l, z(\varepsilon^1_m)_p\rangle_{\mathbb{L}}| \|\varphi_{p,\varepsilon^1_m}\|_{\varepsilon^1_m} \\
&= \sum_{p=1}^n |\delta_{l,p} - \langle z_l, z(\varepsilon^1_m)_p\rangle_{\mathbb{L}}| (\lambda_{p,\varepsilon^1_m} + 1)^{1/2}.
\end{aligned}$$

Since, by estimate (4.10), $\langle z_l, z(\varepsilon^1_m)_p\rangle_{\mathbb{L}} \to \delta_{l,p}$ and for each $p \in [1..n]$, the sequence $(\lambda_{p,\varepsilon^1_m} + 1)_m$ stays bounded as $m \to \infty$, we see that (5)(c) of condition (FSpec) holds. For $u \in \mathbb{R}^n = H^0_1$ and $m \in \mathbb{N}$ we have

$$\begin{aligned}
\langle J_{\varepsilon^1_m} u, \varphi_{l,\varepsilon^1_m}\rangle_{H^{\varepsilon^1_m}} &= \langle J_{\varepsilon^1_m} u, \varphi_{l,\varepsilon^1_m}\rangle_{L^2} \\
&= \sum_{p=1}^n \langle u, z(\varepsilon^1_m)_p\rangle_{\mathbb{L}} \langle \varphi_{p,\varepsilon^1_m}, \varphi_{l,\varepsilon^1_m}\rangle_{L^2} = \langle u, z(\varepsilon^1_m)_l\rangle_{\mathbb{L}}.
\end{aligned}$$

Thus $\langle J_{\varepsilon^1_m} u, \varphi_{l,\varepsilon^1_m}\rangle_{H^{\varepsilon^1_m}} \to \langle u, z_l\rangle_{H^0}$ as $m \to \infty$. Therefore (5)(d) of condition (FSpec) holds. □

For the rest of this section assume the following nonlinear convergence hypothesis:

ASSUMPTION 4.3. (a) For each $\varepsilon \in [0,\varepsilon_0[$ the function $g_\varepsilon\colon [0,1]\times\mathbb{R}\to\mathbb{R}$ is continuous and such that for each $M \in]0,\infty[$ there exists a $L_M \in]0,\infty[$ such that for $|s| \le M$ and $|s'| \le M$

$$|g_\varepsilon(x,s) - g_\varepsilon(x,s')| \le L_M|s-s'|, \quad \text{for all } x\in[0,1],\ \varepsilon\in[0,\varepsilon_0[.$$

(b) There is an $\varepsilon'_8 \in]0,\varepsilon_0[$ such that

$$\sup_{\varepsilon\in[0,\varepsilon'_8]}\ \sup_{x\in[0,1]} |g_\varepsilon(x,0)| < \infty.$$

(c) For each $x\in[0,1]$ and $s\in\mathbb{R}$, $g_\varepsilon(x,s)\to g_0(x,s)$ as $\varepsilon\to 0$.

Let $\varepsilon \in]0,\varepsilon_0[$. Note that each $u\in H^1_{\mathrm{per}}$ is (uniquely represented by) a continuous function. So the map $\widehat{g}_\varepsilon(u)\colon [0,1]\to\mathbb{R}$ defined by

$$\widehat{g}_\varepsilon(u)(x) = g_\varepsilon(x,u(x)), \quad x\in[0,1]\,,$$

is continuous and bounded. Moreover, $\widehat{g}_\varepsilon(u)$ is Lebesgue measurable and so it lies in $L^2(0,1)$. Therefore for each $\varepsilon \in]0,\varepsilon_0[$ we obtain a well defined map $f_\varepsilon\colon H^1_{\mathrm{per}}\to L^2$ given by $f_\varepsilon(u)=\widehat{g}_\varepsilon(u)$, $u\in H^1_{\mathrm{per}}$. Moreover define $f_0\colon\mathbb{R}^n\to\mathbb{R}^n$ by $f_0(u) = (f_0(u)_j)_{j\in[1..n]}$, where

$$f_0(u)_j = \frac{1}{L_j}\int_{K_j} g_0(x,u_j)\,dx,$$

$u=(u_j)_{j\in[1..n]}$, for $j\in[1..n]$.

In the next result we will establish, for the present case, the validity of condition (Conv) introduced in [2].

THEOREM 4.4. *Let* $(H^\varepsilon,\langle\cdot,\cdot\rangle_{H^\varepsilon},A_\varepsilon,J_\varepsilon)_{\varepsilon\in[0,\varepsilon'_7]}$ *be as Theorem* 4.2. *There exists an* $\varepsilon'_9\in]0,\varepsilon'_7]$ *such that the family* $(f_\varepsilon)_{\varepsilon\in[0,\varepsilon'_9]}$ *satisfies condition* (Conv).

PROOF. Let $\varepsilon'_9=\min\{\varepsilon'_2,\varepsilon'_7,\varepsilon'_8\}$. Part (1) of condition (Conv) has just been proved. Let $M\in]0,\infty[$ be arbitrary. Let $\varepsilon\in]0,\varepsilon'_9]$ and u, $v\in H^\varepsilon_1$ be such that $|u|_{H^\varepsilon_1}$, $|v|_{H^\varepsilon_1}\le M$. It follows from Lemma 3.3 that

$$\sup_{x\in[0,1]}|u(x)|\le C'_2M \quad\text{and}\quad \sup_{x\in[0,1]}|v(x)|\le C'_2M.$$

Hence

$$\int_0^1 |g_\varepsilon(x,u(x)) - g_\varepsilon(x,v(x))|^2\,dx \le L^2_{\widetilde{M}}\int_0^1|u(x)-v(x)|^2\,dx \le L^2_{\widetilde{M}}\|u-v\|^2_\varepsilon,$$

where $\widetilde{M}=C'_2M$. This implies that

$$|f_\varepsilon(u)-f_\varepsilon(v)|_{H^\varepsilon}\le L_{\widetilde{M}}|u-v|_{H^\varepsilon_1}, \quad \text{for all } \varepsilon\in]0,\varepsilon'_9].$$

Moreover, let $u, v \in H_1^0$ satisfy $|u|_{H_1^0}$, $|v|_{H_1^0} \le M$.

$$\begin{aligned}\|f_0(u) - f_0(v)\|_{\mathbb{L}}^2 &= \sum_{j=1}^n L_j (f_0(u)_j - f_0(v)_j)^2 \\ &= \sum_{j=1}^n L_j \frac{1}{L_j^2} \bigg(\int_{K_j} (g_0(x, u_j) - g_0(x, v_j))\, dx \bigg)^2 \\ &\le \sum_{j=1}^n \frac{1}{L_j} \bigg(\int_{K_j} |g_0(x, u_j) - g_0(x, v_j)|\, dx \bigg)^2 \\ &\le \sum_{j=1}^n \frac{L_{M'}^2}{L_j} \bigg(\int_{K_j} |u_j - v_j|\, dx \bigg)^2 \\ &\le L_{M'}^2 \sum_{j=1}^n L_j |u_j - v_j|^2 = L_{M'}^2 \|u - v\|_{\mathbb{L}}^2 \le L_{M'}^2 \|u - v\|_0^2,\end{aligned}$$

where $M' = M\Big(\min_{j\in[1..n]} L_j\Big)^{-1/2}$. This implies that

$$|f_0(u) - f_0(v)|_{H^0} \le L_{M'} |u - v|_{H_1^0}.$$

It follows that part (3) of condition (Conv) holds.

Let C be as in formula (4.1). Let $\varepsilon \in \,]0, \varepsilon_9']$ be arbitrary. Then

$$\begin{aligned}\|f_\varepsilon(J_\varepsilon(u))\|_{L^2} &\le \|f_\varepsilon(J_\varepsilon(u)) - f_\varepsilon(0)\|_{L^2} + \|f_\varepsilon(0)\|_{L^2} \\ &\le L_M \|J_\varepsilon(u)\|_\varepsilon + \|f_\varepsilon(0)\|_{L^2} \\ &\le L_M C\|u\|_{\mathbb{L}} + \|f_\varepsilon(0)\|_{L^2} \le L_M C \|u\|_{\mathbb{L}} + K,\end{aligned}$$

where $M = C\|u\|_{\mathbb{L}}$ and $K = \sup_{\varepsilon\in[0,\varepsilon_9']} \sup_{x\in[0,1]} |g_\varepsilon(x, 0)|$. This implies that statement (4) of condition (Conv) holds.

To complete the proof we need to show that (2) of condition (Conv) holds. To this end we will use [2, Theorem 2.2], which holds in the present case in view of Theorem 4.2. We claim that:

Let $u \in H_1^0 = \mathbb{R}^n$ and $t \in \,]0, \infty[$. Then

$$\lim_{\varepsilon\to 0^+} \big|e^{-tA_\varepsilon} f_\varepsilon(J_\varepsilon u) - J_\varepsilon(e^{-tA_0} f_0(u))\big|_{H_1^\varepsilon} = 0. \tag{4.11}$$

Let $(\varepsilon_m)_m$ be a null sequence in $]0, \varepsilon_9']$. Notice that $J_{\varepsilon_m} u \in H^{\varepsilon_m}$ for all $m \in \mathbb{N}$. It follows from (4) of condition (Conv) that

$$\sup_{m\in\mathbb{N}} |f_{\varepsilon_m}(J_{\varepsilon_m} u)|_{H^{\varepsilon_m}} < \infty. \tag{4.12}$$

Theorem 2.5 implies there are a subsequence $(\varepsilon_m^1)_m$ of $(\varepsilon_m)_m$ and a sequence $\widetilde{Z} = (z_l)_{l\in[1..n]}$ in $\mathbb{R}^n$, where z_l is an eigenvector corresponding to $\lambda_{l,0}$, such that

$$T_{\varepsilon_m^1, \widetilde{Z}} = \sup_{l,j\in[1..n]} \sup_{x\in K_{j,\varepsilon_m^1}} |\varphi_{l,\varepsilon_m^1}(x) - z_{l,j}| \to 0, \quad \text{as } m \to \infty. \tag{4.13}$$

Let $l \in [1\,.\,.\,n]$. We will show that

$$\left\langle f_{\varepsilon^1_m}(J_{\varepsilon^1_m}u), \varphi_{l,\varepsilon^1_m}\right\rangle_{L^2} \to \langle u, z_l\rangle_{\mathbb{L}} \quad \text{as } m \to \infty.$$

For each $m \in \mathbb{N}$ we have

$$\left\langle f_{\varepsilon^1_m}(J_{\varepsilon^1_m}u), \varphi_{l,\varepsilon^1_m}\right\rangle = \int_0^1 g_{\varepsilon^1_m}(x, (J_{\varepsilon^1_m}u)(x))\varphi_{l,\varepsilon^1_m}(x)\,dx =: \sum_{j=1}^n \int_{K_j} T_j(x)\,dx,$$

where $T_j(x) = g_{\varepsilon^1_m}\big(x, (J_{\varepsilon^1_m}u)(x)\big)\varphi_{l,\varepsilon^1_m}(x)$, $x \in K_j$, $j \in [1\,.\,.\,n]$. For $m \in \mathbb{N}$, $x \in K_j$ and $j \in [1\,.\,.\,n]$ we have

$$\begin{aligned} T_j(x) &= \big(g_{\varepsilon^1_m}(x, (J_{\varepsilon^1_m}u)(x)) - g_{\varepsilon^1_m}(x, u_j)\big)\varphi_{l,\varepsilon^1_m}(x) \\ &\quad + g_{\varepsilon^1_m}(x, u_j)\big(\varphi_{l,\varepsilon^1_m}(x) - z_{l,j}\big) + \big(g_{\varepsilon^1_m}(x, u_j) - g_0(x, u_j)\big)z_{l,j} + g_0(x, u_j)z_{l,j} \\ &=: S^j_{1,m}(x) + S^j_{2,m}(x) + S^j_{3,m}(x) + S^j_{4,m}(x). \end{aligned}$$

Let $M \in \,]0, \infty[$ be a positive constant such that for all $\varepsilon \in \,]0, \varepsilon'_9]$, $j \in [1\,.\,.\,n]$, $x \in [0,1]$ and $m \in \mathbb{N}$

$$|J_\varepsilon(u)(x)| \le M, \qquad |\varphi_{l,\varepsilon}(x)| \le M,$$
$$|u_j| \le M, \qquad |g_\varepsilon(x, u_j)| \le M.$$

Therefore,

$$|S^j_{1,m}(x)| \le L_M |J_{\varepsilon^1_m}u(x) - u_j| M, \quad \text{for all } j \in [1\,.\,.\,n],\ x \in [0,1] \text{ and } m \in \mathbb{N},$$
$$|S^j_{2,m}(x)| \le M |\varphi_{l,\varepsilon^1_m}(x) - z_{l,j}|, \qquad \text{for all } j \in [1\,.\,.\,n],\ x \in [0,1] \text{ and } m \in \mathbb{N}.$$

Recall that $J_{\varepsilon^1_m} = I_{\varepsilon^1_m, Z(\varepsilon^1_m)}$. Therefore

$$J_{\varepsilon^1_m}u(x) = \sum_{p=1}^n \langle u, z(\varepsilon^1_m)_p\rangle_{\mathbb{L}}\varphi_{p,\varepsilon^1_m}(x), \quad \text{for } x \in [0,1] \text{ and } m \in \mathbb{N}.$$

Let $j \in [1\,.\,.\,n]$. Since $u_j = \sum_{p=1}^n \langle u, z_p\rangle_{\mathbb{L}} z_{p,j}$ we obtain

$$J_{\varepsilon^1_m}u(x) - u_j = \sum_{p=1}^n (\langle u, z(\varepsilon^1_m)_p\rangle_{\mathbb{L}} - \langle u, z_p\rangle_{\mathbb{L}})\varphi_{p,\varepsilon^1_m}(x) + \sum_{p=1}^n \langle u, z_p\rangle_{\mathbb{L}}(\varphi_{p,\varepsilon^1_m}(x) - z_{p,j}).$$

It follows from (4.3), (4.8) and (4.13) that

$$\sup_{x \in K_{j,\varepsilon^1_m}} |J_{\varepsilon^1_m}u(x) - u_j| \to 0, \quad \text{as } m \to \infty \quad \text{and} \quad \sup_{m \in \mathbb{N}} \sup_{x \in K_j} |J_{\varepsilon^1_m}u(x) - u_j| < \infty.$$

Since $\mathbf{m}(K_j \setminus K_{j,\varepsilon^1_m}) \to 0$ as $m \to \infty$ it follows that

$$\int_{K_j} S^j_{1,m}(x)\,dx \to 0, \quad \text{as } m \to \infty.$$

Similarly we show that

$$\sup_{x \in K_{j,\varepsilon^1_m}} |S^j_{2,m}(x)| \to 0 \quad \text{as } m \to \infty \quad \text{and} \quad \sup_{m \in \mathbb{N}} \sup_{x \in K_j} |S^j_{2,m}(x)| < \infty.$$

Hence

$$\int_{K_j} S^j_{2,m}(x)\,dx \to 0 \quad \text{as } m \to \infty.$$

Since $g_\varepsilon(x,s) \to g_0(x,s)$ as $\varepsilon \to 0$ and

$$\sup_{m\in\mathbb{N}} \sup_{x\in K_j} |g_{\varepsilon^1_m}(x,u_j)| < \infty,$$

the Lebesgue Dominated Convergence Theorem implies that

$$\int_{K_j} S^j_{3,m}(x)\,dx \to 0 \quad \text{as } m \to \infty.$$

Finally

$$\int_{K_j} S^j_{4,m}(x)\,dx = \int_{K_j} g_0(x,u_j) z_{l,j}\,dx = L_j f_0(u)_j z_{l,j}.$$

Thus

$$\sum_{j=1}^n \int_{K_j} S^j_{4,m}(x)\,dx = \langle f_0(u), z_l\rangle_{\mathbb{L}}$$

and so

$$\big\langle f_{\varepsilon^1_m}(J_{\varepsilon^1_m}u), \varphi_{l,\varepsilon^1_m}\big\rangle_{L^2} \to \langle f_0(u), z_l\rangle_{\mathbb{L}} \quad \text{as } m \to \infty.$$

This together with (4.12) and [2, Theorem 2.2] imply that

$$\big|e^{-tA_{\varepsilon_m}} f_{\varepsilon_m}(J_{\varepsilon_m}u) - J_{\varepsilon_m}(e^{-tA_0} f_0(u))\big|_{H^{\varepsilon_m}_1} \to 0 \quad \text{as } m \to \infty.$$

This proves claim (4.11) and completes the proof. □

For each $\varepsilon \in]0, \varepsilon'_9]$, consider the abstract parabolic equation

$$\dot{u} = -A_\varepsilon u + f_\varepsilon(u) \tag{4.14}$$

on H^1_{per}. This equation generates a local semiflow π_ε on H^1_{per}. Equation (4.14) is an abstract formulation of the periodic boundary value problem

$$\begin{cases} u_t = (a_\varepsilon u_x)_x + g_\varepsilon(x,u) & \text{for } 0 < x < 1,\ t > 0, \\ u(t,0) = u(t,1) & \text{for } t \geq 0. \end{cases} \tag{E_ε, P}$$

Moreover, we may also consider the system of ordinary differential equations

$$\dot{z} = -A_0 z + f_0(z) \tag{4.15}$$

on $\mathbb{R}^n$. This system generates a local (semi)flow π_0 on $\mathbb{R}^n$. For $\varepsilon \in]0, \varepsilon'_9]$, let $Q_\varepsilon\colon H^\varepsilon_1 \to H^\varepsilon_1$ be the H^ε_1-orthogonal projection of H^ε_1 onto (its closed subspace) $J_\varepsilon(H^0_1)$. Moreover, let $R_\varepsilon\colon J_\varepsilon(H^0_1) \to H^0_1$ be the inverse of $J_\varepsilon\colon H^0_1 \to J_\varepsilon(H^0_1)$.

We can now state the following Conley index continuation principle:

Theorem 4.5. *Let N be a closed and bounded isolating neighbourhood of an invariant set S_0 relative to π_0. For $\varepsilon \in]0,\varepsilon'_9]$ and, for every $\eta \in]0,\infty[$, set*

$$N_{\varepsilon,\eta} := \{u \in H_1^\varepsilon \mid R_\varepsilon Q_\varepsilon u \in N \text{ and } |(I-Q_\varepsilon)u|_{H_1^\varepsilon} \le \eta\}$$

and $S_{\varepsilon,\eta} := \mathrm{Inv}_{\pi_\varepsilon}(N_{\varepsilon,\eta})$ i.e. $S_{\varepsilon,\eta}$ is the largest π_ε-invariant set in $N_{\varepsilon,\eta}$. Then, for every $\eta \in]0,\infty[$, there exists an $\varepsilon^{\mathrm{c}} = \varepsilon^{\mathrm{c}}(\eta) \in]0,\varepsilon'_9]$ such that, for every $\varepsilon \in]0,\varepsilon^{\mathrm{c}}]$, the set $N_{\varepsilon,\eta}$ is a strongly admissible isolating neighbourhood of $S_{\varepsilon,\eta}$ relative to π_ε and

$$h(\pi_\varepsilon, S_{\varepsilon,\eta}) = h(\pi_0, S_0).$$

Here, as usual, $h(\pi,S)$ denotes the Conley index of an isolated invariant set S relative to a local semiflow π. Furthermore, for every $\eta \in]0,\infty[$, the family $(S_{\varepsilon,\eta})_{\varepsilon\in[0,\varepsilon^{\mathrm{c}}(\eta)]}$ of invariant sets, where $S_{0,\eta} = S_0$, is upper semicontinuous at $\varepsilon = 0$ with respect to the family $|\cdot|_{H_1^\varepsilon}$ of norms i.e.

$$\lim_{\varepsilon\to 0^+} \sup_{w\in S_{\varepsilon,\eta}} \inf_{u\in S_0} |w - J_\varepsilon u|_{H_1^\varepsilon} = 0.$$

The family $(S_{\varepsilon,\eta})_{\varepsilon\in]0,\varepsilon^{\mathrm{c}}(\eta)]}$ is asymptotically independent of η, i.e whenever η_1 and $\eta_2 \in]0,\infty[$ then there is an $\varepsilon' \in]0,\min(\varepsilon^{\mathrm{c}}(\eta_1),\varepsilon^{\mathrm{c}}(\eta_2))]$ such that $S_{\varepsilon,\eta_1} = S_{\varepsilon,\eta_2}$ for $\varepsilon \in]0,\varepsilon']$.

Proof. This is an application of the abstract result [2, Theorem 2.4] using Theorems 4.2 and 4.4. □

Remark 4.6. Note that

$$\sup_{\varepsilon\in]0,\varepsilon^{\mathrm{c}}(\eta)]} \sup_{w\in N_{\varepsilon,\eta}} |w|_{H_1^\varepsilon} < \infty \quad \text{and} \quad \sup_{\varepsilon\in]0,\varepsilon^{\mathrm{c}}(\eta)]} \sup_{u\in N} |J_\varepsilon u|_{H_1^\varepsilon} < \infty.$$

In particular, by Lemma 3.3, we also have that

$$\lim_{\varepsilon\to 0^+} \sup_{w\in S_{\varepsilon,\eta}} \inf_{u\in S_0} |w - J_\varepsilon u|_{L^\infty} = 0. \tag{4.16}$$

The embedding J_ε, $\varepsilon \in]0,\varepsilon'_9]$, is somewhat artificial. A more natural embedding can be defined by the map

$$\Theta\colon \mathbb{R}^n \to L^\infty, \qquad u = (u_j)_{j\in[1..n]} \mapsto \sum_{j=1}^n u_j 1_{K_j}.$$

This map is clearly linear injective. Note that

$$u = \sum_{p=1}^n \langle u, z(\varepsilon)_p\rangle_{\mathbb{L}} z(\varepsilon)_p, \quad u \in \mathbb{R}^n \text{ and } \varepsilon \in]0,\varepsilon'_9]. \tag{4.17}$$

It follows that

$$\Theta u(x) := (\Theta u)(x) = u_j \quad \text{for } j \in [1\,..\,n] \text{ and } x \in K_{j,\varepsilon},$$

so

$$J_\varepsilon u(x) - \Theta u(x) = \sum_{p=1}^{n} \langle u, z(\varepsilon)_p \rangle_{\mathbb{L}} (\varphi_{p,\varepsilon}(x) - z(\varepsilon)_{p,j}) \tag{4.18}$$

and so

$$\sup_{j\in[1..n]} \sup_{x\in K_{j,\varepsilon}} |J_\varepsilon u(x) - \Theta u(x)| \le n|u|_{\mathbb{L}} T_{\varepsilon,Z(\varepsilon)}. \tag{4.19}$$

Proposition 4.7. *Under the assumptions of Theorem* 4.5 *the following upper semicontinuity results hold:*

$$\lim_{\varepsilon\to 0^+} \sup_{w\in S_{\varepsilon,\eta}} \inf_{u\in S_0} \sup_{j\in[1..n]} \sup_{x\in K_{j,\varepsilon}} |w(x) - \Theta u(x)| = 0 \tag{4.20}$$

and, for all $r \in [1,\infty[$,

$$\lim_{\varepsilon\to 0^+} \sup_{w\in S_{\varepsilon,\eta}} \inf_{u\in S_0} |w - \Theta u|_{L^r} = 0. \tag{4.21}$$

Proof. For $w \in S_{\varepsilon,\eta}$ and $u \in S_0$ we have

$$\begin{aligned} \sup_{j\in[1..n]} \sup_{x\in K_{j,\varepsilon}} &|w(x) - \Theta u(x)| \\ &\le \sup_{j\in[1..n]} \sup_{x\in K_{j,\varepsilon}} |w(x) - J_\varepsilon u(x)| + \sup_{j\in[1..n]} \sup_{x\in K_{j,\varepsilon}} |J_\varepsilon u(x) - \Theta u(x)| \\ &\le \sup_{j\in[1..n]} \sup_{x\in K_{j,\varepsilon}} |w(x) - J_\varepsilon u(x)| + nC_N T_{\varepsilon,Z(\varepsilon)} \\ &\le |w - J_\varepsilon u|_{L^\infty} + nC_N T_{\varepsilon,Z(\varepsilon)}, \end{aligned} \tag{4.22}$$

where $C_N = \sup_{u\in N} |u|_{\mathbb{L}} < \infty$. Thus

$$\begin{aligned} \sup_{w\in S_{\varepsilon,\eta}} \inf_{u\in S_0} \sup_{j\in[1..n]} \sup_{x\in K_{j,\varepsilon}} &|w(x) - \Theta u(x)| \\ &\le nC_N T_{\varepsilon,Z(\varepsilon)} + \sup_{w\in S_{\varepsilon,\eta}} \inf_{u\in S_0} |w - J_\varepsilon u|_{L^\infty}, \end{aligned} \tag{4.23}$$

so (4.20) follows from (4.16) and (4.8). Now estimate (4.21) follows from estimate (4.20) and Remark 4.6. □

Finally, we have the following homology index braid continuation principle:

Theorem 4.8. *Assume the hypotheses of Theorem* 4.5 *and for every* $\eta \in]0,\infty[$ *let* $\varepsilon^c(\eta) \in]0,\varepsilon'_9]$ *be as in that theorem. Let* $(P,\prec)$ *be a finite poset. Let* $(M_{p,0})_{p\in P}$ *be a* $\prec$*-ordered Morse decomposition of* S_0 *relative to* π_0*. For each* $p \in P$*, let* $V_p \subset N$ *be closed in* $\mathbb{R}^n$ *and such that*

$$M_{p,0} = \mathrm{Inv}_{\pi_0}(V_p) \subset \mathrm{Int}_{H^0_1}(V_p).$$

(*Such sets* V_p, $p \in P$, *exist.*) *For* $\varepsilon \in]0,\varepsilon'_9]$, *for every* $\eta \in]0,\infty[$ *and* $p \in P$ *set* $M_{p,\varepsilon,\eta} := \mathrm{Inv}_{\pi_\varepsilon}(V_{p,\varepsilon,\eta})$, *where*

$$V_{p,\varepsilon,\eta} := \{\, u \in H^\varepsilon_1 \mid R_\varepsilon Q_\varepsilon u \in V_p \text{ and } |(I - Q_\varepsilon)u|_{H^\varepsilon_1} \le \eta \,\}.$$

Then, for every $\eta \in]0,\infty[$, there is an $\widetilde{\varepsilon} = \widetilde{\varepsilon}(\eta) \in]0,\varepsilon^{\mathrm{c}}(\eta)]$ such that for every $\varepsilon \in]0,\widetilde{\varepsilon}]$ and $p \in P$, $M_{p,\varepsilon,\eta} \subset \mathrm{Int}_{H_1^\varepsilon}(V_{p,\varepsilon,\eta})$ and the family $(M_{p,\varepsilon,\eta})_{p\in P}$ is a $\prec$-ordered Morse decomposition of $S_{\varepsilon,\eta}$ relative to π_ε and the homology index braids of $(\pi_0, S_0, (M_{p,0})_{p\in P})$ and $(\pi_\varepsilon, S_{\varepsilon,\eta}, (M_{p,\varepsilon,\eta})_{p\in P})$, $\varepsilon \in]0,\widetilde{\varepsilon}]$, are isomorphic and so they determine the same collection of C-connection matrices. For each $p \in P$, the family $(M_{p,\varepsilon,\eta})_{\varepsilon\in[0,\widetilde{\varepsilon}(\eta)]}$, where $M_{p,0,\eta} = M_{p,0}$, is upper semicontinuous at $\varepsilon = 0$ with respect to the family $|\cdot|_{H_1^\varepsilon}$ of norms and the family $(M_{p,\varepsilon,\eta})_{\varepsilon\in]0,\widetilde{\varepsilon}(\eta)]}$ is asymptotically independent of η.

Proof. This is an application of the abstract result [2, Theorem 2.5] using Theorems 4.2 and 4.4. □

Remark 4.9. Of course, the analogue of Proposition 4.7 holds for each of the families $(M_{p,\varepsilon,\eta})_{\varepsilon\in[0,\widetilde{\varepsilon}(\eta)]}$.

Appendix

Let α, $\beta \in \mathbb{R}$ be arbitrary with $\alpha < \beta$. Let $H^1_{\mathrm{per}}(\alpha,\beta)$ be the set of all $\varphi \in H^1(\alpha,\beta)$ with $u(\alpha) = u(\beta)$, where $u = u_\varphi \in C([\alpha,\beta])$ is the unique continuous representative of φ.

Let $a\colon [\alpha,\beta] \to \mathbb{R}$ be continuous and positive. Define the bilinear form $b = b_a\colon H^1_{\mathrm{per}}(\alpha,\beta) \times H^1_{\mathrm{per}}(\alpha,\beta) \to \mathbb{R}$ by

$$(\varphi,\psi) \mapsto \int_{]\alpha,\beta[} a\varphi'\psi'\,dx.$$

Define $D = D_b$ be the set of all $\varphi \in H^1_{\mathrm{per}}(\alpha,\beta)$ for which there is a $w = w_\varphi \in L^2(\alpha,\beta)$ such that

$$b(\varphi,\psi) = \langle w,\psi\rangle_{L^2(\alpha,\beta)} \quad \text{for all } \psi \in H^1_{\mathrm{per}}(\alpha,\beta).$$

Then, for $\varphi \in D$ the element $w = w_\varphi$ is uniquely defined and writing $A\varphi = w$ we obtain a map $A\colon D \to L^2(\alpha,\beta)$, called the map defined by the pair $(b, \langle\cdot\,,\,\cdot\rangle_{L^2(\alpha,\beta)})$ and denote D by $D(A)$.

It is easy to prove that D is the set of all $\varphi \in H^1_{\mathrm{per}}(\alpha,\beta)$ such that $a\varphi' \in H^1_{\mathrm{per}}(\alpha,\beta)$ and then $A\varphi = -(a\varphi')'$.

Moreover, if $a(\alpha) = a(\beta)$, then the following conditions are equivalent for each $\varphi \in H^1(\alpha,\beta)$:

(1) $\varphi \in D$ and $A\varphi$ has a continuous representative $\widehat{w}$.
(2) The continuous representative u of φ lies in $C^1([\alpha,\beta])$ and in classical sense, $u(\alpha) = u(\beta)$, $u'(\alpha) = u'(\beta)$ and $(au')'(x) = -\widehat{w}(x)$ for all $x \in [\alpha,\beta]$.

This follows by an application of [1, Theorem 8.2].

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