Topological Methods in Nonlinear Analysis
Volume 53, No. 2, 2019, 683–695
DOI: 10.12775/TMNA.2019.017

ON DIRECTIONAL DERIVATIVES FOR CONE-CONVEX FUNCTIONS

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Abstract. We investigate the relationship between the existence of directional derivatives for cone-convex functions with values in a Banach space Y and isomorphisms between Y and c_0.

1. Introduction

Starting from the seminal results of Asplund, many efforts are devoted to characterisations of Banach spaces in terms of differentiability properties of some classes of functions. In the present paper we investigate the relationships between directional differentiability of cone-convex functions and the properties of their image spaces. We prove sufficient conditions ensuring that a Banach space Y does not contain an isomorphic subspace of c_0 (does not contain a copy of c_0). These sufficient conditions are expressed in terms of directional differentiability of cone-convex functions taking values in Y.

Our aim is to relate the existence of directional derivatives for K-convex functions $F\colon X \to Y$, where cone K is normal, to the existence of isomorphisms between image space Y and c_0.

Isomorphisms of a Banach space Y and the space c_0 have been investigated e.g. in [4], [10], [17], [18]. For example in [18] such spaces were investigated in

2010 *Mathematics Subject Classification.* Primary 46B10; Secondary 46B20.
Key words and phrases. Directional derivative; cone isomorphism; convex mappings; cone convex mappings.

terms of strongly summing sequences. In [17] it is proved that c_0 is embeddable in Banach space X if and only if c_0^+ is embeddable in X.

In [10], [9] we can find interesting results for Banach lattices.

Theorem 1.1 ([9, Theorem 1.c.7]). *A Banach lattice does not contain a subspace isomorphic to c_0 if and only if it is weakly sequentially complete.*

Taking into account our main result (Theorem 5.1) we get a characterization of weakly sequentially complete Banach lattices Y in terms of directional differentiability of K-convex functions, where cone $K \subset Y$ is normal.

The organization of the paper is as follows. In Section 2 we present basic notions and facts about cone-convex functions. Section 3 is devoted to normal cones in Banach spaces. In Section 4 we present basic construction of cone-convex functions (cf. [3]) which is used in our main result. Section 5 contains the main result.

2. Notations and preliminaries

Let X be a linear space over reals. Let Y be a normed space over reals and let Y^* be the norm dual of Y. In the definition below, the limit is taken with respect to the norm topology.

Definition 2.1. Let $A \subset X$. The function $F\colon A \to Y$ is *directionally differentiable* at $x_0 \in A$ in the direction $h \neq 0$ such that $x_0 + th \in A$ for all t sufficiently small if the limit

$$F'(x_0; h) := \lim_{t \downarrow 0} \frac{F(x_0 + th) - F(x_0)}{t}$$

exists. The element $F'(x_0; h)$ is called the *directional derivative* of F at x_0 in the direction h. If $F'(x_0, h)$ exists for any direction $h \in X$ we say that F is *directionally differentiable* at x_0.

A sequence $\{y_n\}$ in a Banach space Y is *weak Cauchy*, if $\lim_{n\to\infty} y^*(y_n)$ exists for every $y \in Y^*$. We say that Y is *weakly sequentially complete*, if every weak Cauchy sequence weakly converges in Y.

A nonempty subset K of Y is called a *cone* if $\lambda K \subset K$ for every $\lambda \geq 0$ and $K + K \subset K$. The relation $x \leq_K y$ ($y \geq_K x$) is defined as follows

$$x \leq_K y \ (y \geq_K x) \ \Leftrightarrow \ y - x \in K.$$

We say that $K \subset Y$ is *generating*, if $Y = K - K$, i.e. for every $y \in Y$ there are $k_1, k_2 \in K$ such that $y = k_1 - k_2$.

The *dual cone* [8] of a cone K is defined as

$$K^* = \{y^* \in Y^* : y^*(y) \geq 0 \text{ for all } y \in K\}. \tag{2.1}$$

In the space $c_0 := \Big\{x = (x_1, x_2, \ldots),\ x_i \in \mathbb{R},\ \lim_{i\to\infty} x_i = 0\Big\}$, the cone c_0^+,

$$c_0^+ := \{x = (x_1, x_2, \ldots) \in c_0 : x_i \geq 0,\ i = 1, 2, \ldots\},$$

is closed convex pointed (i.e. $K \cap (-K) = \{0\}$) and generating in c_0 i.e. $c_0 = c_0^+ - c_0^+$. We also have $(c_0^+)^* = \ell_1^+$.

Definition 2.2. Let $K \subset Y$ be a cone. Let $A \subset X$ be a convex set. We say that a function $F\colon X \to Y$ is *K-convex* on A if, for all $x, y \in A$ and for all $\lambda \in [0, 1]$,

$$\lambda F(x) + (1 - \lambda)F(y) - F(\lambda x + (1 - \lambda)y) \in K.$$

Some properties of K-convex functions can be found in e.g. [3], [6], [13]. The following characterization is given in [13] for finite dimensional case.

Lemma 2.3 ([3, Lemma 3.3]). *Let $A \subset X$ be a convex subset of X. Let $K \subset Y$ be a closed convex cone and let $F\colon X \to Y$ be a function. The following conditions are equivalent.*

(a) *The function F is K-convex on A.*
(b) *For any $u^* \in K^*$ the composite function $u^* \circ F\colon A \to \mathbb{R}$ is convex.*

3. Normal cones

In a normed space Y a cone K is normal (see [16]) if there is a number $C > 0$ such that

$$0 \leq_K x \leq_K y \ \Rightarrow\ \|x\| \leq C\|y\|.$$

Some useful characterizations of normal cones are given in the following lemmas.

Lemma 3.1 ([14, Proposition 1.3]). *Let Y be an ordered normed space with the positive cone K. The following assertions are equivalent.*

(a) *$K \subset Y$ is normal.*
(b) *For any two nets $\{x_\beta : \beta \in I\}$ and $\{y_\beta : \beta \in I\}$, if $0 \leq_K x_\beta \leq_K y_\beta$ for all $\beta \in I$ and $\{y_\beta\}$ converges to 0, then $\{x_\beta\}$ converges to 0.*

For lattice cones in Riesz spaces defined as in [2] we get the following lemma.

Lemma 3.2 ([2, Lemma 2.39]). *Every lattice cone in Riesz space is normal closed and generating.*

In some infinite dimensional spaces there are pointed generating cones which are not normal.

Example 3.3 ([2, Example 2.41]). Let $Y = C^1[0, 1]$ be the real vector space of all continuously differentiable functions on $[0, 1]$ and let cone K be defined as

$$K := \{x \in C^1[0, 1] : x(t) \geq 0 \text{ for all } t \in [0, 1]\}.$$

Let us consider the norm

$$\|x\| = \|x\|_\infty + \|x'\|_\infty,$$

where x' denotes the derivative of $x \in Y$. Cone K is closed and generating but it is not normal. Let $x_n := t^n$ and $y_n := 1$, we have $0 \leq_K x_n \leq_K 1$. There is no constant $c > 0$ such that the inequality $\|x_n\| = \|t^n\|_\infty + \|nt^{n-1}\|_\infty = n + 1 \leq c = c\|y\|$ holds for all n.

Some interesting results (see e.g. [11], [19], [15]) for closed convex cones are using the concept of a basis of a space.

Definition 3.4 ([1, Definition 1.1.1]). A sequence $\{x_n\} \subset Y$ in an infinite-dimensional Banach space Y is said to be a *basis* of Y if for each $x \in Y$ there is a unique sequence of scalars $\{a_n\}_{n\in\mathbb{N}}$ such that

$$x = \sum_{n=1}^{\infty} a_n x_n.$$

For basis $\{x_n\}$ we can define the cone associated to the basis $\{x_n\}$.

Definition 3.5 [19, Definition 10.2]). Let $\{x_n\}$ be a basis of a Banach space Y. The set

$$K_{\{x_n\}} := \left\{ y \in Y : y = \sum_{i=1}^{\infty} \alpha_i x_i \in Y \text{ where } \alpha_i \geq 0, \text{ for } i = 1, 2, \dots \right\}$$

is called the cone associated to the basis $\{x_n\}$.

Cone $K_{\{x_n\}}$ is a closed and convex and coincides with the cone generated by $\{x_n\}$, i.e. it is the smallest cone containing $\{x_n\}$.

For sequence $\{x_n\}$ of a Banach space Y functionals $\{x_n^*\}$ are called *biorthogonal functionals* if $x_k^*(x_j) = 1$ for $k = j$, and $x_k^*(x_j) = 0$ otherwise, for any $k, j \in \mathbb{N}$ and $x = \sum\limits_{i=1}^{\infty} x_i^*(x)x_i$ for each $x \in X$. The sequence $\{x_n, x_n^*\}$ is called the *biorthogonal system* ([1, Definition 1.1.2]).

It is easy to see that $\{e_i\}$, where $e_i = (\underbrace{0, \dots, 1, 0, \dots}_{i})$, $i = 1, 2, \dots$ is a basis for c_0 and $\{e_i^*\}$, $e_i^* := e_i$ are biorthogonal functionals.

We also have $c_0^+ = K_{\{e_i\}}$ and $(c_0^+)^* = l_1^+ = K_{\{e_i^*\}}$.

Definition 3.6 ([1, Definition 3.1.1]). A basis $\{x_n\}$ of a Banach space Y is called *unconditional* if for each $x \in Y$ the series $\sum\limits_{n=1}^{\infty} x_n^*(x)x_n$ converges unconditionally.

A basis $\{x_n\}$ is *conditional* if it is not unconditional. A sequence $\{x_n\} \subset X$ is *complete* (see [20]) if $\overline{\operatorname{span}\{x_n\}} = X$.

Theorem 3.7 ([1, Proposition 3.1.3]). *Let $\{x_n\}$ be a complete sequence in a Banach space Y such that $x_n \neq 0$ for every n. Then the following statements are equivalent.*

(a) *$\{x_n\}$ is an unconditional basis for Y.*

(b) *For all $N \geq 1$, for all $c_1, \dots, c_N$ and for all $\varepsilon_1, \dots, \varepsilon_N = \pm 1$, there exists $C_1 \geq 1$ such that*

$$\Big\| \sum_{n=1}^{N} \varepsilon_n c_n x_n \Big\| \leq C_1 \Big\| \sum_{n=1}^{N} c_n x_n \Big\|. \tag{3.1}$$

Example 3.8. The sequence $\{b_i\} \subset c_0$ defined as

$$b_i = \frac{1}{i} e_i, \quad i = 1, 2, \dots, \tag{3.2}$$

($\{e_i\}$ is the canonical basis for c_0) is an unconditional basis.

Since $\|x\| = \sup_i |x^i|$ for $x = (x^1, x^2, \dots) \in c_0$, inequality (3.1) is satisfied with $C_1 = 1$

$$\begin{aligned} &\Big\| \varepsilon_1 c_1 (1, 0, \dots) + \dots + \varepsilon_N c_N \Big(0, 0, \dots, \frac{1}{N}, 0, \dots\Big) \Big\| \\ &\qquad = \Big\| \Big(\varepsilon_1 c_1, \dots, \varepsilon_N c_N \frac{1}{N}, 0, \dots\Big) \Big\| = \Big\| \Big(c_1, \dots, c_N \frac{1}{N}, 0, \dots\Big) \Big\|. \end{aligned}$$

In [11, 19] we can find a characterization of normal cones in terms of unconditional basis.

Theorem 3.9 ([19, Theorem 16.3]). *Let $\{x_n, f_n\}$ be a complete biorthogonal system in a Banach space Y. The following are equivalent.*

(a) *$\{x_n\}$ is an unconditional basis for Y.*

(b) *$K_{\{x_n\}}$ is normal and generating.*

Example 3.10. Let us consider conditional basis $\{x_n\}$ for c_0 given by Gelbaum [7], i.e. $x_n := (1, 1, \dots, 1, 0, \dots) = \sum_{i=1}^{n} e_i$, ($\{e_i\}$ is the canonical basis for c_0), $n = 1, 2, \dots$ All calculations can be found in [19, Example 14.1, p. 424]. From Theorem 3.9 cone $K_{\{x_n\}} \subset c_0$, where

$$x_n = (1, 1 \dots, 1, \underbrace{1}_{n}, 0, 0, 0 \dots), \quad n = 1, 2 \dots$$

is generating pointed and not normal in c_0.

Now let us present some facts about cone isomorphisms.

Definition 3.11 ([5], [15]). Let X and Y be normed spaces ordered by cones $P \subset X$ and $K \subset Y$, respectively. We say that P is *conically isomorphic* to K if there exists an additive, positively homogeneous, one-to-one map i of P onto K

such that i and i^{-1} are continuous in the induced topologies. Then we also say that i is a conical isomorphism of P onto K.

Proposition 3.12. *Let X be a linear space and let Y, Z be Banach spaces. Let K and P be convex cones in Z and Y, respectively. Let function $F\colon X \to K$ be K-convex. If there exists a conical isomorphism $i\colon K \to P$, where cone K is generating in Z, then the function $\overline{F}\colon X \to P$, where*

$$\overline{F} := i \circ F$$

is a P-convex function.

Proof. In view of the fact that K is a generating cone in Z, the conical isomorphism $i\colon K \to P$ can be extended to the function $G\colon Z \to P - P$ defined as

$$G(x) = i(x^1) - i(x^2), \quad \text{where } x = x^1 - x^2,\ x^1,\ x^2 \in K.$$

Let us assume that for $x \in Z$ we have $x = x^1 - x^2$ and $x = y^1 - y^2$, where $x^1, x^2, y^1, y^2 \in K$. We have $x^1 + y^2 = y^1 + x^2$. Since i is additive we get $i(x^1) + i(y^2) = i(y^1) + i(x^2)$, i.e. G is uniquely defined.

Function $G\colon Z \to P - P$ is linear. Indeed, let us take $x, y \in Z$, since K is generating $x = x^1 - x^2$, $y = y^1 - y^2$, where $x^1, x^2, y^1, y^2 \in K$.

$$\begin{aligned} G(x+y) &= G(x^1 - x^2 + y^1 - y^2) = i(x^1 + y^1) - i(x^2 + y^2) \\ &= i(x^1) - i(x^2) + i(y^1) - i(y^2) = G(x) + G(y). \end{aligned}$$

Let us take $\lambda < 0$. We have

$$G(\lambda x) = G(\lambda x^1 - \lambda x^2) = i(-\lambda x^2) - (-\lambda x^1) = -\lambda i(x^2) + \lambda i(x^1) = \lambda G(x).$$

For $\lambda \geq 0$ the calculations are analogous.

Now let us take $x_1, x_2 \in X$ and $\lambda \in [0,1]$. By Definition 2.2 and the linearity of G,

$$F(\lambda x_1 + (1-\lambda)x_2) \leq_K \lambda F(x_1) + (1-\lambda)F(x_2),$$

i.e.

$$\lambda F(x_1) + (1-\lambda)F(x_2) - F(\lambda x_1 + (1-\lambda)x_2) \in K.$$

By the definition of G

$$\begin{aligned} &G(\lambda F(x_1) + (1-\lambda)F(x_2) - F(\lambda x_1 + (1-\lambda)x_2)) \\ &\qquad\qquad i(\lambda F(x_1) + (1-\lambda)F(x_2) - F(\lambda x_1 + (1-\lambda)x_2)) \in P. \end{aligned}$$

Furthermore,

$$\begin{aligned} G(\lambda F(x_1) &+ (1-\lambda)F(x_2) - F(\lambda x_1 + (1-\lambda)x_2)) \\ &= \lambda G(F(x_1)) + (1-\lambda)G(F(x_2)) - G(F(\lambda x_1 + (1-\lambda)x_2)) \\ &= \lambda i(F(x_1)) + (1-\lambda)i(F(x_2)) - i(F(\lambda x_1 + (1-\lambda)x_2)) \geq_P 0. \end{aligned}$$

The latter inequality is equivalent to

$$\overline{F}(\lambda x_1 + (1-\lambda)x_2) \leq_P \lambda \overline{F}(x_1) + (1-\lambda)\overline{F}(x_2)$$

which completes the proof. □

Proposition 3.13 ([17, Proposition 1]). *Suppose that X, Y are normed spaces ordered by the cones P, Q, respectively. If the cones P and Q are isomorphic, we have*: *P is normal if and only if Q is normal.*

From Proposition 3.13 and Example 3.8 we get the following corollary.

Corollary 3.14. *Every Banach space isomorphic to c_0 contains a cone which is closed convex and generating but not normal.*

Let J be a James space, i.e. $J := \left\{ x = (x_n)_{n\in\mathbb{N}} : \lim_{n\to 0} x_n = 0 \right\}$ with the norm

$$\|x\| = \sup \left\{ \left(\sum_{i=1}^{n} (x_{m_{2i-1}} - x_{m_{2i}})^2 \right)^{1/2} : 0 = m_0 < m_1 < \ldots < m_{n+1} \right\} < \infty.$$

An interesting result is the fact that James space does not contain an isomorphic copy of c_0 or l_1.

In [4] Pełczyński and Bessaga proved the following theorem.

Theorem 3.15 ([11, Theorem 6.4], [4]). *A separable Banach space having the space J of James as a subspace* (*e.g.* $C[0,1]$) *does not have an unconditional basis.*

It is easy to find not normal cones in infinite dimensional Banach spaces (see e.g. [19]).

4. Useful constructions of convex functions

In this section we recall constructions of some convex functions which will be used in the sequel, c.f. [3]. Let us start with the following lemma.

Lemma 4.1 ([3, Lemma 4.2]). *Let $\{a_m\}, \{t_m\} \subset \mathbb{R}$ be sequences with $\{t_m\}$ decreasing* ($t_m > t_{m+1}$, $m \in \mathbb{N}$). *The function $g\colon \mathbb{R} \to \mathbb{R}$ defined as*

$$g(r) := \sup_m f_m(r),$$

where

$$f_m(x) := a_m + \frac{x - t_m}{t_{m+1} - t_m}(a_{m+1} - a_m) \quad \textit{for } x \in \mathbb{R},$$

is convex on $\mathbb{R}$ with $g(t_m) = a_m$ if and only if

$$\frac{a_{m+1} - a_m}{t_{m+1} - t_m} \geq \frac{a_{m+2} - a_{m+1}}{t_{m+2} - t_{m+1}} \quad \textit{for } m \in \mathbb{N}. \tag{4.1}$$

Proof. ($\Leftarrow$) Proved in [3].

($\Rightarrow$) If, for some $m \in \mathbb{N}$ the inequality converse to (4.1) would hold, then

$$\frac{a_{m+1} - a_m}{t_{m+1} - t_m} < \frac{a_{m+2} - a_{m+1}}{t_{m+2} - t_{m+1}}.$$

We get

$$(a_{m+1} - a_m)\left(1 - \frac{x - t_m}{t_{m+1} - t_m}\right) + \frac{x - t_{m+1}}{t_{m+2} - t_{m+1}}\,(a_{m+2} - a_{m+1}) > 0$$

and it would be $f_{m+1} > f_m$ for $x \geq t_m$. By taking $x = t_m \geq t_{m+1}$ we get

$$f_{m+1}(t_m) > f_m(t_m) = a_m,$$

contradictory. □

Let Y be a Banach space and $\{y_i\}$ be an arbitrary sequence of elements of Y. Let $\{t_i\}$ be a sequence of positive reals tending to zero.

Let $\overline{F}\colon X_h := \{x \in X : x = \beta h,\ \beta \geq 0\} \to Y$ be a function defined as in [3], i.e. for $r > 0$

$$\overline{F}(rh) := \sum_{i=1}^{\infty} \overline{F}_i(rh), \tag{4.2}$$

where $\overline{F}_i\colon \{x \in X : x = \beta h,\ \beta \geq 0\} \to Y$ is defined as

$$\overline{F}_1(rh) := \begin{cases} y_1 t_1 + \dfrac{r - t_1}{t_2 - t_1}\,(y_2 t_2 - y_1 t_1) & \text{if } t_2 < r, \\ 0 & \text{if } r \leq t_2, \end{cases}$$

and, for $i \geq 2$,

$$\overline{F}_i(rh) := \begin{cases} y_i t_i + \dfrac{r - t_i}{t_{i+1} - t_i}\,(y_{i+1} t_{i+1} - y_i t_i) & \text{if } t_{i+1} < r \leq t_i, \\ 0 & \text{if } r \notin (t_{i+1}, t_i]. \end{cases}$$

Observe that for $r = t_k$ we have

$$\overline{F}(t_k h) = \overline{F}_k(t_k h) = y_k t_k.$$

The following proposition is a simple consequence of Lemma 4.1.

Proposition 4.2. *Let $K \subset Y$ be a closed convex cone with the dual $K^* \subset Y^*$. Let $\{y_i\} \subset Y$ be a sequence in Y. The function $\overline{F}$ defined by* (4.2) *with $t_k = 1/k$, $k = 1, 2, \dots$, is K-convex on X_h if and only if*

$$y^*(2y_{k+1} - y_k - y_{k+2}) \leq 0 \quad \text{for all } y^* \in K^*,\ k = 1, 2, \dots \tag{4.3}$$

Proof. Let us observe that for $x \in X_h$, i.e. $x = rh, r > 0$ and any $y^* \in K^*$ we have $y^*(\overline{F}(rh)) = g(r)$, where $g(r)$ is defined as in Lemma 4.1 for

$$a_k := y^*\left(\overline{F}\left(\frac{1}{k}h\right)\right) = \frac{1}{k}\,y^*(y_k), \quad k = 1, 2, \dots$$

Indeed, let $r \in (0,1]$, then there is $k \in \mathbb{N}$ such that $r \in (1/(k+1), 1/k]$ and

$$\begin{aligned} y^*(\overline{F}(rh)) &= y^*(\overline{F}_k(rh)) \\ &= y^*\left(y_k \frac{1}{k} + \frac{r - \frac{1}{k}}{\frac{1}{k+1} - \frac{1}{k}} \left(y_{i+1} \frac{1}{k+1} - y_i \frac{1}{k}\right)\right) = f_k(r) = g(r). \end{aligned}$$

For $r > 1$, $y^*(\overline{F}(rh)) = f_1(r) = g(r)$. By Lemma 4.1, the function g is convex if and only if inequality (4.1) holds, i.e.

$$\begin{aligned} \frac{\frac{1}{k+2}\, y^*(y_{k+2}) - \frac{1}{k+1}\, y^*(y_{k+1})}{\frac{1}{k+2} - \frac{1}{k+1}} &\le \frac{\frac{1}{k+1}\, y^*(y_{k+1}) - \frac{1}{k}\, y^*(y_k)}{\frac{1}{k+1} - \frac{1}{k}} && \equiv \\ k y^*(y_{k+1}) - (k+1) y^*(y_k) &\le (k+1) y^*(y_{k+2}) - (k+2) y^*(y_{k+1}) && \equiv \\ (k+1) y^*(2y_{k+1} - y_k - y_{k+2}) &\le 0, \quad k = 1,2,\dots && \equiv \\ y^*(2y_{k+1} - y_k - y_{k+2}) &\le 0, \quad k = 1,2,\dots \end{aligned}$$

By Lemma 2.3, the function $\overline{F}$ is K-convex on X_h. □

Proposition 4.3. *Let $\{y_i\} \subset Y$ be a sequence in a Banach space Y. If $\{b_k\} \subset Y$ defined as*

$$2y_{k+1} - y_k - y_{k+2} =: b_k, \quad k = 1,2,\dots, \tag{4.4}$$

forms an unconditional basis in Y, the function $\overline{F}\colon X \to Y$ defined by (4.2) *with $t_k = 1/k$, $k = 1,2,\dots$, is $(-K_{\{b_k\}})$-convex, where*

$$K_{\{b_k\}} := \left\{ y \in Y : y = \sum_{i=1}^{\infty} a_i b_i,\ a_i \ge 0,\ i = 1,2,\dots \right\}$$

is a closed generating and normal cone in Y.

Proof. In view of Theorem 3.9, the cone

$$K_{\{b_k\}} = \left\{ y \in Y : y = \sum_{i=1}^{\infty} \alpha_i b_i \in Y : \alpha_i \ge 0,\ i = 1,2,\dots \right\}$$

is normal and generating in Y. Let us observe that dual cone is defined as

$$K^* := K_{\{b_i^*\}},$$

where $\{b_i, b_i^*\}$ is the biorthogonal system. Inequality (4.3) is satisfied because

$$y^*(b_k) = -\alpha_k \le 0 \quad \text{for some } \alpha_k \ge 0.$$

By Proposition 4.2, the function $\overline{F}$ is $(-K_{\{b_k\}})$-convex. □

Corollary 4.4. *Let $Y = c_0$ and let $\{b_i\} \subset c_0$ be defined as in Example* 3.8, *i.e. $b_i = e_i/i$, $i = 1,2,\dots$ Then there exists a sequence $\{y_k\} \subset c_0$ satisfying* (4.4) *which is not weakly convergent.*

Proof. Let us prove by induction that (4.4) is equivalent to

$$y_k = (k-1)y_2 - (k-2)y_1 - \sum_{i=1}^{k-2}(k-i-1)b_i. \tag{4.5}$$

Let us assume that equality (4.5) holds for all $n \leq k$. We have

$$\begin{aligned} y_{k+1} &= 2y_k - y_{k-1} - b_{k-1} \\ &= 2\Big[(k-1)y_2 - (k-2)y_1 - \sum_{i=1}^{k-2}(k-i-1)b_i\Big] - y_{k-1} - b_{k-1} \\ &= 2\Big[(k-1)y_2 - (k-2)y_1 - \sum_{i=1}^{k-2}(k-i-1)b_i\Big] \\ &\quad - \underbrace{\Big[(k-2)y_2 - (k-3)y_1 - \sum_{i=1}^{k-3}(k-i-2)b_i\Big]}_{y_{k-1}} - b_{k-1} \\ &= ky_2 - (k-1)y_1 - \sum_{i=1}^{k-1}(k-i)b_i \end{aligned}$$

which proves (4.5) for $k+1$.

Let us take $y_1 = (0, 0, \ldots)$ and $y_2 = (1, 1/2, 1/3, \ldots)$. By (4.5), we get

$$\begin{aligned} y_3 &= \Big(1, 1, \frac{2}{3}, \frac{2}{4}, \frac{2}{5}, \frac{2}{6}, \ldots\Big), \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ y_k &= \Big(\underbrace{1, 1, \ldots, 1}_{k-1}, \frac{k-1}{k}, \frac{k-1}{k+1}, \ldots\Big). \end{aligned}$$

The sequence $\{y_k\}$ forms a weakly Cauchy sequence in c_0 without weak limit. □

5. Main result

Now we are ready to formulate our main result.

Theorem 5.1. *Let X be a linear space and Y be a Banach space. If for every closed convex and normal cone $K \subset Y$ every K-convex function $F\colon X \to Y$ is directionally differentiable at $x_0 = 0$, then in Y there is no subspace isomorphic to c_0.*

Proof. The proof is based on Propositions 4.2, 4.4 and Corollary 4.4. We proceed by contradiction. We assume that there exists a subspace $Z \subset Y$ isomorphic to c_0, i.e. there is a continuous one-to-one linear mapping $i\colon c_0 \to Z$. Basing ourselves on this assumption we construct a cone-convex function defined on a linear space X which is not directionally differentiable at $x_0 = 0$.

First let us define a cone convex function with values in the space c_0. Let us take $h \in X$, $h \neq 0$ and $t_k := 1/k$, $k = 1, 2, \ldots$ Let $\{y_k\} \subset c_0$ be defined as in Corollary 4.4, i.e. $y_1 = (0, 0, \ldots)$, $y_2 = (1, 1/2, 1/3, \ldots)$ and $y_k = (\underbrace{1, 1, \ldots, 1}_{k-1}, (k-1)/k, (k-1)/(k+1), \ldots)$.

Let $\overline{F}\colon \{x \in X : x = \beta h,\ \beta \geq 0\} \to c_0$ be defined as follows. For $r > 0$

$$\overline{F}(rh) := \sum_{i=1}^{\infty} \overline{F}_i(rh),$$

where $\overline{F}_i\colon \{x \in X : x = \beta h,\ \beta \geq 0\} \to c_0$ is defined as

$$\overline{F}_1(rh) := \begin{cases} t_1 y_1 + \dfrac{r - t_1}{t_2 - t_1}\,(t_2 y_2 - t_1 y_1) & \text{if } r \geq t_2, \\ 0 & \text{if } r < t_2, \end{cases}$$

and, for $i \geq 2$,

$$\overline{F}_i(rh) := \begin{cases} t_i y_i + \dfrac{r - t_i}{t_{i+1} - t_i}(t_{i+1} y_{i+1} - t_i y_i) & \text{if } t_{i+1} < r \leq t_i, \\ 0 & \text{if } r \notin (t_{i+1}, t_i]. \end{cases}$$

We can see that $\overline{F}\colon \to c_0$ satisfies

$$\overline{F}\left(\frac{1}{k}\, h\right) = \frac{1}{k}\, y_k, \quad k = 1, 2, \ldots$$

Let $\{b_i\} \subset Y$ be defined as in Example 3.8, i.e.

$$b_i = \frac{1}{i}\, e_i = \Bigg(0, \ldots, 0, \underbrace{\frac{1}{i}}_{i}, 0, \ldots\Bigg), \quad i = 1, 2, \ldots$$

Sequence $\{b_i\}$ forms unconditional basis for c_0 and satisfies (4.4), i.e.

$$2y_{k+1} - y_k - y_{k+2} =: b_k, \quad k = 1, 2, \ldots$$

From Proposition 4.3, cone K defined as $K := -K_{\{b_k\}}$ is normal and generating in c_0. Furthermore, function $\overline{F}$ is K-convex. By assumption, there exists an isomorphism i between c_0 and the subspace $Z \subset Y$.

Let us define $F\colon X \to Y$ by the formula $F := i \circ \overline{F}$. Applying Proposition 3.12 for $P = i(K)$ we get that function F is $i(K)$-convex. The directional derivative for the function F at $x_0 = 0$ is equal to

$$\lim_{k\to\infty} \frac{F(t_k h)}{t_k} = \lim_{k\to\infty} \frac{i(\overline{F}(t_k h))}{t_k} = \lim_{k\to\infty} i\left(\frac{\overline{F}(t_k h)}{t_k}\right) = \lim_{k\to\infty} i(y_k).$$

By the fact that i is an isomorphism and by Proposition 3.13, cone $i(K)$ is closed normal and generating in Z. Since $\{y_k\}$ is not weakly convergent (Corollary 4.4), the function F is $i(K)$-convex and is not directionally differentiable at $x_0 = 0$. $\square$

If Y is weakly sequentially complete Banach space, then every K-convex function, where $K \subset Y$ is closed convex and normal is directionally differentiable at $x_0 \in X$, $h \in X \setminus \{0\}$ (see [21]). From Theorem 1.1 we get a characterization of weakly sequentially complete Banach lattices in terms of existence of directional derivative for K-convex functions.

Theorem 5.2. *Banach lattice Y is weakly sequentially complete if and only if, for every closed convex normal cone $K \subset Y$, every K-convex function $f : X \to Y$ is directionally differentiable for all $x_0 \in X$.*

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Manuscript received December 7, 2017

accepted November 22, 2018

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