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STRONG CONVERGENCE OF BI-SPATIAL RANDOM ATTRACTORS FOR PARABOLIC EQUATIONS ON THIN DOMAINS WITH ROUGH NOISE

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Abstract. This article concerns bi-spatial random dynamics for the stochastic reaction-diffusion equation on a thin domain, where the noise is described by a general stochastic process instead of the usual Wiener process. A bi-spatial attractor is obtained when the non-initial state space is the p-times Lebesgue space, meanwhile, measurability of the attractor in the Banach space is proved by using measurability of both cocycle and absorbing set. Finally, the p-norm convergence of attractors is obtained when the thin domain collapses onto a lower dimensional domain. The method of symbolical truncation is applied to provide some uniformly asymptotic estimates.

1. Introduction

The subject of a thin domain problem is to consider both existence and convergence of an attractor when the equation is defined on a thin domain, which collapses onto a lower dimensional domain. Some pioneered works were given by Hale, Raugel and Sell (see [16], [31]), with notable developments for a large number of (deterministic) dissipative equations (see [1], [3], [4], [14], [19], [30], and the references therein).

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Such a thin domain problem was generalized to the stochastic PDE (see [6], [9], [10]). In particular, D. Li et al. [20], [21] had investigated the following stochastic reaction-diffusion equation with Neumann boundary conditions

$$(1.1)\quad \begin{cases} d\widetilde{u}^\varepsilon - \Delta\widetilde{u}^\varepsilon\,dt + \lambda\widetilde{u}^\varepsilon\,dt = (F(t,x,\widetilde{u}^\varepsilon) + G(t,x))\,dt + h(x)\,dW, & t \geq \tau, \\ \dfrac{\partial\widetilde{u}^\varepsilon}{\partial\nu_\varepsilon} = 0, \quad \text{on } \partial\mathcal{O}_\varepsilon, \qquad \widetilde{u}^\varepsilon(\tau,x) = \widetilde{u}^\varepsilon_\tau(x), & x \in \mathcal{O}_\varepsilon,\ \tau \in \mathbb{R}, \end{cases}$$

where $\lambda > 0$, ν_ε is the unit outward normal vector on $\partial\mathcal{O}_\varepsilon$ for $\varepsilon \in (0,1]$. The $n+1$-dimensional thin domain $\mathcal{O}_\varepsilon$ is given by

$$\mathcal{O}_\varepsilon = \{x = (x^*, x_{n+1}) : x^* = (x_1,\ldots,x_n) \in Q,\ 0 < x_{n+1} < \varepsilon g(x^*)\},$$

where Q is a bounded smooth domain in $\mathbb{R}^n$ and $g \in C^2(\overline{Q},(0,+\infty))$.

In this article, we use a general stochastic process W to replace the Wiener process used in [20], [21]. Let

$$\Omega = \bigg\{\omega \in C(\mathbb{R},\mathbb{R}) : \omega(0) = 0,\ \lim_{t\to\pm\infty}\frac{\omega(t)}{t} = 0\bigg\},$$

and take the Frechét metric

$$(1.2)\qquad \varrho(\omega,\omega^*) = \sum_{k=1}^{\infty}\frac{1}{2^k}\frac{\rho_k(\omega,\omega^*)}{1+\rho_k(\omega,\omega^*)},$$

where ϱ_k is the metric in $C([-k,k],\mathbb{R})$. Then, $(\Omega,\mathcal{F})$ is a measurable space, where $\mathcal{F} = \mathfrak{B}(\Omega)$ is the Borel algebra on (Ω,ϱ). We denote a group $\{\theta_t : t \in \mathbb{R}\}$ of self-mappings on Ω by $\theta_t\omega(\,\cdot\,) = \omega(t+\,\cdot\,) - \omega(t)$ for $(\omega,t) \in \Omega\times\mathbb{R}$.

Now, we take a general probability measure P on $(\Omega,\mathcal{F})$ such that $W(t,\omega) := \omega(t)$ $(t \in \mathbb{R})$ is a stochastic process on the probability space $(\Omega,\mathcal{F},P)$, meanwhile, it ensures that θ_t is measure preserving and ergodic with respect to P.

We remark here that one can obtain different stochastic processes from different probability measures. In particular, by [8], one can obtain the usual Wiener process by taking P a Wiener measure, which is widely used in the literature (see [5], [7], [12], [33] and the references therein). In fact, the above class of processes contains any continuous stochastic process with $\lim_{t\to\pm\infty} W(t)/t = 0$, such as the Wong–Zakai-type noise used in the more recent paper [35].

The subject of this article is to consider strong attraction and strong convergence of the L^2-attractor. More precisely, we will prove the existence of a bi-spatial random attractor $\mathcal{A}_\varepsilon$ for equation (1.1) in (L^2,L^p), where $p > 2$. Also, we consider the p-norm convergence from $\mathcal{A}_\varepsilon$ to the attractor $\mathcal{A}_0$ of the following limiting equation:

$$(1.3)\quad \begin{cases} du^0 - \dfrac{1}{g}\displaystyle\sum_{i=1}^{n}(gu^0_{y_i})_{y_i}\,dt + \lambda u^0\,dt = (F_0(t,y^*,u^0) + G_0(t))\,dt + h_0\,dW, \\ \dfrac{\partial u^0}{\partial\nu_0} = 0 \quad \text{on } \partial Q, \qquad u^0(\tau,y^*) = u^0_\tau(y^*), \quad y^* \in Q,\ t \geq \tau,\ \tau \in \mathbb{R}, \end{cases}$$

where $F_0(t,y^*,u^0)=F(t,(y^*,0),u^0)$, $G_0(t,y^*)=G(t,(y^*,0))$, $h_0(y^*)=h(y^*,0)$ and ν_0 is the unit outward normal vector on ∂Q.

In Section 2 some abstract existence results given in Li et al. [24] can be applied to the thin-domain problem if we make a transformation from the varying thin domain to a fixed domain. Under such a fixed domain, we can show that the random dynamical system has an (L^2,L^p)-attractor, see Theorem 4.6.

However, the abstract result on upper semi-continuity of the attractor cannot simply be applied to the thin domain problem. In fact, in Section 5, we consider the convergence from a $n+1$-dimensional function to the lower dimensional average function. This convergence together with some priori estimates in L^p can help us to prove directly the upper semi-continuity from $\mathcal{A}_\varepsilon$ to $\mathcal{A}_0$ under the p-norm, see Theorem 5.2.

It is worth pointing out that random invariant manifolds and random attractors in such a Banach space had been considered by [23], [27], [28], [34], [39], [40], where the non-thin domain problem had been investigated.

Another issue is measurability of the pullback attractor in L^p, which is a main subject different from deterministic pullback attractors (see [22], [29], [36]). However, the random attractor is still the omega-limit set of the absorbing set under the solution operator (cocycle). So, in Section 3, we show that the solution operator is $\mathcal{F}$-measurable in both state spaces L^2 and L^p, which leads to the measurability of the attractor.

2. Transformation of the thin domain and well-posedness

2.1. Assumptions. Let $\widetilde{\mathcal{O}}=Q\times(0,\gamma_2)$ and $\widehat{\mathcal{O}}=Q\times[0,\gamma_2)$, where $\gamma_2\geq\gamma_1>0$ such that $\gamma_1\leq g(x^*)\leq\gamma_2$ for all $x^*\in\overline{Q}$. Note that $u\in L^\infty(\widehat{\mathcal{O}})$ if and only if $u\in L^\infty(\widetilde{\mathcal{O}})$ with the same norms.

Assumption 2.1. The nonlinearity $f\colon\mathbb{R}\times\widehat{\mathcal{O}}\times\mathbb{R}\to\mathbb{R}$ is continuous and satisfies the following conditions: for all $x\in\widehat{\mathcal{O}}$ and $t,s\in\mathbb{R}$,

$$f(t,x,s)s\leq-\alpha_1|s|^p+\psi_1(t,x),\tag{2.1}$$

$$|f(t,x,s)|\leq\alpha_2|s|^{p-1}+\psi_2(t,x),\tag{2.2}$$

$$\frac{\partial f(t,x,s)}{\partial s}\leq\beta,\qquad\left|\frac{\partial f(t,x,s)}{\partial s}\right|\leq\alpha_3|s|^{p-2}+\psi_3(t,x),\tag{2.3}$$

$$\left|\frac{\partial f(t,x,s)}{\partial x}\right|\leq\psi_4(t,x),\tag{2.4}$$

where $p>2$, $\alpha_i,\beta>0$, $\psi_1\in L^1_{\mathrm{loc}}\cap L^2_{\mathrm{loc}}(\mathbb{R},L^\infty(\widetilde{\mathcal{O}}))$, $\psi_2,\psi_3,\psi_4\in L^2_{\mathrm{loc}}(\mathbb{R},L^\infty(\widetilde{\mathcal{O}}))$.

Assumption 2.2. $G\in L^2_{\mathrm{loc}}(\mathbb{R},L^\infty(\widetilde{\mathcal{O}}))$ and $h\in C^2(\overline{Q}\times[0,\gamma_2])$.

Assumption 2.3. Tempered conditions: for any $\tau \in \mathbb{R}$ and $\sigma > 0$,

$$\int_{-\infty}^{\tau} e^{1/4\lambda s}\big(\|G(s)\|_\infty^2 + \|\psi_1(s)\|_\infty + \|\psi_2(s)\|_\infty^2 + \|\psi_4(s)\|_\infty^2\big)\, ds < \infty, \tag{2.5}$$

$$e^{\sigma r}\int_{-\infty}^{0} e^{1/4\lambda s}\big(\|G(s+r)\|_\infty^2 + \|\psi_1(s+r)\|_\infty + \|\psi_4(s+r)\|_\infty^2\big)\, ds \to 0, \tag{2.6}$$

as $r \to -\infty$, where we use $\|\cdot\|_\infty$ to denote the norm in $L^\infty(\widetilde{\mathcal{O}})$.

Assumption 2.4. By the same method as defining F_0, G_0 and h_0 in the limiting equation (1.3), we define the restrictions $\psi_{j,0}$ $(j = 1, \ldots, 4)$. Then, we assume $\psi_{1,0} \in L^1_{\mathrm{loc}} \cap L^2_{\mathrm{loc}}(\mathbb{R}, L^\infty(Q))$ and $\psi_{2,0}, \psi_{3,0}, \psi_{4,0} \in L^2_{\mathrm{loc}}(\mathbb{R}, L^\infty(Q))$.

2.2. Transformation of the thin domain. We consider a transformation T_ε from $\mathcal{O}_\varepsilon$ onto $\mathcal{O} = Q \times (0,1)$, defined by

$$(y^*, y_{n+1}) = T_\varepsilon(x^*, x_{n+1}) = \left(x^*, \frac{x_{n+1}}{\varepsilon g(x^*)}\right) \quad \text{for all } x = (x^*, x_{n+1}) \in \mathcal{O}_\varepsilon.$$

Then, the bijective mapping T_ε has the Jacobian matrix:

$$J = \frac{\partial(y_1, \ldots, y_{n+1})}{\partial(x_1, \ldots, x_{n+1})} = \begin{pmatrix} I & 0 \\ -\dfrac{y_{n+1}}{g}(g_{y_1}, \ldots, g_{y_n}) & \dfrac{1}{\varepsilon g(y^*)} \end{pmatrix}$$

with the positive determinant $|J| = 1/\varepsilon g(y^*)$. By [17], [21], we have $\nabla_x \widetilde{u}(x) = J^* \nabla_y u(y)$ and

$$\Delta_x \widetilde{u}(x) = |J| \mathrm{div}_y\big(|J|^{-1} J J^* \nabla_y u(y)\big) = \frac{1}{g}\mathrm{div}_y(\Upsilon_\varepsilon u(y)),$$

where $u(y) = \widetilde{u}(x)$ $(y = T_\varepsilon x \in \mathcal{O})$, J^* is the transport of J and Υ_ε is the operator given by

$$\Upsilon_\varepsilon u(y) = \begin{pmatrix} g u_{y_1} - g_{y_1} y_{n+1} u_{y_{n+1}} \\ \vdots \\ g u_{y_n} - g_{y_n} y_{n+1} u_{y_{n+1}} \\ -\displaystyle\sum_{i=1}^{n} y_{n+1} g_{y_i} u_{y_i} + \frac{1}{\varepsilon^2 g}\left(1 + \sum_{i=1}^{n} (\varepsilon y_{n+1} g_{y_i})^2\right) u_{y_{n+1}} \end{pmatrix}. \tag{2.7}$$

We can rewrite the problem (1.1) as an equation defined on $\mathcal{O}$:

$$\begin{cases} du^\varepsilon - \dfrac{1}{g}\,\mathrm{div}_y(\Upsilon_\varepsilon u^\varepsilon)\, dt + \lambda u^\varepsilon\, dt \\ \qquad\qquad = (F_\varepsilon(t, y, u^\varepsilon) + G_\varepsilon(t, y))\, dt + h_\varepsilon(y)\, dW, \\ \Upsilon_\varepsilon u^\varepsilon \cdot \nu = 0, \quad \text{on } \partial\mathcal{O}, \qquad u^\varepsilon(\tau, y) = \widetilde{u}^\varepsilon_\tau(T_\varepsilon^{-1}(y)), \quad y \in \mathcal{O},\ \tau \in \mathbb{R}. \end{cases} \tag{2.8}$$

where ν is the unit outward normal vector on $\partial\mathcal{O}$, and

$$\begin{aligned} F_\varepsilon(t,y^*,y_{n+1},u) &= F(t,y^*,\varepsilon g(y^*)y_{n+1},u),\\ G_\varepsilon(t,y^*,y_{n+1}) &= G(t,y^*,\varepsilon g(y^*)y_{n+1}),\\ h_\varepsilon(y^*,y_{n+1}) &= h(y^*,\varepsilon g(y^*)y_{n+1}). \end{aligned}$$

We take the equivalent norms on $X = L^2(\mathcal{O})$ and $Y = L^p(\mathcal{O})$ by

$$\|u\|_g^2 = \int_{\mathcal{O}} gu^2\,dy, \quad u\in X \quad \text{and} \quad \|v\|_p^p = \int_{\mathcal{O}} g|v|^p\,dy, \quad v\in Y.$$

Also, we consider a family of new norms and bilinear forms on $Z = H^1(\mathcal{O})$:

$$\|u\|^2_{H^1_\varepsilon} = a_\varepsilon(u,u) + \|u\|_g^2 \quad \text{and} \quad a_\varepsilon(u,v) = (J^*\nabla_y u, J^*\nabla_y v)_g,$$

for $u,v\in Z$. It is necessary to make clear the uniformness of the norm equivalences in small ε, which slightly generalizes the results in [16], [17].

Lemma 2.5. *There exist $\varepsilon_0\in(0,1)$ and $\eta_1,\eta_2>0$ such that, for all $\varepsilon\in(0,\varepsilon_0]$,*

$$\eta_1\|u\|^2_{H^1} \le \eta_1\bigg(\|u\|^2_{H^1} + \frac{\|u_{y_{n+1}}\|^2}{\varepsilon^2}\bigg) \le \|u\|^2_{H^1_\varepsilon} \le \eta_2\bigg(\|u\|^2_{H^1} + \frac{\|u_{y_{n+1}}\|^2}{\varepsilon^2}\bigg). \tag{2.9}$$

Proof. Let

$$\gamma_3 = \max_{y\in\overline{Q}} \sum_{i=1}^n g^2_{y_i}(y) \quad \text{and} \quad \varepsilon_0 = \frac{1}{1+\sqrt{2\gamma_3}}.$$

Then, for all $\varepsilon\in(0,\varepsilon_0]$,

$$\begin{aligned} \|u\|^2_{H^1_\varepsilon} &= \|u\|_g^2 + \int_{\mathcal{O}} g\bigg(\sum_{i=1}^n\bigg(u_{y_i} - \frac{y_{n+1}}{g}\,g_{y_i}u_{y_{n+1}}\bigg)^2 + \frac{1}{\varepsilon^2g^2}\,u^2_{y_{n+1}}\bigg)\\ &\ge \|u\|_g^2 + \frac{\gamma_1}{2}\sum_{i=1}^n\|u_{y_i}\|^2 + \int_{\mathcal{O}}\frac{1}{g}\,u^2_{y_{n+1}}\bigg(\frac{1}{\varepsilon^2} - \sum_{i=1}^n g^2_{y_i}\bigg)\\ &\ge \|u\|_g^2 + \frac{\gamma_1}{2}\sum_{i=1}^n\|u_{y_i}\|^2 + \int_{\mathcal{O}}\frac{1}{2\varepsilon^2 g}\,u^2_{y_{n+1}}\\ &\ge \bigg(\gamma_1\|u\|^2 + \frac{\gamma_1}{2}\sum_{i=1}^n\|u_{y_i}\|^2 + \frac{1}{4\gamma_2\varepsilon_0^2}\,\|u_{y_{n+1}}\|^2\bigg) + \frac{1}{4\gamma_2}\frac{\|u_{y_{n+1}}\|^2}{\varepsilon^2}. \end{aligned}$$

By taking $\eta_1 = \min\{\gamma_1/2, 1/(4\gamma_2)\}$, we obtain the second inequality in (2.9). It is similar to prove the third inequality by taking $\eta_2 = \max\{2\gamma_2, 2/\gamma_1\}$ with the same ε_0. The first inequality is obvious. □

Now, we define an unbounded operator on X by

$$A_\varepsilon u = -\frac{1}{g}\,\mathrm{div}_y(\Upsilon_\varepsilon u), \quad \text{and so} \quad (A_\varepsilon u, v)_g = a_\varepsilon(u,v), \quad \text{for } u\in D(A_\varepsilon),\ v\in Y.$$

where $D(A_\varepsilon) = \{u \in H^2(\mathcal{O}) : \Upsilon_\varepsilon u \cdot \nu = 0 \text{ on } \partial\mathcal{O}\}$. Therefore, equations (2.8) can be rewritten as an abstract equation on X.

$$\begin{cases} \dfrac{du^\varepsilon}{dt} + A_\varepsilon u^\varepsilon + \lambda u^\varepsilon = F_\varepsilon(t,y,u^\varepsilon) + G_\varepsilon(t,y) + h_\varepsilon \dfrac{dW}{dt}, \\ u^\varepsilon(\tau) = u^\varepsilon_\tau, \quad y \in \mathcal{O},\ t \geq \tau. \end{cases} \tag{2.10}$$

2.3. Well posedness of solutions. We use a transformation of variables: $v^\varepsilon(t,\tau,\omega,v_\tau) = u^\varepsilon(t,\tau,\omega,u_\tau) - h_\varepsilon z(\theta_t\omega)$, where

$$z(\omega) = -\lambda \int_{-\infty}^0 e^{\lambda s}\omega(s)\,ds, \quad \omega \in \Omega. \tag{2.11}$$

It is easy to see the mapping $t \to z(\theta_t\omega)$ is continuous for each $\omega \in \Omega$. By $\lim\limits_{t\to\pm\infty} \omega(t)/t = 0$ and (2.11), it follows from [2, Proposition 4.1.3] that there exists another tempered random variable $r(\omega)$ such that

$$\widehat{z}(\theta_t\omega) := |z(\theta_t\omega)| + |z(\theta_t\omega)|^{2p} \leq e^{\lambda/2|t|} r(\omega), \quad \text{for all } t \in \mathbb{R},\ \omega \in \Omega. \tag{2.12}$$

Then, the equation (2.10) can be translated into a random equation:

$$\begin{cases} \dfrac{dv^\varepsilon}{dt} + A_\varepsilon v^\varepsilon + \lambda v^\varepsilon = f_\varepsilon(t,y,v^\varepsilon + h_\varepsilon z(\theta_t\omega)) + G_\varepsilon(t,y) - A_\varepsilon h_\varepsilon z(\theta_t\omega), \\ v^\varepsilon(\tau,\tau,\omega,v_\tau) = v_\tau \quad y \in \mathcal{O},\ t \geq \tau. \end{cases} \tag{2.13}$$

The following well-posedness of problem (2.13) can be found in [21].

Lemma 2.6. *For any $\tau \in \mathbb{R}$, $\omega \in \Omega$, $v_\tau \in X$ and $\varepsilon \in (0,\varepsilon_0)$, problem* (2.13) *has a unique solution*

$$v^\varepsilon(\,\cdot\,,\tau,\omega,v_\tau) \in C\big([\tau,\infty),X\big) \cap L^p\big((\tau,\tau+T),Y\big) \cap L^2\big((\tau,\tau+T),Z\big) \tag{2.14}$$

for every $T > 0$. Moreover, this solution continuously depends on v_τ and t.

3. Lusin continuity in samples and random cocycle

In this section, we prove $\mathcal{F}$-measurability (actually Lusin continuity) of the solution mapping from Ω to X. The following result generalizes the corresponding result given in [11] from the Wiener process to a general process. Let

$$\Omega_i = \big\{\omega \in \Omega : |\omega(t)| \leq i e^{\lambda|t|/2}, \text{for all } t \in \mathbb{R}\big\}, \quad \text{for all } i \in \mathbb{N}. \tag{3.1}$$

Lemma 3.1.

(a) $\Omega = \bigcup\limits_{i=1}^{\infty} \Omega_i$ *and $\{\Omega_i\}$ is an increasing sequence of closed sets in (Ω,ϱ).*

(b) *For each $I \in \mathbb{N}$, the mapping $\omega \mapsto z(\theta_t\omega)$ is continuous on (Ω_I,ϱ), uniformly in t on a compact intervals. More precisely, for any $[a,b] \subset \mathbb{R}$,*

$$\sup_{t\in[a,b]} |z(\theta_t\omega_k) - z(\theta_t\omega_0)| \to 0, \quad \text{as } \varrho(\omega_k,\omega_0) \to 0,\ \omega_k,\omega_0 \in \Omega_I. \tag{3.2}$$

Proof. (a) Given any $\omega \in \Omega$, we know $\lim_{t\to\pm\infty} \omega(t)/t = 0$, which implies

$$\lim_{t\to\pm\infty} \frac{\omega(t)}{e^{\lambda|t|/2}} = \lim_{t\to\pm\infty} \frac{\omega(t)}{t} \cdot \frac{t}{e^{\lambda|t|/2}} = 0.$$

Hence, by the continuity of $t \to \omega(t)$, there is $i_0 = i_0(\omega) \in \mathbb{N}$ such that $|\omega(t)| \le i_0 e^{\lambda|t|/2}$ for all $t \in \mathbb{R}$, which means $\omega \in \Omega_{i_0}$. Therefore, $\Omega = \bigcup_{i=1}^{\infty} \Omega_i$. The other assertions are obvious.

(b) Assume $[a,b] \subset [-n_0, n_0]$ with $n_0 \in \mathbb{N}$. Let $n_1 > n_0$, since $\omega_k, \omega_0 \in \Omega_I$, it follows from (3.1), we can find

$$\begin{aligned} M_k := & \left| \int_{-\infty}^{-n_0} e^{\lambda s}(\omega_k(s) - \omega_0(s))\, ds \right| \\ \le & \int_{-\infty}^{-n_1} e^{\lambda s}|\omega_k(s) - \omega_0(s)|\, ds + \int_{-n_1}^{-n_0} e^{\lambda s}|\omega_k(s) - \omega_0(s)|\, ds \\ \le & \int_{-\infty}^{-n_1} e^{\lambda s} 2I e^{-\lambda s/2}\, ds + \rho_{n_1}(\omega_k, \omega_0) \int_{-n_1}^{-n_0} e^{\lambda s} q\, ds \\ \le & \frac{4I}{\lambda} e^{-\lambda n_1} + \frac{1}{\lambda} \rho_{n_1}(\omega_k, \omega_0). \end{aligned}$$

Let $k, n_1 \to \infty$, we have $M_k \to 0$. Suppose $t \in [a,b] \subset [-n_0, n_0]$, by (2.11), we have

$$\begin{aligned} |z(\theta_t\omega_k) - z(\theta_t\omega_0)| &= \lambda \left| \int_{-\infty}^{0} e^{\lambda s}(\omega_k(s+t) - \omega_0(s+t) - \omega_k(t) + \omega_0(t))\, ds \right| \\ &\le \lambda \left| \int_{-\infty}^{0} e^{\lambda s}(\omega_k(s+t) - \omega_0(s+t))\, ds \right| + |\omega_k(t) - \omega_0(t)| \\ &\le \lambda e^{-\lambda t} \left| \int_{-\infty}^{t} e^{\lambda s}(\omega_k(s) - \omega_0(s))\, ds \right| + \rho_{n_0}(\omega_k, \omega_0) \\ &\le \lambda e^{-\lambda t} \left(M_k + \int_{-n_0}^{t} e^{\lambda s}|\omega_k(s) - \omega_0(s)|\, ds \right) + \rho_{n_0}(\omega_k, \omega_0) \\ &\le \lambda e^{\lambda n_0} M_k + \big(e^{2\lambda n_0} + 1\big)\rho_{n_0}(\omega_k, \omega_0), \end{aligned}$$

which converges to zero as $k \to \infty$ uniformly in $t \in [a,b]$. □

Lemma 3.2. *For each $I \in \mathbb{N}$, the mapping $\omega \to v^{\varepsilon}(t, \tau, \omega, v_\tau)$ is continuous from (Ω_I, ϱ) to $(X, \|\cdot\|_g)$, where v is the solution of equation* (2.13).

Proof. We omit the superscript $^{\varepsilon}$ when there is no ambiguity. Let $\omega_k, \omega_0 \in \Omega_I$ such that $\rho(\omega_k, \omega_0) \to 0$ as $k \to \infty$. We denote by $v_k := v(t, \tau, \omega_k, v_\tau)$, $v_0 := v(t, \tau, \omega_0, v_\tau)$ and $V_k := v_k - v_0$, where $t \in [\tau, \tau + T]$ with $T > 0$. By (2.13), we have

$$\begin{aligned} \frac{dV_k}{dt} + \lambda V_k + A_\varepsilon V_k = {} & F_\varepsilon(t, y, v_k + h_\varepsilon z(\theta_t\omega_k)) \\ & - F_\varepsilon(t, y, v_0 + h_\varepsilon z(\theta_t\omega_0)) - A_\varepsilon h_\varepsilon(z(\theta_t\omega_k) - z(\theta_t\omega_0)) \end{aligned} \tag{3.3}$$

with the initial data $V_k(\tau) = v_\tau - v_\tau = 0$. We multiply (3.3) with gV_k and then integrate over $\mathcal{O}$ to obtain

$$\frac{1}{2}\frac{d}{dt}\|V_k\|_g^2 + \lambda\|V_k\|_g^2 + a_\varepsilon(V_k, V_k) = J_1 + J_2. \tag{3.4}$$

By the mean valued theorem and the condition (2.3),

$$\begin{aligned} J_1 :&= \big(F_\varepsilon(t, y, v_k + h_\varepsilon z(\theta_t\omega_k)) - F_\varepsilon(t, y, v_0 + h_\varepsilon z(\theta_t\omega_0)), V_k\big)_g \\ &= \int_{\mathcal{O}} g\,\frac{\partial F_\varepsilon}{\partial s}(V_k + h_\varepsilon(z(\theta_t\omega_k) - z(\theta_t\omega_0)))V_k\,dy \\ &\le \beta\|V_k\|_g^2 + C|z(\theta_t\omega_k) - z(\theta_t\omega_0)|\int_{\mathcal{O}} g\big(|\psi_3| + |v_k|^{p-2} + |v_0|^{p-2}\big)|V_k|\,dy \\ &\le \beta\|V_k\|_g^2 + CZ_k^2\|\psi_3(t)\|_\infty^2 + CZ_k(1 + \|v_0\|_p^p + \|v_k\|_p^p), \end{aligned}$$

where $Z_k = \sup\limits_{t\in[\tau,\tau+T]} |z(\theta_t\omega_k) - z(\theta_t\omega_0)|$, and we have used the facts: $h_\varepsilon \in L^\infty(\mathcal{O})$ and $\sup\limits_k \sup\limits_{t\in[\tau,\tau+T]} |z(\theta_t\omega_k)| < +\infty$. While

$$\begin{aligned} J_2 :=& -(A_\varepsilon h_\varepsilon(z(\theta_t\omega_k) - z(\theta_t\omega_0)), V_k)_g = -a_\varepsilon(h_\varepsilon(z(\theta_t\omega_k) - z(\theta_t\omega_0)), V_k) \\ \le& \frac{1}{2}\,a_\varepsilon(V_k, V_k) + \frac{1}{2}\,Z_k^2 a_\varepsilon(h_\varepsilon, h_\varepsilon) \le \frac{1}{2}\,a_\varepsilon(V_k, V_k) + \frac{1}{2}\,Z_k^2 a_\varepsilon\|h_\varepsilon\|_{H_\varepsilon^1}^2 \\ \le& \frac{1}{2}\,a_\varepsilon(V_k, V_k) + \frac{\eta_2}{2}\,Z_k^2\bigg(\|h_\varepsilon\|_{H^1}^2 + \frac{1}{\varepsilon^2}\bigg\|\frac{\partial}{\partial y_{n+1}}\,h(y^*, \varepsilon g(y^*)y_{n+1})\bigg\|^2\bigg) \\ \le& \frac{1}{2}\,a_\varepsilon(V_k, V_k) + CZ_k^2. \end{aligned}$$

The above estimates yield

$$\frac{d}{dt}\,\|V_k\|_g^2 \le C\|V_k\|_g^2 + CZ_k\big(1 + \|v_0\|_p^p + \|v_k\|_p^p\big) + CZ_k^2\big(1 + \|\psi_3(t)\|_\infty^2\big). \tag{3.5}$$

By the Gronwall inequality over $[\tau, t]$ with $t \in [\tau, \tau + T]$, we find

$$\begin{aligned} \|V_k(t)\|_g^2 \le& Ce^{CT}\bigg(Z_k\int_\tau^{\tau+T}(1 + \|v_0(s)\|_p^p + \|v_k(s)\|_p^p)\,ds \\ &+ Z_k^2\int_\tau^{\tau+T}\big(1 + \|\psi_3(s)\|_\infty^2\big)\,ds\bigg) \\ \le& C\bigg(Z_k + Z_k^2 + Z_k\int_\tau^{\tau+T}\|v_k(s)\|_p^p\,ds\bigg), \end{aligned}$$

where we have used the facts: $\psi_3 \in L^2_{\rm loc}\big(\mathbb{R}, L^\infty\big(\widetilde{\mathcal{O}}\big)\big)$ and $v_0 \in L^p_{\rm loc}(\mathbb{R}, L^p(\mathcal{O}))$. By an energy inequality on v_k (see [20, (47)]),

$$\begin{aligned} &\frac{d}{dt}\|v_k\|_g^2 + \lambda\|v_k\|_g^2 + c\|v_k\|_p^p \\ &\qquad\le C(1 + |z(\theta_t\omega_k)|)^p + c\big(\|G(t)\|_\infty^2 + \|\psi_1(t)\|_\infty + \|\psi_2(t)\|_\infty^2\big). \end{aligned}$$

The Gronwall inequality implies that

$$\begin{aligned} e^{-\lambda T} \int_\tau^{\tau+T} \|v_k(s)\|_p^p \, ds \|v_k\|_p^p \\ \leq C\|v_\tau\|_g^2 + C \int_\tau^{\tau+T} \left(1 + \|G(s)\|_\infty^2 + \|\psi_1(s)\|_\infty + \|\psi_2(s)\|_\infty^2\right) ds < +\infty. \end{aligned}$$

By Lemma 3.1 (b), we know $Z_k \to 0$, and thus $\|V_k(t)\|_g^2 \to 0$ as $k \to \infty$, uniformly in $t \in [\tau, \tau + T]$. □

Corollary 3.3. *$\omega \to v^\varepsilon(t, \tau, \omega, v_\tau)$ is $(\mathcal{F}, \mathfrak{B}(X))$ measurable, for $X = L^2(\mathcal{O})$.*

Proof. By Lemma 3.1 (a) and the countable additivity of P, it is easy to see $\lim_{i\to\infty} P(\Omega_i) = P(\Omega) = 1$. Then Lemma 3.2 implies Lusin/basic continuity of the mapping, which further implies the needed measurability. □

Next, we need to prove that the solution mapping is $\mathcal{F}$-measurable in $Y = L^p(\mathcal{O})$. In this case, we recall the concept of a *quasi-continuous* mapping, which is introduced by Li and Guo [25] and developed by Gess [15].

Let M be a Polish space and $\mathcal{X}$ a separable Banach space. A mapping $\Phi\colon M \mapsto \mathcal{X}$ is said to be *quasi-continuous* if $\Phi m_i \rightharpoonup \Phi m$ weakly in $\mathcal{X}$, whenever $\{\Phi m_i\}_{i=1}^\infty$ is bounded in $\mathcal{X}$ and $m_i \to m$ in M. The following result can be found in a recent article by Cui, Langa and Li [11].

Lemma 3.4.

(a) (Measurability) *Φ is $(\mathfrak{B}(M), \mathfrak{B}(\mathcal{X}))$ measurable if $\Phi\colon M \mapsto \mathcal{X}$ is quasi-continuous.*

(b) (Inheritability) *Let $\mathcal{Y} \hookrightarrow \mathcal{X}$ and $\mathcal{X}^* \hookrightarrow \mathcal{Y}^*$ densely. Then, $\Phi\colon M \mapsto \mathcal{Y}$ is quasi-continuous if $\Phi\colon M \mapsto \mathcal{X}$ is quasi-continuous and $\Phi(M) \subset \mathcal{Y}$.*

Lemma 3.5. *For $t > \tau$, the solution mapping $\omega \to v^\varepsilon(t, \tau, \omega, v_\tau)$ is $(\mathcal{F}, \mathfrak{B}(Y))$ measurable, where $Y = L^p(\mathcal{O})$.*

Proof. By Lemma 3.2, the solution mapping is continuous from (Ω_I, ρ) to X for each $I \in \mathbb{N}$, and so it is quasi-continuous from Ω_I to X. By Lemma 2.6, $v(t, \tau, \omega, v_\tau) \in Y$ for $t > \tau$ and $v_\tau \in X$. Since $Y \hookrightarrow X$ and $X^* \hookrightarrow Y^*$ densely, it follows from inheritability given in Lemma 3.4 (b) that the solution mapping is quasi-continuous from Ω_I to Y. Then, by the measurability of a quasi-continuous mapping (see Lemma 3.4 (a)), the solution mapping is $(\mathfrak{B}(\Omega_I), \mathfrak{B}(Y))$ measurable for each $I \in \mathbb{N}$. By Lemma 3.1, each Ω_I is closed in Ω and $\bigcup_{i=1}^\infty \Omega_i = \Omega$. Therefore, it is easy to prove that the solution mapping is $(\mathcal{F}, \mathfrak{B}(Y))$ measurable. □

Now, we define a family of mappings $\phi_\varepsilon\colon \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \to X$ by

$$\phi_\varepsilon(t, \tau, \omega, v_\tau) = v^\varepsilon(t + \tau, \tau, \theta_{-\tau}\omega, v_\tau).$$

Recall that the concept of *random cocycle* which is given by Wang [32].

Definition 3.6. A mapping $\phi\colon \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \mapsto X$ is called a *random cocycle* on X if

(a) ϕ is $(\mathfrak{B}(\mathbb{R}^+) \times \mathfrak{B}(\mathbb{R}) \times \mathcal{F} \times \mathfrak{B}(X), \mathfrak{B}(X))$ measurable;
(b) it holds the cocycle property: for all $t, s \in \mathbb{R}^+$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\phi(t+s,\tau,\omega) = \phi(t,\tau+s,\theta_s\omega)\phi(s,\tau,\omega), \quad \phi(0,\tau,\omega) = \mathrm{id}_X.$$

Applying Lemmas 2.6, 3.2, 3.5 and Corollary 3.3, we have proved the following result.

Theorem 3.7. *For each $\varepsilon \in (0,\varepsilon_0]$, ϕ_ε is a continuous random cocycle on X. Its restriction on Y is a quasi-continuous random cocycle on Y.*

Finally, we take a universe $\mathfrak{D}$ of all set-valued mappings $\mathcal{D}\colon \mathbb{R} \times \Omega \to 2^X \setminus \emptyset$ such that, for any $\gamma > 0$,

$$\lim_{t\to+\infty} e^{-\gamma t}\|\mathcal{D}(\tau - t, \theta_{-t}\omega)\|_X^2 = 0, \quad \tau \in \mathbb{R},\ \omega \in \Omega,$$

where $\|D\|$ denote the supremum of norms for all elements, and $X = L^2(\mathcal{O})$. It is similar to define the universe $\mathfrak{D}_0$ on $L^2(Q)$.

4. Random attractors in p-times Lebesgue space

We need the following basic estimates for the solution $v^\varepsilon(s, \tau - t, \theta_{-\tau}\omega, v_0)$ in X (see [20]).

Lemma 4.1. [20]. *Let ε_0 be the positive number given in Lemma 2.5. Then, for each $\mathcal{D} \in \mathfrak{D}$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$, there exist $T = T(\mathcal{D},\tau,\omega) \geq 2$ such that for all $t \geq T$, $v_0 \in \mathcal{D}(\tau - t, \theta_{-t}\omega)$ and $\varepsilon \in (0,\varepsilon_0)$,*

$$\|v^\varepsilon(\tau,\tau-t,\theta_{-\tau}\omega,v_0)\|_{H^1_\varepsilon}^2 + \int_{\tau-t}^{\tau} e^{\lambda s}\|u^\varepsilon(s)\|_p^p\,ds \leq c_1\rho_1(\tau,\omega) \tag{4.1}$$

where ρ_1 is tempered and given by

$$\rho_1(\tau,\omega) = r(\omega) + \int_{-\infty}^{0} e^{\lambda s}(1 + \Psi(s+\tau))\,ds,$$

with $\Psi(s) = \|G(s)\|_\infty^2 + \|\psi_1(s)\|_\infty + \|\psi_2(s)\|_\infty^2 + \|\psi_4(s)\|_\infty^2$ and $r(\omega)$ is given in (2.12).

The following Gronwall-type lemma will be used frequently, which can be founded in [26].

Lemma 4.2. *Let z, z_1 be nonnegative locally integrable such that $\dot z + az \leq z_1$. Then, for any $\tau \in \mathbb{R}$ and $\mu > 0$,*

$$z(\tau) \leq \frac{1}{\mu}\int_{\tau-\mu}^{\tau} e^{a(r-\tau)}z(r)\,dr + \int_{\tau-\mu}^{\tau} e^{a(r-\tau)}z_1(r)\,dr. \tag{4.2}$$

LEMMA 4.3. *For any $\mathcal{D} \in \mathfrak{D}$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$, there exist $T \geq 2$ such that*

$$\sup_{s\in[\tau-1,\tau]} \sup_{t\geq T} \sup_{\varepsilon\in(0,\varepsilon_0)} \|v^\varepsilon(s,\tau-t,\theta_{-\tau}\omega,v_0)\|_p^p \leq c_2\rho_2(\tau,\omega), \tag{4.3}$$

whenever $v_0 \in \mathcal{D}(\tau-t,\theta_{-t}\omega)$, where ρ_2 is a finite function given by

$$\rho_2(\tau,\omega) = \big(1+e^{\lambda(1-\tau)}\big)\rho_1(\tau,\omega) + \int_{-\infty}^0 e^{\lambda s}\|\psi_1(s+\tau)\|_\infty^2\,ds.$$

PROOF. We multiply (2.13) with $g|v|^{p-2}v$ and integrating over $\mathcal{O}$ to obtain

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\|v\|_p^p + \lambda\|v\|_p^p + \int_{\mathcal{O}} gA_\varepsilon v\cdot|v|^{p-2}v\,dy \\ = \big(F_\varepsilon(t,y,u),|v|^{p-2}v\big)_g + \big(G_\varepsilon(t,y),|v|^{p-2}v\big)_g - \big(A_\varepsilon h_\varepsilon z(\theta_t\omega),|v|^{p-2}v\big)_g.\end{aligned} \tag{4.4}$$

The Laplace term is non-negative. Indeed,

$$\begin{aligned}&\int_{\mathcal{O}} gA_\varepsilon v\cdot|v|^{p-2}v\,dy \\ &\quad= -\frac{1}{\varepsilon}\int_{\mathcal{O}_\varepsilon}\Delta_x\widetilde{v}|\widetilde{v}|^{p-2}\widetilde{v}\,dx = \frac{1}{\varepsilon}\int_{\mathcal{O}_\varepsilon}\nabla_x\widetilde{v}\cdot\nabla_x\big(|\widetilde{v}|^{p-2}\widetilde{v}\big)\,dx \\ &\quad= \frac{p-2}{\varepsilon}\int_{\mathcal{O}_\varepsilon}\nabla_x\widetilde{v}\cdot|\widetilde{v}|^{p-4}|\widetilde{v}|^2\nabla_x\widetilde{v}\,dx + \frac{1}{\varepsilon}\int_{\mathcal{O}_\varepsilon}\nabla_x\widetilde{v}\cdot|\widetilde{v}|^{p-2}\nabla_x\widetilde{v}\,dx \\ &\quad= \frac{p-1}{\varepsilon}\int_{\mathcal{O}_\varepsilon}|\widetilde{v}|^{p-2}|\nabla_x\widetilde{v}|^2\,dx \geq 0.\end{aligned}$$

In order to estimate the nonlinear term in (4.4), we use the conditions (2.1) and (2.2) to obtain

$$\begin{aligned}F_\varepsilon(t,y,u)v &= F(t,y^*,\varepsilon g(y^*)y_{n+1},u)u - F(t,y^*,\varepsilon g(y^*)y_{n+1},u)h_\varepsilon z(\theta_t\omega) \\ &\leq -\alpha_1|u|^p + \psi_1(t) + \big(\alpha_2|u|^{p-1} + |\psi_2(t)|\big)|h_\varepsilon z(\theta_t\omega)| \\ &\leq -\frac{\alpha_1}{2^p}|v|^p + c|h_\varepsilon z(\theta_t\omega)|^p + |\psi_1(t)| + \big(\alpha_2|u|^{p-1} + |\psi_2(t)|\big)|h_\varepsilon z(\theta_t\omega)| \\ &\leq -\frac{\alpha_1}{2^{p+1}}|v|^p + |\psi_1(t)| + |\psi_2(t)h_\varepsilon z(\theta_t\omega)| + c|h_\varepsilon z(\theta_t\omega)|^p,\end{aligned}$$

where $\psi_1(t) = \psi_1(t,y^*,\varepsilon g(y^*)y_{n+1})$, and it is similar for $\psi_2(t)$. Hence,

$$\begin{aligned}\int_{\mathcal{O}} gF_\varepsilon(t,y,u)v|v|^{p-2}\,dy &\leq -\frac{\alpha_1\gamma_1}{2^{p+1}}\int_{\mathcal{O}}|v|^{2p-2}\,dy \\ &\quad+ c\gamma_2\int_{\mathcal{O}}(|\psi_1(t)| + |\psi_2(t)h_\varepsilon z(\theta_t\omega)| + |h_\varepsilon z(\theta_t\omega)|^p)|v|^{p-2}\,dy.\end{aligned}$$

By the Young inequality $ab^{p-2} \leq \eta b^{2p-2} + C(\eta)a^\mu$, where $\mu = 2-2/p$ such that $1 \leq \mu < 2$, we have

$$\begin{aligned}c\gamma_2|\psi_1(t)||v|^{p-2} &\leq \frac{\alpha_1\gamma_1}{2^{p+4}}|v|^{2p-2} + c|\psi_1(t)|^\mu \\ &\leq \frac{\alpha_1\gamma_1}{2^{p+4}}|v|^{2p-2} + c\big(|\psi_1(t)| + |\psi_1(t)|^2\big).\end{aligned}$$

Similarly, by $h \in C^2(\overline{Q} \times [0,\gamma_2])$ and so $h \in L^\infty\big(\widetilde{\mathcal{O}}\big)$,

$$c\gamma_2|h_\varepsilon z(\theta_t\omega)|^p|v|^{p-2} \le \frac{\alpha_1\gamma_1}{2^{p+4}}|v|^{2p-2} + c\big(|z(\theta_t\omega)|^p + |z(\theta_t\omega)|^{2p}\big) \le \frac{\alpha_1\gamma_1}{2^{p+4}}|v|^{2p-2} + c\widehat{z}(\theta_t\omega),$$

where $\widehat{z}(\theta_t\omega)$ is given in (2.12). By the generalized Young inequality: $abc \le \eta a^{(2p-2)/(p-2)} + C(\eta)b^2 + C(\eta)c^{2p-2}$, we have

$$|v|^{p-2}(c\gamma_2|\psi_2(t)|)|h_\varepsilon z(\theta_t\omega)| \le \frac{\alpha_1\gamma_1}{2^{p+4}}|v|^{2p-2} + c|\psi_2(t)|^2 + c\widehat{z}(\theta_t\omega).$$

All above estimates imply that

$$\begin{aligned}\int_{\mathcal{O}} gF_\varepsilon(t,y,u)v|v|^{p-2}\,dy \\ \le -\frac{\alpha_1\gamma_1}{2^{p+2}}\|v\|_{2p-2}^{2p-2} + c\big(\|\psi_1(t)\|_\infty + \|\psi_1(t)\|_\infty^2 + \|\psi_2(t)\|_\infty^2\big) + c\widehat{z}(\theta_t\omega).\end{aligned} \tag{4.5}$$

where $\|\cdot\|_\infty$ denotes the norm in $L^\infty\big(\widetilde{\mathcal{O}}\big)$. The second term on the right side of (4.4) is controlled by

$$\begin{aligned}\int_{\mathcal{O}} gG_\varepsilon(t,y)|v|^{p-2}v\,dy &\le \frac{\alpha_1\gamma_1}{2^{p+4}}\int_{\mathcal{O}}|v|^{2p-2}\,dy + c\int_{\mathcal{O}} G_\varepsilon^2(t,y)\,dy \\ &\le \frac{\alpha_1\gamma_1}{2^{p+4}}\|v\|_{2p-2}^{2p-2} + c\|G(t)\|_\infty^2.\end{aligned} \tag{4.6}$$

The final term of (4.4) is bounded by

$$\begin{aligned}-(A_\varepsilon h_\varepsilon z(\theta_t\omega), |v|^{p-2}v)_g &= \int_{\mathcal{O}} gz(\theta_t\omega)A_\varepsilon h_\varepsilon\cdot|v|^{p-2}v\,dy \\ &\le \gamma_2\int_{\mathcal{O}} gz(\theta_t\omega)A_\varepsilon h_\varepsilon\cdot|v|^{p-2}v\,dy \le \frac{\alpha_1\gamma_1}{2^{p+4}}\|v\|_{2p-2}^{2p-2} + c\widehat{z}(\theta_t\omega),\end{aligned} \tag{4.7}$$

where, by $h \in C^2(\overline{Q} \times [0,\gamma_2])$, we have

$$\begin{aligned}\|A_\varepsilon h_\varepsilon\|_g^2 = \int_{\mathcal{O}} g|A_\varepsilon h_\varepsilon\|^2\,dy &= \int_{\mathcal{O}_\varepsilon} g|\Delta_x h(x)|^2\,dx \\ &\le \int_{Q\times[0,\gamma_2]} g|\Delta_x h(x)|^2\,dx < +\infty.\end{aligned}$$

By (4.4)–(4.7), there are constants $c_1, c_2 > 0$ such that

$$\frac{d}{dt}\|v\|_p^p + \lambda\|v\|_p^p + c_1\|v\|_{2p-2}^{2p-2} \le c_2\big(\widehat{\Psi}(t) + \widehat{z}(\theta_t\omega)\big). \tag{4.8}$$

where $\widehat{\Psi}(t) = \|\psi_1(t)\|_\infty + \|\psi_1(t)\|_\infty^2 + \|\psi_2(t)\|_\infty^2 + \|G(t)\|_\infty^2$. For each $s \in [\tau-1,\tau]$, we apply the Gronwall-type inequality (4.2) with $\mu = s-(\tau-2) \ge 1$ and replace

ω by $\theta_{-\tau}\omega$ in (4.8), the result is

$$\begin{aligned}&\|v(s,\tau-t,\theta_{-\tau}\omega,v_0)\|_p^p\\&\le\int_{\tau-2}^{s}e^{\lambda(\sigma-s)}\|v(\sigma,\tau-t,\theta_{-\tau}\omega,v_0)\|_p^p\,d\sigma+c\int_{\tau-2}^{s}e^{\lambda(\sigma-s)}\big(\widehat{z}(\theta_{\sigma-\tau}\omega)+\widehat{\Psi}(\sigma)\big)\,d\sigma\\&\le e^{\lambda(1-\tau)}\int_{\tau-t}^{\tau}e^{\lambda\sigma}\|v(\sigma,\tau-t,\theta_{-\tau}\omega,v_0)\|_p^p\,d\sigma+c\int_{-\infty}^{\tau}e^{\lambda(\sigma-s)}\big(\widehat{z}(\theta_{\sigma-\tau})+\widehat{\Psi}(\sigma)\big)\,d\sigma.\end{aligned}$$

for all $t\ge T\ge 2$ with the same entry time T as given in Lemma 4.1. Note that $\widehat{\Psi}(\sigma)\le\Psi(\sigma)+\|\psi_1(\sigma)\|_\infty^2$. By (4.1) in Lemma 4.1, we obtain (4.3) as required.□

Lemma 4.4. *Let $T:=T(\mathcal{D},\tau,\omega)\ge 1$ be the entry time, given in Lemmas* 4.1 *and* 4.3*, for any $(\mathcal{D},\tau,\omega)\in\mathfrak{D}\times\mathbb{R}\times\Omega$. Then*

$$\lim_{K\to\infty}\sup_{\varepsilon\in(0,\varepsilon_0]}\sup_{t\ge T}\int_{\mathcal{O}(|v^\varepsilon|\ge K)}|v^\varepsilon(\tau,\tau-t,\theta_{-\tau}\omega,v_0)|^p\,dy=0,\tag{4.9}$$

uniformly in $v_0\in\mathcal{D}(\tau-t,\theta_{-t}\omega)$, where $\mathcal{O}(|v^\varepsilon|\ge K)=\mathcal{O}_K\cup\mathcal{O}_{-K}$ with

$$\begin{aligned}\mathcal{O}_K=\mathcal{O}_K^\varepsilon(s,\tau-t)&=\{y\in\mathcal{O}:v^\varepsilon(s,\tau-t,\theta_{-\tau}\omega,v_0)(y)\ge K\},\\ \mathcal{O}_{-K}&=\{y\in\mathcal{O}:v^\varepsilon(s,\tau-t,\theta_{-\tau}\omega,v_0)(y)\le -K\}.\end{aligned}$$

Proof. We first show that

$$\lim_{K\to\infty}\sup_{s\in[\tau-1,\tau]}\sup_{\varepsilon\in(0,\varepsilon_0]}\sup_{t\ge T}\sup_{v_0\in\mathcal{D}(\tau-t,\theta_{-t}\omega)}|\mathcal{O}_K^\varepsilon(s,\tau-t,v_0)|=0,\tag{4.10}$$

where $|\mathcal{O}_K|$ denotes the Lebesgue measure. For this end, by Lemma 4.3, we know that

$$|\mathcal{O}_K^\varepsilon(s,\tau-t)|K^p\le\int_{\mathcal{O}_K}|v^\varepsilon(s,\tau-t)|^p\,dy\le\int_{\mathcal{O}}|v^\varepsilon(s,\tau-t)|^p\,dy\le C<+\infty,$$

hereafter, we denote by $C=C(\tau,\omega)$ and denote by c a constant. Letting $K\to+\infty$ in the above inequality yields (4.10).

On the other hand, by the continuity of $s\to z(\theta_s\omega)$, we have

$$\sup_{s\in[-1,0]}|z(\theta_s\omega)|\|h\|_{L^\infty(Q\times[0,\gamma_2])}=K_1<+\infty.$$

By the condition (2.1), we can take $K_2>0$ such that

$$F(s,x,u)\le-\alpha_1u^{p-1}+\psi_1(s,x)u^{-1},\quad\text{if } u>K_2.\tag{4.11}$$

Now, let K be large enough such that $K\ge K_1+K_2+1$, and take the inner product of (2.13) with $g(v-K)_+^{p-1}$ in $L^2(\mathcal{O})$, where $w_+:=\max\{w,0\}$. The result is

$$\begin{aligned}\frac1p\frac{d}{ds}\|(v-K)_+\|_p^p&+\lambda\big(v,(v-K)_+^{p-1}\big)_g+\big(A_\varepsilon v,(v-K)_+^{p-1}\big)_g\\&=(F_\varepsilon(s,y,u),(v-K)_+^{p-1})_g+(G_\varepsilon(s,y),(v-K)_+^{p-1})_g\\&\qquad-\big(A_\varepsilon h_\varepsilon z(\theta_{s-\tau}\omega),(v-K)_+^{p-1}\big)_g.\end{aligned}\tag{4.12}$$

for all $s \in [\tau-1,\tau]$. It is easy to see that

$$(4.13)\qquad (A_\varepsilon v,(v-K)_+^{p-1})_g \ge 0, \qquad \lambda\int_{\mathcal{O}} gv(v-K)_+^{p-1}\,dy \ge \lambda\|(v-K)_+\|_p^p.$$

If $v \ge K$, then

$$u = v + h_\varepsilon(y)z(\theta_{s-\tau}\omega) \ge v - |h_\varepsilon(y)z(\theta_{s-\tau}\omega)| \ge v - K_1 \ge K_2.$$

By (4.11),

$$\begin{aligned} F(s,x,u) &\le -\alpha_1 u^{p-1} + \psi_1(s,x)u^{-1} \\ &\le -\frac{\alpha_1}{2^p}v^{p-1} + |\psi_1(s,x)|u^{-1} + c|h_\varepsilon z(\theta_{s-\tau}\omega)|^{p-1}. \end{aligned}$$

Therefore, we obtain the following estimates of the nonlinearity,

$$\begin{aligned} (4.14)\qquad &\int_{\mathcal{O}_K^\varepsilon} gF_\varepsilon(s,y^*,\varepsilon g(y^*)y_{n+1},u)(v-K)_+^{p-1}\,dy \\ &\le -\frac{\alpha_1\gamma_1}{2^p}\int_{\mathcal{O}_K^\varepsilon} v^{p-1}(v-K)_+^{p-1}\,dy + \gamma_2\int_{\mathcal{O}_K^\varepsilon}|\psi_1(s)|(v-K)_+^{p-2}\,dy \\ &\quad + c\int_{\mathcal{O}_K^\varepsilon}|h_\varepsilon z(\theta_{s-\tau}\omega)|^{p-1}(v-K)_+^{p-1}\,dy \\ &\le -\frac{\alpha_1\gamma_1}{2^{p+1}}\int_{\mathcal{O}_K^\varepsilon} v^{p-1}(v-K)_+^{p-1}\,dy \\ &\quad + c\int_{\mathcal{O}_K^\varepsilon}|\psi_1(s)|^{2-2/p}\,dy + c\int_{\mathcal{O}_K^\varepsilon}|h_\varepsilon z(\theta_{s-\tau}\omega)|^{2p-2}\,dy \\ &\le -\frac{\alpha_1\gamma_1}{2^{p+1}}\int_{\mathcal{O}_K^\varepsilon} v^{p-1}(v-K)_+^{p-1}\,dy \\ &\quad + c(\|\psi_1(s)\|_\infty + \|\psi_1(s)\|_\infty^2)|\mathcal{O}_K^\varepsilon| + c\widehat{z}(\theta_{s-\tau}\omega)|\mathcal{O}_K^\varepsilon|. \end{aligned}$$

where $\psi_1(s) = \psi_1(s,y^*,\varepsilon g(y^*)y_{n+1})$ and $\|\cdot\|_\infty$ denotes the norm in $L^\infty(\widetilde{\mathcal{O}})$. Similarly, we have

$$\big(G_\varepsilon(s,y),(v-K)_+^{p-1}\big)_g \le \frac{\alpha_1\gamma_1}{2^{p+1}}\int_{\mathcal{O}_K^\varepsilon} v^{p-1}(v-K)_+^{p-1}\,dy + c\|G(s)\|_\infty^2|\mathcal{O}_K^\varepsilon|.$$

By using $A_\varepsilon h_\varepsilon \in L^2(Q)$, we have

$$\begin{aligned} (4.15)\qquad -\big(A_\varepsilon h_\varepsilon z(\theta_{s-\tau}\omega),(v-K)_+^{p-1}\big)_g &= \int_{\mathcal{O}_K^\varepsilon} gA_\varepsilon h_\varepsilon z(\theta_{s-\tau}\omega)(v-K)_+^{p-1}\,dy \\ &\le \frac{\alpha_1\gamma_1}{2^{p+1}}\int_{\mathcal{O}_K^\varepsilon} v^{p-1}(v-K)_+^{p-1}dy + c\widehat{z}(\theta_{s-\tau}\omega)|\mathcal{O}_K^\varepsilon|. \end{aligned}$$

By (4.12)–(4.15), we can obtain that

$$\begin{aligned} (4.16)\qquad &\frac{d}{ds}\|(v-K)_+\|_p^p + C_2\int_{\mathcal{O}_K^\varepsilon} v^{p-1}(v-K)_+^{p-1}\,dy \\ &\le C_3(\|\psi_1(s)\|_\infty + \|\psi_1(s)\|_\infty^2 + \|G(s)\|_\infty^2 + \widehat{z}(\theta_{s-\tau}\omega))|\mathcal{O}_K^\varepsilon|. \end{aligned}$$

where C_2, C_3 are positive and independent of K and ε. Note that

$$\int_{\mathcal{O}_K^\varepsilon} v^{p-1}(v-K)_+^{p-1}\,dy \geq \int_{\mathcal{O}_K^\varepsilon} v^{p-2}(v-K)_+^{p}\,dy \geq K^{p-2}\|(v-K)_+\|_p^p,$$

then, (4.16) can be rewritten as follows:

$$\frac{d}{ds}\|(v-K)_+\|_p^p + C_2\int_{\mathcal{O}_K^\varepsilon} v^{p-1}(v-K)_+^{p-1}\,dy \leq C_3\big(\|\psi_1(s)\|_\infty + \|\psi_1(s)\|_\infty^2 + \|G(s)\|_\infty^2 + \widehat{z}(\theta_{s-\tau}\omega)\big)|\mathcal{O}_K^\varepsilon|. \tag{4.17}$$

By the Gronwall-type inequality (4.2) in Lemma 4.2 with $\mu = 1$, we have

$$\begin{aligned}\|(v(\tau)-K)_+\|_p^p &\leq \int_{\tau-1}^{\tau} e^{C_2K^{p-2}(s-\tau)}\|(v(s)-K)_+\|_p^p\,ds \\ &\quad + C_3|\mathcal{O}_K^\varepsilon|\int_{\tau-1}^{\tau}(\|\psi_1(s)\|_\infty + \|\psi_1(s)\|_\infty^2 + \|G(s)\|_\infty^2 + \widehat{z}(\theta_{s-\tau}\omega))\,ds \\ &\leq \int_{\tau-1}^{\tau} e^{C_2K^{p-2}(s-\tau)}\|(v(s)-K)_+\|_p^p\,ds + C_4|\mathcal{O}_K^\varepsilon|,\end{aligned}$$

in the last step, we have used $\psi_1, G \in L^2_{\rm loc}\big(\mathbb{R}, L^\infty\big(\widetilde{\mathcal{O}}\big)\big)$ and the continuity of $\widehat{z}(\theta_\cdot\omega)$. Since $\|(v-K)_+\|_p^p \leq \|v\|_p^p$, it follows from Lemma 4.3 that

$$\sup_{s\in[\tau-1,\tau]}\ \sup_{t\geq T}\ \sup_{\varepsilon\in(0,\varepsilon_0]} \|(v^\varepsilon(s,\tau-t,\theta_{-\tau}\omega)-K)_+\|_p^p \leq C_5.$$

Therefore, by (4.10), as $K\to\infty$,

$$\|(v^\varepsilon(\tau,\tau-t,\theta_{-\tau}\omega,v_0)-K)_+\|_p^p \leq \frac{C_5}{C_2K^{p-2}} + C_4|\mathcal{O}_K^\varepsilon| \to 0,$$

uniformly in $\varepsilon\in(0,\varepsilon_0]$, $t\geq T$ and $v_0\in\mathcal{D}(\tau-t,\theta_{-t}\omega)$. Note that $v\leq 2(v-K)$ if $v\geq 2K$. We have

$$\int_{\mathcal{O}_{2K}^\varepsilon} |(v^\varepsilon(\tau,\tau-t,\theta_{-\tau}\omega,v_0)|^p dy \leq 2\gamma_1^{-1}\|(v-K)_+\|_p^p \to 0,$$

as $K\to+\infty$, uniformly in $\varepsilon\in(0,\varepsilon_0]$, $t\geq T$ and $v_0\in\mathcal{D}(\tau-t,\theta_{-t}\omega)$. Similarly, the above uniform convergence holds true on $\mathcal{O}_{-2K}$. □

We give the following concept of a *bi-spatial random attractor*, which is slightly different from the concept given in [24] because we require that the $\mathcal{F}$-measurability of the attractor holds true in both initial and terminate spaces.

Definition 4.5. A bi-parametric set $\mathcal{A} = \{\mathcal{A}(\tau,\omega)\}$ is said to be a (X,Y)-random attractor for a random cocycle ϕ if

(a) $\omega\to\mathcal{A}(\tau,\omega)$ is $\mathcal{F}$-measurable in X and in Y respectively;
(b) $\mathcal{A}\in\mathfrak{D}$, and $\mathcal{A}(\tau,\omega)$ is compact in $X\cap Y$;
(c) $\mathcal{A}$ is invariant, i.e. $\phi(s,\tau,\omega)\mathcal{A}(\tau,\omega) = \mathcal{A}(\tau+s,\theta_s\omega)$ for $s\geq 0$;

(d) $\mathcal{A}$ is pullback attracting in Y, i.e. for every $\mathcal{D} \in \mathfrak{D}$,

$$\lim_{t\to+\infty} \mathrm{dist}_Y(\phi(t,\tau-t,\theta_{-t}\omega)\mathcal{D}(\tau-t,\theta_{-t}\omega),\mathcal{A}(\tau,\omega)) = 0.$$

Theorem 4.6. *For each $\varepsilon \in (0,\varepsilon_0]$, the cocycle ϕ_ε, generated by the problem (2.13), has a unique $\mathfrak{D}$-pullback (X,Y)-random attractor $\mathcal{A}_\varepsilon = \{\mathcal{A}_\varepsilon(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$, where $X = L^2(\mathcal{O})$ and $Y = L^p(\mathcal{O})$.*

Proof. By Lemma 4.1, a random absorbing set is given by

$$\mathcal{K}(\tau,\omega) = \{u \in L^2(\mathcal{O}) : \|u\|^2 \le c_1\rho_1(\tau,\omega)\}, \text{for all } \tau \in \mathbb{R},\ \omega \in \Omega.$$

It is obvious that $\mathcal{K} \in \mathfrak{D}$, and the absorption is uniform in $\varepsilon \in (0,\varepsilon_0]$. Next, we need to show that ϕ_ε is asymptotically compact in Y.

In fact, we prove the stronger *eventual compactness* in Y. Let $(\mathcal{D},\tau,\omega) \in \mathfrak{D} \times \mathbb{R} \times \Omega$ and $\varepsilon \in (0,\varepsilon_0]$ be fixed, we define a decreasing family of sets by

$$B_\varepsilon(T) := \bigcup_{t\ge T} \phi_\varepsilon(t,\tau-t,\theta_{-t}\omega)\mathcal{D}(\tau-t,\theta_{-t}\omega), \quad \text{for all } T > 0. \tag{4.18}$$

Let $T_0 = T_0(\mathcal{D},\tau,\omega)$ be the entry time given in Lemmas 4.1 and 4.4. By Lemma 4.4, for each $\eta > 0$, we can find a $K = K(\eta) > 0$ such that

$$\int_{\mathcal{O}(|v|\ge K)} |v(y)|^p\,dy < \eta^p, \quad \text{for all } v \in B_\varepsilon(T_0). \tag{4.19}$$

On the other hand, by Lemmas 4.1, $B_\varepsilon(T_0)$ is bounded in $H^1_\varepsilon(\mathcal{O})$ and so in $H^1(\mathcal{O})$ (by Lemma 2.5), which implies that $B_\varepsilon(T_0)$ is pre-compact in $L^2(\mathcal{O})$. Hence, $B_\varepsilon(T_0)$ has a finite net in $L^2(\mathcal{O})$ with the same radius $(K^{(2-p)/2}\eta^{p/2}$ and the finite centers $v_k \in B_\varepsilon(T)$, $k = 1,\ldots,m$. That is, for any $v \in B_\varepsilon(T_0)$, we can find a center v_k such that

$$\|v - v_k\|^2 \le K^{2-p}\eta^p. \tag{4.20}$$

We will prove $\|v-v_k\|_p \le c\eta$, by dividing the domain into four parts: $\mathcal{O} = \bigcup_{j=1}^{4} \mathcal{O}_j$, where,

$$\mathcal{O}_1 = \mathcal{O}(|v| \ge K) \cap \mathcal{O}(|v_k| \le K), \qquad \mathcal{O}_2 = \mathcal{O}(|v| \le K) \cap \mathcal{O}(|v_k| \ge K),$$
$$\mathcal{O}_3 = \mathcal{O}(|v| \ge K) \cap \mathcal{O}(|v_k| \ge K), \qquad \mathcal{O}_4 = \mathcal{O}(|v| \le K) \cap \mathcal{O}(|v_k| \le K).$$

Note that $|v| \ge K \ge |v_k|$ on $\mathcal{O}_1$, and $|v| \le K \le |v_k|$ on $\mathcal{O}_2$. By (4.19), we have

$$\int_{\mathcal{O}_1} |v-v_k|^p\,dy \le 2^p \int_{\mathcal{O}_1} (|v|^p + |v_k|^p)\,dy \le 2^{p+1} \int_{\mathcal{O}(|v|\ge K)} |v|^p\,dy \le 2^{p+1}\eta^p,$$
$$\int_{\mathcal{O}_2} |v-v_k|^p\,dy \le 2^{p+1} \int_{\mathcal{O}(|v_k|\ge K)} |v_k|^p\,dy \le 2^{p+1}\eta^p.$$

By (4.19) again, we have

$$\int_{\mathcal{O}_3} |v-v_i|^p\,dy \le 2^p\bigg(\int_{\mathcal{O}(|v|\ge K)} |v|^p\,dy + \int_{\mathcal{O}(|v_k|\ge K)} |v_k|^p\,dy\bigg) \le 2^{p+1}\eta^p.$$

On the other hand, by (4.20), we have

$$\int_{\mathcal{O}_4} |v - v_k|^p \, dy \le (2K)^{p-2} \int_{\mathcal{O}_4} |v - v_k|^2 \, dy \le (2K)^{p-2} \|v - v_k\|^2 \le 2^{p-2}\eta^p.$$

By the estimates mentioned above, $\|v - v_k\|_p^p \le 2^{p+3}\eta^p$, which implies that $B_\varepsilon(T_0)$ has a finite 16η-net in $L^p(\mathcal{O})$ with the same centers v_k, $k = 1, \dots, m$. Therefore, $B_\varepsilon(T_0)$ is pre-compact in $L^p(\mathcal{O})$ and so ϕ_ε is eventually compact in $L^p(\mathcal{O})$ as required.

By the abstract existence result of bi-spatial attractors given in [26] (see [24] in the autonomous case), we know that ϕ_ε has a (X, Y)-attractor $\mathcal{A}_\varepsilon$, except for $\mathcal{F}$-measurability in Y. By Lemma 3.5, the cocycle ϕ_ε is $\mathcal{F}$-measurable in Y. By Lemma 4.3, ϕ_ε has a $\mathfrak{D}$-pullback absorbing set $\mathcal{K}_p$ in $L^p(\mathcal{O})$ given by

$$\mathcal{K}_p(\tau, \omega) = \{u \in L^p(\mathcal{O}) : \|u\|_p^p \le c_2\rho_2(\tau, \omega)\}, \quad \text{for all } \tau \in \mathbb{R}, \ \omega \in \Omega.$$

It is obvious that $\mathcal{K}_p$ is a random set in $L^p(\mathcal{O})$ in view of the measurability of the mapping $\omega \to \rho_2(\tau, \omega)$. Then, it follows from [11, Theorem 19] that the attractor $\mathcal{A}_\varepsilon$ is $\mathcal{F}$-measurable in $L^p(\mathcal{O})$. Therefore, $\mathcal{A}_\varepsilon$ is indeed a (X, Y)-random attractor in the sense of Definition 4.5. □

In order to consider the limiting equation (1.3) on Q, we define an operator A_0 by

$$D(A_0) = \left\{ u \in H^2(Q) : \frac{\partial u}{\partial \nu_0} = 0 \text{ on } \partial Q \right\},$$

and, for $u \in D(A_0)$,

$$A_0 u = -\frac{1}{g} \sum_{i=1}^{n} (g u_{y_i})_{y_i}, \qquad (A_0 u, v)_g = a_0(u, v) = \int_Q g \nabla u \cdot \nabla v \, dy^*.$$

Let u^0 is a solution of problem (1.3). Then, $v^0(t, \tau, \omega, v_\tau^0) = u^0(t, \tau, \omega, u_\tau^0) - h_0(y^*) z(\theta_t \omega)$ satisfies the following equation:

$$\text{(4.21)} \qquad \begin{cases} \dfrac{dv^0}{dt} + A_0 v^0 + \lambda v^0 = f_0(t, y^*, u^0) + G_0(t, y^*) - A_0 h_0(y^*) z(\theta_t \omega), \\ v^0(\tau) = v_\tau^0, \quad y^* \in Q, \ t \ge \tau, \end{cases}$$

and the solution determines a continuous random cocycle $\phi_0(t, \tau, \omega, u_\tau^0)$ on $L^2(Q)$.

Theorem 4.7. *Under the Assumption* 2.4*, the cocycle* ϕ_0*, generated by equation* (4.21)*, has a unique* $\mathfrak{D}_0$*-pullback* $(L^2(Q), L^p(Q))$ *random attractor* $\mathcal{A}_0 \in \mathfrak{D}_0$.

5. Upper semicontinuity of bi-spatial random attractors

For a function defined on $\mathcal{O}$, we consider its average function with respect to the $n+1$-th variable, by using the average operator $\mathcal{M} \colon L^2(\mathcal{O}) \mapsto L^2(Q)$,

$$(\mathcal{M}u)(y^*) = \int_0^1 u(y^*, y_{n+1}) \, dy_{n+1}.$$

Conversely, for a function u defined on Q, we regard that u is identical to the function $\widehat{u}(y^*, y_{n+1}) = u(y^*)$, $(y^*, y_{n+1}) \in \mathcal{O} = Q \times (0,1)$. The following result can be found in [18]: If $u \in H^1(\mathcal{O})$, then $\mathcal{M}u \in H^1(Q)$ and

$$\|u - \mathcal{M}u\|_{L^2(\mathcal{O})} \leq c\,\varepsilon \|u\|_{H^1_\varepsilon(\mathcal{O})}. \tag{5.1}$$

We need some convergence assumptions for both source and force.

Assumption 5.1. There exist two functions $\mu_1(\,\cdot\,), \mu_2(\,\cdot\,) \in L^2_{\rm loc}(\mathbb{R})$ such that

$$\begin{aligned} \|f_\varepsilon(t,\,\cdot\,,s) - f_0(t,\,\cdot\,,s)\|_{L^2(\mathcal{O})} &\leq \mu_1(t)\varepsilon, \quad \text{for all } t, s \in \mathbb{R}, \\ \|G_\varepsilon(t,\,\cdot\,) - G_0(t,\,\cdot\,)\|_{L^2(\mathcal{O})} &\leq \mu_2(t)\varepsilon, \quad \text{for all } t \in \mathbb{R}. \end{aligned}$$

Since $h \in C^2(\overline{Q} \times [0,\gamma_2]$, by the mean valued theorem, we have the same convergence from h_ε to h_0 as

$$\sup_{y \in \mathcal{O}} |h_\varepsilon(y) - h_0(y^*)| \leq c\varepsilon.$$

Then, under the Assumption 5.1, the following convergence of the cocycle ϕ_ε can be found in [20, Theorem 2.2]: Suppose $\|v_0^\varepsilon\|_{H^1_\varepsilon(\mathcal{O})}$ is bounded with respect to $\varepsilon \in (0, \varepsilon_0]$, then

$$\lim_{\varepsilon \to 0} \big\|\phi_\varepsilon(t,\tau,\omega)v_0^\varepsilon - \phi_0(t,\tau,\omega)\mathcal{M}v_0^\varepsilon\big\|_{L^2(\mathcal{O})} = 0, \tag{5.2}$$

for each $t \geq 0$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$.

By using the above convergence, [20, Theorem 2.3] further proved the following convergence of the random attractor in $L^2(\mathcal{O})$:

$$\lim_{\varepsilon \to 0} \operatorname{dist}_{L^2(\mathcal{O})}(\mathcal{A}_\varepsilon(\tau,\omega), \mathcal{A}_0(\tau,\omega)) = 0. \tag{5.3}$$

Our main result in this section is to show that the convergence (5.3) holds true in the stronger topology. This type of semi-continuity is different from the semi-continuity come from the varying densities of noise (see [13], [37], [38]).

Theorem 5.2. *The random attractor $\mathcal{A}_\varepsilon$ is upper semi-continuous in $L^p(\mathcal{O})$ at $\varepsilon = 0$, that is*

$$\lim_{\varepsilon \to 0} \operatorname{dist}_{L^p(\mathcal{O})}(\mathcal{A}_\varepsilon(\tau,\omega), \mathcal{A}_0(\tau,\omega)) = 0, \quad \textit{for all } \tau \in \mathbb{R},\ \omega \in \Omega. \tag{5.4}$$

Proof. We split the proof into three parts.

Part 1. We show that any sequence $z_k \in \mathcal{A}_{\varepsilon_k}(\tau,\omega)$ is pre-compact in $L^p(\mathcal{O})$, where $\varepsilon_k \to 0$. For this end, we assume without lose of generality that $\varepsilon_k \in (0,\varepsilon_0]$ for all $k \in \mathbb{N}$. By Lemma 4.1, each cocycle ϕ_{ε_k} has a collective absorbing set $\mathcal{K} \in \mathfrak{D}$ defined by

$$\mathcal{K}(\tau,\omega) := \big\{u \in L^2(\mathcal{O}) : \|u\|^2 \leq c_1\rho_1(\tau,\omega)\big\}. \tag{5.5}$$

Then, the invariance of $\mathcal{A}_{\varepsilon_k}$ and the absorption of $\mathcal{K}$ implies that

$$\bigcup_{k\in\mathbb{N}} \mathcal{A}_{\varepsilon_k}(s,\widehat{\omega}) \subset \mathcal{K}(s,\widehat{\omega}), \quad \text{for all } s\in\mathbb{R},\ \widehat{\omega}\in\Omega.$$

Let T be the same entry time given in Lemmas 4.1 and 4.4 when $\mathcal{K}$ is absorbed by itself. By the invariance of $\mathcal{A}_{\varepsilon_k}$ and the above inclusion, we know that

$$z_k \in \phi_{\varepsilon_k}(T,\tau-T,\theta_{-T}\omega)\mathcal{K}(\tau-T,\theta_{-T}\omega), \quad \text{for all } k\in\mathbb{N}.$$

By Lemma 4.4, for each $\delta>0$ there is a $R=R(\delta)$ such that

$$\sup_{k\in\mathbb{N}} \int_{\mathcal{O}(|z_k|\geq R)} |z_k|^p\,dy \leq \delta^p. \tag{5.6}$$

By Lemma 4.1, we know

$$\sup_k \|\phi_{\varepsilon_k}(T,\tau-T,\theta_{-T}\omega)\mathcal{K}(\tau-T,\theta_{-T}\omega)\|^2_{H^1_{\varepsilon_k}(\mathcal{O})} \leq c_1\rho_1(\tau,\omega),$$

which, together with the first inequality in Lemma 2.5, implies that

$$\sup_k \|z_k\|^2_{H^1(\mathcal{O})} \leq \sup_k \frac{1}{\eta_1}\|z_k\|^2_{H^1_{\varepsilon_k}(\mathcal{O})} \leq c\rho_0(\tau,\omega).$$

Then, by the Sobolev compact embedding, the sequence $\{z_k\}_{k=1}^\infty$ has a convergent subsequence (not relabeled) in $L^2(\mathcal{O})$. In particular, $\{z_k\}_{k=1}^\infty$ is a Cauchy sequence in $L^2(\mathcal{O})$. Then, there is a $k_0\in\mathbb{N}$ such that

$$\|z_k-z_m\|^2_{L^2(\mathcal{O})} \leq R^{2-p}\delta^p, \quad \text{for all } k,m\geq k_0. \tag{5.7}$$

By the similar method as given in the proof of Theorem 4.6, we split the domain $\mathcal{O}=\bigcup_{j=1}^4 \mathcal{O}_j$ with

$$\mathcal{O}_1=\mathcal{O}(|z_k|\geq R)\cap\mathcal{O}(|z_m|\leq R), \qquad \mathcal{O}_2=\mathcal{O}(|z_k|\leq R)\cap\mathcal{O}(|z_m|\geq R),$$
$$\mathcal{O}_3=\mathcal{O}(|z_k|\geq R)\cap\mathcal{O}(|z_m|\geq R), \qquad \mathcal{O}_4=\mathcal{O}(|z_k|\leq R)\cap\mathcal{O}(|z_m|\leq R).$$

By (5.6), we can calculate as follows:

$$\int_{\mathcal{O}_1}|z_k-z_m|^p dy \leq 2^{p+1}\int_{\mathcal{O}(|z_k|\geq R)}|z_k|^p\,dy \leq 2^{p+1}\delta^p,$$
$$\int_{\mathcal{O}_2}|z_k-z_m|^p\,dy \leq 2^{p+1}\int_{\mathcal{O}(|z_m|\geq R)}|z^i|^p\,dy \leq 2^{p+1}\delta^p,$$
$$\int_{\mathcal{O}_3}|z_k-z^i|^p\,dy \leq 2^p\left(\int_{\mathcal{O}(|z_k|\geq R)}|z_k|^p\,dy + \int_{\mathcal{O}(|z_m|\geq R)}|z^i|^p\,dy\right) \leq 2^{p+1}\delta^p.$$

By (5.7)

$$\int_{\mathcal{O}_4}|z_k-z_m|^p\,dy \leq (2R)^{p-2}\int_{\mathcal{O}}|z_k-z_m|^2\,dy \leq (2R)^{p-2}R^{2-p}\eta^p \leq 2^{p+1}\delta^p.$$

Hence, $\|z_k-z_m\|_p^p \leq 2^{p+3}\delta^p$ and so $\|z_k-z_m\|_p\leq 4\delta$. Therefore, the subsequence $\{z_k\}_{k=1}^\infty$ is a Cauchy sequence and thus convergent in $L^p(\mathcal{O})$ as required.

Part 2. We construct an absorbing set $\mathcal{B} \subset H^1(\mathcal{O})$ such that $\mathcal{B}_0 = \overline{\mathcal{M}(\mathcal{B})}$ is a closed tempered set in $L^2(Q)$ and so $\mathcal{B}_0 \in \mathfrak{D}_0$ is attracted by the attractor $\mathcal{A}_0$ under the topology of $L^p(Q)$. For this end, we define two bi-parametric sets in $H^1(\mathcal{O})$ and in $L^2(Q)$ respectively.

$$\begin{aligned}\mathcal{B}(\tau,\omega) &= \bigg\{u \in H^1(\mathcal{O}) : u \in \mathcal{K}(\tau,\omega),\ \|u\|^2_{H^1(\mathcal{O})} \leq \frac{c_1}{\eta_1}\,\rho_1(\tau,\omega)\bigg\},\\ \mathcal{B}_0(\tau,\omega) &= \overline{\{\mathcal{M}u : u \in \mathcal{B}(\tau,\omega)\}},\end{aligned}$$

where the over-line denotes the closure in $L^2(Q)$ and $\mathcal{K}$ is the absorbing set given by (5.5). Since $\mathcal{B}(\tau,\omega) \subset \mathcal{K}(\tau,\omega)$, we have $\mathcal{B} \in \mathfrak{D}$. By Lemmas 2.5 and 4.1, for any $\varepsilon \in (0,\varepsilon_0]$ and $\mathcal{D} \in \mathfrak{D}$,

$$\begin{aligned}&\|\phi_\varepsilon(t,\tau-t,\theta_{-t}\omega)\mathcal{D}(\tau-t,\theta_{-t}\omega)\|^2_{H^1(\mathcal{O})}\\ &\qquad\leq \eta_1^{-1}\|\phi_\varepsilon(t,\tau-t,\theta_{-t}\omega)\mathcal{D}(\tau-t,\theta_{-t}\omega)\|^2_{H^1_\varepsilon(\mathcal{O})} \leq c_1\eta_1^{-1}\rho_1(\tau,\omega),\end{aligned}$$

provided t is large enough. Hence, $\mathcal{B} \in \mathfrak{D}$ is still a $\mathfrak{D}$-pullback absorbing set. On the other hand, by (5.1) and by Lemma 2.5 again, we have, for all $u \in \mathcal{B}(\tau,\omega)$,

$$\|u-\mathcal{M}u\|^2_{L^2(\mathcal{O})} \leq c\varepsilon^2\|u\|^2_{H^1_\varepsilon(\mathcal{O})} \leq c\,\varepsilon^2\,\frac{\eta_2}{\varepsilon^2}\,\|u\|^2_{H^1(\mathcal{O})} \leq c\,\frac{\eta_2}{\eta_1}\,\rho_1(\tau,\omega),$$

Hence, for all $u \in \mathcal{B}(\tau,\omega)$,

$$\|\mathcal{M}u\|^2_{L^2(Q)} \leq 2\big(\|u\|^2_{L^2(\mathcal{O})} + \|u-\mathcal{M}u\|^2_{L^2(\mathcal{O})}\big) \leq c\rho_1(\tau,\omega).$$

Since $\rho_1(\tau,\omega)$ is a tempered random variable, the above estimate yields $\mathcal{B}_0 \in \mathfrak{D}_0$ (we can not prove $\mathcal{K}_0 \in \mathfrak{D}_0$, where $\mathcal{K}_0 = \mathcal{M}(\mathcal{K})$ was used in [20], [21]).

Now, by Theorem 4.7, the bi-spatial attractor $\mathcal{A}_0$ attracts $\mathcal{B}_0 \in \mathfrak{D}_0$ under the topology of $L^p(Q)$. More precisely, for each $\delta > 0$, there is a $T_0 = T_0(\delta) > 0$ such that for all $t \geq T_0$,

$$\operatorname{dist}_{L^p(Q)}(\phi_0(t,\tau-t,\theta_{-t}\omega)\mathcal{B}_0(\tau-t,\theta_{-t}\omega),\mathcal{A}_0(\tau,\omega)) < \delta. \tag{5.8}$$

Part 3. We argue the convergence of random attractors in $L^p(\mathcal{O})$ by contradiction. Suppose (5.4) is not true, then, there exist $\delta > 0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$, $\varepsilon_k \to 0$ and $z_k \in \mathcal{A}_{\varepsilon_k}(\tau,\omega)$ such that

$$\operatorname{dist}_{L^p(\mathcal{O})}(z_k,\mathcal{A}_0(\tau,\omega)) \geq \delta, \quad \text{for all } k \in \mathbb{N}.$$

By Part 1, there is a $z \in L^p(\mathcal{O})$ such that, passing to a subsequence,

$$\lim_{k\to\infty}\|z_k - z\|_{L^p(\mathcal{O})} = 0 \quad\text{and}\quad \operatorname{dist}_{L^p(\mathcal{O})}(z,\mathcal{A}_0(\tau,\omega)) \geq \delta. \tag{5.9}$$

By Part 2, $\mathcal{B}$ is an absorbing set, which, together with the invariance of $\mathcal{A}_{\varepsilon_k}$, implies that

$$\bigcup_k \mathcal{A}_{\varepsilon_k}(s,\widetilde{\omega}) \subset \mathcal{B}(s,\widetilde{\omega}), \quad \text{for all } s \in \mathbb{R},\ \text{for all } \widetilde{\omega} \in \Omega. \tag{5.10}$$

By Part 2 again, $\mathcal{B} \in \mathfrak{D}$ and so $\mathcal{B}$ can absorb itself. In this case, we let $T = T(\mathcal{B}) > 0$, independent of ε_k, be an entry time such that $T \geq T_0$, where $T_0 = T_0(\delta)$ is the attraction time given in (5.8) when $\mathcal{A}_0$ attracts $\mathcal{B}_0$.

Now, for each $k \in \mathbb{N}$, by the invariance of $\mathcal{A}_{\varepsilon_k}$, there are $\widehat{z}_k \in \mathcal{A}_{\varepsilon_k}(\tau - T, \theta_{-T}\omega)$ such that

$$z_k = \phi_{\varepsilon_k}(T, \tau - T, \theta_{-T}\omega)\widehat{z}_k.$$

By Lemma 4.1 and (5.10), there exists another entry time $\widehat{T} = \widehat{T}(\mathcal{B}, \tau - T, \theta_{-T}\omega)$ such that, for all $t \geq \widehat{T}$ and $k \in \mathbb{N}$,

$$\begin{aligned} \|\widehat{z}_k\|_{H^1_{\varepsilon_k}} &\leq \|\phi_{\varepsilon_k}(t, \tau - T - t, \theta_{-t}\theta_{-T}\omega)\mathcal{A}_{\varepsilon_k}(\tau - T - t, \theta_{-t}\theta_{-T}\omega)\|_{H^1_{\varepsilon_k}} \\ &\leq \|\phi_{\varepsilon_k}(t, \tau - T - t, \theta_{-t}\theta_{-T}\omega)\mathcal{B}(\tau - T - t, \theta_{-t}\theta_{-T}\omega)\|_{H^1_{\varepsilon_k}} \\ &\leq c_1\rho_1(\tau - T, \theta_{-T}\omega). \end{aligned} \tag{5.11}$$

This means that $\|\widehat{z}_k\|_{H^1_{\varepsilon_k}}$ is bounded in k, which together with (5.2) give

$$\|\phi_{\varepsilon_k}(T, \tau - T, \theta_{-T}\omega)\widehat{z}_k - \phi_0(T, \tau - T, \theta_{-T}\omega)\mathcal{M}\widehat{z}_k\|_{L^2(\mathcal{O})} \to 0, \quad \text{as } k \to \infty.$$

that is

$$\|z_k - \phi_0(T, \tau - T, \theta_{-T}\omega)\mathcal{M}\widehat{z}_k\|_{L^2(\mathcal{O})} \to 0, \quad \text{as } k \to \infty.$$

By (5.9) and by the Hölder inequality, we have

$$\|z_k - z\|^2_{L^2(\mathcal{O})} \leq |\mathcal{O}|\|z_k - z\|^p_{L^p(\mathcal{O})} \to 0, \quad \text{as } k \to \infty.$$

Then, we have

$$\|z - \phi_0(T, \tau - T, \theta_{-T}\omega)\mathcal{M}\widehat{z}_k\|_{L^2(\mathcal{O})} \to 0, \quad \text{as } k \to \infty. \tag{5.12}$$

Once more, we consider the sequence $\widehat{z}_k \in \mathcal{A}_{\varepsilon_k}(\tau - T, \theta_{-T}\omega)$. By (5.11), $\|\widehat{z}_k\|_{H^1_{\varepsilon_k}(\mathcal{O})}$ is bounded in k, which together with (5.1) imply that

$$\|\widehat{z}_k - \mathcal{M}\widehat{z}_k\|_{L^2(\mathcal{O})} \leq c\varepsilon_k\|\widehat{z}_k\|_{H^1_{\varepsilon_k}(\mathcal{O})} \leq C\varepsilon_k \to 0.$$

By Part 1, $\{\widehat{z}_k\}$ has a convergent subsequence (denoted by itself) in $L^p(\mathcal{O})$ and thus in $L^2(\mathcal{O})$. Then, the above convergence shows that the corresponding subsequence $\{\mathcal{M}\widehat{z}_k\}$ is a Cauchy sequence in $L^2(\mathcal{O})$ and thus in $L^2(Q)$. So, there is a $\widehat{z}_0 \in L^2(Q)$ such that

$$\mathcal{M}\widehat{z}_k \to \widehat{z}_0 \quad \text{in } L^2(Q) \text{ as } k \to \infty.$$

By the continuity of the operator $\phi_0 \colon L^2(Q) \mapsto L^2(Q)$, we have

$$\phi_0(T, \tau - T, \theta_{-T}\omega)\mathcal{M}\widehat{z}_k \to \phi_0(T, \tau - T, \theta_{-T}\omega)\widehat{z}_0 \quad \text{in } L^2(Q),$$

and so in $L^2(\mathcal{O})$ by expending the domain. This together with (5.12) implies that $z = \phi_0(T, \tau - T, \theta_{-T}\omega)\widehat{z}_0$ in $L^2(\mathcal{O})$. So, $z = \phi_0(T, \tau - T, \theta_{-T}\omega)\widehat{z}_0$ almost everywhere on $\mathcal{O}$, which implies

$$z = \phi_0(T, \tau - T, \theta_{-T}\omega)\widehat{z}_0 \quad \text{in } L^p(\mathcal{O}).$$

By (5.10), we know $\widehat{z}_k \in \mathcal{A}_{\varepsilon_k}(\tau - T, \theta_{-T}\omega) \subset \mathcal{B}(\tau - T, \theta_{-T}\omega)$. Then, by the construction in Part 2, it follows that $\mathcal{M}\widehat{z}_k \in \mathcal{B}_0(\tau - T, \theta_{-T}\omega)$ for all $k \in \mathbb{N}$. Hence, the limit $\widehat{z}_0 \in \mathcal{B}_0(\tau - T, \theta_{-T}\omega)$ in view of the closedness of $\mathcal{B}_0$. By (5.8) in Part 2 and by $T \geq T_0$, we have

$$\mathrm{dist}_{L^p(\mathcal{O})}(z, \mathcal{A}_0(\tau,\omega)) = \mathrm{dist}_{L^p(Q)}\big(\phi_0(T, \tau - T, \theta_{-T}\omega)\widehat{z}_0, \mathcal{A}_0(\tau,\omega)\big) < \delta.$$

This gives a contradiction with (5.9). □

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