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ABOUT POSITIVE $W^{1,\Phi}_{\mathrm{loc}}(\Omega)$-SOLUTIONS TO QUASILINEAR ELLIPTIC PROBLEMS WITH SINGULAR SEMILINEAR TERM

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Abstract. This paper deals with the existence, uniqueness and regularity of positive $W^{1,\Phi}_{\mathrm{loc}}(\Omega)$-solutions of singular elliptic problems on a smooth bounded domain with Dirichlet boundary conditions involving the Φ-Laplacian operator. The proof of the existence is based on a variant of the generalized Galerkin method that we developed inspired by ideas of Browder [4] and a comparison principle. By the use of a kind of Moser's iteration scheme we show the $L^\infty(\Omega)$-regularity for positive solutions.

1. Introduction

The paper concerns the existence, uniqueness and regularity of $W^{1,\Phi}_{\mathrm{loc}}(\Omega)$-solutions to the singular elliptic problem

$$(1.1)\qquad -\mathrm{div}(\phi(|\nabla u|)\nabla u) = \frac{a(x)}{u^\alpha} \quad \text{in } \Omega, \qquad u > 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

where $\Omega \subset \mathbb{R}^N$, with $N \geq 2$, is a bounded domain with smooth boundary $\partial\Omega$, a is a non-negative function, $0 < \alpha < \infty$ and $\phi\colon (0,\infty) \to (0,\infty)$ is of class C^1 and satisfies

(ϕ_1) (i) $t\phi(t) \to 0$ as $t \to 0$,

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(ii) $t\phi(t) \to \infty$ as $t \to \infty$,

(ϕ_2) $t\phi(t)$ is strictly increasing in $(0,\infty)$,

(ϕ_3) there exist $\ell, m \in (1,N)$ such that

$$\ell - 1 \leq \frac{(t\phi(t))'}{\phi(t)} \leq m-1, \quad t>0.$$

We extend $s \mapsto s\phi(s)$ to $\mathbb{R}$ as an odd function. It follows that the function

$$\Phi(t) = \int_0^t s\phi(s)\,ds, \quad t \in \mathbb{R}$$

is even and it is actually an N-function. Due to the nature of the operator

$$\Delta_\Phi u := \operatorname{div}(\phi(|\nabla u|)\nabla u)$$

we shall work in the framework of Orlicz and Orlicz–Sobolev spaces namely $L_\Phi(\Omega)$, $L_{\widetilde{\Phi}}(\Omega)$ and $W_0^{1,\Phi}(\Omega)$.

We recall some basic notation on these spaces along with bibliographycal references in the Apendix.

In the last years many research papers have been devoted to the study of singular problems like (1.1). In [23], Karlin and Nirenberg studied the singular integral equation

$$u(x) = \int_0^1 G(x,y)\,\frac{1}{u(y)^\alpha}\,dy, \quad 0 \leq x \leq 1,$$

where $\alpha > 0$ and $G(x,y)$ is a suitable potential. In [11], Crandall, Rabinowitz and Tartar, addressed a class of singular problems which included as a special case, the model problem

$$-\Delta u = \frac{a(x)}{u^\alpha} \quad \text{in } \Omega, \qquad u>0 \quad \text{in } \Omega, \qquad u=0 \quad \text{on } \partial\Omega, \tag{1.2}$$

where $\alpha > 0$ and $a\colon \Omega \to [0,\infty)$ is a suitable L^1-function. A broad literature on problems like (1.2) is available to date. We would like to mention [24], [36], [38] and their references. We would like to refer the reader to the very recent paper by Orsina and Petitta [31] who dealt with the problem

$$-\Delta u = \frac{\mu}{u^\alpha} \quad \text{in } \Omega, \qquad u>0 \quad \text{in } \Omega, \qquad u=0 \quad \text{on } \partial\Omega,$$

μ is a nonnegative bounded Radon measure. Other kinds of operators have been addressed and we mention Canino, Sciunzi and Trombetta [7], Chu-Wenjie [9] and De Cave [13] for problems involving the p-Laplacian operator

$$-\operatorname{div}\big(|\nabla u|^{p-2}\,\nabla u\big) = \frac{a(x)}{u^\alpha} \quad \text{in } \Omega, \qquad u>0 \quad \text{in } \Omega, \qquad u=0 \quad \text{on } \partial\Omega;$$

Qihu Zhang [37] and Liu, Zhang and Zhao [28] for $p(x)$-Laplacian operator,

$$-\operatorname{div}\big(|\nabla u|^{p(x)-2}\,\nabla u\big) = \frac{a(x)}{u^\alpha} \quad \text{in } \Omega, \qquad u>0 \quad \text{in } \Omega, \qquad u=0 \quad \text{on } \partial\Omega;$$

Boccardo, Orsina [3] and Bocardo, Casado-Díaz [2] for the problem

$$-\mathrm{div}(M(x)\nabla u) = \frac{a(x)}{u^\alpha} \quad \text{in } \Omega, \qquad u>0 \quad \text{in } \Omega, \qquad u=0 \quad \text{on } \partial\Omega,$$

where M is a suitable matrix, Lazer and McKenna [26]; Gonçalves and Santos [18], Hu and Wang [22] for problems involving the Monge–Ampére operator, e.g.

$$\det(D^2u) = \frac{a(x)}{(-u)^\gamma} \quad \text{in } \Omega, \qquad u<0 \quad \text{in } \Omega, \qquad u=0 \quad \text{on } \partial\Omega,$$

where $a \in C^\infty(\overline{\Omega})$, $a>0$ and $\gamma>1$. Finally, Canino, Montoro, Sciunzi and Squassina [5] considered issues of existence and uniqueness for the fractional p-Laplacian operator.

To the best of our knowledge singular problems like (1.1) in the presence of the operator Δ_Φ were never studied and the main results of this paper (see Section 2) namely Theorems 2.1, 2.3 as well as Corollary 2.2 are new.

Other problems which are special cases of (1.1) are

$$-\Delta_p u - \Delta_q u = a(x)u^{-\alpha} \quad \text{in } \Omega, \qquad u>0 \quad \text{in } \Omega, \qquad u=0 \quad \text{on } \partial\Omega,$$

where $\phi(t) = t^{p-2} + t^{q-2}$ with $1<p<q<N$,

$$-\sum_{i=1}^{N} \Delta_{p_i} u = a(x)u^{-\alpha} \quad \text{in } \Omega, \qquad u>0 \quad \text{in } \Omega, \qquad u=0 \quad \text{on } \partial\Omega.$$

where $\phi(t) = \sum\limits_{j=1}^{N} t^{p_j-2}$, $1<p_1<\ldots<p_N<\infty$ and $\sum\limits_{j=1}^{N} \dfrac{1}{p_j} > 1$,

$$\begin{cases} -\mathrm{div}(a(|\nabla u|^p)|u|^{p-2}\nabla u) = a(x)u^{-\alpha} & \text{in } \Omega, \\ u>0 & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $\phi(t) = a(t^p)t^{p-2}$, $2 \le p < N$ and $a\colon (0,\infty) \to (0,\infty)$ is a suitable $C^1(\mathbb{R}^+)$-function.

We also refer the reader to the paper [29], where the operator Δ_Φ is employed. The operator Δ_Φ appears in applied mathematics, for instance in Plasticity, see e.g. Fukagai and Narukawa [16] and references therein. We refer the reader to [33] for problems involving general operators.

2. Main results

In this work, we will consider that $u \in W^{1,\Phi}_{\mathrm{loc}}(\Omega)$ is a solution of the problem (1.1) if $u>0$ in Ω and $(u-\varepsilon)^+ \in W^{1,\Phi}_0(\Omega)$ for each $\varepsilon>0$. Besides, let us denote by $d(x) = \inf\limits_{y\in\partial\Omega} |x-y|$ the distance of the point $x \in \Omega$ to the boundary of Ω.

Theorem 2.1. *Assume that* (ϕ_1)–(ϕ_3) *and* $a \in L^1(\Omega)$ *hold. Then there is* u *such that* $u^{(\alpha-1+\ell)/\ell} \in W_0^{1,\ell}(\Omega)$, $u \geq Cd$ *almost everywhere in* Ω, *for some* $C > 0$, *and*:

(a) $u \in W_0^{1,\Phi}(\Omega)$, *and*

$$\int_\Omega \phi(|\nabla u|)\nabla u \nabla \varphi \, dx = \int_\Omega \frac{a(x)}{u^\alpha}\varphi \, dx, \quad \varphi \in W_0^{1,\Phi}(\Omega), \tag{2.1}$$

provided additionally that either $ad^{-\alpha} \in L_{\widetilde{\Phi}}(\Omega)$ *or* $0 < \alpha \leq 1$ *and* $a \in L^{\ell^*/(\ell^*+\alpha-1)}(\Omega)$,

(b) $u \in W_{\mathrm{loc}}^{1,\Phi}(\Omega)$, *and*

$$\int_\Omega \phi(|\nabla u|)\nabla u \nabla \varphi \, dx = \int_\Omega \frac{a(x)}{u^\alpha}\varphi \, dx, \quad \varphi \in C_0^\infty(\Omega) \tag{2.2}$$

provided in addition that $\alpha \geq 1$.

Next we will present some regularity results:

Corollary 2.2. *Under the conditions of the Theorem* 2.1, *we have that*:

(a) $u \in C(\overline{\Omega})$ *if* $a \in L^\infty(\Omega)$,

(b) $u \in L^\infty(\Omega)$ *if either* $a \in L^q(\Omega) \cap L^{\ell^*/(\ell^*+\alpha-1)}(\Omega)$ *and* $0 < \alpha \leq 1$ *or* $a \in L^q(\Omega)$ *and* $\alpha > 1$, *where* $N/\ell < q \leq q(\alpha)$ *with*

$$q(s) := \begin{cases} \ell^*/s & \text{if } 0 < s \leq 1, \\ (\ell^* + (\alpha-1)\ell^*/\ell)/s & \text{if } s > 1, \end{cases} \tag{2.3}$$

(c) *there exists a* **unique** *solution to the problem* (1.1) *both in the sense of* (2.1) *and in the sense of* (2.2).

We are going to take advantage of our techniques to show the existence results to the singular-convex problem

$$\begin{aligned} -\mathrm{div}(\phi(|\nabla u|)\nabla u) &= \frac{a(x)}{u^\alpha} + b(x)u^\gamma && \text{in } \Omega, \\ u &> 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{2.4}$$

where $\alpha, \gamma > 0$.

Theorem 2.3. *Assume* $(\phi_1) - (\phi_3)$ *and* $0 \leq \gamma < \ell - 1$. *Assume in addition that* $ad^{-\alpha} \in L_{\widetilde{\Phi}}(\Omega)$ *and* $0 \leq b \in L^\sigma(\Omega)$ *for some* $\sigma > \ell/(\ell - \gamma - 1)$. *Then problem* (2.4) *admits a weak solution* $u \in W_0^{1,\Phi}(\Omega)$ *such that* $u \geq Cd$ *in* Ω *for some constant* $C > 0$. *Besides,* $u \in L^\infty(\Omega)$ *if* $b \in L^\infty(\Omega)$, *and either* $a \in L^q(\Omega) \cap L^{\ell^*/(\ell^*+\alpha-1)}(\Omega)$ *with* $0 < \alpha \leq 1$ *or* $a \in L^q(\Omega)$ *with* $\alpha > 1$, *where* $N/\ell < q \leq q(\alpha + \gamma)$ *and* $q(s)$ *was defined in* (2.3).

REMARK 2.4. We note that:

(a) solutions studied in both Theorems may be found by variational arguments in some particular cases,

(b) if Ψ is an N-function such that $\Phi < \Psi << \Phi_*$, then the conditions $ad^{-\alpha} \in L_{\widetilde{\Psi}}(\Omega)$ and $a \in L^{\widetilde{\Phi}}_{\mathrm{loc}}(\Omega)$ could be used in our results, instead of $ad^{-\alpha} \in L_{\widetilde{\Phi}}(\Omega)$ and $a \in L^{\infty}_{\mathrm{loc}}(\Omega)$, respectively.

3. A family of auxiliary problems

In this section, we are going to "regularize" Problem (2.4) by considering a perturbation by small $\varepsilon > 0$ of the singular term in (2.4). Of course a regularized form of problem (1.1) corresponds to $b = 0$. Let us consider

$$\begin{cases} -\Delta_\Phi u = \dfrac{a_\varepsilon(x)}{(u+\varepsilon)^\alpha} + b_\varepsilon(x)\, u^\gamma & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

for each $\varepsilon > 0$ given, where the $L^\infty(\Omega)$-functions are defined by

$$a_\varepsilon(x) = \min\{a(x), 1/\varepsilon\}, \quad b_\varepsilon(x) = \min\{b(x), 1/\varepsilon\}, \quad x \in \Omega.$$

Consider the map $A := A_\varepsilon \colon W^{1,\Phi}_0(\Omega) \times W^{1,\Phi}_0(\Omega) \to \mathbb{R}$, defined by

$$A(u,\varphi) := \int_\Omega \bigg[\phi(|\nabla u|)\nabla u \nabla\varphi\, dx - \frac{a_\varepsilon(x)\varphi}{(|u|+\varepsilon)^\alpha} - b_\varepsilon(x)(u^+)^\gamma \varphi\bigg]\, dx, \tag{3.2}$$

Thus, finding a weak solution of (3.1) means to find $u \in W^{1,\Phi}_0(\Omega)$ such that

$$A(u,\varphi) = 0 \quad \text{for each } \varphi \in W^{1,\Phi}_0(\Omega). \tag{3.3}$$

PROPOSITION 3.1. *For each $u \in W^{1,\Phi}_0(\Omega)$, the functional $A(u,\,\cdot\,)$ is linear and continuous. In particular, the operator $T := T_\varepsilon \colon W^{1,\Phi}_0(\Omega) \to W^{-1,\widetilde{\Phi}}(\Omega)$ defined by*

$$\langle T(u), \varphi\rangle = A(u,\varphi), \quad \textit{for } u,\varphi \in W^{1,\Phi}_0(\Omega),$$

is linear and continuous, and satisfies

$$\|T(u)\|_{W^{-1,\widetilde{\Phi}}} \leq 2\|\phi(|\nabla u|)\nabla u\|_{\widetilde{\Phi}} + \frac{C}{\varepsilon}\,\|a_\varepsilon\|_{\widetilde{\Phi}} + C\|b_\varepsilon|u|^\gamma\|_{\widetilde{\Phi}}. \tag{3.4}$$

PROOF. Let $u,\varphi \in W^{1,\Phi}_0(\Omega)$. We shall use below the Hölder inequality and the embedding $W^{1,\Phi}_0(\Omega) \hookrightarrow L_\Phi(\Omega)$:

$$\begin{aligned} |A(u,\varphi)| &\leq \int_\Omega \bigg[\phi(|\nabla u|)|\nabla u||\nabla\varphi| + \frac{a_\varepsilon(x)|\varphi|}{\varepsilon^\alpha} + b_\varepsilon(x)(u^+)^\gamma|\varphi|\bigg]\, dx \\ &\leq 2\|\phi(|\nabla u|)\nabla u\|_{\widetilde{\Phi}}\|\varphi\| + \frac{2}{\varepsilon^\alpha}\|a_\varepsilon\|_{\widetilde{\Phi}}\|\varphi\|_\Phi + 2\|b_\varepsilon|u|^\gamma\|_{\widetilde{\Phi}}\|\varphi\|_\Phi \\ &\leq \bigg(2\|\phi(|\nabla u|)\nabla u\|_{\widetilde{\Phi}} + \frac{C}{\varepsilon^\alpha}\,\|a_\varepsilon\|_{\widetilde{\Phi}} + C\|b_\varepsilon|u|^\gamma\|_{\widetilde{\Phi}}\bigg)\|\varphi\|. \end{aligned} \tag{3.5}$$

It is sufficient to show that $\|b_\varepsilon|u|^\gamma\|_{\widetilde{\Phi}} < \infty$. Indeed, by using the embedding $L_\Phi(\Omega) \hookrightarrow L^\ell(\Omega)$ and $\gamma \in (0, \ell - 1)$ it follows by Lemma A.3 that

$$\begin{aligned}\int_\Omega \widetilde{\Phi}(b_\varepsilon(x)|u^\gamma|)\,dx &\leq \max\big\{\|b_\varepsilon\|_\infty^{\ell/(\ell-1)}, \|b_\varepsilon\|_\infty^{m/(m-1)}\big\}\int_\Omega \widetilde{\Phi}(|u|^\gamma)\,dx \\ &\leq C\bigg(\int_{u\leq 1} + \int_{u\geq 1}\bigg)\widetilde{\Phi}(|u|^\gamma)\,dx \\ &\leq C\bigg(|\Omega| + \int_{u\geq 1}|u|^{\gamma\ell/(\ell-1)}\,dx\bigg) \leq C\bigg(|\Omega| + \int_{u\geq 1}|u|^\ell\,dx\bigg) \\ &\leq C\bigg(|\Omega| + \int_\Omega |u|^\ell\,dx\bigg) \leq C(|\Omega| + \|u\|^\ell),\end{aligned}$$

where $C = C(b, \Phi, \varepsilon) > 0$ is a constant. So $A(u, \,\cdot\,)$ is linear and continuous. The claims about T are now immediate.

By Proposition 3.1 the problem of finding a weak solution of (3.1) reduces to finding $u = u_\varepsilon \in W_0^{1,\Phi}(\Omega) \setminus \{0\}$ such that $T(u_\varepsilon) = 0$. This ends the proof. $\square$

4. Applied generalized Galerkin method

In order to find $u = u_\varepsilon \in W_0^{1,\Phi}(\Omega) \setminus \{0\}$ such that $T(u_\varepsilon) = 0$, we shall employ a Galerkin-like method inspired by arguments found in [4]. We are going to constrain the operator T to finite dimensional subspaces. As a first step take a $\omega \in W_0^{1,\Phi}(\Omega)$ such that

$$a\omega \neq 0 \quad \text{and} \quad a\omega \in L^1(\Omega), \tag{4.1}$$

Let $F \subset W_0^{1,\Phi}(\Omega)$ be a finite dimensional subspace such that $\omega \in F$. Now, consider the map $T_F\colon F \to F'$ given by $T_F = I_F' \circ T \circ I_F$, where

$$I_F\colon (F, \|\cdot\|) \to (W_0^{1,\Phi}(\Omega), \|\cdot\|), \quad I_F(u) = u$$

and let I_F' be the adjoint of I_F. So, we have that $T_F = T|_F$, because

$$\langle T_F u, v\rangle = \langle I_F' \circ T \circ I_F u, v\rangle = \langle T \circ I_F u, I_F v\rangle = \langle Tu, v\rangle, \quad u, v \in F,$$

that is, for $u, v \in F$,

$$\langle T_F(u), v\rangle := \int_\Omega \bigg[\phi(|\nabla u|)\nabla u \nabla v - \frac{a_\varepsilon(x)v}{(|u| + \varepsilon)^\alpha} - b_\varepsilon(x)(u^+)^\gamma v\bigg]\,dx. \tag{4.2}$$

The result below, which is a consequence of the Brouwer Fixed Point Theorem (see [27]), will play a central role in solving the finite dimensional equation $T_F(u) = 0$.

Proposition 4.1. *Assume that $S\colon \mathbb{R}^s \to \mathbb{R}^s$ is a continuous map such that $(S(\eta), \eta) > 0$, $|\eta| = r$ for some $r > 0$, where $(\,\cdot\,,\,\cdot\,)$ is the usual inner product in $\mathbb{R}^s$ and $|\cdot|$ is its corresponding norm. Then, there is $\eta_0 \in B_r(0)$ such that $S(\eta_0) = 0$.*

Proposition 4.2. *The operator T_F is continuous.*

PROOF. Let $(u_n) \subseteq F$ be a sequence such that $u_n \to u$ in F. Since, the operator $-\Delta_\Phi \colon W^{1,\Phi}_0(\Omega) \to W^{-1,\widetilde{\Phi}}(\Omega)$ given by

$$\langle -\Delta_\Phi u, v\rangle := \int_\Omega \phi(|\nabla u|)\nabla u \nabla v \, dx, \quad u, v \in W^{1,\Phi}_0(\Omega),$$

is continuous (see [16, Lemma 3.1]), we have that $\Delta_\Phi|_F$ is also continuous.

To complete our proof, it remains to show that $T_F - \Delta_\Phi|_F$ is continuous. By applying Lemma A.4 and the embedding $L_\Phi(\Omega) \hookrightarrow L^\ell(\Omega)$, it follows, by passing to a subsequence if necessary, that

(1) $u_n \to u$ almost everywhere in Ω;
(2) there is $h \in L^\ell(\Omega)$ such that $|u_n| \le h$.

Then, for each $v \in W^{1,\Phi}_0(\Omega)$,

$$\frac{a_\varepsilon(x)v}{(|u_n| + \varepsilon)^\alpha} \to \frac{a_\varepsilon(x)v}{(|u| + \varepsilon)^\alpha}, \qquad b_\varepsilon(x)(u_n^+)^\gamma v \to b_\varepsilon(x)(u^+)^\gamma v \quad \text{a.e. in } \Omega.$$

On the other hand, since $\widetilde{\Phi}$ is increasing, we obtain

$$\begin{aligned}\text{(4.3)}\quad \widetilde{\Phi}\bigg(\bigg|\frac{a_\varepsilon(x)}{(|u_n| + \varepsilon)^\alpha} - \frac{a_\varepsilon(x)}{(|u| + \varepsilon)^\alpha}\bigg|\bigg) \\ \le \widetilde{\Phi}\bigg(\frac{a_\varepsilon(x)}{(|u_n| + \varepsilon)^\alpha} + \frac{a_\varepsilon(x)}{(|u| + \varepsilon)^\alpha}\bigg) \le \widetilde{\Phi}\bigg(\frac{2a_\varepsilon(x)}{\varepsilon^\alpha}\bigg) \in L^1(\Omega),\end{aligned}$$

because $0 \le a_\varepsilon \le 1/\varepsilon$. So, by Lebesgue's Theorem,

$$\int_\Omega \widetilde{\Phi}\bigg(\bigg|\frac{a_\varepsilon(x)}{(|u_n| + \varepsilon)^\alpha} - \frac{a_\varepsilon(x)}{(|u| + \varepsilon)^\alpha}\bigg|\bigg)\, dx \to 0,$$

and as a consequence of $\widetilde{\Phi} \in \Delta_2$, we have

$$\bigg\|\frac{a_\varepsilon(x)}{(|u_n| + \varepsilon)^\alpha} - \frac{a_\varepsilon(x)}{(|u| + \varepsilon)^\alpha}\bigg\|_{\widetilde{\Phi}} \to 0.$$

By applying the Hölder's inequality, we find that, for each $v \in W^{1,\Phi}_0(\Omega)$,

$$\bigg|\int_\Omega \bigg(\frac{a_\varepsilon(x)}{(|u_n| + \varepsilon)^\alpha} - \frac{a_\varepsilon(x)}{(|u| + \varepsilon)^\alpha}\bigg) v\, dx\bigg| \le 2\bigg\|\frac{a_\varepsilon(x)}{(|u_n| + \varepsilon)^\alpha} - \frac{a_\varepsilon(x)}{(|u| + \varepsilon)^\alpha}\bigg\|_{\widetilde{\Phi}} \|v\|_\Phi \to 0.$$

Estimating as in (4.3), we have

$$\begin{aligned}\widetilde{\Phi}(b_\varepsilon|(u_n^+)^\gamma - (u^+)^\gamma|) &\le \widetilde{\Phi}\bigg(2|b_\varepsilon|_\infty \frac{(u_n^+)^\gamma + (u^+)^\gamma}{2}\bigg) \\ &\le C\Big(\widetilde{\Phi}((u_n^+)^\gamma) + \widetilde{\Phi}((u^+)^\gamma)\Big) \le C\big(|u|^\ell + |h|^\ell + 2\big) \in L^1(\Omega),\end{aligned}$$

for some $C = C(a, \Phi, \varepsilon) > 0$. Arguing as above, we obtain

$$\int_\Omega b_\varepsilon(x)\big[(u_n^+)^\gamma - (u^+)^\gamma\big] v\, dx \to 0$$

showing that T_F is continuous. $\square$

Proposition 4.3. *There exists $0 \neq u = u_F = u_{\varepsilon,F} \in F$ such that $T_F(u) = 0$ for each $\varepsilon > 0$ sufficiently small.*

Proof. Let $s := \dim F$ be the dimension of the subspace F, and set $F = \operatorname{span}\{e_1, \ldots, e_s\}$. That is, each $u \in F$ is uniquely expressed as

$$u = \sum_{j=1}^{s} \xi_j e_j, \quad \xi = (\xi_1, \ldots, \xi_s) \in \mathbb{R}^s.$$

Set $|\xi| := \|u\|$ and consider the map $i = i_F : (\mathbb{R}^s, |\cdot|) \to (F, \|\cdot\|)$ given by $i(\xi) = u$. So, it follows by Proposition 4.2 and the fact that i is an isometry that the operator $S_F \colon \mathbb{R}^s \to \mathbb{R}^s$ given by

$$S_F := i' \circ T_F \circ i \tag{4.4}$$

is continuous as well, where i' is the adjoint of i. Besides, by setting $u := i(\xi)$ for $\xi \in \mathbb{R}^s$, it follows from (ϕ_3) and the embeddings $W_0^{1,\Phi}(\Omega) \hookrightarrow L_\Phi(\Omega) \hookrightarrow L^\ell(\Omega) \hookrightarrow L^{\gamma+1}(\Omega)$ that

$$\begin{aligned}(S_F\xi, \xi) &= (i' \circ T_F \circ i(\xi), \xi) = \langle T_F(u), u\rangle \\ &\geq \int_\Omega \bigg[\phi(|\nabla u|)|\nabla u|^2 - \frac{a_\varepsilon(x)|u|}{\varepsilon^\alpha} - b_\varepsilon(x)|u|^{\gamma+1}\bigg]\,dx \\ &\geq \ell \int_\Omega \Phi(|\nabla u|)\,dx - \frac{1}{\varepsilon^\alpha}\|a_\varepsilon\|_{\widetilde{\Phi}}\|u\|_\Phi - |b_\varepsilon|_\infty |u|_{\gamma+1}^{\gamma+1} \\ &\geq \ell \min\{\|u\|^\ell, \|u\|^m\} - C_1\|u\| - C_2\|u\|^{\gamma+1},\end{aligned} \tag{4.5}$$

for some positive constants $C_1 = C_1(\varepsilon)$ and $C_2 = C_2(\varepsilon)$. So, we can choose an $r_0 = r_0(\varepsilon) > 1$ such that $\ell r_0^\ell - C_1 r_0 - C_2 r_0^{\gamma+1} > 0$. More specifically, for each ξ such that $|\xi| = r_0$, we have $(S_F\xi, \xi) > 0$.

By the above, from Proposition 4.1 it follows that there exists a $\xi_F \in \overline{B}_{r_0}(0)$ such that $S_F(\xi_F) = 0$, that is, letting $u = u_F = i(\xi_F)$, it follows from (4.4), that

$$\langle T_F(u), v\rangle = (S_F(\xi_F), \eta) = 0 \quad \text{for all } v \in F,$$

where $v = i(\eta)$, and hence $T_F(u) = 0$. As a consequence of this, we have

$$\int_\Omega \bigg[\phi(|\nabla u|)\nabla u \nabla v - \frac{a_\varepsilon(x)v}{(|u| + \varepsilon)^\alpha} - b_\varepsilon(x)(u^+)^\gamma v\bigg]\,dx = 0 \quad \text{for all } v \in F.$$

We claim that $u = u_\varepsilon \neq 0$ for enough small $\varepsilon > 0$. Indeed, otherwise by taking $v = w$ and using Lebesgue's Theorem, we obtain

$$\int_\Omega a(x)w\,dx = \lim_{\varepsilon \to 0} \int_\Omega a_\varepsilon(x)w\,dx = 0,$$

but this is impossible by (4.1). $\square$

The result below is a direct consequence of the above proved proposition.

Corollary 4.4. *The number $r_0 > 0$ and the function $u_F \in F$ found above satisfy: $\|u_F\| \le r_0$, $T_F(u_F) = 0$, and $r_0 > 0$ does not depends on subspace $F \subset W_0^{1,\Phi}(\Omega)$ with $0 < \dim F < \infty$. Besides, we can choose it independent of $\varepsilon > 0$ as well if $0 < \alpha \le 1$, $a \in L^{\ell^*/(\ell^*+\alpha-1)}(\Omega)$, and $b \in L^\sigma(\Omega)$ for some $\sigma > \ell/(\ell-\gamma-1)$.*

Proof. The first part of it was proved above. To show that r_0 does not depends on $\varepsilon > 0$, we just redo the estimatives in (4.5) by using the hypotheses on a and b. $\square$

Our aim below is to build a non-zero vector $u_\varepsilon \in W_0^{1,\Phi}(\Omega)$ such that $T(u_\varepsilon) = 0$, where T was given by Proposition 3.1. This will provide us with some $u_\varepsilon \in W_0^{1,\Phi}(\Omega)$ such that

$$\int_\Omega \left[\phi(|\nabla u|)\nabla u \nabla\varphi - \frac{a_\varepsilon(x)\varphi}{(|u|+\varepsilon)^\alpha} - b_\varepsilon(x)(u^+)^\gamma\varphi\right] dx = 0, \quad \varphi \in W_0^{1,\Phi}(\Omega). \tag{4.6}$$

In this direction we have

Lemma 4.5. *There is a non-zero vector $u_\varepsilon \in W_0^{1,\Phi}(\Omega)$ such that $T(u_\varepsilon) = 0$ or equivalently* (4.6) *holds true.*

Proof. Let w as in (4.1) and set

$$\mathcal{A} = \big\{F \subset W_0^{1,\Phi}(\Omega) \mid F \text{ is a finite dimensional subspace of } W_0^{1,\Phi}(\Omega) \text{ and } \omega \in F\big\}.$$

We assume that $\mathcal{A}$ is partially ordered by set inclusion. Take $F_0 \in \mathcal{A}$ and set

$$V_{F_0} = \big\{u_F \in F \mid F \in \mathcal{A},\ F_0 \subset F,\ T_F(u_F) = 0 \text{ and } \|u_F\| \le r_0\big\}.$$

Note that, by Proposition 4.3 and Corolary 4.4, $V_{F_0} \neq \emptyset$.

Since $V_{F_0} \subset \overline{B_{r_0}}(0)$, then $\overline{V}^\sigma_{F_0} \subset \overline{B_{r_0}}(0)$, where $\overline{V}^\sigma_{F_0}$ denotes the weak closure of V_{F_0}. As a matter of this fact, $\overline{V}^\sigma_{F_0}$ is weakly compact.

Consider the family $\mathcal{B} := \big\{\overline{V}^\sigma_F \mid F \in \mathcal{A}\big\}$.

Claim $\mathcal{B}$ has the finite intersection property.

Proof of Claim. Indeed, let $\{\overline{V}^\sigma_{F_1}, \dots, \overline{V}^\sigma_{F_p}\}$ be a finite subfamily of $\mathcal{B}$ and set $F := \operatorname{span}\{F_1, \dots, F_p\}$. By the very definition of V_{F_i}, we have that $u_F \in \overline{V}^\sigma_{F_i}$, $i = 1, \dots, p$, that is

$$\bigcap_{i=1}^p \overline{V}^\sigma_{F_i} \neq \emptyset. \qquad \square$$

Since $\overline{B}_{r_0}$ is weakly compact, it follows that (cf. [30, Theorem 26.9])

$$W := \bigcap_{F\in\mathcal{A}} \overline{V}^\sigma_F \neq \emptyset.$$

Let $u_\varepsilon \in W$. Then $u_\varepsilon \in \overline{V}^\sigma_F$ for each $F \in \mathcal{A}$.

Take $F_0 \in \mathcal{A}$ such that $\operatorname{span}\{\omega, u_\varepsilon\} \subset F_0$. Since $u_\varepsilon \in \overline{V}^{\sigma}_{F_0}$, it follows by [14, Theorem 1.5] and the definition of V_{F_0} that there are sequences $(u_n) = (u_{n,\varepsilon}) \subset V_{F_0}$ and $(F_n) = (F_{n,\varepsilon}) \subset \mathcal{A}$ such that $u_n \rightharpoonup u_\varepsilon$ in $W_0^{1,\Phi}(\Omega)$, $u_n \in F_n$, $\|u_n\| \le r_0$, $F_0 \subset F_n$, and for each $v \in F_n$

$$\int_\Omega \phi(|\nabla u_n|)\nabla u_n \nabla v\, dx = \int_\Omega \bigg(\frac{a_\varepsilon(x)}{(|u_n| + \varepsilon)^\alpha} + b_\varepsilon(x)(u_n^+)^\gamma\bigg) v\, dx. \tag{4.7}$$

Now, by eventually taking subsequences and using $W_0^{1,\Phi}(\Omega) \overset{\text{comp}}{\hookrightarrow} L_\Phi(\Omega)$, we obtain that $u_n \to u_\varepsilon$ in $L_\Phi(\Omega)$, $u_n \to u_\varepsilon$ almost everywhere in Ω and $(|u_n|)$ is bounded away by some function in $L_\Phi(\Omega)$. Set $v_n = u_n - u_\varepsilon$ and note that $v_n \in F_n$, because $u_n \in F_n$ and $u_\varepsilon \in F_0 \subset F_n$ in (4.7). Then

$$\begin{aligned}\lim\langle -\Delta_\Phi(u_n), u_n - u_\varepsilon\rangle &\\ &= \lim \int_\Omega \bigg(\frac{a_\varepsilon(x)}{(|u_n| + \varepsilon)^\alpha} + b_\varepsilon(x)\big(u_n^+\big)^\gamma\bigg)(u_n - u_\varepsilon)\, dx \\ &\le \lim \int_\Omega \bigg(\frac{a_\varepsilon(x)}{\varepsilon^\alpha} + b_\varepsilon(x)|u_n|^\gamma\bigg)|u_n - u_\varepsilon\, |dx.\end{aligned} \tag{4.8}$$

As $W_0^{1,\Phi}(\Omega) \overset{\text{comp}}{\hookrightarrow} L_\Phi(\Omega)$, we have

$$\bigg| \int_\Omega \frac{a_\varepsilon(x)}{\varepsilon^\alpha}\,(u_n - u_0)\, dx\bigg| \le q\frac{1}{\varepsilon^\alpha}\, \|a_\varepsilon\|_{\widetilde{\Phi}}\, \|u_n - u_\varepsilon\|_\Phi \to 0.$$

Recalling that $\gamma < \ell - 1$, $W_0^{1,\Phi}(\Omega) \hookrightarrow L^\ell(\Omega)$ and (u_n) is bounded in $L^\ell(\Omega)$, we get

$$\begin{aligned}\int_\Omega b_\varepsilon(x)|u_n|^\gamma |u_n - u_\varepsilon|\, dx &\le |b_\varepsilon|_\infty \bigg(\int_\Omega |u_n|^{\gamma\ell/(\ell-1)}\, dx\bigg)^{(\ell-1)/\ell} |u_n - u_\varepsilon|_\ell \\ &\le |b_\varepsilon|_\infty \bigg(|\Omega| + \int_\Omega |u_n|^\ell\, dx\bigg)^{(\ell-1)/\ell} |u_n - u_\varepsilon|_\ell \to 0.\end{aligned}$$

Now, by using the facts above, it follows from (4.7) that

$$\lim\langle -\Delta_\Phi(u_n), u_n - u_\varepsilon\rangle \le 0,$$

and a consequence of this, we have that $u_n \to u_\varepsilon$ in $W_0^{1,\Phi}(\Omega)$, because $-\Delta_\Phi$ satisfies the condition (S_+) (see [8, Proposition A.2]). So, passing to a subsequence if necessary, we have

(1) $\nabla u_n \to \nabla u_\varepsilon$ almost everywhere in Ω,
(2) there is $h \in L_\Phi(\Omega)$ such that $|\nabla u_n| \le h$.

Since $\varphi \in W_0^{1,\Phi}(\Omega)$, it follows of the fact that $t\phi(t)$ is nondecreasing in $[0, \infty)$ and (2), that

$$\begin{aligned}|\phi(|\nabla u_n|)\nabla u_n \nabla\varphi| &\le \phi(|\nabla u_n|)|\nabla u_n||\nabla\varphi| \le \phi(h)h|\nabla\varphi| \\ &\le \widetilde{\Phi}(\phi(h)h) + \Phi(|\nabla\varphi|) \le \Phi(2h) + \Phi(|\nabla\varphi|) \in L^1(\Omega),\end{aligned}$$

that is, it follows by the Lebesgue Theorem, that

$$\int_\Omega \phi(|\nabla u_n|)\nabla u_n \nabla\varphi\, dx \to \int_\Omega \phi(|\nabla u_\varepsilon|)\nabla u_\varepsilon \nabla\varphi\, dx.$$

Now, by passing to the limit in (4.7) and using the above information, we get that u_ε satisfies (4.6), that is, $T_\varepsilon(u_\varepsilon) = T(u_\varepsilon) = 0$ for each $\varepsilon > 0$, since $\varphi \in W^{1,\Phi}_0(\Omega)$ was taken arbitrarily. By arguments as in the proof of Proposition 4.3 we infer that $u_\varepsilon \not\equiv 0$. $\square$

LEMMA 4.6. *The function $u_\varepsilon \in C^{1,\alpha_\varepsilon}(\overline{\Omega})$, for some $0 < \alpha_\varepsilon \le 1$, and it is a solution of* (3.1).

PROOF. By Lemma 4.5, it remains to show that $u_\varepsilon > 0$. Set $-u^-_\varepsilon$ as a test function in (4.6). So, it follows by Remark A.1 (see Appendix), that

$$\ell\int_\Omega \Phi(|\nabla u^-_\varepsilon|)\, dx \le \int_\Omega \phi(|\nabla u^-_\varepsilon|)|\nabla u^-_\varepsilon|^2\, dx = -\int_\Omega \frac{a_\varepsilon(x)}{(|u^-_\varepsilon| + \varepsilon)^\alpha} u^-_\varepsilon\, dx,$$

which implies that $u^-_\varepsilon \equiv 0$. So, for all $\varphi \in W^{1,\Phi}_0(\Omega)$, u_ε satisfies

$$\int_\Omega \phi(|\nabla u_\varepsilon|)\nabla u_\varepsilon\nabla\varphi = \int_\Omega \frac{a_\varepsilon(x)}{(u_\varepsilon+\varepsilon)^\alpha}\varphi\, dx + \int_\Omega b_\varepsilon(x) u^\gamma_\varepsilon \varphi\, dx. \tag{4.9}$$

Finally, for each $p \in (m, \ell^*)$, it follows that

$$|f(x,t)| := \frac{a_\varepsilon(x)}{(|t|+\varepsilon)^\alpha} + b_\varepsilon(x)(t^+)^\gamma \le C_\varepsilon\big(1 + |t|^{p-1}\big) \quad \text{and} \quad \lim_{t\to\infty}\frac{t^p}{\Phi_*(\lambda t)} = 0$$

for each $\varepsilon, \lambda > 0$ given. So, by [35, Corollary 3.1], $u_\varepsilon \in C^{1,\alpha_\varepsilon}(\overline{\Omega})$ for some $0 < \alpha_\varepsilon \le 1$. Now, by summing up the term $u_\varepsilon\phi(u_\varepsilon)$ to both sides of (4.9) and applying [8, Proposition 5.2] we infer that $u_\varepsilon > 0$. In conclusion, u_ε is a solution of (3.1). $\square$

5. Comparison of solutions and estimates

Let $n \ge 1$ be an integer and take $\varepsilon = 1/n$. Let $u_n \in W^{1,\Phi}_0(\Omega) \cap C^{1,\alpha_n}(\overline{\Omega})$, for some $\alpha_n \in (0,1]$, denotes the solution of (3.1), both for $b = 0$ and $b \ge 0$ not identically null, given by Lemma 4.6, that is,

$$\begin{cases} -\Delta_\Phi u_n = \dfrac{a_n(x)}{(u_n + 1/n)^\alpha} + b_n(x)u^\gamma & \text{in } \Omega,\\ u_n > 0 & \text{in } \Omega,\\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.1}$$

We have the following result on comparison of solutions.

LEMMA 5.1. *The following inequalities hold*:

(a) $u_n + 1/n \ge u_1$ *for each integer* $n \ge 1$,
(b) $u_1 \ge Cd$ *in* Ω *for some* $C > 0$ *which independs of* n.

Proof. First, we consider $b=0$ in (5.1), that is,

$$-\Delta_\Phi u_n = \frac{a_n(x)}{(u_n+1/n)^\alpha} \quad \text{in } \Omega, \qquad u_n>0 \quad \text{in } \Omega, \qquad u_n=0 \quad \text{on } \partial\Omega. \tag{5.2}$$

So, by (5.2), we have

$$\operatorname{div}(\phi(|\nabla u_1|)\nabla u_1) - \frac{a_1(x)}{(u_1+1)^\alpha} \geq 0 \quad \text{in } \Omega, \tag{5.3}$$

in the weak sense. On the other hand, since

$$\frac{a_n(x)}{(w_n+1/n)^\alpha} \geq \frac{a_1(x)}{((w_n+1/n)+1)^\alpha} \quad \text{in } \Omega.$$

we get by (5.2) that

$$\operatorname{div}(\phi(|\nabla (u_n+1/n)|)\nabla (u_n+1/n)) - \frac{a_1(x)}{((u_n+1/n)+1)^\alpha} \leq 0 \quad \text{in } \Omega, \tag{5.4}$$

in the weak sense, (test finctions are taken non-negative). By applying Theorem 2.4.1 in [32] to (5.3) and (5.4), we obtain $u_n+1/n \geq u_1$.

Now, since $\partial\Omega$ is smooth, it follows by [17, Lemma 14.16] that the distance function $x \mapsto d(x)$ satisfies

$$d \in C^2(\overline{\Omega}), \quad d>0 \quad \text{on } \overline{\Omega}_\delta \quad \text{and} \quad \frac{\partial d}{\partial \eta} < 0 \quad \text{on } \overline{\Omega} \setminus \Omega_\delta,$$

where $\Omega_\delta = \{x \in \overline{\Omega} \mid d(x) > \delta\}$ for some $\delta > 0$, and η stands for the exterior unit normal to $\partial\Omega$.

Now, since $u_1 \in W_0^{1,\Phi}(\Omega) \cap C^{1,\alpha_1}(\overline{\Omega})$ is a solution of

$$-\Delta_\Phi u = \frac{a_1(x)}{(u+1)^\alpha} \quad \text{in } \Omega, \qquad u>0 \quad \text{in } \Omega, \qquad u=0 \quad \text{on } \partial\Omega, \tag{5.5}$$

it follows by [35, Lemma 4.2] that

$$\frac{\partial u_1}{\partial \eta} < 0 \quad \text{on } \overline{\Omega} \setminus \Omega_\delta.$$

So there is a constant $C>0$ such that

$$\frac{\partial u_1}{\partial \eta} \leq C \frac{\partial d}{\partial \eta} \quad \text{on } \overline{\Omega} \setminus \Omega_\delta,$$

and as a consequence

$$Cd(x) \leq u_1(x) \quad \text{for } x \in \Omega. \tag{5.6}$$

This ends the proof of Lemma 5.1 for $b=0$. If b is not identically null, we redo the above proof by considering (5.3) and obtaining (5.4) as a consequence of b be non-negative. $\square$

We have the following estimates.

LEMMA 5.2. *Let $u_n \in C^{1,\alpha_n}(\overline{\Omega})$ be a solution of (5.2). Then there is a constant $C > 0$ such that*

$$\big\| \big[(u_n + 1/n)^{(\alpha+\ell-1)/\ell} - (1/n)^{(\alpha+\ell-1)/\ell}\big] \big\|_{1,\ell} \leq C, \quad \textit{for all integer } n \geq 1.$$

where $\|\cdot\|_{1,\ell}$ above is the norm of $W_0^{1,\ell}$.

PROOF. At first notice that

$$u_n,\ \big[(u_n + 1/n)^{\alpha} - (1/n)^{\alpha}\big] \in W_0^{1,\Phi}(\Omega) \cap C^{1,\alpha_n}(\overline{\Omega}) \subset W_0^{1,\ell}(\Omega).$$

By estimating, we get

$$\begin{aligned}
\ell\alpha\Phi(1) &\int_{|\nabla u_n|\geq 1} |\nabla u_n|^{\ell}\bigg(u_n + \frac{1}{n}\bigg)^{\alpha-1} dx \\
&\leq \ell\alpha\Phi(1)\bigg[\int_{|\nabla u_n|<1} |\nabla u_n|^{m}\bigg(u_n + \frac{1}{n}\bigg)^{\alpha-1} dx \\
&\qquad + \int_{|\nabla u_n|\geq 1} |\nabla u_n|^{\ell}\bigg(u_n + \frac{1}{n}\bigg)^{\alpha-1} dx\bigg] \\
&= \ell\alpha\Phi(1)\int_{\Omega} \min\big\{|\nabla u_n|^{\ell}, |\nabla u_n|^{m}\big\}\bigg(u_n + \frac{1}{n}\bigg)^{\alpha-1} dx.
\end{aligned}$$

Applying Remark A.1 and Lemma A.2 and using $[(u_n + 1/n)^{\alpha} - (1/n)^{\alpha}]$ as a test function in (5.2), we find

$$\begin{aligned}
\text{(5.7)}\qquad \ell\alpha\Phi(1) &\int_{|\nabla u_n|\geq 1} |\nabla u_n|^{\ell}(u_n + 1/n)^{\alpha-1}\, dx \\
&\leq \ell\alpha \int_{\Omega} \Phi(|\nabla u_n|)(u_n + 1/n)^{\alpha-1}\, dx \\
&\leq \alpha \int_{\Omega} \phi(|\nabla u_n|)|\nabla u_n|^{2}(u_n + 1/n)^{\alpha-1}\, dx \\
&= \int_{\Omega} \frac{a_n(x)[(u_n + 1/n)^{\alpha} - (1/n)^{\alpha}]}{(u_n + 1/n)^{\alpha}}\, dx \leq |a|_1.
\end{aligned}$$

When $\alpha \leq 1$, it follows from Lemma 5.1, that

$$\begin{aligned}
\text{(5.8)}\quad \ell\alpha\Phi(1) &\int_{|\nabla u_n|\leq 1} |\nabla u_n|^{\ell}\bigg(u_n + \frac{1}{n}\bigg)^{\alpha-1} dx \\
&\leq \ell\alpha\Phi(1)\bigg[|\Omega| + C^{\alpha-1}\int_{\Omega} d(x)^{\alpha-1}\bigg] := D,
\end{aligned}$$

which is finite, by a well known result, cf. Lazer and McKenna [25].

From (5.7) and (5.8) it follows that

$$\int_{\Omega} \bigg|\nabla\bigg(\bigg(u_n + \frac{1}{n}\bigg)^{(\alpha-1+\ell)/\ell}\bigg)\bigg|^{\ell} dx \leq \bigg(\frac{\alpha+\ell-1}{\ell}\bigg)^{\ell} \frac{1}{\ell\alpha\Phi(1)}(\|a\|_1 + D),$$

because

$$\bigg|\nabla\bigg(\bigg(u_n+\frac{1}{n}\bigg)^{(\alpha+\ell-1)/\ell}\bigg)\bigg|^\ell=\bigg(\frac{\alpha+\ell-1}{\ell}\bigg)^\ell|\nabla u_n|^\ell\bigg(u_n+\frac{1}{n}\bigg)^{\alpha-1}.$$

Hence, $[(u_n+1/n)^{(\alpha+\ell-1)/\ell}-(1/n)^{(\alpha+\ell-1)/\ell}]$ is bounded in $W_0^{1,\ell}(\Omega)$.

When $\alpha>1$, we have

$$\begin{aligned}\ell\alpha\Phi(1)\int_{|\nabla u_n|\leq 1}|\nabla u_n|^\ell\bigg(u_n+\frac{1}{n}\bigg)^{\alpha-1}\,dx\qquad\qquad\\ \leq\ell\alpha\Phi(1)\bigg[|\Omega|+\int_{u_n>1}\bigg(u_n+\frac{1}{n}\bigg)^{\alpha-1}\,dx\bigg].\end{aligned}\tag{5.9}$$

Summing up (5.7) and (5.9), we obtain a positive constant C such that

$$\int_\Omega\bigg|\nabla\bigg(\bigg(u_n+\frac{1}{n}\bigg)^{(\alpha-1+\ell)/\ell}\bigg)\bigg|^\ell\leq C\bigg(1+\int_{u_n>1}\bigg(u_n+\frac{1}{n}\bigg)^{\alpha-1}\bigg).\tag{5.10}$$

Now, by picking ε such that $0<\varepsilon<\ell-\ell(\alpha-1)/(\alpha+\ell-1)$, it follows from (5.10), using $u_n>1$ and of the embbeding $W_0^{1,\ell}(\Omega)\hookrightarrow L^\ell(\Omega)\hookrightarrow L^{\ell-\varepsilon}(\Omega)$, that

$$\begin{aligned}\bigg\|\nabla\bigg(\bigg(u_n+\frac{1}{n}\bigg)^{(\alpha-1+\ell)/\ell}\bigg)\bigg\|_\ell^\ell&\leq C\bigg(1+\int_{u_n>1}\bigg(\bigg(u_n+\frac{1}{n}\bigg)^{(\alpha+\ell-1)/\ell}\bigg)^{\ell-\varepsilon}dx\bigg)\\&\leq C\bigg(1+\bigg\|\nabla\bigg(\bigg(u_n+\frac{1}{n}\bigg)^{(\alpha+\ell-1)/\ell}\bigg)\bigg\|_\ell^{\ell-\varepsilon}\bigg),\end{aligned}$$

for some $C>0$. That is, $[(u_n+1/n)^{(\alpha+\ell-1)/\ell}-(1/n)^{(\alpha+\ell-1)/\ell}]$ is bounded in $W_0^{1,\ell}(\Omega)$ as well. □

6. Proof of the main results

We begin proving Theorem 2.1 that treats about existence of positive solution to the pure singular problem (1.1).

6.1. Pure singular problem – existence of solutions.

Proof of (a) of Theorem 2.1. Assume first that $ad^{-\alpha}\in L_{\widetilde{\Phi}}(\Omega)$. Since $u_n\in W_0^{1,\Phi}(\Omega)$ satisfies (5.2), it follows from Remark A.1, Lemma A.2, (5.6) and Hölder inequality, that

$$\begin{aligned}\ell\zeta_0(\|\nabla u_n\|_\Phi)\leq\ell\int_\Omega\Phi(|\nabla u_n|)\,dx&\leq\int_\Omega\phi(|\nabla u_n|)|\nabla u_n|^2\,dx\\&=\int_\Omega\frac{a_n(x)}{(u_n+1/n)^\alpha}u_n\,dx\leq C\int_\Omega\frac{a(x)}{d^\alpha}|u_n|\,dx\\&=C\bigg(\int_{\Omega/\Omega_\delta}+\int_{\Omega_\delta}\bigg)\frac{a(x)}{d^\alpha}|u_n|\,dx\end{aligned}\tag{6.1}$$

$$\begin{aligned}&\le C\int_\Omega |u_n|\,dx + C\int_\Omega \frac{a(x)}{d^\alpha(x)}|u_n|\,dx\\ &\le C\|u_n\|_\Phi + 2C\left\|\frac{a}{d^\alpha}\right\|_{\widetilde{\Phi}}\|u_n\|_\Phi,\end{aligned}$$

where we used $a_n \le a$ just above. It follows from our assumptions and from $W_0^{1,\Phi}(\Omega) \overset{\mathrm{cpt}}{\hookrightarrow} L_\Phi(\Omega)$, that $(u_n) \subset W_0^{1,\Phi}(\Omega)$ is bounded. If $0 < \alpha \le 1$ and $a \in L^{\ell^*/(\ell^*+\alpha-1)}(\Omega)$, then the boundedness of (u_n) in $W_0^{1,\Phi}(\Omega)$ is a consequence of Corollary 4.4. So, in both cases, up to subsequences, there exist $u \in W_0^{1,\Phi}(\Omega)$ and $\theta \in L_\Phi(\Omega)$ such that

(1) $u_n \rightharpoonup u$ in $W_0^{1,\Phi}(\Omega)$,
(2) $u_n \to u$ in $L_\Phi(\Omega)$,
(3) $u_n \to u$ almost everywhere in Ω,
(4) $0 \le u_n \le \theta$.

As a first consequence of these facts, it follows from Lemma 5.1 and (3) that $u \ge Cd$ almost everywhere in Ω.

Now, by using $u_n - u$ as a test function in (5.2) and following similar arguments as in (4.8), we get

$$\begin{aligned}\langle -\Delta_\Phi u_n, u_n - u\rangle &\le \left|\int_\Omega \frac{a_n(x)}{(u_n+1/n)^\alpha}(u_n-u)\,dx\right|\\ &\le \left[C + 2\left\|\frac{a}{d^\alpha}\right\|_{\widetilde{\Phi}}\right]\|u_n-u\|_\Phi\end{aligned} \tag{6.2}$$

for some $C > 0$ independent of n. Since, the operator $-\Delta_\Phi$ is of the type S_+, it follows from (2) and (6.2) that $u_n \to u$ in $W_0^{1,\Phi}(\Omega)$.

To finish our proof, given $\varphi \in W_0^{1,\Phi}(\Omega)$, it follows from Lemma 5.1, that

$$\left|\frac{a_n}{(u_n+1/n)^\alpha}\,\varphi\right| \le \frac{a}{d^\alpha}\left(\frac{d}{u_n+1/n}\right)^\alpha|\varphi| \le C\,\frac{a}{d^\alpha}|\varphi| \in L^1(\Omega),$$

that is, by passing to the limit in (5.2), we obtain that u is a solution of (1.1). □

We were not able to employ the above arguments in the proof of (b) of Theorem 2.1, because in this case we do not know if a/d^α belongs to $L_{\widetilde{\Phi}}(\Omega)$, that is, the sequence (u_n) is likely not bounded in $W_0^{1,\Phi}(\Omega)$. Instead, it is possible to show that (u_n) is bounded in $W^{1,\Phi}_{\mathrm{loc}}(\Omega)$. This was done by applying Lemma 5.2.

PROOF OF (b) OF THEOREM 2.1. Given $U \subset\subset \Omega$, let $\delta_U = \min\{d(x) \mid x \in U\} > 0$. So, it follows from Lemma 5.1, that $u_n + 1/n \ge C\delta_U := C_U > 0$ in U, that is, for $n > 1$ enough big, we can take $(u_n + 1/n - C_U)^+$ as a test function in (5.2), to obtain

$$\begin{aligned}
(6.3)\qquad \int_U \phi(|\nabla u_n|)|\nabla u_n|^2 &\leq \int_{u_n+1/n\geq C_U} \phi(|\nabla u_n|)|\nabla u_n|^2\,dx \\
&\leq \int_{u_n+1/n\geq C_U} \frac{a(x)}{(u_n+1/n)^{\alpha-1}}\,dx \\
&\leq \frac{1}{C_U^{\alpha-1}}\int_\Omega a\,dx < \infty,
\end{aligned}$$

because $a \in L^1(\Omega)$, and $\alpha \geq 1$. So, from Remark A.1 and Lemma A.2 it follows that $(u_n) \subset W^{1,\Phi}(U)$ is bounded. That is, there exist $(u^U_{n_1}), u^U \in W^{1,\Phi}(U)$ such that $u^U_{n_1} \rightharpoonup u^U$ in $W^{1,\Phi}(U)$, $u^U_{n_1} \to u^U$ in $L_\Phi(U)$, $u^U_{n_1}(x) \to u^U(x)$ almost everywher in U. In particular, from Lemma 5.1 and of the pointwise convergence it follows that $u \geq Cd$ almost everywhere in U. Hence, by using a Cantor diagonalization argument applied to an exhaustion U_k of Ω with $U_k \subset\subset U_{k+1} \subset\subset \Omega$, we show that there is $u \in W^{1,\Phi}_{\mathrm{loc}}(\Omega)$ such that $u_k \to u$ in $W^{1,\Phi}_{\mathrm{loc}}(\Omega)$ and $u \geq Cd$ almost everywhere in Ω.

Now, we are going to show that this u satisfies the equation in (1.1). Given $\varphi \in C^\infty_0(\Omega)$, let $\Theta \subset\subset \Omega$ be the support of φ. So, by very above information, we have that

(a) $u_n \rightharpoonup u$ in $W^{1,\Phi}(\Theta)$,
(b) $u_n \to u$ in $L_\Phi(\Theta)$,
(c) $u_n(x) \to u(x)$ almost everywhere in Θ,

and there exists $\theta \in L_\Phi(\Theta)$ such that $u_n \leq \theta$ in Θ. So, by using $\varphi(u_n - u)$ as a test function in (5.2), $L_\Phi(\Theta) \hookrightarrow L^1(\Theta)$, and (b) above, we obtain

$$\begin{aligned}
\bigg|\int_\Theta \phi(|\nabla u_n|)\nabla u_n \nabla(\varphi(u_n-u))\bigg|\,dx &\leq \frac{1}{c_d^\alpha}\int_\Theta a_n|\varphi(u_n-u)|\,dx \\
&\leq C_\varphi \|a\|_{L_{\widetilde\Phi(\Theta)}}\|u_n-u\|_{L_\Phi(\Theta)} \to 0,
\end{aligned}$$

where $\Theta \subset\subset \Omega$ is the support of φ. That is,

$$(6.4)\qquad \int_\Theta \phi(|\nabla u_n|)\nabla u_n \nabla(u_n-u)\varphi = \int_\Theta \phi(|\nabla u_n|)\nabla u_n \nabla\varphi(u_n-u) + o_n(1).$$

In addition Holder's inequality, (b) above and the property $\widetilde\Phi(\phi(t)t) \leq \Phi(2t)$ for $t > 0$ imply that

$$\begin{aligned}
\bigg|\int_\Theta \phi(|\nabla u_n|)\nabla u_n \nabla\varphi(u_n-u)\bigg|\,dx &\leq C_\varphi\int_\Theta \phi(|\nabla u_n|)|\nabla u_n||u_n-u|\,dx \\
&\leq C_\varphi\|\phi(|\nabla u_n|)|\nabla u_n|\|_{L_{\widetilde\Phi}(\Theta)}\|u_n-u\|_{L_\Phi(\Theta)} \to 0 \\
&\leq C_\varphi\|u_n-u\|_{L_\Phi(\Theta)} \to 0,
\end{aligned}$$

and using this information in (6.4), we obtain that

$$(6.5)\qquad \int_\Omega \phi(|\nabla u_n|)\nabla u_n \nabla(u_n-u)\varphi\,dx = o_n(1).$$

Besides, we note that

$$\bigg|\int_\Theta \phi(|\nabla u|)\nabla u\nabla(u_n-u)\varphi\,dx\bigg| \le \bigg|\int_\Theta \phi(|\nabla u|)\nabla u\nabla[\varphi(u_n-u)]\varphi\,dx\bigg| + \bigg|\int_\Theta \phi(|\nabla u|)\nabla u\nabla\varphi(u_n-u)\,dx\bigg|,$$

and the first integral on the right side goes to zero, due to (a) above, and the second one converges to zero due to (b) above. That is,

$$\bigg|\int_\Theta \phi(|\nabla u|)\nabla u\nabla(u_n-u)\varphi\,dx\bigg| \to 0. \tag{6.6}$$

So, it follows from (4.13) and (4.15), that

$$\int_\Theta (\phi(|\nabla u_n|)\nabla u_n - \phi(|\nabla u|)\nabla u, \nabla u_n - \nabla u)\varphi\,dx \to 0.$$

As a consequence of this, together with the Lemma 6 in [12], we have that $\nabla u_n(x)\to\nabla u(x)$ almost everywhere in Θ, i.e.

$$\phi(|\nabla u_n(x)|)\nabla u_n(x) \to \phi(|\nabla u(x)|)\nabla u(x) \quad \text{a.e. in } \Theta.$$

In addition, since $(\phi(|\nabla u_n|)\nabla u_n) \subset (L_{\widetilde{\Phi}}(\Theta))^N$ is bounded, from Lemma 2 in [19] it follows that

$$\phi(|\nabla u_n|)\nabla u_n \rightharpoonup \phi(|\nabla u|)\nabla u \quad \text{in } \big(W^{1,\Phi}(\Theta)\big)^N.$$

Now, passing to limit in (5.2), we obtain that $u\in W^{1,\Phi}_{\mathrm{loc}}(\Omega)$ satisfies

$$\int_\Omega \phi(|\nabla u|)\nabla u\nabla\varphi\,dx = \int_\Omega \frac{a(x)}{u^\alpha}\varphi\,dx.$$

Besides, Lemma 5.2 implies that

$$u_n^{(\alpha-1-\ell)/\ell} \rightharpoonup v \quad \text{in } W^{1,\ell}_0(\Omega),$$

that is, $u^{(\alpha-1-\ell)/\ell}\in W^{1,\ell}_0(\Omega)$ as well. □

Below, we take advantage of the former arguments to show the existence of solutions to problem (2.4). The greatest effort is made to show L^∞-regularity of its solutions.

6.2. Convex singular problem. Regularity of solutions.

PROOF OF THEOREM 2.3. Since $0<\gamma<\ell-1$ and $0\le a\in L^q(\Omega)$ for some $q>\ell/(\ell-\gamma-1)$, it follows by arguments similar to those used in the proof of Theorem 2.1 that there exist both a sequence of approximating solutions still denoted by (u_n) and a corresponding solution $u\in W^{1,\Phi}_0(\Omega)$ to problem (2.4) such that $u\ge Cd$ in Ω for some constant $C>0$.

CLAIM. $u\in L^\infty(\Omega)$.

The proof of this Claim uses arguments driven by a Moser Iteration Scheme. Parts of our argument were motivated by reading of [21]. However our proof in the present paper is self-contained. In order to show the Claim, set

$$\beta_1 := (\ell+\alpha-1)q' > 0, \quad \beta_k^* := \beta_k + \beta_1, \quad \beta_{k+1} := \frac{\ell^*}{\ell q'}\,\beta_k^*, \quad \delta := \frac{\ell^*}{q'\ell},$$

where $1/q' + 1/q = 1$.

We point out that $\delta > 1$ because $q > N/\ell$. In addition,

$$\beta_k^* = \big(2\delta^{k-1} + \delta^{k-2} + \ldots + 1\big)\beta_1 = \frac{2\delta^k - \delta^{k-1} - 1}{\delta - 1}\,\beta_1, \tag{6.7}$$

$$\beta_k = \frac{2\delta^k - \delta^{k-1} - \delta}{\delta - 1}\,\beta_1, \tag{6.8}$$

and, since $\delta > 1$, $\beta_k \nearrow \infty$.

Now, taking k_0 such that $\beta_{k_0}, \beta_{k_0} + q'(\alpha-1) > 1$, we have that $u_n^{\beta_k/(q'+\alpha)}$ is a test function for each $k \geq k_0$ and using it in (4.6), we obtain

$$\begin{aligned} \frac{\beta_k}{q'}\int_\Omega \phi(|\nabla u_n|)|\nabla u_n|^2 u_n^{\beta_k/q'+\alpha-1}\,dx \\ &\leq \int_\Omega \bigg(\frac{a_n u_n^{\beta_k/q'+\alpha}}{(u_n+1/n)^\alpha} + b u_n^{\beta_k/q'+\alpha+\gamma}\bigg)\,dx \\ &\leq \int_\Omega \big(a u_n^{\beta_k/q'} + b u_n^{\beta_k/q'+\alpha+\gamma}\big)\,dx \\ &\leq \|a\|_q \|u_n\|_{\beta_k}^{\beta_k/q'} + \|b\|_\infty \|u_n\|_{\beta_k}^{\beta_k/q'} \|u_n^{\alpha+\gamma}\|_q. \end{aligned} \tag{6.9}$$

We claim that $\|u_n^{\alpha+\gamma}\|_q$ is bounded.

Indeed, if $(\alpha+\gamma)q \leq 1$, it follows that $\alpha \leq 1$, because $q > N/\ell > 1$. In this case, it follows from Corollary 4.4 that u_n is bounded in $W_0^{1,\Phi}(\Omega)$. In particular, there exists $\theta_0 \in L^1(\Omega)$ such that $u_n \leq \theta_0$, that is,

$$\|u_n^{\alpha+\gamma}\|_q \leq (|\Omega| + \|\theta_0\|_1)^{1/q}.$$

If $(\alpha+\gamma)q > 1$ we distinguish between two cases: $\alpha > 1$ and $\alpha \leq 1$.

In the case $\alpha > 1$, we find by using that $((u_n+1/n)^{(\ell+\alpha-1)/\ell})$ is bounded in $W^{1,\ell}(\Omega)$ and $W^{1,\ell}(\Omega) \hookrightarrow L^{\ell^*}(\Omega)$ that

$$\|u_n\|_{\ell^*+(\alpha-1)\ell^*/\ell}^{1+(\alpha-1)/\ell} = \bigg(\int_\Omega u_n^{\ell^*+(\alpha-1)\ell^*/\ell}\,dx\bigg)^{1/\ell^*} = \big\|u_n^{(\ell+\alpha-1)/\ell)}\big\|_{\ell^*} \leq C,$$

that is, by using our assumption $q \leq q(\alpha+\gamma)$, it follows from its definition (see (2.3)) that $(\alpha+\gamma)q \leq \ell^* + (\alpha-1)\ell^*/\ell$. So,

$$\|u_n\|_{(\alpha+\gamma)q} \leq C, \tag{6.10}$$

because $L^{\ell^*+(\alpha-1)\ell^*/\ell}(\Omega) \hookrightarrow L^{(\alpha+\gamma)q}(\Omega)$.

If $\alpha \le 1$, then again we have that u_n is bounded in $W_0^{1,\Phi}(\Omega)$. So, the embedding $W_0^{1,\Phi}(\Omega) \hookrightarrow L^{(\gamma+\alpha)q}(\Omega)$, see (2.3) again, implies that

$$\|u_n^{\alpha+\gamma}\|_q = \|u_n\|_{(\alpha+\gamma)q}^{\alpha+\gamma} \le \kappa \|u_n\|^{\alpha+\gamma} \le C,$$

for some $\kappa, C > 0$.

Thus, in both cases, in view of (6.9) and the estimates just above, we see that there exists a constant $c_0 > 0$ such that

$$\frac{\beta_k}{q'} \int_\Omega \phi(|\nabla u_n|)|\nabla u_n|^2 u_n^{\beta_k/q'+\alpha-1}\, dx \le (\|a\|_q + \|b\|_\infty c_0)\|u_n\|_{\beta_k}^{\beta_k/q'}. \tag{6.11}$$

On the other hand, it follows by Lemma A.2 that

$$\begin{aligned}\frac{\beta_k}{q'} \int_\Omega \phi(|\nabla u_n|)|\nabla u_n|^2 u_n^{\beta_k/q'+\alpha-1}\, dx \\ \le \frac{\ell\Phi(1)}{q'}\beta_k \int_{|\nabla u_n|\ge 1} |\nabla u_n|^\ell u_n^{\beta_k/q'+\alpha-1}\end{aligned} \tag{6.12}$$

and so it follows from (6.11) and (6.12), that

$$\begin{aligned}&\frac{\ell\Phi(1)}{q'}\beta_k \int_\Omega |\nabla u_n|^\ell u_n^{\beta_k/q'+\alpha-1}\, dx \\ &\le \frac{\ell\Phi(1)}{q'}\,\beta_k \int_{|\nabla u_n|<1} |\nabla u_n|^\ell u_n^{\beta_k/q'+\alpha-1}\, dx + (\|a\|_q + \|b\|_\infty c_0)\|u_n\|_{\beta_k}^{\beta_k/q'} \\ &\le \frac{\ell\Phi(1)}{q'}\,\beta_k \int_\Omega u_n^{\beta_k/q'+\alpha-1}\, dx + (\|a\|_q + \|b\|_\infty c_0)\|u_n\|_{\beta_k}^{\beta_k/q'}.\end{aligned} \tag{6.13}$$

Our next objective is to show that

$$\int_\Omega |\nabla u_n|^\ell u_n^{(\beta_k+(\alpha-1)q')/q'}\, dx \le B\|u_n\|_{\beta_k}^{\beta_k/q'}, \tag{6.14}$$

for some constant $B > 0$. To do this, we are going to consider two cases again: $\alpha \le 1$ and $\alpha > 1$.

If $\alpha \le 1$, the we notice that $L^{\beta_k}(\Omega) \hookrightarrow L^{\beta_k/q'+\alpha-1}(\Omega)$. Hence

$$\begin{aligned}\int_\Omega u_n^{\beta_k/q'+\alpha-1}\, dx = \|u_n\|_{\beta_k/q'+\alpha-1}^{\beta_k/q'+\alpha-1} \\ \le |\Omega|^{1-1/q'+(1-\alpha)/\beta_k}\|u_n\|_{\beta_k}^{\beta_k/q'}\|u_n\|_{\beta_k}^{\alpha-1}.\end{aligned} \tag{6.15}$$

On the other hand, since $u_1 \le u_n$, we have

$$\|u_1\|_{\beta_k} \le \|u_n\|_{\beta_k}, \tag{6.16}$$

and by the embedding $L^{\beta_k}(\Omega) \hookrightarrow L^1(\Omega)$ we get

$$\|u_1\|_1 \le |\Omega|^{1-1/\beta_k}\|u_1\|_{\beta_k}. \tag{6.17}$$

Combining (6.16) and (6.17) we have

$$\|u_n\|_{\beta_k}^{\alpha-1} \le |\Omega|^{(1-\alpha)(1-1/\beta_k)}\|u_1\|_1^{\alpha-1}. \tag{6.18}$$

So, by (6.15) and (6.18), we infer that

$$\int_\Omega u_n^{\beta_k/(q'+\alpha-1)}\,dx \le |\Omega|^{2-\alpha-1/q'}\|u_1\|_1^{\alpha-1}\|u_n\|_{\beta_k}^{\beta_k/q'}. \tag{6.19}$$

Now, by applying (6.19) in (6.13), we get

$$\begin{aligned}\frac{\ell\Phi(1)}{q'}\,\beta_k\int_\Omega|\nabla u_n|^\ell u_n^{\beta_k/q'+\alpha-1}\,dx \\ \le \frac{\ell\Phi(1)}{q'}\,|\Omega|^{2-\alpha-1/q'}\|u_1\|_1^{\alpha-1}\beta_k\|u_n\|_{\beta_k}^{\beta_k/q'}+(\|a\|_q+\|b\|_\infty c_0)\|u_n\|_{\beta_k}^{\beta_k/q'}.\end{aligned} \tag{6.20}$$

Let $\alpha>1$. By Hölder inequality, $(\alpha-1)q<(\alpha+\gamma)q$ and (6.10), we have

$$\begin{aligned}\int_\Omega u_n^{\beta_k/q'+\alpha-1}\,dx &\le \|u_n\|_{\beta_k}^{\beta_k/q'}\bigg(\int_\Omega u_n^{(\alpha-1)q}\,dx\bigg)^{1/q}\\ &\le \|u_n\|_{\beta_k}^{\beta_k/q'}\bigg(|\Omega|+\int_{[u_n\ge1]}u_n^{(\alpha-1)q}\,dx\bigg)^{1/q}\\ &\le \|u_n\|_{\beta_k}^{\beta_k/q'}\big(|\Omega|+\|u_n\|_{(\alpha+\gamma)q}^{(\alpha+\gamma)q}\big)^{1/q}\le(|\Omega|+C)^{1/q}\|u_n\|_{\beta_k}^{\beta_k/q'}.\end{aligned} \tag{6.21}$$

Now, by applying (6.21) in (6.13), we get

$$\begin{aligned}\frac{\ell\Phi(1)}{q'}\,\beta_k\int_\Omega|\nabla u_n|^\ell u_n^{\beta_k/q'+\alpha-1}\,dx \\ \le \frac{\ell\Phi(1)}{q'}\beta_k(|\Omega|+C)^{1/q}\|u_n\|_{\beta_k}^{\beta_k/q'}+(\|a\|_q+\|b\|_\infty c_0)\|u_n\|_{\beta_k}^{\beta_k/q'}.\end{aligned} \tag{6.22}$$

So, it follows from (6.20) (the case $\alpha\le1$) and (6.22) (the case $\alpha>1$) that the inequality (6.14) is true for $B>0$ defined by

$$B:=\begin{cases}\dfrac{q'}{\ell\Phi(1)}\bigg(\dfrac{\ell\Phi(1)}{q'}\,|\Omega|^{2-\alpha-1/q'}\|u_1\|_1^{\alpha-1}+\|a\|_q+\|b\|_\infty c_0\bigg) & \text{if } 0<\alpha\le1,\\ \dfrac{q'}{\ell\Phi(1)}\bigg(\dfrac{\ell\Phi(1)}{q'}(|\Omega|+C)^{1/q}+\|a\|_q+\|b\|_\infty c_0\bigg) & \text{if } \alpha>1.\end{cases}$$

This shows the inequality (6.14). Now, since

$$\bigg(\frac{\ell q'}{\beta_k+\beta_1}\bigg)^\ell\int_\Omega\big|\nabla\big(u_n^{(\beta_k+\beta_1)/(\ell q')}\big)\big|^\ell\,dx=\int_\Omega|\nabla u_n|^\ell\,u_n^{(\beta_k+q'(\alpha-1))/q'}\,dx,$$

it follows from (6.14) and $W_0^{1,\ell}(\Omega)\hookrightarrow L^{\ell^*}(\Omega)$ that, for some $\mu>0$,

$$\|u_n\|_{\beta_{k+1}}^{\beta_k^*/q'}=\big\|u^{\beta_k^*/(\ell q')}\big\|_{\ell^*}^\ell\le\mu^\ell B\bigg(\frac{\beta_k^*}{\ell q'}\bigg)^\ell\|u_n\|_{\beta_k}^{\beta_k/q'}, \tag{6.23}$$

Set $F_{k+1}:=\beta_{k+1}\ln(\|u_n\|_{\beta_{k+1}})$. So, it follows from the last inequality, that

$$\begin{aligned}F_{k+1}&\le\frac{\beta_{k+1}q'}{\beta_k^*}\bigg(\ell\ln\mu+\ell\ln\bigg(\frac{\beta_k^*}{\ell q'}\bigg)+\ln B+\frac{\beta_k}{q'}\ln(\|u_n\|_{\beta_k})\bigg)\\ &\le\ell^*\ln\big(\mu B\beta_k^*\big)+\frac{\ell^*}{q'\ell}\,F_k=\lambda_k+\delta F_k,\end{aligned} \tag{6.24}$$

where $\lambda_k := \ell^* \ln\left(\mu B \beta_k^*\right)$.

Now, by using (6.7) and (6.8), we can infer that

$$\lambda_k = b + \ell^* \ln(2\delta^{k-1} + \delta^{k-2} + \ldots + 1),$$

where $b := \ell^* \ln(\mu B \beta_1)$, that is,

$$F_k \le \delta^{k-1} F_1 + \lambda_{k-1} + \delta\lambda_{k-2} + \ldots + \delta^{k-2}\lambda_1.$$

So

$$\frac{F_k}{\beta_k} \le \frac{\delta^{k-1}F_1 + \lambda_{k-1} + \delta\lambda_{k-2} + \ldots + \delta^{k-2}\lambda_1}{\dfrac{2\delta^k - \delta^{k-1} - \delta}{\delta - 1}\beta_1} = \frac{F_1 + \dfrac{\lambda_{k-1}}{\delta^{k-1}} + \dfrac{\lambda_{k-2}}{\delta^{k-2}} + \ldots + \dfrac{\lambda_1}{\delta}}{\dfrac{2\delta - 1 - 1/\delta^{k-1}}{\delta - 1}\beta_1}. \tag{6.25}$$

Since

$$\frac{\lambda_n}{\delta^n} = \frac{b}{\delta^n} + \frac{\ell^*}{\delta^n}\ln\left(\frac{2\delta^n - \delta^{n-1} - 1}{\delta - 1}\right) \le \frac{b}{\delta^n} + \frac{\ell^*}{\delta^n}\ln\left(\frac{2\delta^n}{\delta - 1}\right),$$

it follows from (6.25), that

$$\begin{aligned}
\frac{F_k}{\beta_k} &\le \frac{F_1 + b\left(\dfrac{1}{\delta^{k-1}} + \ldots + \dfrac{1}{\delta}\right) + \ell^*\left(\dfrac{1}{\delta^{k-1}}\ln\left(\dfrac{2\delta^{k-1}}{\delta - 1}\right) + \ldots + \dfrac{1}{\delta}\ln\left(\dfrac{2\delta}{\delta - 1}\right)\right)}{\dfrac{2\delta - 1 - 1/\delta^{k-1}}{\delta - 1}\beta_1} \\
&\le \frac{F_1 + \dfrac{b}{\delta - 1} + \ell^*\left(\dfrac{1}{\delta^{k-1}}\ln\left(\dfrac{2\delta^{k-1}}{\delta - 1}\right) + \ldots + \dfrac{1}{\delta}\ln\left(\dfrac{2\delta}{\delta - 1}\right)\right)}{\dfrac{2\delta - 1 - 1/\delta^{k-1}}{\delta - 1}\beta_1} \\
&\le \frac{F_1 + \dfrac{b}{\delta - 1} + \ell^*\left[\ln\dfrac{2}{\delta - 1}\left(\dfrac{1}{\delta^{k-1}} + \ldots + \dfrac{1}{\delta}\right) + \ln\delta\left(\dfrac{k-1}{\delta^{k-1}} + \ldots \dfrac{1}{\delta}\right)\right]}{\dfrac{2\delta - 1 - 1/\delta^{k-1}}{\delta - 1}\beta_1} \\
&\le \frac{F_1 + \dfrac{b}{\delta - 1} + \ell^*\left[\dfrac{1}{\delta - 1}\ln\dfrac{2}{\delta - 1} + \ln\delta\displaystyle\sum_{n=1}^{\infty}\dfrac{n}{\delta^n}\right]}{\dfrac{2\delta - 1 - 1/\delta^{k-1}}{\delta - 1}\beta_1} \to d_0.
\end{aligned}$$

Now, going back to the definition of F_k, we obtain

$$|u_n(x)| \le \|u_n\|_\infty = \limsup_{k\to\infty}\|u_n\|_{\beta_k} \le \limsup_{k\to\infty} e^{F_k/\beta_k} \le e^{d_0}$$

for all $x \in \Omega$, and

$$|u(x)| = \lim_{n\to\infty}|u_n(x)| \le e^{d_0}$$

for almost every $x \in \Omega$, because $u_n(x) \to u(x)$ almost everywhere in Ω, that is, $u \in L^\infty(\Omega)$. □

Proof of Corollary 2.2. (a) In this case, we have $a_n = a$ for n large enough. So, as a consequence of the Comparison Principle, like at the end of the proof in Lemma 5.1, we have that $u_{n+1} \geq u_n$. Besides this, if we assume that

$$\Omega_0 := \left\{ x \in \Omega \;\middle|\; u_{n+1}(x) + \frac{1}{n+1} > u_n(x) + \frac{1}{n} \right\} \subset\subset \Omega,$$

is not empty, then we would obtain $-\Delta_\Phi(u_{n+1} + 1/(n+1)) \leq -\Delta_\Phi(u_n + 1/n)$ in Ω_0, that is

$$u_{n+1}(x) + \frac{1}{n+1} \leq u_n(x) + \frac{1}{n} \quad \text{in } \Omega_0.$$

This is impossible. So, we have

$$0 \leq u_n - u_k \leq \frac{1}{k} - \frac{1}{n} \quad \text{in } \Omega.$$

Since $(u_n) \subset C^1(\overline{\Omega})$, we obtain that u_n converges uniformly to u, that is, $u \in C(\overline{\Omega})$.

(b) It just follows from the same arguments as those used in the proof of Theorem 2.3 by taking $b = 0$.

(c) This proof is based on the ideas from [6]. If $0 < u, v \in W_0^{1,\Phi}(\Omega)$ are two solutions of the problem (1.1), then the claim is immediately true. So, let us assume that $0 < u, v \in W_{\mathrm{loc}}^{1,\Phi}(\Omega)$ are two solutions of the problem (1.1). Now, by defining $\mathcal{C}_v := \{w \in W_0^{1,\Phi}(\Omega) \mid 0 \leq w \leq v\}$ and $J_\varepsilon \colon \mathcal{C}_v \to \overline{\mathbb{R}}$ by

$$J_\varepsilon(w) := \int_\Omega \left[\Phi(|\nabla w|) - a(x) \int_0^{w(x)} \frac{1}{(s+\varepsilon)^\alpha} \right] dx,$$

we obtain that J_ε is weakly lower semicontinuous and coercive on the convex and closed set $\mathcal{C}_v$. Therefore, there is a $w = w_\varepsilon \in \mathcal{C}_v$ such that

$$J_\varepsilon(w) = \inf_{\mathcal{C}_v} J_\varepsilon,$$

that is, by defining $\sigma \colon [0,1] \to \mathbb{R}$ by $\sigma(t) = J_\varepsilon(t\psi + (1-t)w)$ for $\psi \in \mathcal{C}$, we get

$$\sigma(0) = J_\varepsilon(w) = \min\{J(w) \mid w \in \mathcal{C}\} \leq \sigma(t) \quad \text{for all } t \in [0,1].$$

In other words, we have that

$$0 \leq \sigma'(0) = \langle J'_\varepsilon(w), \psi - w \rangle \quad \text{for all } \psi \in \mathcal{C}.$$

This leads us, after some manipulations, to

$$\int_\Omega \phi(|\nabla w|) \nabla w \nabla \varphi \, dx \geq \int_\Omega \frac{a(x)}{(w+\varepsilon)^\alpha} \varphi \, dx \tag{6.26}$$

for all $\varphi \in W_0^{1,\Phi}(\Omega) \cap L_c^\infty(\Omega)$ with $\varphi \geq 0$.

Now, since $w \in W_0^{1,\Phi}(\Omega)$, it follows from our definition of zero-boundary condition, that $(u - w - \varepsilon)^+ \in W_0^{1,\Phi}(\Omega)$, and $T_\tau((u - w - \varepsilon)^+) \in W_0^{1,\Phi}(\Omega)$ with $\operatorname{supp}(T_\tau((u - w - \varepsilon)^+))) \subset \Omega$ for each $\tau > 0$ given, where $T_\tau(s) := \min\{s, \tau\}$

for $s \geq 0$, and $T_\tau(-s) = -T_\tau(s)$ for $s < 0$. So, by using that u is a $W^{1,p(x)}_{\mathrm{loc}}(\Omega)$-solution for (1.1) and $w \in W^{1,\Phi}_0(\Omega)$ satisfies (6.26), we obtain

$$\int_\Omega (\phi(|\nabla u|)\nabla u - \phi(|\nabla w|)\nabla w)\nabla T_\tau((u-w-\varepsilon)^+) \\ \leq \int_\Omega \left[\frac{a(x)}{u^\alpha} - \frac{a(x)}{(w+\varepsilon)^\alpha}\right] T_\tau((u-w-\varepsilon)^+) \leq 0,$$

that is

$$\int_{[u\geq w+\varepsilon]} (\phi(|\nabla(u-\varepsilon)|)\nabla(u-\varepsilon) - \phi(|\nabla w|)\nabla w)\nabla T_\tau((u-w-\varepsilon)^+) \leq 0$$

for each $\tau > 0$ given. Now, by passing $\tau \to \infty$ and using the fact that Φ is a strictly convex function, we obtain

$$0 \leq \int_{[u\geq w+\varepsilon]} [\phi(|\nabla(u-\varepsilon)|)\nabla(u-\varepsilon) - \phi(|\nabla w|)\nabla w][\nabla(u-\varepsilon) - \nabla w] \leq 0,$$

which implies that $\nabla(u-w-\varepsilon)^+ = 0$ almost everywhere in Ω. Since $(u-w-\varepsilon)^+ \in W^{1,\Phi}_0(\Omega)$, we obtain that $|[u \geq w+\varepsilon]| = 0$, that is,

$$u \leq w + \varepsilon \leq v + \varepsilon \quad \text{a.e. in } \Omega.$$

for each $\varepsilon > 0$. By redoing the above arguments with $\mathcal{C}_u$ in the place of $\mathcal{C}_v$, we obtain that $u = v$ in Ω. □

Appendix A. On Orlicz–Sobolev spaces

In this section we present for the reader's convenience several results/notation used in the paper. The reader is referred to [1], [34] regarding basics on Orlicz–Sobolev spaces. The usual norm on $L_\Phi(\Omega)$ is (Luxemburg norm)

$$\|u\|_\Phi = \inf\left\{\lambda > 0 \;\middle|\; \int_\Omega \Phi\left(\frac{u(x)}{\lambda}\right) dx \leq 1\right\},$$

while the Orlicz–Sobolev norm of $W^{1,\Phi}(\Omega)$ is

$$\|u\|_{1,\Phi} = \|u\|_\Phi + \sum_{i=1}^N \left\|\frac{\partial u}{\partial x_i}\right\|_\Phi.$$

We denote by $W^{1,\Phi}_0(\Omega)$ the closure of $C^\infty_0(\Omega)$ with respect to the Orlicz–Sobolev norm of $W^{1,\Phi}(\Omega)$. We remind that

$$\widetilde{\Phi}(t) = \max_{s\geq 0}\{ts - \Phi(s)\}, \quad \text{for } t \geq 0.$$

It turns out that Φ and $\widetilde{\Phi}$ are N-functions satisfying the Δ_2-condition, (cf. [34, p. 22]). In addition, $L_\Phi(\Omega)$ and $W^{1,\Phi}(\Omega)$ are reflexive and Banach spaces.

REMARK A.1. It is well known that (ϕ_3) implies that the condition

$(\phi_3)'$ $\ell \leq \phi(t)\, t^2/\Phi(t) \leq m$, $t > 0$,

is verified. Furthermore, under this condition, $\Phi, \widetilde{\Phi} \in \Delta_2$.

By the Poincaré Inequality (see e.g. [19]), i.e., the inequality

$$\int_\Omega \Phi(u)\,dx \le \int_\Omega \Phi(2d_\Omega|\nabla u|)\,dx,$$

where $d_\Omega = \operatorname{diam}(\Omega)$, it follows that

$$\|u\|_\Phi \le 2d_\Omega\|\nabla u\|_\Phi \quad \text{for all } u \in W_0^{1,\Phi}(\Omega).$$

As a consequence, we have that $\|u\| := \|\nabla u\|_\Phi$ defines a norm in $W_0^{1,\Phi}(\Omega)$ that is equivalent to $\|\cdot\|_{1,\Phi}$. Let Φ_* be the inverse of the function

$$t \in (0,\infty) \mapsto \int_0^t \frac{\Phi^{-1}(s)}{s^{(N+1)/N}}\,ds$$

which can be extended to $\mathbb{R}$ by $\Phi_*(t) = \Phi_*(-t)$ for $t \le 0$.

We say that an N-function Ψ grows essentially more slowly (grows more slowly) than Υ, denoted by $\Psi \ll \Upsilon$ ($\Psi < \Upsilon$), if

$$\lim_{t\to\infty} \frac{\Psi(\lambda t)}{\Phi_*(t)} = 0 \quad \text{for each } \lambda > 0$$

($\Psi(t) \le \Upsilon(kt)$ for all $t \ge t_0$ for some $k, t_0 > 0$).

The imbeddings below (cf. [1]) were used in this paper. First, we have

$$W_0^{1,\Phi}(\Omega) \overset{\text{cpt}}{\hookrightarrow} L_\Psi(\Omega) \quad \text{if } \Phi < \Psi \ll \Phi_*,$$

and in particular, $W_0^{1,\Phi}(\Omega) \overset{\text{cpt}}{\hookrightarrow} L_\Phi(\Omega)$, because $\Phi \ll \Phi_*$ (cf. [20, Lemma 4.14]). Furthermore,

$$W_0^{1,\Phi}(\Omega) \overset{\text{cont}}{\hookrightarrow} L_{\Phi_*}(\Omega).$$

Besides, it is worth mentioning that, if $(\phi_1) - (\phi_2)$ and $(\phi_3)'$ are satisfied (cf. [10, Lemma D.2]), then

$$L_\Phi(\Omega) \overset{\text{cont}}{\hookrightarrow} L^\ell(\Omega).$$

In this text we use the notation $L^\Psi_{\text{loc}}(\Omega)$ in the sense that $u \in L^\Psi_{\text{loc}}(\Omega)$ if and only if $u \in L_\Psi(\Omega)$ for all $U \subset\subset \Omega$.

Lemma A.2. (cf. [15]) *Assume that ϕ satisfies conditions (ϕ_1)–(ϕ_3). Set*

$$\zeta_0(t) = \min\{t^\ell, t^m\} \quad \textit{and} \quad \zeta_1(t) = \max\{t^\ell, t^m\}, \quad t \ge 0.$$

Then Φ satisfies

$$\begin{gathered}\zeta_0(t)\Phi(\rho) \le \Phi(\rho t) \le \zeta_1(t)\Phi(\rho), \qquad \rho, t > 0,\\ \zeta_0(\|u\|_\Phi) \le \int_\Omega \Phi(u)\,dx \le \zeta_1(\|u\|_\Phi), \quad u \in L_\Phi(\Omega).\end{gathered}$$

Lemma A.3 (cf. [15]). *Assume that ϕ satisfies (ϕ_1)–(ϕ_3) and $1 < \ell, m < N$ hold. Set*

$$\zeta_2(t) = \min\{t^{\widetilde{\ell}}, t^{\widetilde{m}}\} \quad \textit{and} \quad \zeta_3(t) = \max\{t^{\widetilde{\ell}}, t^{\widetilde{m}}\}, \quad t \ge 0,$$

where $\widetilde{m} = m/(m-1)$ *and* $\widetilde{\ell} = \ell/(\ell-1)$. *Then*

$$\begin{aligned} \widetilde{\ell} \le \frac{t^2\,\widetilde{\Phi}'(t)}{\widetilde{\Phi}(t)} &\le \widetilde{m}, && t>0,\\ \zeta_2(t)\,\widetilde{\Phi}(\rho) \le \widetilde{\Phi}(\rho t) &\le \zeta_3(t)\,\widetilde{\Phi}(\rho), && \rho,t>0,\\ \zeta_2(\|u\|_{\widetilde{\Phi}}) \le \int_\Omega \widetilde{\Phi}(u)\,dx &\le \zeta_3(\|u\|_{\widetilde{\Phi}}), && u\in L_{\widetilde{\Phi}}(\Omega). \end{aligned}$$

Lemma A.4. *Let Let Φ be an N-function satisfying Δ_2. Let $(u_n)\subset L_\Phi(\Omega)$ be a sequence such that $u_n\to u$ in $L_\Phi(\Omega)$. Then there is a subsequence $(u_{n_k})\subseteq(u_n)$ such that*

(a) *$u_{n_k}(x)\to u(x)$ for almost every $x\in\Omega$,*
(b) *there is $h\in L_\Phi(\Omega)$ such that $|u_{n_k}|\le h$ almost everywhere in Ω.*

Proof (Sketch). We have that $\int_\Omega \Phi(u_n-u)\,dx\to 0$. By [1] $L_\Phi(\Omega)\hookrightarrow L^1(\Omega)$. So, there is a subsequence, we keep the notation, and $\widetilde{h}\in L^1(\Omega)$ such that $u_n\to u$ almost everywehere in Ω and $\Phi(u_n-u)\le\widetilde{h}$ almost everywhere in Ω. Since Φ is convex, increasing and satisfies Δ_2, we have

$$\Phi(|u_n|)\le C\Phi\left(\frac{|u_n-u|+|u|}{2}\right)\le\frac{C}{2}\left[\Phi(|u_n-u|)+\Phi(|u|)\right]\le\frac{C}{2}\left[\widetilde{h}+\Phi(|u|)\right],$$

that is

$$|u_n|\le\Phi^{-1}\left(\frac{C}{2}\big(\widetilde{h}+\Phi(|u|)\big)\right):=h\in L_\Phi(\Omega),$$

because $\widetilde{h}\in L^1(\Omega)$, $\Phi(|u|)\in L^1(\Omega)$, and

$$\int_\Omega\Phi(h)\,dx=\int_\Omega\Phi\left(\Phi^{-1}\left(\frac{K}{2}\big(\widetilde{h}+\Phi(|u|)\big)\right)\right)dx=\int_\Omega\left(\frac{K}{2}\big(\widetilde{h}+\Phi(|u|)\big)\right)dx<\infty,$$

showing that $h\in L_\Phi(\Omega)$. □

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