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EXTREME PARTITIONS OF A LEBESGUE SPACE AND THEIR APPLICATION IN TOPOLOGICAL DYNAMICS

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Abstract. It is shown that any topological action Φ of a countable orderable and amenable group G on a compact metric space X and every Φ-invariant probability Borel measure μ admit an extreme partition ζ of X such that the equivalence relation R_ζ associated with ζ contains the asymptotic relation $A(\Phi)$ of Φ. As an application of this result and the generalized Glasner theorem it is proved that $A(\Phi)$ is dense for the set $E_\mu(\Phi)$ of entropy pairs.

1. Introduction

In the paper we consider topological dynamical systems on a compact metric space being actions of a countable amenable and orderable (CAO) group.

The simplest class of CAO groups applied in topological dynamics is formed by the groups $\mathbb{Z}^d$, $d \geq 1$. One can show that all finitely generated, torsion-free nilpotent groups are CAO ([3], [7]).

In our further considerations we shall assume that the given compact metric space is equipped with a Borel measure invariant with respect to the considered action. Measurable partitions of the space form a useful tool in the theory of dynamical systems.

In particular extreme partitions play an important role in the entropy theory. The existence of extreme partitions for $\mathbb{Z}^d$-actions has been proved by Rokhlin

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and Sinai ($d = 1$) and by the second author (for arbitrary $d \geq 2$) (cf. [12], [8]). Golodets and Sinelshchikov ([7]) have shown this for actions of an arbitrary finitely generated, torsion-free nilpotent group.

Blanchard, Host and Ruette ([1]) proved a strengthened form of the Rokhlin-Sinai theorem for Lebesgue spaces being compact metric spaces equipped with a Borel probability invariant measure. Namely, they showed the existence of an extreme partition (the authors used the name "excellent") such that the equivalence relation associated with this partition is a subset of the asymptotic relation.

The aim of our paper is to extend the results of [1] to the case of any CAO group. In [2] we applied measurable partitions which are "almost" extreme to study the asymptotic relation of a topological dynamical system (X, Φ) where Φ is a topological action of a CAO group G on a compact metric space X. It is shown among other things that Φ is deterministic if $A(\Phi)$ is a diagonal and that $A(\Phi)$ is dense if Φ admits an invariant measure with full support and completely positive entropy.

Our first main result, Theorem 3.6, says that for any topological action Φ and any Φ-invariant measure μ there exists an extreme partition ζ of X such that the equivalence relation R_ζ associated with ζ is contained in $A(\Phi)$. Applying this and the generalized Glasner theorem (Proposition 3.8) we relate $A(\Phi)$ to the set $E_\mu(\Phi)$. Namely, we show (Theorem 3.9) that $E_\mu(\Phi)$ is contained in the closure of $A(\Phi)$. As consequences of this we obtain the two results of [2] mentioned above.

It is worth to point out that $A(\Phi)$ may be used to characterize zero entropy actions (see [4] for $G = \mathbb{Z}$).

2. Preliminaries

Let (X, d) be a compact metric space and suppose μ is a Borel probability measure on X. We assume X is equipped with the σ-algebra $\mathcal{B}$ being the completion of the Borel σ-algebra with respect to μ. The extension of μ to $\mathcal{B}$ will be also denoted by μ. We associate with μ its support $\operatorname{Supp}\mu$ and the set $S(\mu) = \{(x, x) : x \in \operatorname{Supp}\mu\}$.

For a σ-algebra $\mathcal{A} \subset \mathcal{B}$ we denote by $\mu \underset{\mathcal{A}}{\times} \mu$ the relative self product of μ with respect to $\mathcal{A}$, i.e.

$$\big(\mu \underset{\mathcal{A}}{\times} \mu\big)(A \times B) = \int_X \mathbb{E}(\mathbf{1}_A|\mathcal{A})\mathbb{E}(\mathbf{1}_B|\mathcal{B})\, d\mu, \quad \text{for } A, B \in \mathcal{B}.$$

We denote by $\mathcal{M}(X)$ the lattice of measurable partitions of $(X, \mathcal{B}, \mu)$. For the definition and basic properties of $\mathcal{M}(X)$ we refer the reader to [10].

Let $\mathcal{F}(X) \subset \mathcal{M}(X)$ denote the set of finite partitions. For any $\xi \in \mathcal{M}(X)$ we denote by $R_\xi \subset X \times X$ the equivalence relation determined by ξ and by $\widehat{\xi}$

the σ-algebra of ξ-sets, i.e. measurable unions of elements of ξ. We denote by $\mathcal{N}$ the σ-algebra corresponding to the trivial partition ν_X of X.

Let $\xi, \eta \in \mathcal{M}(X)$. The relation $\xi \prec \eta$ means that any atom of η is included in some atom of ξ. If $\xi \prec \eta$ then obviously $\widehat{\xi} \subset \widehat{\eta}$.

For a countable family $\{\xi_t : t \in T\} \subset \mathcal{M}(X)$ we denote by $\bigvee_{t\in T} \xi_t$ its join. It is known ([10]) that $\bigvee_{t\in T} \xi_t \in \mathcal{M}(X)$. Moreover, if the elements of ξ_t, $t \in T$ are Borel sets then the elements of $\bigvee_{t\in T} \xi_t$ are so.

Let $\langle G, \cdot \rangle$ be a countable amenable group equipped with a set $\Gamma \subset G$ called an algebraic past satisfying the following conditions:

- $\Gamma \cap \Gamma^{-1} = \emptyset$,
- $\Gamma \cup \Gamma^{-1} \cup \{e\} = G$,
- $\Gamma \cdot \Gamma \subset \Gamma$,
- $g\Gamma g^{-1} \subset \Gamma$,

where e is the unity element of G, $g \in G$.

For a finite set $A \subset G$ we denote by $|A|$ the number of elements of A.

It is well known that the amenability of G is equivalent to the existence of a Følner sequence (A_n) of finite subsets of G, i.e. a sequence satisfying the condition

$$\lim_{n\to\infty} \frac{|g \cdot A_n \cap A_n|}{|A_n|} = 1 \quad \text{for any } g \in G.$$

It is also known (cf. [9]) that every countable amenable group has a Følner sequence (A_n) such that

$$A_n^{-1} = A_n, \quad A_n \subset A_{n+1}, \quad n \geq 1, \quad \bigcup_{n=1}^{\infty} A_n = G.$$

The existence of an algebraic past in G is equivalent to the fact that G is orderable, i.e. there exists in G a linear order $<$ compatible with the group operation. We have $\Gamma = \{g \in G : g < e\}$. It is well-known that all free groups are orderable and abelian groups are orderableif and only if they are torsion free ([5]).

Let $\mathcal{H}(X)$ be the group of all homeomorphisms of X and let Φ be a topological action of G on X, i.e. a homomorphism of G into $\mathcal{H}(X)$. Let Φ^g, $g \in G$ be the homeomorphism corresponding to g. We denote by $\mathcal{P}(X, \Phi)$ the set of all Φ-invariant probability measures. Given a measure $\mu \in \mathcal{P}(X, \Phi)$ and a partition $\xi \in \mathcal{F}(X)$ we denote be $H_\mu(\xi)$ the entropy of ξ and we use the symbols $h_\mu(\Phi)$ and $\pi_\mu(\Phi)$ for the entropy and the Pinsker σ-algebra of Φ, respectively.

We call a pair of points $(x, x') \in X \times X$, $x \neq x'$ a measure-theoretic entropy pair for Φ if for any Borel partition $\xi = \{F, F^c\}$ of X such that $x \in \mathrm{Int}(F) \neq \emptyset \neq \mathrm{Int}(F^c) \ni x'$ it holds $h_\mu(\Phi, \xi) > 0$. We denote by $E_\mu(\Phi)$ the set of measure-theoretic entropy pairs for Φ.

Let us denote $\lambda_\mu = \mu \underset{\pi_\mu(\Phi)}{\times} \mu$ and $\Lambda_\mu =$ Supp λ_μ. For a given topological G-action Φ on X the relation

$$\mathbf{A}(\Phi) = \Big\{(x, x') \in X \times X : \lim_{g \in \Gamma^{-1}} d(\Phi^g x, \Phi^g x') = 0\Big\}$$

is said to be the asymptotic relation of Φ, where the above limit has the following meaning:

$$\forall \varepsilon > 0 \ \exists g_0 \in \Gamma^{-1} \ \forall g > g_0 \quad d(\Phi^g x, \Phi^g x') < \varepsilon.$$

It is clear that $\mathbf{A}(\Phi)$ is an equivalence relation.

3. Main results

Let $\mu \in \mathcal{P}(X, \Phi)$ be fixed. From now on, up to Proposition 3.8, we omit the subscript μ in the notation of entropies H_μ, h_μ and the Pinsker σ-algebra π_μ. For a partition $\xi \in \mathcal{M}(X)$ and a set $A \subset G$ we define

$$\xi(A) = \bigvee_{g \in A} \Phi^g \xi.$$

We put $\xi^- = \xi(\Gamma)$, $\xi_\Phi = \xi(G)$.

Let $\sigma \in \mathcal{M}(X)$ be totally invariant, i.e. $\Phi^g \sigma = \sigma$, $g \in G$. We will make use of the following result given in [2].

Proposition 3.1. *For any Følner sequence (A_n) in G and any $\xi \in \mathcal{F}(X)$ it holds*

$$\lim_{n \to \infty} \frac{1}{|A_n|} H(\xi(A_n) | \widehat{\sigma}) = H\big(\xi | \widehat{\xi}^- \vee \widehat{\sigma}\big).$$

We call this limit the *mean σ-relative entropy of ξ with respect to* Φ and we denote it by $h(\xi, \Phi | \sigma)$.

By $\pi(\Phi|\sigma)$ we denote the relative Pinsker σ-algebra of Φ with respect to σ, i.e. the join of all partitions $\xi \in \mathcal{F}(X)$ with $h(\xi, \Phi|\sigma) = 0$. For the trivial partition $\sigma = \nu_X$ of X we have $h(\xi, \Phi|\nu) = h(\xi, \Phi)$ and $\pi(\Phi|\sigma) = \pi(\Phi)$.

Applying the methods given in [2] to the relative mean entropy instead of the mean entropy one can prove the following relative version of the generalized Pinsker formula ([2, Lemma 2]) and its corollary.

Lemma 3.2 (Relative Pinsker formula). *For any $\xi, \eta \in \mathcal{F}(X)$ we have*

$$h(\xi \vee \eta, \Phi | \sigma) = h(\xi, \Phi | \sigma) + H\big(\eta | \widehat{\eta}^- \vee \widehat{\xi}_\Phi \vee \sigma\big).$$

Corollary 3.3. *For any $\xi, \eta, \zeta \in \mathcal{F}(X)$ with $\xi \preceq \eta$ we have*

$$\lim_{g \in \Gamma} H\big(\xi | \widehat{\eta}^- \vee \Phi^g \widehat{\zeta}^- \vee \widehat{\sigma}\big) = H(\xi | \widehat{\eta}^- \vee \widehat{\sigma}).$$

In the sequel we shall also need the following

Remark 3.4. $\pi(\Phi | \pi(\Phi)) = \pi(\Phi)$.

PROOF. If $\xi \in \mathcal{F}(X)$ is measurable with respect to $\pi(\Phi)$ then $H(\xi|\pi(\Phi)) = 0$. Hence $0 = H(\xi|\xi^- \vee \pi(\Phi)) = h(\xi, \Phi|\pi(\Phi))$, i.e. ξ is measurable with respect to $\pi(\Phi|\pi(\Phi))$.

Let now $\xi \in \mathcal{F}(X)$ be measurable with respect to $\pi(\Phi|\pi(\Phi))$, i.e. $h(\xi, \Phi|\pi(\Phi)) = 0$. Let $\eta_n \in \mathcal{F}(X)$, $n \in \mathbb{N}$, be such that $\eta_n \nearrow \pi(\Phi)$. Therefore $h(\eta_n, \Phi) = 0$, $n \in \mathbb{N}$. Hence

$$H(\eta_n|\eta_n^- \vee \xi_\Phi) = 0, \quad n \in \mathbb{N}.$$

Applying the generalized Pinsker formula we get, for $n \in \mathbb{N}$,

$$\begin{aligned} h(\xi, \Phi) &= h(\xi, \Phi) + H(\eta_n|\eta_n^- \vee \xi_\Phi) = h(\xi \vee \eta_n, \Phi) + H(\xi \vee \eta_n|\xi^- \vee \eta_n^-) \\ &= H(\xi|\xi^- \vee \eta_n^-) + H(\eta_n|\eta_n^- \vee \xi \vee \xi^-) = H(\xi|\xi^- \vee \eta_n^-). \end{aligned}$$

Taking the limit as $n \to \infty$ we get by the assumption

$$h(\xi, \Phi) = H(\xi|\xi^- \vee \pi(\Phi)) = h(\xi, \phi|\pi(\Phi)) = 0,$$

i.e. ξ is measurable with respect to $\pi(\Phi)$ which finishes the proof. □

DEFINITION 3.5. A partition $\zeta \in \mathcal{M}(X)$ is said to be *extreme* for Φ if

(a) $\Phi^g \zeta \preceq \zeta$, $g \in \Gamma$,

(b) $\bigvee_{g \in G} \Phi^g \widehat{\zeta} = \mathcal{B}$,

(c) $\bigcap_{g \in G} \Phi^g \widehat{\zeta} = \pi(\Phi)$.

Now we shall show our first main result.

THEOREM 3.6. *For any measure $\mu \in \mathcal{P}(X, \Phi)$ there exists an extreme partition $\zeta \in \mathcal{M}(X)$ with*

(d) $R_\zeta \subset \mathbf{A}(\Phi)$.

PROOF. We start (as in the proof of Theorem 4.2 from [2]) with a sequence $(\alpha_n) \subset \mathcal{F}(X)$ of Borel measurable partitions such that

$$\alpha_n \preceq \alpha_{n+1}, \quad n \in \mathbb{N} \quad \text{and} \quad \operatorname{diam} \alpha_n \to 0 \quad \text{as } n \to \infty. \tag{3.1}$$

Let $\sigma \in \mathcal{M}(X)$ be a totally invariant partition. Applying a relativized technique of Rokhlin (cf. [2]) with respect to $\widehat{\sigma}$, we get a new sequence $(\xi_p) \subset \mathcal{F}(X)$ given by $\xi_p = \bigvee_{k=1}^{p} \Phi^{g_k^{-1}} \alpha_k$ where $(g_k) \subset G$ is chosen in such a way that

$$H\big(\xi_p|\widehat{\xi}_p^{-\vee}\widehat{\sigma}\big) - H(\xi_p|\widehat{\xi}_{p+t}^- \vee \widehat{\sigma}) < \frac{1}{p} \quad \text{for any } p, t \geq 1. \tag{3.2}$$

We consider the following measurable partitions

$$\xi = \bigvee_{p=1}^{+\infty} \xi_p, \quad \eta = \xi^-.$$

Taking in (3.2) limit as $t \to \infty$ we obtain

$$H\big(\xi_p|\widehat{\xi_p^-} \vee \widehat{\sigma}\big) - H(\xi_p|\widehat{\eta} \vee \widehat{\sigma}) \leq \frac{1}{p}, \quad p \geq 1. \tag{3.3}$$

It is clear that

$$\Phi^g \eta \preceq \eta, \quad g \in \Gamma. \tag{3.4}$$

As a consequence of (3.1) we obtain

$$\bigvee_{g \in G} \Phi^g \widehat{\eta} = \mathcal{B} \tag{3.5}$$

in the same way as in the proof of Theorem 4.2 from [2].

Now we shall show that

$$\bigcap_{g \in G} \Phi^g(\widehat{\eta} \vee \widehat{\sigma}) \subset \pi(\Phi|\sigma). \tag{3.6}$$

Let $\alpha \in \mathcal{F}(X)$ be measurable with respect to $\bigcap_{g \in G} \Phi^g(\widehat{\eta} \vee \widehat{\sigma})$. Applying Lemma 3.2 we have

$$\begin{aligned} h(\alpha \vee \xi_p, \Phi|\sigma) &= h(\alpha, \Phi|\sigma) + H\big(\xi_p|\widehat{\xi_p^-} \vee \widehat{\alpha}_\Phi \vee \widehat{\sigma}\big) \\ &= h(\xi_p, \Phi|\sigma) + H\big(\alpha|\widehat{\alpha}^- \vee (\widehat{\xi}_p)_\Phi \vee \widehat{\sigma}\big). \end{aligned}$$

Hence

$$h(\alpha, \Phi|\sigma) = h(\xi_p, \Phi|\sigma) - H(\xi_p|\widehat{\xi_p^-} \vee \widehat{\alpha}_\Phi \vee \widehat{\sigma}) + H(\alpha|\widehat{\alpha}^- \vee (\widehat{\xi}_p)_\Phi \vee \widehat{\sigma}).$$

Using the inclusion $\widehat{\alpha}_\Phi \subset \bigcap_{g \in G} \Phi^g(\widehat{\eta} \vee \widehat{\sigma}) \subset \widehat{\eta} \vee \widehat{\sigma}$ and applying the inequality (3.3) we have

$$\begin{aligned} h(\alpha, \Phi|\sigma) &\leq h(\xi_p, \Phi|\sigma) - H\big(\xi_p|\widehat{\eta} \vee \widehat{\sigma}\big) + H\big(\alpha|\widehat{\alpha}^- \vee (\widehat{\xi}_p)_\Phi \vee \widehat{\sigma}\big) \\ &= H(\xi_p|\widehat{\xi_p^-} \vee \widehat{\sigma}) - H(\xi_p|\widehat{\eta} \vee \widehat{\sigma}) + H\big(\alpha|\widehat{\alpha}^- \vee (\widehat{\xi}_p)_\Phi \vee \widehat{\sigma}\big) \\ &\leq \frac{1}{p} + H\big(\alpha|\widehat{\alpha}^- \vee (\widehat{\xi}_p)_\Phi \vee \widehat{\sigma}\big). \end{aligned}$$

Taking the limit as $p \to \infty$ and applying (3.5) we obtain $h(\alpha, \Phi|\sigma) = 0$, i.e. α is measurable with respect to $\pi(\Phi|\sigma)$, which proves (3.6). Now, by the same reasoning as in the proof of Theorem 4.2 from [2], we get $R_\eta \subset A(\Phi)$.

Let now $\sigma = \pi(\Phi)$ and $\zeta = \eta \vee \sigma = \eta \vee \pi(\Phi)$. From (3.4) and (3.5) it follows that (a) and (b) are satisfied and

$$\pi(\Phi) = \sigma \subset \bigcap_{g \in G} \Phi^g \widehat{\zeta} \subset \pi(\Phi|\sigma) = \pi(\Phi|\pi(\Phi)) = \pi(\Phi),$$

i.e. (c) is also true. Since $R_\zeta \subset R_\eta$ we have $R_\zeta \subset A(\Phi)$ which finishes the proof. □

To prove Proposition 3.8 we shall need the following

REMARK 3.7. For any σ-algebra $\mathcal{A}$ and a measure $\mu \in \mathcal{P}(X, \Phi)$ we have

$$\Delta \cap \operatorname{Supp} \mu \underset{\mathcal{A}}{\times} \mu = S(\mu).$$

PROOF. Assume first that $(x, x) \in \Delta \cap \operatorname{Supp} \mu \underset{\mathcal{A}}{\times} \mu$ and $(x, x) \notin S(\mu)$, i.e. $x \notin \operatorname{Supp} \mu$. Therefore there exists an open neighbourhood U of x such that $\mu(U) = 0$. Therefore

$$0 = \mu(U) = \int_X \mathbb{E}(\mathbf{1}_U | \mathcal{A}) d\mu,$$

thus $\mathbb{E}(\mathbf{1}_U | \mathcal{A}) = 0$ for μ-almost every $x \in X$. But by the assumption we have

$$0 < \big(\mu \underset{\mathcal{A}}{\times} \mu\big)(U \times U) = \int_X \mathbb{E}^2(\mathbf{1}_U | \mathcal{A})\, d\mu = \int_X \mathbb{E}(\mathbf{1}_U | \mathcal{A})\, d\mu.$$

This contradiction gives $x \in \operatorname{Supp} \mu$.

Now let $(x, x) \in S(\mu)$ and $(x, x) \notin \operatorname{Supp} \mu \underset{\mathcal{A}}{\times} \mu$. Then there exists an open set $G \subset X \times X$ such that $(x, x) \in G$ and $(\mu \underset{\mathcal{A}}{\times} \mu)(G) = 0$. Let $U \subset X$ be an open neighbourhood of x such that $U \times U \subset G$. Thus

$$0 = \big(\mu \underset{\mathcal{A}}{\times} \mu\big)(U \times U) = \int_X \mathbb{E}(\mathbf{1}_U | \mathcal{A})\, d\mu = \mu(U)$$

which gives us a contradiction since $x \in \operatorname{Supp} \mu$. □

PROPOSITION 3.8. *For any measure* $\mu \in \mathcal{P}(X, \Phi)$

$$\Lambda_\mu = E_\mu(\Phi) \cup S(\mu).$$

PROOF. The idea of the proof is the same as in the proof of Theorem 1 in [6]. We give here the sketch of the proof for the convenience of the reader.

First one shows the inclusions

$$\Lambda_\mu \subset E_\mu(\Phi) \cup \Delta \subset \Lambda_\mu \cup \Delta. \tag{3.7}$$

The proof of the first inclusion is based on the observation that if Q is a Borel set and $\xi = \{Q, Q^c\}$ is the partition of X induced by Q then the equality $h_\mu(\xi, \Phi) = 0$ is equivalent to the measurability of Q with respect to the Pinsker σ-algebra $\pi_\mu(\Phi)$.

In order to show the second inclusion one proves that if $x, y \in X$, $x \neq y$ and A, B are Borel sets such that $x \in A$, $y \in B$, $\lambda_\mu(A \times B) = 0$ then one can find a Borel set Q with

$$A \subset Q, \quad B \subset Q^c, \quad h_\mu(\eta, \Phi) = 0, \quad \text{where } \eta = \{Q, Q^c\}.$$

Namely, one takes $Q = A$ in the case $\mu(A) = 0$ and $Q = A \cup (F \setminus B)$ when $\mu(A) > 0$ where $F = \{\mathbb{E}(\mathbf{1}_A | \pi_\mu(\Phi)) > 0\}$.

Proposition 3.8 easily follows now from (3.7) and Remark 3.7 for $\mathcal{A} = \pi_\mu(\Phi)$. □

Theorem 3.9. *For any measure $\mu \in \mathcal{P}(X,\Phi)$ the set $A(\Phi)$ of asymptotic pairs is dense in the set $E_\mu(\Phi)$ of entropy pairs, i.e. $E_\mu(\Phi) \subset \overline{A(\Phi)}$.*

Proof. Let $\mu \in \mathcal{P}(X,\Phi)$ and $\zeta \in \mathcal{M}(X)$ be an extreme partition given in Theorem 3.6. By (c) we have

$$\bigcap_{g\in G} \Phi^g\widehat{\zeta} = \pi_\mu(\Phi).$$

Let $g \in G$ and let λ_g denote the relative product measure $\lambda_g = \mu \underset{\Phi^g\widehat{\zeta}}{\times} \mu$. Since the net $(\Phi^g\widehat{\zeta})_{g\in G}$ of sub-σ-algebras of $\mathcal{B}$ is decreasing, the martingale convergence theorem (cf. [2, Theorem 3.4]) implies that the measure $\lambda_\mu = \mu \underset{\pi_\mu(\Phi)}{\times} \mu$ is the weak limit of $(\lambda_g,\ g \in G)$, i.e.

$$\lambda_\mu = \lim_{g\in G} \lambda_g. \tag{3.8}$$

Since we deal with a closed set, we have

$$\lambda_\mu(\overline{A(\Phi)}) \geq \limsup_{g\in G} \lambda_g(\overline{A(\Phi)}).$$

Applying Lemma 6 of [1] we obtain $\big(\mu \underset{\widehat{\zeta}}{\times} \mu\big)(R_\zeta) = 1$. Therefore applying (d) we get $\big(\mu \underset{\widehat{\zeta}}{\times} \mu\big)(\overline{A(\Phi)}) = 1$. The fact that $A(\Phi)$ is $\Phi \times \Phi$-invariant implies

$$\big(\mu \underset{\Phi^g\widehat{\zeta}}{\times} \mu\big)(\overline{A(\Phi)}) = 1, \quad \text{for any } g \in G.$$

Thus (3.8) gives $\lambda_\mu(\overline{A(\Phi)}) = 1$, i.e. $\Lambda_\mu \subset \overline{A(\Phi)}$. Hence by Proposition 3.8 we get $E_\mu(\Phi) \subset \overline{A(\Phi)}$ which completes the proof. $\square$

As an easy consequence of Theorem 3.9 we have the following

Corollary 3.10 ([2, Proposition 3]). *If $\mu \in \mathcal{P}(X,\Phi)$ has full support and the dynamical system $(X,\mathcal{B},\mu,\Phi)$ has completely positive entropy, i.e. $\pi_\mu(\Phi) = \mathcal{N}$ then $A(\Phi)$ is a dense subset of $X \times X$.*

Proof. The assumptions and Remark 3.7 imply

$$\Lambda_\mu = \operatorname{Supp} \mu \underset{\pi_\mu(\Phi)}{\times} \mu = \operatorname{Supp} \mu \times \mu = X \times X \quad \text{and} \quad S(\mu) = \Delta.$$

Therefore Theorem 3.9 and Propositon 3.8 give

$$\overline{A(\Phi)} \supset E_\mu(\Phi) \cup \Delta = E_\mu(\Phi) \cup S(\mu) = \Lambda_\mu = X \times X. \qquad \square$$

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