ON POSITIVE VISCOSITY SOLUTIONS
OF FRACTIONAL LANE–EMDEN SYSTEMS

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ABSTRACT. In this paper we discuss the existence, nonexistence and uniqueness of positive viscosity solution for the following coupled system involving fractional Laplace operator on a smooth bounded domain Ω in $\mathbb{R}^n$:

$$\begin{cases} (-\Delta)^s u = v^p & \text{in } \Omega, \\ (-\Delta)^s v = u^q & \text{in } \Omega, \\ u = v = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

By means of an appropriate variational framework and a Hölder regularity result for suitable weak solutions of the above system, we prove that such a system admits at least one positive viscosity solution for any $0 < s < 1$, provided that $p, q > 0$, $pq \neq 1$ and the couple (p, q) is below the critical hyperbole

$$\frac{1}{p+1} + \frac{1}{q+1} = \frac{n-2s}{n}$$

whenever $n > 2s$. Moreover, by using the maximum principles for the fractional Laplace operator, we show that uniqueness occurs whenever $pq < 1$. Lastly, assuming Ω is star-shaped, by using a Rellich type variational identity, we prove that no such a solution exists if (p, q) is on or above the critical hyperbole. A crucial point in our proofs is proving, given a critical point $u \in W_0^{s,(p+1)/p}(\Omega) \cap W^{2s,(p+1)/p}(\Omega)$ of a related functional, that there is a function v in an appropriate Sobolev space (Proposition 2.1) so that (u, v) is a weak solution of the above system and a bootstrap argument can be applied successfully in order to establish its Hölder regularity (Proposition 3.1). The difficulty is caused mainly by the absence of a L^p Calderón–Zygmund theory with $p > 1$ associated to the operator $(-\Delta)^s$ for $0 < s < 1$.

1. Introduction and main results

This work is devoted to the study of the existence, uniqueness and nonexistence of positive viscosity solutions for nonlocal elliptic systems on bounded domains which will be described henceforth.

The fractional Laplace operator (or fractional Laplacian) of order $2s$, with $0 < s < 1$, denoted by $(-\Delta)^s$, is defined as

$$(-\Delta)^s u(x) = C(n,s) \quad \text{P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} \, dy,$$

or equivalently,

$$(1.1) \qquad (-\Delta)^s u(x) = -\frac{1}{2} C(n,s) \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}} \, dy$$

for all $x \in \mathbb{R}^n$, where P.V. denotes the principal value of the first integral and

$$C(n,s) = \left(\int_{\mathbb{R}^n} \frac{1 - \cos(\zeta_1)}{|\zeta|^{n+2s}} \, d\zeta \right)^{-1}$$

with $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{R}^n$. It is important to mention that the convergence of the integral of the right-hand side of (1.1) relies on a suitable decaying of the function u. Indeed, it is well known that $(-\Delta)^s u(x)$ is finite for every function $u \in H^s(\mathbb{R}^n)$ (see [15]), where $H^s(\mathbb{R}^n)$ is the Sobolev space defined by

$$\left\{ u \in L^2(\mathbb{R}^n) : \frac{|u(x) - u(y)|}{|x - y|^{n/2+s}} \in L^2(\mathbb{R}^n \times \mathbb{R}^n) \right\}.$$

Notice also that $(-\Delta)^s u$ interpolates the Laplace operator in $\mathbb{R}^n$ in the sense that, for any function $u \in C_0^\infty(\mathbb{R}^n)$,

$$\lim_{s \to 1^-} (-\Delta)^s u(x) = -\Delta u(x)$$

pointwise in $\mathbb{R}^n$. Fractional Laplace operators arise naturally and have applications in different areas such as physics, ecology, finance, chemistry and probability, see [3].

A closely related operator, but different from $(-\Delta)^s$, is the spectral fractional Laplace operator $\mathcal{A}^s$ which is defined in terms of the Dirichlet spectra of the Laplace operator on Ω. Roughly, for a L^2-orthonormal basis of eigenfunctions (ψ_k) corresponding to eigenvalues (μ_k) of the Laplace operator with zero Dirichlet boundary values on $\partial \Omega$, the operator $\mathcal{A}^s$ is defined as $\mathcal{A}^s u = \sum_{k=1}^{\infty} a_k \mu_k^s \psi_k$, where a_k, $k \geq 1$, are the coefficients of the expansion $u = \sum_{k=1}^{\infty} a_k \psi_k$.

After the work [10] on the characterization for any $0 < s < 1$ of the operator $(-\Delta)^s$ in terms of a Dirichlet-to-Neumann map associated to a suitable extension problem, a great deal of attention has been dedicated in the last years to

nonlinear nonlocal problems of the kind

(1.2)
$$\begin{cases} (-\Delta)^s u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where Ω is a smooth bounded open subset of $\mathbb{R}^n$, $n \geq 1$ and $0 < s < 1$.

Several works have been focused on existence [19], [35], [36], [37], [39], nonexistence [16], [39], regularity [8], [33] and symmetry [5] of viscosity solutions, among other qualitative properties (e.g. [18]). For developments related to (1.2) involving the operator $\mathcal{A}^s$, we refer to [4], [7], [9], [11], among others.

An especially important problem (1.2) occurs for the power function $f(x,u) = u^p$ with $p > 0$, which is well known as fractional Lane–Emden problem. Recently, it has been proved in [38] that this problem admits at least one positive viscosity solution for $1 < p < (n+2s)/(n-2s)$. The nonexistence has been established in [34] whenever $p \geq (n+2s)/(n-2s)$ and Ω is star-shaped. These results were known long before for $s = 1$, see the classical references [2], [23], [30].

We here are interested in studying the following vector counterpart of the fractional Lane–Emden problem:

(1.3)
$$\begin{cases} (-\Delta)^s u = v^p & \text{in } \Omega, \\ (-\Delta)^s v = u^q & \text{in } \Omega, \\ u = v = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

for $p, q > 0$, which inspires the title of this work.

For $s = 1$, the system (1.3) and a number of its generalizations have been widely investigated in the literature during the three last decades, see for instance the survey [20] and references therein. In particular, the notions of superlinearity, sublinearity and criticality (supercriticality, subcriticality) (in function of a certain hyperbole) have been introduced in [17], [27]. Namely, the system (1.3) has a superlinear behavior when $pq > 1$, sublinear when $pq < 1$ and critical (supercritical, subcritical) when $n \geq 3$ and (p,q) is on (above, below) the hyperbole, known as critical hyperbole,

$$\frac{1}{p+1} + \frac{1}{q+1} = \frac{n-2}{n}.$$

The sublinear case has been considered in [17] where the existence and uniqueness of positive classical solution has been proved. The superlinear and subcritical case has been completely covered in the works [13], [21], [22], [24] where the existence of at least one positive classical solution has been established. The nonexistence of positive classical solutions has been established in [27] on star-shaped domains. Finally, when $pq = 1$, the behavior of (1.3) is resonant and the related eigenvalue problem has been studied in [28].

In this work we focus on the existence and nonexistence of positive viscosity solution of (1.3) for any $0 < s < 1$. We determine the precise set of positive exponents p and q for which the problem (1.3) admits always a positive viscosity solution. In particular, we extend the above-mentioned results corresponding to the fractional Lane–Emden problem for $0 < s \leq 1$ and to the Lane–Emden system involving Laplace operator. As a byproduct, the notions of sublinearity, superlinearity and criticality (supercriticality, subcriticality) associated to the problem (1.3) are naturally extended for any $0 < s < 1$.

The ideas involved in our proofs base on variational methods, C^α regularity of weak solutions and an integral variational identity satisfied by positive viscosity solutions of (1.3). We shall introduce a variational framework in order to establish the existence of nontrivial nonnegative weak solution of (1.3) in a suitable sense. In our formulation, the function u arises as a nonzero critical point of a functional defined on an appropriate space of functions and then, in a natural way, we construct a function v so that the couple (u, v) is a weak solution of (1.3). The construction of v is a strategic step in our approach (see Proposition 2.1), once we do not have a Calderón–Zygmund theory available for fractional operators. Using a C^α regularity result (see Proposition 3.1) for weak solutions of (1.3) and maximum principles for fractional Laplace operators, we then deduce that the couple (u, v) is a positive viscosity solution of (1.3). Moreover, we prove its uniqueness when $pq < 1$. The key tool used in the nonexistence proof is a Rellich type variational identity (see Proposition 4.1) to positive viscosity solutions of (1.3). The proof of the C^α regularity consists in first showing that weak solutions of (1.3) belong to $L^\infty(\Omega) \times L^\infty(\Omega)$ and then applying to each equation the C^α regularity up to the boundary proved recently in [33]. The proof of the variational identity to Lane–Emden systems uses the Pohozaev variational identity to fractional elliptic equations obtained recently in [34].

Other questions have recently been discussed in some papers which mention ours. We quote for example the works [12] on asymptotic behavior of minimal energy solutions, [42] on symmetry of solutions and [43] on Liouville type theorems on half spaces for fractional systems. Related systems also have been investigated by using other methods. We refer to the work [26] for systems involving different operators $(-\Delta)^s$ and $(-\Delta)^t$ in each one of equations and to the work [11] for systems involving the spectral fractional operator $\mathcal{A}^s$.

In order to state our three main theorems, we should first introduce the concept of positive viscosity solution to (1.3). A couple $(u, v) \in (H^s(\mathbb{R}^n))^2$ of continuous functions in $\mathbb{R}^n$ is said to be a viscosity subsolution (supersolution) of (1.3) if each point $x_0 \in \Omega$ admits a neighborhood U with $\overline{U} \subset \Omega$ such that for any $\varphi, \psi \in C^2(\overline{U})$ satisfying $u(x_0) = \varphi(x_0)$, $v(x_0) = \psi(x_0)$, $u \geq (\leq)$ φ and

$v \geq (\leq) \ \psi$ in U, the functions

$$(1.4) \qquad \overline{u} = \begin{cases} \varphi & \text{in } U, \\ u & \text{in } \mathbb{R}^n \setminus U, \end{cases} \quad \text{and} \quad \overline{v} = \begin{cases} \psi & \text{in } U, \\ v & \text{in } \mathbb{R}^n \setminus U, \end{cases}$$

satisfy

$$(-\Delta)^s \overline{u}(x_0) \leq \ (\geq) \ |v(x_0)|^{p-1} v(x_0) \quad \text{and} \quad (-\Delta)^s \overline{v}(x_0) \leq \ (\geq) \ |u(x_0)|^{q-1} u(x_0).$$

The couple (u, v) is said to be a viscosity solution of (1.3) if it is simultaneously a viscosity subsolution and supersolution. If further u and v are positive in Ω and nonnegative in $\mathbb{R}^n$, we say that (u, v) is a positive viscosity solution.

THEOREM 1.1 (Sublinear case). *Let Ω be a smooth bounded open subset of $\mathbb{R}^n$, $n \geq 1$ and $0 < s < 1$. Assume that $p, q > 0$ and $pq < 1$. Then the problem (1.3) admits a unique positive viscosity solution.*

THEOREM 1.2 (Superlinear-subcritical case). *Let Ω be a smooth bounded open subset of $\mathbb{R}^n$, $n \geq 1$ and $0 < s < 1$. Assume that $p, q > 0$, $pq > 1$ and*

$$(1.5) \qquad \frac{1}{p+1} + \frac{1}{q+1} > \frac{n-2s}{n}.$$

Then the problem (1.3) admits at least one positive viscosity solution.

THEOREM 1.3 (Critical and supercritical cases). *Let Ω be a smooth bounded open subset of $\mathbb{R}^n$, $n > 2s$ and $0 < s < 1$. Assume that Ω is star-shaped, $p, q > 0$ and*

$$(1.6) \qquad \frac{1}{p+1} + \frac{1}{q+1} \leq \frac{n-2s}{n}.$$

Then the problem (1.3) admits no positive viscosity solution.

Positive viscosity solutions provided in Theorems 1.1 and 1.2 are indeed classical. In fact, in the subcritical case, such solutions belongs to $C^\alpha(\mathbb{R}^n) \times C^\alpha(\mathbb{R}^n)$ for some $\alpha \in (0, 1)$, thanks to Proposition 3.1 of Section 3. In this case, the right-hand side of system (1.3) is in $C^\gamma(\overline{\Omega}) \times C^\gamma(\overline{\Omega})$ for some $\gamma \in (0, 1)$. Using this fact, by Theorem 2.5 of [31], it follows that viscosity solutions of (1.3) are classical. Notice also that the nonexistence result provided in Theorem 1.3 clearly holds for classical solutions, since these ones are more restrictive than viscosity solutions.

For dimensions $n > 2s$, the hyperbole

$$(1.7) \qquad \frac{1}{p+1} + \frac{1}{q+1} = \frac{n-2s}{n}$$

is named critical hyperbole associated to the Lane–Emden system (1.3), see the dashed black curve in the figure 1. As we shall see in this work, it appears naturally in the context of Sobolev embedding and integral variational identity.

Note also that the curve (p, q) given by the hyperbole $pq = 1$ splits the behavior of (1.3) into sublinear and superlinear, see the dashed red curve in the Figure 1.
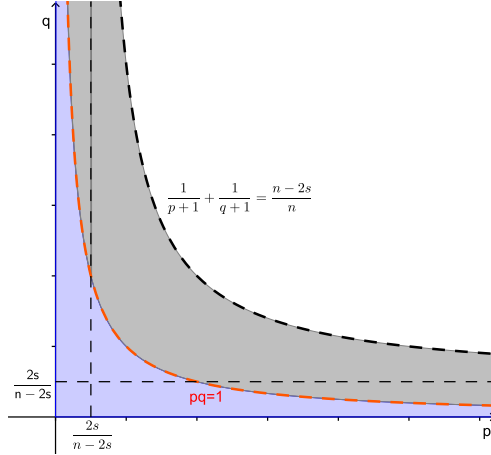

FIGURE 1. The existence range of couples (p, q) when $n > 2s$.

The remainder of paper is organized into six sections. In Section 2 we briefly recall some definitions and facts dealing with fractional Sobolev spaces and introduce the variational framework and the concept of weak solutions to be used in the existence proofs. In Section 3 we prove the C^α regularity of weak solutions of (1.3) in the subcritical case regarding the hyperbole (1.7). In Section 4 we establish a Rellich type variational identity to positive viscosity solutions of (1.3). In Section 5 we prove Theorem 1.1 by using a direct minimization approach, the referred C^α regularity and maximum principles satisfied by the fractional Laplacian $(-\Delta)^s$. In Section 6 we prove Theorem 1.2 by using the mountain pass theorem and again the same regularization result. Finally, in Section 7 we prove Theorem 1.3 by applying the variational identity obtained in Section 4.

2. Preliminaries and the variational setting

In this section we recall the definition of fractional Sobolev spaces on bounded open subsets of $\mathbb{R}^n$ (see [15] for more details) and present the variational formulation to be used in the proofs of Theorems 1.1 and 1.2.

We start by fixing a parameter $0 < s < 1$. Let Ω be an open subset of $\mathbb{R}^n$ with $n \geq 1$. For any $r \in (1, +\infty)$, consider the fractional Sobolev space $W^{s,r}(\Omega)$ defined by

$$(2.1) \qquad W^{s,r}(\Omega) := \left\{ u \in L^r(\Omega) : \frac{|u(x) - u(y)|}{|x - y|^{n/r+s}} \in L^r(\Omega \times \Omega) \right\},$$

which is a Banach space interpolating $L^r(\Omega)$ and $W^{1,r}(\Omega)$ induced with the norm

$$(2.2) \qquad \|u\|_{W^{s,r}(\Omega)} := \left(\int_\Omega |u|^r \, dx + \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^r}{|x-y|^{n+sr}} \, dx \, dy \right)^{1/r},$$

where the term

$$[u]_{W^{s,r}(\Omega)} := \left(\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^r}{|x-y|^{n+sr}} dx dy \right)^{1/r}$$

is called Gagliardo semi-norm of u.

Let $s \in \mathbb{R} \setminus \mathbb{N}$ with $s \geq 1$. The space $W^{s,r}(\Omega)$ is defined as

$$W^{s,r}(\Omega) = \{u \in W^{[s],r}(\Omega) : D^j u \in W^{s-[s],r}(\Omega), \text{ for all } j, \ |j| = [s]\},$$

where $[s]$ is the largest integer smaller than s, j denotes the n-uple $(j_1, \ldots, j_n) \in \mathbb{N}^n$ and $|j|$ denotes the sum $j_1 + \ldots + j_n$. The space $W^{s,r}(\Omega)$ endowed with the norm

$$(2.3) \qquad \|u\|_{W^{s,r}(\Omega)} = \left(\|u\|_{W^{[s],r}(\Omega)}^r + [u]_{W^{s-[s],r}(\Omega)}^r \right)^{1/r}$$

is a reflexive Banach space. Clearly, if $s = m$ is an integer, the space $W^{s,r}(\Omega)$ coincides with the Sobolev space $W^{m,r}(\Omega)$.

Let $W_0^{s,r}(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{W^{s,r}(\Omega)}$ defined in (2.3). For $0 < s \leq 1$, we have

$$W_0^{s,r}(\Omega) = \{u \in W^{s,r}(\mathbb{R}^n) : u = 0 \text{ in } \mathbb{R}^n \setminus \Omega\}$$

and $W_0^{s,2}(\Omega) = H_0^s(\Omega)$.

We are ready to introduce the variational framework associated to (1.3). Let Ω be a smooth bounded open subset of $\mathbb{R}^n$, $n \geq 1$ and $0 < s < 1$. In order to inspire our formulation, assume that the couple (u,v) of nonnegative functions is roughly a solution of (1.3). From the first equation, we have $v = \left((-\Delta)^s u\right)^{1/p}$. Plugging this equality into the second equation, we obtain

$$(2.4) \qquad \begin{cases} (-\Delta)^s \left((-\Delta)^s u\right)^{1/p} = u^q & \text{in } \Omega, \\ u \geq 0 & \text{in } \mathbb{R}^n, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

The basic idea in trying to solve (2.4) is considering the functional $\Phi \colon E_p^s \to \mathbb{R}$ defined by

$$(2.5) \qquad \Phi(u) = \frac{p}{p+1} \int_\Omega |(-\Delta)^s u|^{(p+1)/p} \, dx - \frac{1}{q+1} \int_\Omega (u^+)^{q+1} \, dx,$$

where E_p^s denotes the reflexive Banach space $W_0^{s,(p+1)/p}(\Omega) \cap W^{2s,(p+1)/p}(\Omega)$ as endowed with the norm

$$\|u\|_{E_p^s} := \left(\int_\Omega |(-\Delta)^s u|^{(p+1)/p} \, dx \right)^{p/(p+1)}.$$

In the case that E_p^s is continuously embedded in $L^{q+1}(\Omega)$, the Gateaux derivative of Φ at $u \in E_p^s$ in the direction $\varphi \in E_p^s$ is given by

$$\Phi'(u)\varphi = \int_\Omega \left|(-\Delta)^s u\right|^{1/p-1}(-\Delta)^s u(-\Delta)^s \varphi \, dx - \int_\Omega (u^+)^q \varphi \, dx.$$

This is the case when the couple (p, q) is below the critical hyperbole (1.7).

As we briefly shall see, weak solutions of (1.3) can be constructed in an appropriate product space starting from critical points of Φ in E_p^s. Since we do not have a $W^{2,p}$ theory available for fractional Laplace operators, the following result plays an essential role in the proof of the regularity result of the next section.

PROPOSITION 2.1. *Let Ω be a smooth bounded open subset of $\mathbb{R}^n$, $n \geq 1$ and $0 < s < 1$. Let u be a critical point of Φ in E_p^s. Assume that the couple (p, q) satisfies $p, q > 0$ and (1.5). Then, there exists a function $v \in H_0^s(\Omega)$ such that (u, v) is a nonnegative weak solution of the problem (1.3) in the sense that*

$$\int_\Omega u(-\Delta)^s \varphi \, dx = \int_\Omega v^p \varphi \, dx, \qquad \int_\Omega v(-\Delta)^s \varphi \, dx = \int_\Omega u^q \varphi \, dx$$

for all $\varphi \in C_0^\infty(\Omega)$.

PROOF. Sobolev embeddings used here can be found in [14]. Firstly, we claim that $(u^+)^q \in H^{-s}(\Omega)$ for all $u \in E_p^s$. In fact, note that $W^{2s,(p+1)/p}(\Omega) \hookrightarrow L^\infty(\Omega)$ for $n \leq 2s$, so that readily $(u^+)^q \in H^{-s}(\Omega)$. For $n > 2s$, we have $L^{(2_s^*)'}(\Omega) \subset H^{-s}(\Omega)$, where $2_s^* = 2n/(n-2s)$. So, for $0 < p \leq 2s/(n-2s)$ and $q > 0$, we derive $W^{2s,(p+1)/p}(\Omega) \hookrightarrow L^r(\Omega)$ for all $r \geq 1$, so that $(u^+)^q \in H^{-s}(\Omega)$. If $p > 2s/(n-2s)$ and $q \leq (n+2s)/(n-2s)$, we have $(u^+)^q \in H^{-s}(\Omega)$, since E_p^s is continuously embedded in $L^{q+1}(\Omega)$ and $(q+1)/q \geq (2_s^*)'$. There remains only the cases when $p > 2s/(n-2s)$ and $q > (n+2s)/(n-2s)$. By the condition (1.5), we get $p < (n+2s)/(n-2s)$. Since $u \in L^{q+1}(\Omega)$ and $q > (n+2s)/(n-2s)$, we have $u^+ \in H^{-s}(\Omega)$. So, by Proposition 3.4 of [19], the problem

$$\begin{cases} (-\Delta)^s w_1 = u^+ & \text{in } \Omega, \\ w_1 = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

admits a unique nonnegative weak solution $w_1 \in H_0^s(\Omega)$. For $2s < n \leq 6s$, we have $q + 1 \geq n/(2s)$. Then, $u^+ \in L^{q(2_s^*)'}(\Omega)$, so that $(u^+)^q \in H^{-s}(\Omega)$. Furthermore, Sobolev embedding gives that $u^+ \in L^{n(p+1)/(np-2s(p+1))}(\Omega)$. Therefore, by Proposition 1.4 of [33], we have $w_1 \in L^{n(p+1)/(np-4s(p+1))}(\Omega)$. Since $w_1 \in H^{-s}(\Omega)$, again by Proposition 3.4 of [19], the problem

$$\begin{cases} (-\Delta)^s w_2 = w_1 & \text{in } \Omega, \\ w_2 = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

admits a unique nonnegative weak solution $w_2 \in H_0^s(\Omega)$. If $6s < n \leq 10s$, we have $n(p+1)/(np-4s(p+1)) \geq n/(2s)$ and then $u^+ \in L^{q(2_s^*)'}(\Omega)$. Consequently, $(u^+)^q \in H^{-s}(\Omega)$. Proceeding inductively, we get $(u^+)^q \in H^{-s}(\Omega)$ for any $n > 2s$. Thanks to this fact and Proposition 3.4 of [19], for any $u \in E_p^s$, it follows that the problem

$$\begin{cases} (-\Delta)^s v = (u^+)^q & \text{in } \Omega, \\ v = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

admits a unique nonnegative weak solution $v \in H_0^s(\Omega)$. Again using the condition (1.5), Proposition 1.4 of [33] and Sobolev embedding theorem, we see that $v \in L^{p+1}(\Omega)$.

On the other hand, if u is a critical point of Φ in E_p^s and v is a weak solution of the above equation in $H_0^s(\Omega)$, we have

$$(2.6) \qquad \int_\Omega \left(|(-\Delta)^s u|^{1/p-1}(-\Delta)^s u - v\right)(-\Delta)^s \varphi \, dx = 0$$

for all $\varphi \in E_p^s \cap H_0^s(\Omega)$ by density (see [14]). Notice now that, by Proposition 3.4 of [19], inequality (3.4) of [29] and the standard potential theory (see formula (41) of [41]), the problem

$$\begin{cases} (-\Delta)^s \varphi = f & \text{in } \Omega, \\ \varphi = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

admits a unique weak solution $\varphi \in E_p^s \cap H_0^s(\Omega)$ for any $f \in C_0^\infty(\Omega)$. Since the term in the parenthesis in (2.6) belongs to $L^{p+1}(\Omega)$, it follows that u satisfies pointwise, and so in the weak sense,

$$\begin{cases} (-\Delta)^s u = v^p & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

Lastly, by the maximum principle for s-superharmonic functions in $L^1(\Omega)$ due to Abatangelo (see Lemma 3.3.4 of [1]), one concludes that u is nonnegative.

In summary, starting from a critical point $u \in E_p^s$ of Φ, we were able to construct a nonnegative weak solution (u, v) of the problem (1.3) in the space $E_p^s \times H_0^s(\Omega)$. $\qquad \square$

3. Hölder regularity

In this section we show that weak solutions of (1.3) are Hölder viscosity solutions when the system is subcritical.

PROPOSITION 3.1. *Let Ω be a smooth bounded open subset of $\mathbb{R}^n$, $n \geq 1$ and $0 < s < 1$. Let $(u, v) \in E_p^s \times H_0^s(\Omega)$ be a nonnegative weak solution of the*

problem (1.3)). Assume that the couple (p, q) satisfies $p, q > 0$ and (1.5). Then,
$(u, v) \in C^\alpha(\mathbb{R}^n) \times C^\alpha(\mathbb{R}^n)$ for some $\alpha \in (0, 1)$.

Proof. Firstly, notice that it suffices to prove that $u \in L^\infty(\Omega)$ or $v \in L^\infty(\Omega)$. If this is the case, then both functions belong to $L^\infty(\Omega)$. In fact, applying Proposition 1.4 of [33] to the second (or first) equation of (1.3), it follows readily that $v \in L^\infty(\Omega)$ (or $u \in L^\infty(\Omega)$). Thus, again using Proposition 1.4 of [33] in both equations, we derive the statement that $(u, v) \in C^\alpha(\mathbb{R}^n) \times C^\alpha(\mathbb{R}^n)$ for some $\alpha \in (0, 1)$.

In order to show that $u \in L^\infty(\Omega)$ or $v \in L^\infty(\Omega)$, we analyze separately some different cases depending on the values of n, s, p and q. The Sobolev embeddings to be used here were established in [14].

For $n \leq 2s$, we have $W^{2s, (p+1)/p}(\Omega) \hookrightarrow L^\infty(\Omega)$, so that $u \in L^\infty(\Omega)$. Assume then $n > 2s$. Note that the above conclusion also holds for $0 < p < 2s/(n - 2s)$ and $q > 0$, that is $W^{2s, (p+1)/p}(\Omega) \hookrightarrow L^\infty(\Omega)$ and so $u \in L^\infty(\Omega)$. For $2s/(n - 2s) \leq p \leq 1$ and $q > 1$, we rewrite the problem (1.3) as follows

$$(3.1) \qquad \begin{cases} (-\Delta)^s u = a(x) v^{\frac{p}{2}} & \text{in } \Omega, \\ (-\Delta)^s v = b(x) u & \text{in } \Omega, \\ u = v = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where $a(x) = v(x)^{p/2}$ and $b(x) = u(x)^{q-1}$. Since the couple (p, q) satisfies (1.5), we have $p + 1 < 2n/(n - 2s)$. By Sobolev embedding, $H_0^s(\Omega) \to L^{p+1}(\Omega)$ is bounded, so that $a \in L^{2(p+1)/p}(\Omega)$. Thus, for each fixed $\varepsilon > 0$, we can construct functions $q_\varepsilon \in L^{2(p+1)/p}(\Omega)$, $f_\varepsilon \in L^\infty(\Omega)$ and a constant $K_\varepsilon > 0$ such that

$$a(x) v(x)^{p/2} = q_\varepsilon(x) v(x)^{p/2} + f_\varepsilon(x)$$

and $\|q_\varepsilon\|_{L^{2(p+1)/p}(\Omega)} < \varepsilon$, $\|f_\varepsilon\|_{L^\infty(\Omega)} < K_\varepsilon$.

In fact, consider the set $\Omega_k = \{x \in \Omega : |a(x)| < k\}$, where k is chosen such that

$$\int_{\Omega_k^c} |a(x)|^{2(p+1)/p} \, dx < \frac{1}{2} \varepsilon^{2(p+1)/p}.$$

This condition is clearly satisfied for $k = k_\varepsilon$ sufficiently large.

We now write

$$(3.2) \qquad q_\varepsilon(x) = \begin{cases} \dfrac{1}{m} \, a(x) & \text{for } x \in \Omega_{k_\varepsilon}, \\ a(x) & \text{for } x \in \Omega_{k_\varepsilon}^c, \end{cases}$$

and $f_\varepsilon(x) = (a(x) - q_\varepsilon(x))v(x)^{p/2}$. So,

$$\int_\Omega |q_\varepsilon(x)|^{2(p+1)/p}\, dx = \int_{\Omega_{k_\varepsilon}} |q_\varepsilon(x)|^{2(p+1)/p}\, dx + \int_{\Omega_{k_\varepsilon}^c} |q_\varepsilon(x)|^{2(p+1)/p}\, dx$$

$$= \left(\frac{1}{m}\right)^{2(p+1)/p} \int_{\Omega_{k_\varepsilon}} |a(x)|^{2(p+1)/p}\, dx + \int_{\Omega_{k_\varepsilon}^c} |a(x)|^{2(p+1)/p}\, dx$$

$$< \left(\frac{1}{m}\right)^{2(p+1)/p} \int_{\Omega_{k_\varepsilon}} |a(x)|^{2(p+1)/p}\, dx + \frac{1}{2}\varepsilon^{2(p+1)/p}.$$

Then, for $m = m_\varepsilon > (2^{p/(2(p+1))}/\varepsilon)\|a\|_{L^{2(p+1)/p}(\Omega)}$, we have

$$\|q_\varepsilon\|_{L^{2(p+1)/p}(\Omega)} < \varepsilon.$$

Note also that $f_\varepsilon(x) = 0$ for all $x \in \Omega_{k_\varepsilon}^c$ and, for this choice of m,

$$f_\varepsilon(x) = \left(1 - \frac{1}{m_\varepsilon}\right)a(x)^2 \leq \left(1 - \frac{1}{m_\varepsilon}\right)k_\varepsilon^2$$

for all $x \in \Omega_{k_\varepsilon}$. Therefore,

$$\|f_\varepsilon\|_{L^\infty(\Omega)} \leq \left(1 - \frac{1}{m_\varepsilon}\right)k_\varepsilon^2 := K_\varepsilon.$$

On the other hand, we get $v(x) = (-\Delta)^{-s}(bu)(x)$, where $b \in L^{(q+1)/(q-1)}(\Omega)$. Then,

$$u(x) = (-\Delta)^{-s}\big[q_\varepsilon(x)((-\Delta)^{-s}(bu)(x))^{p/2}\big] + (-\Delta)^{-s}f_\varepsilon(x).$$

By Proposition 1.4 of [33], the assertions (ii) and (iv) below follow directly. In addition, by using Hölder's inequality, we also deduce the statements (i) and (iii). Precisely, for fixed $\gamma > 1$, we have:

(i) The map $w \to b(x)w$ is bounded from $L^\gamma(\Omega)$ to $L^\beta(\Omega)$ for

$$\frac{1}{\beta} = \frac{q-1}{q+1} + \frac{1}{\gamma}.$$

(ii) For any $\theta \geq 1$ in the case that $\beta \geq n/(2s)$ or, for θ given by

$$2s = n\left(\frac{1}{\beta} - \frac{2}{p\theta}\right)$$

in the case that $\beta < n/(2s)$, there exists a constant $C > 0$, depending on β and θ, such that

$$\big\|((-\Delta)^s w)^{p/2}\big\|_{L^\theta(\Omega)} \leq C\|w\|_{L^\beta(\Omega)}^{p/2} \quad \text{for all } w \in L^\beta(\Omega).$$

(iii) The map $w \to q_\varepsilon(x)w$ is bounded from $L^\theta(\Omega)$ to $L^\eta(\Omega)$ with norm given by $\|q_\varepsilon\|_{L^{2(p+1)/p}(\Omega)}$, where $\theta \geq 1$ and η satisfies

$$\frac{1}{\eta} = \frac{p}{2(p+1)} + \frac{1}{\theta}.$$

(iv) For any $\delta \geq 1$ in the case that $\eta \geq n/2s$ or, for δ given by

$$2s = n\left(\frac{1}{\eta} - \frac{1}{\delta}\right)$$

in the case that $\eta < n/(2s)$, the map $w \to (-\Delta)^{-s}w$ is bounded from $L^\eta(\Omega)$ to $L^\delta(\Omega)$.

Joining (i)–(iv) and using that (p,q) satisfies (1.5), one easily checks that $\gamma < \delta$ and, moreover,

$$\|u\|_{L^\delta(\Omega)} \leq \big\|(-\Delta)^{-s}\big[q_\varepsilon(x)((-\Delta)^{-s}(bu))^{p/2}\big]\big\|_{L^\delta(\Omega)} + \|(-\Delta)^{-s}f_\varepsilon\|_{L^\delta(\Omega)}$$

$$\leq C\Big(\|q_\varepsilon\|_{L^{(2(p+1))/p}(\Omega)}\|u\|_{L^\delta(\Omega)}^{p/2} + \|f_\varepsilon\|_{L^\delta(\Omega)}\Big).$$

Since $p \leq 1$, $\|q_\varepsilon\|_{L^{2(p+1)/p}(\Omega)} < \varepsilon$ and $f_\varepsilon \in L^\infty(\Omega)$, we have $\|u\|_{L^\delta(\Omega)} \leq K$ for some constant $K > 0$ independent of u. Proceeding inductively, we get $u \in L^\delta(\Omega)$ for all $\delta \geq 1$. So, Proposition 1.4 of [33] implies that $v \in L^\infty(\Omega)$.

The remainder cases are treated in a similar way by writing $a(x) = v(x)^{p-1}$ if $p > 1$ and $b(x) = u(x)^{\frac{q}{2}}$ if $q \leq 1$ or $b(x) = u(x)^{q-1}$ if $q > 1$. $\qquad\square$

As consequence of Propositions 2.1 and 3.1, Theorem 1 of [38] and the strong maximum principle for $(-\Delta)^s$ (see [40]), we have

COROLLARY 3.2. *Let Ω be a smooth bounded open subset of $\mathbb{R}^n$, $n \geq 1$ and $0 < s < 1$. Assume that the couple (p,q) satisfies $p, q > 0$ and (1.5). If $u \in E_p^s$ is a nonzero critical point of Φ, then there exists a function v such that (u,v) is a positive viscosity solution of (1.3).*

4. Rellich type variational identity

In this section we deduce that positive viscosity solutions of (1.3) satisfy the following integral identity:

PROPOSITION 4.1. (Rellich identity) *Let Ω be a smooth bounded open subset of $\mathbb{R}^n$, $n \geq 1$ and $0 < s < 1$. Then, every positive viscosity solution (u,v) of the problem (1.3) satisfies*

$$(4.1) \quad \Gamma(1+s)^2 \int_{\partial\Omega} \frac{u}{d^s}\frac{v}{d^s}(x \cdot \nu)\, d\sigma = \left(\frac{n}{q+1} + \frac{n}{p+1} - (n-2s)\right)\int_\Omega u^{q+1}\, dx,$$

where ν denotes the unit outward normal to $\partial\Omega$, Γ is the Gamma function, $d(x) = \mathrm{dist}(x, \partial\Omega)$ and

$$\frac{u}{d^s}(x) := \lim_{\varepsilon \to 0^+} \frac{u(x - \varepsilon\nu)}{d^s(x - \varepsilon\nu)} > 0 \quad \text{for all } x \in \partial\Omega.$$

It deserves mention that $u/d^s, v/d^s \in C^\alpha(\overline\Omega)$ for some $\alpha \in (0,1)$ and $u/d^s, v/d^s > 0$ in $\overline\Omega$ (see Theorem 1.2 in [32]). So, the left-hand side of the identity (4.1) is well defined.

PROOF. Let (u, v) be a viscosity solution of (1.3). Then,

(4.2)
$$\begin{cases} (-\Delta)^s(u + v) = v^p + u^q & \text{in } \Omega, \\ u + v = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

and

(4.3)
$$\begin{cases} (-\Delta)^s(u - v) = v^p - u^q & \text{in } \Omega, \\ u - v = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

Applying the Pohozaev variational identity for semilinear problems involving the operator $(-\Delta)^s$ (Theorem 1.1 of [34]), we get

$$-\int_\Omega (x \cdot \nabla u + v)((-\Delta)^s u + (-\Delta)^s v) \, dx$$
$$= -\frac{2s - n}{2} \int_\Omega (u + v)(v^p + u^q) \, dx + \frac{1}{2}\Gamma(1 + s)^2 \int_{\partial\Omega} \left(\frac{u + v}{d^s}\right)^2 (x \cdot \nu) \, d\sigma$$

and

$$-\int_\Omega (x \cdot \nabla u + v)((-\Delta)^s u - (-\Delta)^s v) \, dx$$
$$= -\frac{2s - n}{2} \int_\Omega (u - v)(v^p - u^q) \, dx + \frac{1}{2}\Gamma(1 + s)^2 \int_{\partial\Omega} \left(\frac{u - v}{d^s}\right)^2 (x \cdot \nu) \, d\sigma.$$

Now, subtracting both identities, we obtain

(4.4) $\quad 2 \int_\Omega [(x \cdot \nabla u)(-\Delta)^s v + (x \cdot \nabla v)(-\Delta)^s u] \, dx$

$$(2s - n) \int_\Omega [u(-\Delta)^s v + v(-\Delta)^s u] \, dx - 2\Gamma(1 + s)^2 \int_{\partial\Omega} \frac{u}{d^s} \frac{v}{d^s} (x \cdot \nu) \, d\sigma.$$

Because $v = 0$ in $\mathbb{R}^n \setminus \Omega$, we have

$$\int_\Omega (x \cdot \nabla u)(-\Delta)^s v \, dx = \int_\Omega (x \cdot \nabla u)u^q \, dx$$
$$= \frac{1}{q + 1} \int_\Omega (x \cdot \nabla u^{q+1}) \, dx = -\frac{n}{q + 1} \int_\Omega u^{q+1} \, dx.$$

In a similar way,

$$\int_\Omega (x \cdot \nabla v)(-\Delta)^s u \, dx = -\frac{n}{p + 1} \int_\Omega v^{p+1} \, dx.$$

Plugging these two identities into (4.4), we derive

$$2\Gamma(1 + s)^2 \int_{\partial\Omega} \frac{u}{d^s} \frac{v}{d^s} (x \cdot \nu) \, d\sigma = \left(2s - n + \frac{2n}{q + 1}\right) \int_\Omega u^{q+1} \, dx$$
$$+ \left(2s - n + \frac{2n}{p + 1}\right) \int_\Omega v^{p+1} dx.$$

Since every viscosity solution of (1.3) is also a bounded weak solution, one has

$$\int_\Omega v^{p+1}\,dx = \int_\Omega v(-\Delta)^s u\,dx$$

$$= \int_{\mathbb{R}^n} (-\Delta)^{s/2} u (-\Delta)^{s/2} v\,dx = \int_\Omega u(-\Delta)^s v\,dx = \int_\Omega u^{q+1}\,dx.$$

Thus, the desired conclusion follows directly from this equality. □

5. Proof of Theorem 1.1

We organize the proof of Theorem 1.1 into two parts. We start by proving the existence of a positive viscosity solution. By Corollary 3.2, it suffices to show the existence of a nonzero critical point $u \in E_p^s$ of the functional Φ.

5.1. The existence part. We apply the direct method to the functional Φ on E_p^s. In order to show the coercivity of Φ, note that $q + 1 < (p+1)/p$ because $pq < 1$, so that the embedding $E_p^s \hookrightarrow L^{q+1}(\Omega)$ is continuous. So, there exist constants $C_1, C_2 > 0$ such that

$$\Phi(u) = \frac{p}{p+1}\int_\Omega |(-\Delta)^s u|^{(p+1)/p}\,dx - \frac{1}{q+1}\int_\Omega |u|^{q+1}\,dx$$

$$\geq \frac{p\,C_1}{p+1}\|u\|_{E_p^s}^{(p+1)/p} - \frac{C_2}{q+1}\|u\|_{E_p^s}^{q+1}$$

$$= \|u\|_{E_p^s}^{(p+1)/p}\left(\frac{pC_1}{p+1} - \frac{C_2}{(q+1)\|u\|_{E_p^s}^{(p+1)/p-(q+1)}}\right)$$

for all $u \in E_p^s$. For the existence of C_1 see [29], [41]. Then, Φ is lower bounded and coercive, that is, $\Phi(u) \to +\infty$ as $\|u\|_{E_p^s} \to +\infty$.

Let $(u_k) \subset E_p^s$ be a minimizing sequence of Φ. It is clear that (u_k) is bounded in E_p^s, since Φ is coercive. So, module a subsequence, we have $u_k \rightharpoonup u_0$ in E_p^s. By compactness of the embedding $E_p^s \hookrightarrow L^{q+1}(\Omega)$ (see [14]), we have $u_k \to u_0$ in $L^{q+1}(\Omega)$. Using the assumption $q + 1 < (p+1)/p$, we have

$$\lim_{n\to\infty}\inf \Phi(u_k) = \lim_{k\to\infty}\inf \frac{p}{p+1}\|(-\Delta)^s u_k\|_{L^{(p+1)/p}(\Omega)}^{(p+1)/p} - \frac{1}{q+1}\|u_0\|_{L^{q+1}(\Omega)}^{q+1}$$

$$\geq \frac{p}{p+1}\|(-\Delta)^s u_0\|_{L^{(p+1)/p}(\Omega)}^{(p+1)/p} - \frac{1}{q+1}\|u_0\|_{L^{q+1}(\Omega)}^{q+1} = \Phi(u_0).$$

So, u_0 minimizers Φ on E_p^s. We guarantee that u_0 is non-zero by noting that $\Phi(\varepsilon u_1) < 0$ for any non-zero function $u_1 \in E_p^s$ and $\varepsilon > 0$ small enough. This last statement follows directly from

$$\Phi(\varepsilon u_1) = \frac{p\varepsilon^{(p+1)/p}}{p+1}\int_\Omega |(-\Delta)^s u_1|^{(p+1)/p}\,dx - \frac{\varepsilon^{q+1}}{q+1}\int_\Omega |u_1|^{q+1}\,dx < 0$$

for $\varepsilon > 0$ small enough. This completes the proof of existence. □

5.2. The uniqueness part. The main tools in the proof of uniqueness are the Silvestre's strong maximum principle, a C^α regularity result up to the boundary and a Hopf's lemma adapted to fractional operators.

Let $(u_1, v_1), (u_2, v_2) \in C(\mathbb{R}^n) \times C(\mathbb{R}^n)$ be two positive viscosity solutions of problem (1.3)). Define

$$T = \{t \in (0, 1] : u_1 - \theta u_2, \ v_1 - \theta v_2 \geq 0 \ \text{ in } \overline{\Omega} \ \text{ for all } \ \theta \in [0, t]\}.$$

By Theorem 1.2 of [32], we have $u_i/d^s, v_i/d^s \in C^\alpha(\overline{\Omega})$ for some $\alpha \in (0, 1)$ and both quotients are positive on $\overline{\Omega}$. So, $(u_1 - \theta u_2)/d^s, (v_1 - \theta v_2)/d^s > 0$ on $\partial\Omega$ for $\theta > 0$ small enough and thus the set T is no empty.

Let $t_* = \sup T$ and assume that $t_* < 1$. Clearly,

(5.1) $$u_1 - t_* u_2, \ v_1 - t_* v_2 \geq 0 \quad \text{in } \overline{\Omega}.$$

By (5.1) and the integral representation in terms of the Green function G_Ω of $(-\Delta)^s$ (see [6], [25]), we derive

$$u_1(x) = \int_\Omega G_\Omega(x, y) v_1^p(y) \, dy \geq \int_\Omega G_\Omega(x, y) t_*^p v_2^p(y) \, dy$$

$$= t_*^p \int_\Omega G_\Omega(x, y) v_2^p(y) \, dy = t_*^p u_2(x)$$

for all $x \in \overline{\Omega}$. In a similar way, one gets $v_1 \geq t_*^q v_2$ in $\overline{\Omega}$. Using the assumption $pq < 1$ and the fact that $t_* < 1$, we get

(5.2) $$\begin{cases} (-\Delta)^s(u_1 - t_* u_2) = v_1^p - t_* v_2^p \geq (t_*^{pq} - t_*) v_2^p > 0, \\ (-\Delta)^s(v_1 - t_* v_2) = u_1^q - t_* u_2^q \geq (t_*^{pq} - t_*) u_2^q > 0, \end{cases} \quad \text{in } \Omega.$$

So, by the Silvestre's strong maximum principle (see [40]), one has $u_1 - t_* u_2, v_1 - t_* v_2 > 0$ in Ω. Arguing again as above, we easily deduce that $(u_1 - t_* u_2)/d^s$, $(v_1 - t_* v_2)/d^s > 0$ on $\partial\Omega$, so that $u_1 - (t_* + \varepsilon)u_2, v_1 - (t_* + \varepsilon)v_2 > 0$ in Ω for $\varepsilon > 0$ small enough, contradicting the definition of t_*. Then, $t_* \geq 1$ and, by (5.1), $u_1 - u_2, v_1 - v_2 \geq 0$ in $\overline{\Omega}$. A similar reasoning also produces $u_2 - u_1, v_2 - v_1 \geq 0$ in $\overline{\Omega}$. This completes the proof of uniqueness. $\qquad\square$

6. Proof of Theorem 1.2

By Corollary 3.2, it suffices to show the existence of a nonzero critical point $u \in E_p^s$ of the functional Φ. Assume $p, q > 0$, $pq > 1$ and the assumption (1.5). The proof consists in applying the classical mountain pass theorem of Ambrosetti and Rabinowitz in our variational setting. By well-known embedding theorems (see [14]), (1.5) implies that E_p^s is compactly embedded in $L^{q+1}(\Omega)$.

We first assert that Φ has a local minimum in the origin. Consider the set $\Gamma := \{u \in E_p^s : \|u\|_{E_p^s} = \rho\}$. Then, on Γ, we get

$$\Phi(u) = \frac{p}{p+1} \int_\Omega |(-\Delta)^s u|^{(p+1)/p} \, dx - \frac{1}{q+1} \int_\Omega |u|^{q+1} \, dx$$

$$\geq \frac{p\, C_1}{p+1} \|u\|_{E_p^s}^{(p+1)/p} - \frac{C_2}{q+1} \|u\|_{E_p^s}^{q+1}$$

$$= \rho^{(p+1)/p} \left(\frac{p\, C_1}{p+1} - \frac{C_2}{q+1} \rho^{q+1-(p+1)/p} \right) > 0 = \Phi(0)$$

for fixed $\rho > 0$ small enough, so that the origin $u_0 = 0$ is a local minimum point. For the existence of C_1 see [29], [41]. In particular, $\inf_\Gamma \Phi > 0 = \Phi(u_0)$.

It is clear that Γ is a closed subset of E_p^s and decomposes E_p^s into two connected components, namely $\{u \in E_p^s : \|u\|_{E_p^s} < \rho\}$ and $\{u \in E_p^s : \|u\|_{E_p^s} > \rho\}$.

Let $u_1 = t\overline{u}$, where $t > 0$ and $\overline{u} \in E_p^s$ is a nonzero nonnegative function. Since $pq > 1$, we can choose t large enough so that

$$\Phi(u_1) = \frac{pt^{(p+1)/p}}{p+1} \int_\Omega |(-\Delta)^s \overline{u}|^{(p+1)/p} \, dx - \frac{t^{q+1}}{q+1} \int_\Omega (\overline{u}^+)^{q+1} \, dx < 0.$$

Note that $u_1 \in \{u \in E_p^s : \|u\|_{E_p^s} > \rho\}$. Moreover, $\inf_\Gamma \Phi > \max\{\Phi(u_0), \Phi(u_1)\}$, so that the mountain pass geometry is satisfied.

Now, we show that Φ fulfills the Palais–Smale condition (PS). Let $(u_k) \subset E_p^s$ be a (PS)-sequence, that is,

$$|\Phi(u_k)| \leq C_0 \quad \text{and} \quad |\Phi'(u_k)\varphi| \leq \varepsilon_k \|\varphi\|_{E_p^s}$$

for all $\varphi \in E_p^s$, where $\varepsilon_k \to 0$ as $k \to +\infty$. From these two inequalities and the assumption $pq > 1$, we have

$$C_0 + \varepsilon_k \|u_k\|_{E_p^s} \geq |(q+1)\Phi(u_k) - \Phi'(u_k)u_k|$$

$$\geq \left(\frac{p(q+1)}{p+1} - 1 \right) \int_\Omega |(-\Delta)^s u_k|^{(p+1)/p} \, dx \geq C\|u_k\|_{E_p^s}^{(p+1)/p}$$

and thus (u_k) is bounded in E_p^s. Since E_p^s is compactly imbedded in $L^{q+1}(\Omega)$, one easily checks that (u_k) converges strongly in E_p^s. Thus, by the mountain pass theorem, we obtain a nonzero critical point $u \in E_p^s$. $\qquad\square$

7. Proof of Theorem 1.3

It suffices to assume that Ω is star-shaped with respect to the origin, that is, $(x \cdot \nu) > 0$ for any $x \in \partial\Omega$, where ν is the unit outward normal to $\partial\Omega$ at x.

Arguing by contradiction, assume the problem (1.3) admits a positive viscosity solution (u, v). Then, by Theorem 1.2 of [32], we have

$$2\Gamma(1+s)^2 \int_{\partial\Omega} \frac{u}{d^s} \frac{v}{d^s} (x \cdot \nu) \, d\sigma > 0.$$

On the other hand, the assumption (1.6) is equivalent to

$$\frac{n}{q+1} + \frac{n}{p+1} - (n-2s) \le 0$$

and thus the right-hand side of the Rellich identity (4.1) is non-positive, providing the wished contradiction. Hence, the problem (1.3) admits no positive viscosity solution and we end the proof. $\qquad\square$

References

[1] N. ABATANGELO, *Large s-harmonic functions and boundary blow-up solutions for the fractional Laplacian*, Discrete Contin. Dyn. Syst. **35** (2015), 5555–5607.

[2] A. AMBROSETTI AND P. RABINOWITZ, *Dual variational methods in critical points theory and applications*, J. Funct. Anal. **14** (1973), 349–381.

[3] D. APPLEBAUM, textitLévy processes – from probability to finance and quantum groups, Notices Amer. Math. Soc. **51** (2004), 1336–1347.

[4] B. BARRIOS, E. COLORADO, A. DE PABLO AND U. SANCHEZ, *On some critical problems for the fractional Laplacian operator*, J. Differential Equations **252** (2012), 6133–6162.

[5] M. BIRKNER, J.A. LÓPES-MIMBELA AND A. WAKOLBINGER, *Comparison results and steady states for the Fujita equation with fractional Laplacian*, Ann. Inst. H. Poincaré Anal. Non Linéaire **22** (2005), 83–97.

[6] M. BONFORTE AND V.J. LUIS, *A priori estimates for fractional nonlinear degenerate diffusion equations on bounded domains*, Arch. Ration. Mech. Anal. **218** (2015), 317–362.

[7] C. BRANDLE, E. COLORADO, A. DE PABLO AND U. SANCHEZ, *A concave-convex elliptic problem involving the fractional Laplacian*, Proc. Roy. Soc. Edinburgh **143** (2013), 39–71.

[8] X. CABRÉ AND Y. SIRE, *Nonlinear equations for fractional Laplacians*, I *Regularity, maximum principles, and Hamiltonian estimates*, Ann. Inst. H. Poincaré Anal. Non Linéaire **31** (2014), 23–53.

[9] X. CABRÉ AND J. TAN, *Positive solutions of nonlinear problems involving the square root of the Laplacian*, Adv. Math. **224** (2010), 2052–2093.

[10] L. CAFFARELLI AND L. SILVESTRE, *An extension problem related to the fractional Laplacian*, Comm. Partial Differential Equations **32** (2007), 1245–1260.

[11] W. CHOI, *On strongly indefinite systems involving the fractional Laplacian*, Nonlinear Anal. **120** (2015), 127–153.

[12] W. CHOI AND S. KIM, *Minimal energy solutions to the fractional Lane-Enden system*, I. *Existence and singularity formation*, arXiv: 1610.02853, 2016.

[13] PH. CLÉMENT, D.G. DE FIGUEIREDO AND E. MITIDIERI, *Positive solutions of semilinear elliptic systems*, Comm. Partial Differential Equations **17** (1992), 923–940.

[14] F. DEMENGEL AND G. DEMENGEL, *Functional Spaces for the Theory of Elliptic Partial Differential Equations*, (Universitext)-Springer, 2012.

[15] E. DI NEZZA, G. PALATUCCI AND E. VALDINOCI, *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math. **136** (2012), 521–573.

[16] M.M. FALL AND T. WETH, *Nonexistence results for a class of fractional elliptic boundary value problems*, J. Funct. Anal. **263** (2012), 2205–2227.

[17] P. FELMER AND S. MARTÍNEZ, *Existence and uniqueness of positive solutions to certain differential systems*, Adv. Differential Equations **4** (1998), 575–593.

[18] P. FELMER AND A. QUAAS, *Boundary blow up solutions for fractional elliptic equations*, Asymptot. Anal. **78** (2012), 123–144.

[19] M. Felsinger, M. Kassmann and P. Voigt, *The Dirichlet problem for nonlocal operators*, Math. Z. **279** (2015), 779–809.

[20] D.G. de Figueiredo, *Semilinear elliptic systems*, Nonlinear Funct. Anal. Appl. Diferential Equations, World Sci. Publishing, River Edge (1998), 122–152.

[21] D.G. de Figueiredo and P. Felmer, *On superquadratic elliptic systems*, Trans. Amer. Math. Soc. **343** (1994), 99–116.

[22] D.G. Figueiredo and B. Ruf, *Elliptic systems with nonlinearities of arbitrary growth*, Mediterr. J. Math. **1** (2004), 417–431.

[23] B. Gidas and J. Spruck, *A priori bounds for positive solutions of nonlinear elliptic equations*, Comm. Partial Differential Equations **6** (1981), 883–901.

[24] J. Hulshof and R. van der Vorst, *Diferential Systems with Strongly Indefinite Variational Structure*, J. Funct. Anal. **114** (1993), 32–58.

[25] T. Jakubowski, *The estimates for the Green function in Lipschitz domains for the symmetric stable processes*, Probab. Math. Statist. **22** (2002), 419–441.

[26] E. Leite and M. Montenegro, *A priori bounds and positive solutions for non-variational fractional elliptic systems*, Differential Integral Equations **30** (2017), 947–974.

[27] E. Mitidieri, *A Rellich type identity and applications*, Comm. Partial Differential Equations **18** (1993), 125–151.

[28] M. Montenegro, *The construction of principal spectra curves for Lane-Emden systems and applications*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **29** (2000), 193–229.

[29] F. Petitta, *Some remarks on the duality method for integro-differential equations with measure data*, Adv. Nonlinear Stud. **16** (2016), 115–124.

[30] S.I. Pohozaev, *On the eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$*, Dokl. Akad. Nauk SSSR **165** (1965), 1408–1411.

[31] A. Quaas and A. Xia, *Liouville type theorems for nonlinear elliptic equations and systems involving fractional Laplacian in the half space*, Calc. Var. Partial Differential Equations **526** (2014), 1–19.

[32] X. Ros-Oton and J. Serra, *The Dirichlet problem for the fractional Laplacian: regularity up to the boundary*, J. Math. Pures Appl. **101** (2014), 275–302.

[33] X. Ros-Oton and J. Serra, *The extremal solution for the fractional Laplacian*, Calc. Var. Partial Differential Equations **50** (2014), 723–750.

[34] X. Ros-Oton and J. Serra, *The Pohozaev identity for the fractional Laplacian*, Arch. Ration. Mech. Anal. **213** (2014), 587–628.

[35] R. Servadei and E. Valdinoci, *Mountain pass solutions for non-local elliptic operators*, J. Math. Anal. Appl. **389** (2012), 887–898.

[36] R. Servadei and E. Valdinoci, *Variational methods for non-local operators of elliptic type*, Discrete Contin. Dyn. Syst. **33** (2013), 2105–2137.

[37] R. Servadei and E. Valdinoci, *A Brezis–Nirenberg result for non-local critical equations in low dimension*, Commun. Pure Appl. Anal. **12** (2013), 2445–2464.

[38] R. Servadei and E. Valdinoci, *Weak and viscosity solutions of the fractional Laplace equation*, Publ. Mat. **58** (2014), 133–154.

[39] R. Servadei and E. Valdinoci, *The Brezis–Nirenberg result for the fractional Laplacian*, Trans. Amer. Math. Soc. **367** (2015), 67–102.

[40] L. Silvestre, *Regularity of the obstacle problem for a fractional power of the Laplace operator*, Commun. Pure Appl. Math. **60** (2007), 67–112.

[41] E. Stein, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, New York, 1970.

[42] L. Yan and M. Pei, *Symmetry of solutions for a fractional system*, Sci. China Math. **60** (2017), 1805–1824.

[43] L. Zhanga, M. Yua and J. Heb, *A Liouville theorem for a class of fractional systems in $\mathbb{R}_+^n$*, J. Differential Equations **263** (2017), 6025–6065.

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