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NEW RESULTS OF MIXED MONOTONE OPERATOR EQUATIONS

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Abstract. In this article, we study the existence and uniqueness of fixed points for some mixed monotone operators and monotone operators with perturbation. These mixed monotone operators and monotone operators are e-concave-convex operators and e-concave operators respectively. Without using compactness or continuity, we obtain the existence and uniqueness of fixed points by monotone iterative techniques and properties of cones. Our main results extended and improved some existing results. Also, we applied the results to some differential equations.

1. Introduction and preliminaries

Throughout the paper, E is a real Banach space with norm $\|\cdot\|$. P is a cone in E if it satisfies:

(1) if $x \in P$, $\lambda \geq 0$ then $\lambda x \in P$;
(2) if $x \in P$, $-x \in P$ then $x = \theta$,

where θ is zero in E, $P^+ = P - \{\theta\}$.

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We denote by $\mathring{P}$ the interior set of P and the set $P_h = \{x \in E \mid x \sim h\}$. The Banach space E is partially ordered by a cone $P \subset E$, i.e. $x \leq y$ if and only if $y - x \in P$.

We say that P is a normal cone if there exists a constant $N > 0$ such that for all x, $y \in E$, $\theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$, and the smallest N is called the normality constant of P. For $e \in P^+$, set

$$C_e = \{x \in E \mid \text{there exist positive numbers } \alpha,\ \beta \text{ such that } \alpha e \leq x \leq \beta e\}.$$

For the sake of convenience, we introduce some definitions. For more details see [2].

Definition 1.1. $A\colon P \times P \to P$ is said to be a mixed monotone operator if $A(x, y)$ is increasing in x, and decreasing in y, i.e. u_i, $v_i \in P$ $(i = 1, 2)$, $u_1 \leq u_2$, $v_1 \geq v_2$ imply $A(u_1, v_1) \leq A(u_2, v_2)$.

Definition 1.2. Let $A\colon C_e \times C_e \to C_e$ be an operator and $e \in P^+$. Suppose that there exists an $\eta(u, v, t) > 0$ such that

$$A(tu, t^{-1}v) \geq t(1 + \eta(u, v, t))A(u, v) \quad \text{for all } u, v \in C_e,\ 0 < t < 1.$$

Then A is called an e-concave-convex operator.

Definition 1.3. Let $A\colon P \to P$ be an operator and $e \in P^+$. Suppose that $Ae \in C_e$, there exists a real number $\eta = \eta(x, t) > 0$ such that

$$A(tx) \geq t(1 + \eta)Ax, \quad \text{for all } x \in C_e,\ 0 < t < 1.$$

Then A is called a generalized e-concave operator.

Definition 1.4. $A\colon P \times P \to P$ is a mixed monotone operator. An elements $x \in P$ is called a fixed point of A if $A(x, x) = x$.

Definition 1.5. An operator $B\colon P \to P$ is said to be sub-homogeneous if it satisfies:

$$B(tx) \geq tBx, \quad \text{for all } t \in (0, 1),\ x \in P.$$

Definition 1.6. $A\colon P \times P \to P$ is a mixed monotone operator. If $x, y \in P$, $x \leq y$ such that $x \leq A(x, y)$, $A(y, x) \leq y$, then (x, y) is called a coupled lower-upper fixed point.

Mixed monotone operators, e-concave operators and e-concave-convex operators were introduced by Guo and Lakshmikantham [2]. Thereafter, many authors have investigated mixed monotone operators and obtained meaningful and important results (see [6], [7], [10]–[14]). These results not only have important significance in theory, but also have widespread applications in engineering, chemistry, biology, etc.

In [17], Zhao and Du studied fixed points of generalized e-concave (generalized e-convex) operators and applied the results to the singular boundary value problems for second order differential equations. The main results from their papers is as follows:

Theorem 1.7 (Theorem 1.1 in [17]). *Let $A\colon P \to P$ be an incresing generalized e-concave. Then*:

(a) *A has at most one fixed point in C_e;*

(b) *Suppose P is a normal cone of E and one of the following conditions is satisfied*:

(A1) $\inf\limits_{x \in C_e} \eta(x,t) > 0$;

(A2) *For all $t \in (0,1)$, $\eta(x,t)$ is nonincreasing with respect to $x \in C_e$ and there exists $w_0 \in C_e$ such that $Aw_0 \leq w_0$;*

(A3) *For all $t \in (0,1)$, $\eta(x,t)$ is nondecreasing with respect to $x \in C_e$ and there exists $v_0 \in C_e$ such that $v_0 \leq Av_0$;*

(A4) *For all $t \in (0,1)$, $\eta(x,t)$ is nondecreasing with respect to $x \in C_e$ and there exists $x_0 \in C_e$ such that*

$$\lim_{t \to 0^+} \eta(x_0, t) = +\infty.$$

Then A has a fixed point in C_e;

(c) *If A has a positive fixed point $x^* \in C_e$, then constructing successively the sequence $x_n = Ax_{n-1}$ $(n = 1, 2, \ldots)$, for any initial $x_0 \in C_e$, we have $\|x_n - x^*\|_e \to 0$ $(n \to \infty)$;*

(d) *If A has a positive fixed point $x^* \in C_e$, then*

$$\max\{x \in C_e \mid x \leq Ax\} = \min\{y \in C_e \mid Ay \leq y\} = x^*.$$

In [16], Zhao investigated the existence and uniqueness of fixed points for mixed monotone e-concave-convex operators and applied the results to an integral equation of polynomial type which possesses items of measurable functions. They proved the following theorem:

Theorem 1.8 (Theorem 3.1 in [16]). *Suppose P is a normal cone of a real Banach space E, $e \in P^+$, $A\colon C_e \times C_e \to C_e$ is a mixed monotone and e-concave-convex operator. Assume that one of the following conditions is satisfied*:

(A5) *There exists sequence $\{t_n\} \subset (0,1)$ and $\{s_n\} \subset (0,1)$ such that*

$$t_n \to 0^+, \qquad \inf_{u,v \in C_e} \{\eta(u,v,t_n)\} > 0,$$
$$s_n \to 1^-, \qquad \inf_{u,v \in C_e} \{\eta(u,v,s_n)\} > 0;$$

(A6) *For any $t \in (0,1)$, $\eta(u,v,t)$ is non-increasing with respect to $u \in C_e$, non-decreasing with respect to $v \in C_e$;*

(A7) *For any* $t \in (0,1)$, $\eta(u,v,t)$ *is non-decreasing with respect to* $u \in C_e$, *non-increasing with respect to* $v \in C_e$, *and there exist* $x_0, y_0 \in C_e$, $x_0 \le y_0$ *such that* $\overline{\lim\limits_{t\to 0^+}}\,\eta(x_0,y_0,t) = +\infty$.

Then A has exactly one fixed point. Moreover, constructing successively sequences

$$x_n = A(x_{n-1},y_{n-1}),\quad y_n = A(y_{n-1},x_{n-1}),\quad n=1,2,\ldots,$$

for any initial values $x_0, y_0 \in C_e$, *we have that*

$$\|x_n - x^*\| \to 0,\quad \|y_n - x^*\| \to 0,\quad n\to\infty.$$

In [8], the authors presented the definition of the $t-\eta(t,u,v)$ mixed monotone model operator and gave a new existence and uniqueness theorem of fixed point of these operators. One of the main results of the paper [8] is the following theorem.

Theorem 1.9 (Theorem 2.2 in [8]). *Let P be a normal and solid cone of a real Banach space E, and* $h > \theta$. *For a class of operators* $A = B + \lambda C + D$, *where* $\lambda \ge 0$ *is a constant, we assume that*

(A8) $B\colon P_h \times P_h \to P_h$ *is a mixed monotone operator, and there exists a function* $\alpha\colon P_h\times P_h\times(0,1)\to(0,1)$ *and* $u_0, v_0 \in P_h, u_0 \le v_0$ *such that*
  (a) *for all* $x, y \in P_h, t\in(0,1)$, $B(tx,t^{-1}y) \ge t^{\alpha(t,x,y)}B(x,y)$;
  (b) $u_0 \le B(u_0,v_0)+\lambda C(u_0,v_0)+Du_0$ *and* $B(v_0,u_0)+\lambda C(v_0,u_0)+Dv_0$.

(A9) $C\colon P_h\times P_h\to P_h$ *is a mixed monotone operator, and there exists a function* $\beta\colon(0,+\infty)\to(1,+\infty)$ *such that, for all* $x,y\in P_h$, $t>0$,

$$C(tx,t^{-1}y) \ge t^{\beta(t)}C(x,y);$$

(A10) $D\colon P\to P$ *satisfies the following conditions*:
  (a) $D(x-y) = Dx - Dy$, *for all* $x,y\in P$, $x\ge y$;
  (b) $D(tx) = tD(x)$, *for all* $x\in P$, $t\ge 0$.

Suppose that

$$\gamma(t) = \inf_{x,y\in[u_0,v_0]} t^{\alpha(t,x,y)} > t\left[1+\lambda c(1-t^{\beta(t)-1})\right],\quad t\in(0,1),$$

where $c = \inf\{r \mid C(x,y)\le rB(x,y),\ x,y\in[u_0,v_0]\}$. *Then there exists a unique fixed point* x^* *in* $[u_0,v_0]$ *such that* $A(x^*,x^*) = x^*$. *Moreover, for any initial values* $x_0\in[u_0,v_0]$, *constructing successively the sequences* $x_n = A(x_{n-1},x_{n-1})$, $n=1,2,\ldots$, *we have* $\|x_n - x^*\|\to 0$, *as* $n\to\infty$.

Motivated by the above works, this paper considers the existence and uniqueness of fixed points for monotone e-concave operators and mixed monotone e-concave-convex operators with perturbation. We will consider the following equations

$$\mathcal{A}(x,x) + B(x,x) = x, \tag{1.1}$$

or

$$\mathcal{A}x + Bx = x, \tag{1.2}$$

where $A\colon C_e \times C_e \to C_e$ is a e-concave-convex and mixed monotone operators and $A\colon C_e \to C_e$ is e-concave and increasing operators, and B is an increasing sub-homogeneous operator. We obtain the unique positive solution of (1.1) and (1.2). Our results extend and improve the main results of [17], [16], [8], [5], [9] and [4].

The rest of this paper is organized as follows. In Section 2, we consider the existence and uniqueness of fixed points for monotone e-concave operators or mixed monotone e-concave-convex operators with perturbation. In Section 3, we give an example to demonstrate the application of our theoretical results.

2. Main results

In this section, we consider the existence and uniqueness of fixed points for monotone e-concave operators or mixed monotone e-concave-convex operators with perturbation under appropriate conditions. We always assume that E is a real Banach space with a partial order induced by a normal cone P of E. Take $e \in P^+$ and C_e as given in Section 1. The following lemma is an important result that is used the proofs of our main results.

Lemma 2.1 (see [15]). *Let E be a real ordered Banach space, P is a normal cone in E, $e \in P^+$, and $A\colon C_e \times C_e \to C_e$ a mixed monotone operator. There exists a function $\eta\colon\ : (0,1) \times C_e \times C_e \to (0,+\infty)$ such that, for all $x, y \in C_e$, $t \in (0,1)$, we have*

$$A(tx, t^{-1}y) \geq t[1 + \eta(t,x,y)]A(x,y).$$

If $(u_0, v_0) \in C_e \times C_e$ is coupled lower-upper fixed point of A, and

$$\xi(t) = \inf_{x,y \in [u_0,v_0]} \eta(t,x,y) > 0, \quad t \in (0,1),$$

then A has exactly one fixed point x^* in C_e. Moreover, constructing successively the sequence $x_n = A(x_{n-1}, x_{n-1})$, $n = 1, 2, \ldots$, for any initial value $x_0 \in C_e$, we have $\|x_n - x^*\| \to 0$ as $n \to \infty$.

Now let give our results as follows.

Theorem 2.2. *Let P be a normal cone in E, $P^+ = P - \{\theta\}$, $e \in P^+$. We assume that*:

- (H1) *$A\colon C_e \times C_e \to C_e$ is a mixed monotone and e-concave-convex operator and in addition one of the following three conditions is satisfied:*
  - (L1) *for any $\varepsilon \in (0,1)$, there exists $\delta \in (\varepsilon, 1)$, such that*

$$\inf_{u_0 \leq u,\, v \leq v_0} \eta(u,v,\delta) > 0;$$

(L2) *for any* $t \in (0,1)$, $\eta(u,v,t)$ *is non-increasing with respect to* $u \in C_e$ *and non-decreasing with respect to* $v \in C_e$;

(L3) *for any* $t \in (0,1)$, $\eta(u,v,t)$ *is non-decreasing with respect to* $u \in C_e$ *and non-increasing with respect to* $v \in C_e$;

(H2) $B\colon P \times P \to P$ *is a mixed monotone operator and for all* $t \in (0,1)$, $x, y \in P$, *the operator* B *satisfies* $B\big(tx, t^{-1}y\big) \geq tB(x,y)$;

(H3) $u_0, v_0 \in C_e$, $u_0 \leq v_0$,

$$u_0 \leq A(u_0,v_0) + B(u_0,v_0), \qquad A(v_0,u_0) + B(v_0,u_0) \leq v_0.$$

Then

(a) *the operator equation* $x = A(x,x) + B(x,x)$ *has a unique solution* x^* *in* $[u_0, v_0]$;

(b) *for any initial values* $x_0, y_0 \in [u_0, v_0]$, *constructing successively the sequences*

$$x_n = A(x_{n-1}, y_{n-1}) + B(x_{n-1}, y_{n-1}),$$
$$y_n = A(y_{n-1}, x_{n-1}) + B(y_{n-1}, x_{n-1}),$$

for $n = 1, 2, \ldots$, *we have* $\|x_n - x^*\| \to 0$, $\|y_n - x^*\| \to 0$ *as* $n \to \infty$.

Proof. First we define an operator

$$T(x,y) = A(x,y) + B(x,y), \quad x, y \in [u_0, v_0].$$

Then T is a mixed monotone operator and

$$T(v_0,u_0) = A(v_0,u_0) + B(v_0,u_0) \leq v_0.$$

Since $v_0 \in C_e$, $A(u_0,v_0) \in C_e$, then there exists constant $c > 0$ such that $cA(u_0,v_0) \geq v_0$. Thus

$$T(v_0,u_0) - cA(u_0,v_0) \leq v_0 - cA(u_0,v_0) \leq 0. \tag{2.1}$$

From (2.1), we obtain

$$T(x,y) \leq T(v_0,u_0) \leq cA(u_0,v_0) \leq cA(x,y), \quad x, y \in [u_0, v_0].$$

According to the assumptions (H1) and (H2), for any $t \in (0,1)$, we know

$$\begin{aligned} T(tx, t^{-1}y) &= A(tx, t^{-1}y) + B(tx, t^{-1}y) \\ &\geq t\,[1 + \eta(x,y,t)]\,A(x,y) + tB(x,y) \\ &= tA(x,y) + tB(x,y) + t\eta(x,y,t)A(x,y) \\ &\geq t(A(x,y) + B(x,y)) + t\eta(x,y,t)\,\frac{1}{c}\,T(x,y) \\ &= t\bigg[1 + \frac{1}{c}\,\eta(x,y,t)\bigg]T(x,y). \end{aligned} \tag{2.2}$$

Set

$$u_n = T(u_{n-1}, v_{n-1}), \quad v_n = T(v_{n-1}, u_{n-1}), \quad n = 1, 2, \ldots \tag{2.3}$$

Then $u_0 \le v_0$ and (2.3) implies $u_1 \le v_1$. Noting that there exists t' such that $u_0 \ge t'v_0$, we can get $u_n \ge u_0 \ge t'v_0 \ge t'v_n$, $n = 1, 2, \ldots$ It is clear that

$$u_0 \le u_1 \le \ldots \le u_n \le \ldots \le v_n \le \ldots \le v_1 \le v_0. \tag{2.4}$$

So, if we set

$$t_n = \sup\{t' > 0 \mid u_n \ge t'v_n\}, \quad n = 0, 1, \ldots, \tag{2.5}$$

then we know that, for $n = 0, 1, dots$, $u_n \ge t_n v_n$. Also that $u_{n+1} \ge u_n \ge t_n v_n \ge t_n v_{n+1}$. So we get $t_{n+1} \ge t_n$. Thus we have

$$0 < t_0 \le t_1 \le \ldots \le t_n \le t_{n+1} \le \ldots < 1.$$

So there exists $\lim_{n\to\infty} t_n = t''$, where $0 < t'' \le 1$.

(i) Now we will show $t'' = 1$ under the assumption (L1). Otherwise, we have $0 < t'' < 1$. From (L1), there exists $\delta \in (t'', 1)$ such that

$$\varphi \triangleq \inf_{u_0 \le u,\, v \le v_0} \eta(u, v, \delta) > 0.$$

Applying (2.2) and (2.5) we obtain that

$$\begin{aligned} u_{n+1} = T(u_n, v_n) &\ge T\biggl(t_n v_n, \frac{1}{t_n}\, u_n\biggr) = T\biggl(\frac{t_n}{\delta}\, \delta v_n, \frac{\delta}{t_n}\frac{1}{\delta}\, u_n\biggr) \\ &\ge \frac{t_n}{\delta}\, T\biggl(\delta v_n, \frac{1}{\delta}\, u_n\biggr) \ge t_n\biggl[1 + \frac{1}{c}\, \eta(v_n, u_n, \delta)\biggr]\, T(v_n, u_n) \\ &\ge t_n\biggl(1 + \frac{1}{c}\, \varphi\biggr) T(v_n, u_n) = t_n\biggl(1 + \frac{1}{c}\, \varphi\biggr) v_{n+1}. \end{aligned}$$

Thus, by (2.5), we have $t_{n+1} \ge t_n(1 + \varphi/c)$.

Let $n \to \infty$, then $t'' \ge t''(1 + \varphi/c) > t''$. This is a contradiction. Hence, we know $t'' = 1$.

(ii) Now we shall show that $t'' = 1$ under the assumption (L2). Otherwise, we have $0 < t'' < 1$. Applying (2.2), (2.5) and (L2), we obtain that

$$\begin{aligned} u_{n+1} = T(u_n, v_n) &\ge T\biggl(t_n v_n, \frac{1}{t_n}\, u_n\biggr) = T\biggl(\frac{t_n}{t''}\, t'' v_n, \frac{t''}{t_n}\frac{1}{t''}\, u_n\biggr) \\ &\ge \frac{t_n}{t''}\, T\biggl(t'' v_n, \frac{1}{t''}\, u_n\biggr) \ge t_n\biggl[1 + \frac{1}{c}\, \eta\bigl(v_n, u_n, t''\bigr)\biggr] T(v_n, u_n) \\ &\ge t_n\biggl[1 + \frac{1}{c}\, \eta(v_0, u_0, t'')\biggr] T(v_n, u_n) = t_n\biggl[1 + \frac{1}{c}\, \eta\bigl(v_0, u_0, t''\bigr)\biggr] v_{n+1}. \end{aligned}$$

Thus, by (2.5), we have

$$t_{n+1} \ge t_n\biggl[1 + \frac{1}{c}\, \eta\bigl(v_0, u_0, t''\bigr)\biggr].$$

Let $n \to \infty$, then

$$t'' \geq t'' \left[1 + \frac{1}{c}\,\eta\big(v_0, u_0, t''\big)\right] > t''.$$

This is a contradiction. Hence, we know $t'' = 1$.

(iii) Now we will prove that $t'' = 1$ under the assumption (L3). Otherwise, we have $0 < t'' < 1$. Applying (2.2), (2.5) and (L3), we obtain that

$$\begin{aligned} u_{n+1} = T(u_n, v_n) &\geq T\bigg(t_n v_n, \frac{1}{t_n}\,u_n\bigg) = T\bigg(\frac{t_n}{t''}\,t'' v_n, \frac{t''}{t_n}\frac{1}{t''}\,u_n\bigg) \\ &\geq \frac{t_n}{t''}\,T\bigg(t'' v_n, \frac{1}{t''}\,u_n\bigg) \geq t_n\left[1 + \frac{1}{c}\,\eta\big(v_n, u_n, t''\big)\right] T(v_n, u_n) \\ &\geq t_n\left[1 + \frac{1}{c}\,\eta\big(u_0, v_0, t''\big)\right] T(v_n, u_n) = t_n\left[1 + \frac{1}{c}\,\eta\big(u_0, v_0, t''\big)\right] v_{n+1}. \end{aligned}$$

Thus, by (2.5), we have

$$t_{n+1} \geq t_n\left[1 + \frac{1}{c}\,\eta\big(u_0, v_0, t''\big)\right].$$

Let $n \to \infty$, then

$$t'' \geq t''\left[1 + \frac{1}{c}\,\eta\big(u_0, v_0, t''\big)\right] > t''.$$

This is a contradiction. Hence, we know $t'' = 1$.

Thus for any natural number p, we get that

$$\theta \leq u_{n+p} - u_n \leq v_n - t_n v_n = (1 - t_n) v_n \leq (1 - t_n) v_0, \quad n = 0, 1, \ldots, \tag{2.6}$$

$$\theta \leq v_n - v_{n+p} \leq v_n - u_n \leq v_n - t_n v_n \leq (1 - t_n) v_0, \qquad n = 0, 1, \ldots \tag{2.7}$$

Since the cone P is normal we have, for $n, p = 1, 2, \ldots$,

$$\|u_{n+p} - u_n\| \leq N(1 - t_n)\|v_0\|, \qquad \|v_n - v_{n+p}\| \leq N(1 - t_n)\|v_0\|, \tag{2.8}$$

where N is the normality constant of P. So $\|u_{n+p} - u_n\| \to 0$, $\|v_n - v_{n+p}\| \to 0$ as $n \to \infty$. Hence we know that $\{u_n\}$, $\{v_n\}$ are Cauchy sequences. Because E is complete, there exist u^*, v^* such that $u_n \to u^*$, $v_n \to v^*$ as $n \to \infty$. By (2.4), we know that $u_n \leq u^* \leq v^* \leq v_n$ with $u^*, v^* \in [u_0, v_0]$, and

$$\theta \leq v^* - u^* \leq v_n - u_n \leq (1 - t_n) v_0. \tag{2.9}$$

Then $\|v^* - u^*\| \leq N(1 - t_n)\|v_0\|$. Letting $n \to \infty$, we have $\|v^* - u^*\| \to 0$. Thus $u^* = v^*$. Let $x^* : = u^* = v^*$, then we have

$$u_{n+1} = T(u_n, v_n) \leq T(x^*, x^*) \leq T(v_n, u_n) = v_{n+1}, \quad n = 1, 2, \ldots \tag{2.10}$$

Let $n \to \infty$, we have $x^* = T(x^*, x^*)$. That is, x^* is a fixed point of T in $[u_0, v_0]$.

Now we prove that x^* is the unique fixed point of T in $[u_0, v_0]$. Suppose $\overline{x}$ is another fixed point of T in $[u_0, v_0]$ and $\overline{x} \neq x^*$. Then

$$u_0 \leq T(\overline{x}, \overline{x}) = \overline{x} \leq v_0.$$

Repeating the above iterative procedure (2.6)–(2.10), we have $u_n \leq \overline{x} \leq v_n$. Thus $u^* = \overline{x} = x^* = v^*$.

Now, we construct successively the sequences

$$x_n = T(x_{n-1}, y_{n-1}), \quad y_n = T(y_{n-1}, x_{n-1}), \quad n = 1, 2, \dots$$

for any initial points $x_0, y_0 \in [u_0, v_0]$. Applying the mixed monotonicity of the operator T, we obtain that

$$T(u_0, v_0) \leq T(x_0, y_0) \leq T(v_0, u_0).$$

It means that $u_1 \leq x_1 \leq v_1$. Similarly, $u_1 \leq y_1 \leq v_1$. By applying the same method used in (2.3)–(2.10), we have $u_n \leq x_n \leq v_n$, $u_n \leq y_n \leq v_n$, which implies that $\|x_n - x^*\| \to 0$, $\|y_n - x^*\| \to 0$. □

Remark 2.3. In the Theorem 2.2, if we reduce the operators A and B to the operator of one variable, and reduce $\eta(x, y, t)$ to $\eta(x, t)$ correspondingly, then we obtain the same conlusions. That is, the operator sum equation $x = Ax + Bx$ has a unique solution in C_e, and we have the iterative sequence $x_n = Ax_{n-1} + Bx_{n-1}$ such that $\|x_n - x^*\| \to 0$.

Theorem 2.4. *Let P be a normal cone in E, $P^+ = P - \{\theta\}$, and $e \in P^+$. For a class of operators $T = A + \lambda B + C$, where $\lambda \geq 0$ is a constant, we assume that*:

(H4) *$A\colon C_e \times C_e \to C_e$ is a mixed monotone operator and e-concave-convex operator, and $\inf\limits_{x,y \in [u_0, v_0]} \eta(x, y, t) > 0$;*

(H5) *$B\colon C_e \times C_e \to C_e$ is a mixed monotone operator, and there exists a function $\varphi(t)\colon (0, +\infty) \to (1, +\infty)$ such that*

$$B\left(tx, \frac{y}{t}\right) \geq \varphi(t) B(x, y), \quad \text{where } x, y \in C_e,\ t > 0;$$

(H6) *$C\colon P \times P \to P$ is a mixed monotone operator and for all $t \in (0, 1)$, $x, y \in P$, operator C satisfied $C(tx, t^{-1}y) \geq tC(x, y)$;*

(H7) *$u_0, v_0 \in C_e$, $u_0 \leq v_0$,*

$$u_0 \leq A(u_0, v_0) + \lambda B(u_0, v_0) + C(u_0, v_0),$$
$$A(v_0, u_0) + \lambda B(v_0, u_0) \leq v_0 + C(v_0, u_0).$$

Then, the operator equation $x = T(x, x)$ has a unique solution x^* in $[u_0, v_0]$. Moreover, for any initial values $x_0 \in [u_0, v_0]$, constructing successively the sequences $x_n = T(x_{n-1}, x_{n-1})$, $n = 1, 2, \dots$, we have $\|x_n - x^*\| \to 0$, as $n \to \infty$.

Proof. For any $x, y \in [u_0, v_0]$, since $A(v_0, u_0)$, $B(u_0, v_0) \in C_e$, there exists constant $c' > 0$ such that

$$B(x, y) \geq B(u_0, y) \geq B(u_0, v_0) \geq c' A(v_0, u_0) \geq c' A(v_0, y) \geq c' A(x, y).$$

For any $x, y \in [u_0, v_0]$, since $A(u_0, v_0) \in C_e$ and $v_0 \in C_e$, there exists constant $c > 0$ such that $cv_0 \le A(u_0, v_0)$. Then

$$cT(v_0, u_0) - A(u_0, v_0) \le cv_0 - A(u_0, v_0) \le 0.$$

So we obtain

$$cT(x, y) \le cT(v_0, u_0) \le A(u_0, v_0) \le A(x, y).$$

Hence, for all $x, y \in [u_0, v_0]$, $t \in (0, 1)$, we have

$$\begin{aligned} T(tx, t^{-1}y) &= A\big(tx, t^{-1}y\big) + \lambda B\big(tx, t^{-1}y\big) + C\big(tx, t^{-1}y\big) \\ &\ge t[1 + \eta(x, y, t)]A(x, y) + \lambda\varphi(t)B(x, y) + tC(x, y) \\ &= tA(x, y) + \lambda tB(x, y) + tC(x, y) \\ &\quad + t\eta(x, y, t)A(x, y) + \lambda(\varphi(t) - t)B(x, y) \\ &\ge tT(x, y) + t\eta(x, y, t)cT(x, y) + \lambda(\varphi(t) - t)c'A(x, y) \\ &\ge tT(x, y) + t\eta(x, y, t)cT(x, y) + \lambda(\varphi(t) - t)c'cT(x, y) \\ &\ge t\bigg[1 + \eta(x, y, t)c + \lambda\bigg(\frac{\varphi(t)}{t} - 1\bigg)c'c\bigg]T(x, y). \end{aligned}$$

Let

$$\xi(t) = \eta(x, y, t)c + \lambda\bigg(\frac{\varphi(t)}{t} - 1\bigg)c'c,$$

thus according to $\inf\limits_{x,y\in[u_0,v_0]} \eta(x, y, t) > 0$, we know $\xi(t) > 0$ and

$$T(tx, t^{-1}y) \ge t[1 + \xi(t)]T(x, y).$$

According to Lemma 2.1, the operator equation $x = T(x, x)$ has a unique solution x^* in $[u_0, v_0]$. Moreover, for any initial values $x_0 \in [u_0, v_0]$, constructing successively the sequences $x_n = A(x_{n-1}, x_{n-1})$, $n = 1, 2, \ldots$, we have $\|x_n - x^*\| \to 0$, as $n \to \infty$. □

Remark 2.5. Comparing this result with above Theorem 1.9 (Theorem 2.2 of [8]), we notice three differences. Firstly, the operator B in (A8) of [8] needs the condition $B(tx, t^{-1}y) \ge t^{\alpha(t,x,y)}B(x, y)$, where $t^{\alpha(t,x,y)} \in (0, 1)$. In the proof of Theorem 2.2 in [8], authors let $\eta(x, y, t) = t^{\alpha(t,x,y)-1} - 1 \in (0, 1)$. This means that they changed the condition of the operator B to satisfying $B(tx, t^{-1}y) \ge t[1+\eta(x, y, t)]B(x, y)$, where $0 < t[1+\eta(x, y, t)] < 1$. But, in our Theorem 2.4, we let the operator A also satisfy the condition $A(tx, t^{-1}y) \ge t[1+\eta(x, y, t)]A(x, y)$. Here we need only $\eta(x, y, t) > 0$.

Secondly, we replaced the special function $t^{\beta(t)}$ in (A9) of [8] with the function $\varphi(t)$ in (H5). Obviously, our function is more general.

Finally, we generalize the operator D in (A10) of [8] from one variable to two variables. Meanwhile, we generalize the operator D from homogeneous to subhomogeneous. This means that our Theorem 2.4 improves Theorem 2.2 of [8].

Since Theorem 2.2 of [8] improved the Theorem 2.1 of [5], our Theorem 2.4 also improved Theorem 2.1 of [5].

Taking $B = \theta$ in our Theorem 2.2, we get the following corollary.

Corollary 2.6. *Let P be a normal cone in E, $e \in P^+ = P - \{\theta\}$ and operator $A\colon C_e \times C_e \to C_e$ be a mixed monotone and e-concave-convex. We assume that*:

(H8) *$u_0, v_0 \in C_e$, there have $u_0 \le v_0$, $u_0 \le A(u_0, v_0) \le A(v_0, u_0) \le v_0$;*
(H9) *one of the following conditions is satisfied*
  (L1) *for any $\varepsilon \in (0,1)$, there exists $\delta \in (\varepsilon, 1)$, such that*
$$\inf_{u_0 \le u,\, v \le v_0} \eta(u, v, \delta) > 0;$$
  (L2) *for any $t \in (0,1)$, $\eta(u, v, t)$ is non-increasing with respect to $u \in C_e$ and non-decreasing with respect to $v \in C_e$;*
  (L3) *for any $t \in (0,1)$, $\eta(u, v, t)$ is non-decreasing with respect to $u \in C_e$ and non-increasing with respect to $v \in C_e$.*

Then:

(a) *the operator equation $x = A(x, x)$ has a unique solution x^* in $[u_0, v_0]$;*
(b) *for any initial values $x_0, y_0 \in [u_0, v_0]$, constructing successively the sequences*
$$x_n = A(x_{n-1}, y_{n-1}), \quad y_n = A(y_{n-1}, x_{n-1}), \quad n = 1, 2, \dots,$$
we have $\|x_n - x^*\| \to 0$, $\|y_n - x^*\| \to 0$.

Remark 2.7. Comparing the above Theorem 1.8 (Theorem 3.1 of [16]) with our Corollary 2.6, we can see that Theorem 3.1 of [16] utilizes one of the assumptions (A5)–(A7) to construct a coupled lower-upper fixed point first and then to obtain the existence of a fixed point. But in our Corollary 2.6, the coupled lower-upper fixed point has been given as an assumption. This gives the differences between (A5) and (L1), (A7) and (L3).

We can remove the condition
$$\{t_n\} \subset (0,1), \qquad t_n \to 0^+, \qquad \inf_{u,v \in C_e} \{\eta(u, v, t_n)\} > 0$$
from (A5) in our (L1). Also we can remove the condition that there exist $x_0, y_0 \in C_e$, $x_0 \le y_0$ such that
$$\overline{\lim_{t \to 0^+}} \eta(x_0, y_0, t) = +\infty$$
from (A7) in our (L3). Note, we keep (L2) is the as same as (A6).

Because the above Theorem 1.8 (Theorem 3.1 of [16]) improves Theorem 2.1 and Theorem 3.2 of [9] when $t(1 + \eta(u, v, t)) = t^{\alpha(t)}$ and $t(1 + \eta(u, v, t)) = t^{\alpha(t,u,v)}$, respectively. Consequently, we can make a similar comparison between our Corollary 2.6 and Theorem 2.1 and Theorem 3.2 of [9]. To some extent, our Corollary 2.6 extends Theorem 2.1 and Theorem 3.2 of [9] also.

Remark 2.8. In the Corollary 2.6, if we reduce the operator $A(u,v)$ to $A(u)$, and reduce $\eta(u,v,t)$ to $\eta(u,t)$, then we can obtain the same conclusion as the above Theorem A (Theorem 1.1 of [17]). Theorem 1.1 of [17] improved the main results in [4]. Consequently, our result Corollary 2.6 improved the main results of [4] also.

In the following theorem, we obtain the solution of the nonlinear eigenvalue equation $\lambda x = A(x,x)$ and discuss its dependency on the parameter.

Theorem 2.9. *Assume that the conditions in the above Corollary* 2.6 *are satisfied and* $0 < t[1+\eta(x,y,t)] < 1$ *for all* $t \in (0,1)$*. Then there exists* $\lambda > 0$ *such that the operator equation* $\lambda x = A(x,x)$ *has a unique solution* x_λ *in* $[u_0, v_0]$*. Furthermore, we have the following conclusions*:

- (R1) *if* $t[1+\eta(u,v,t)] > t^{1/2}$, $t \in (0,1)$, *then* x_λ *is strictly decreasing in* λ, *that is,* $0 < \lambda_1 < \lambda_2$ *implies* $x_{\lambda_1} > x_{\lambda_2}$;
- (R2) *if* $t[1+\eta(u,v,t)] > t^\beta$, $t \in (0,1)$, $\beta \in (0,1)$, *then* x_λ *is continuous in* λ, *that is,* $\lambda \to \lambda_0 (\lambda_0 > 0)$ *implies* $\|x_\lambda - x_{\lambda_0}\| \to 0$;
- (R3) *if* $t[1+\eta(u,v,t)] > t^\beta$, $t \in (0,1)$, $\beta \in (0,1/2)$, *then* $\lim_{\lambda\to\infty} \|x_\lambda\| = 0$, $\lim_{\lambda\to 0^+} \|x_\lambda\| = \infty$.

Proof. For any fixed $\lambda > 0$, from corollary 2.6 we know that $A/\lambda\colon C_e \times C_e \to C_e$ is mixed monotone and satisfies

$$\left(\frac{1}{\lambda}A\right)(tx, t^{-1}y) \geq \frac{1}{\lambda}t[1+\eta(x,y,t)]A(x,y) = t[1+\eta(x,y,t)]\left(\frac{1}{\lambda}A\right)(x,y).$$

From (H8), we get that $u_0, v_0 \in C_e$, $u_0 \leq v_0$, $u_0 \leq A(u_0,v_0) \leq A(v_0,u_0) \leq v_0$ and $A(u_0,v_0) \in C_e$, $A(v_0,u_0) \in C_e$. So, there exist $\lambda > 0$ such that

$$u_0 \leq \frac{1}{\lambda}A(u_0,v_0) \leq \frac{1}{\lambda}A(v_0,u_0) \leq v_0.$$

Then, from Corollary 2.6, we know that A/λ has a unique solution x_λ in $[u_0,v_0]$. Thus $\lambda x_\lambda = A(x_\lambda, x_\lambda)$.

(1) First we prove (R1). Suppose $0 < \lambda_1 < \lambda_2$, then we have $x_{\lambda_1}, x_{\lambda_2} \in C_e$. So there exists t such that $x_{\lambda_1} > t x_{\lambda_2}, x_{\lambda_2} > t x_{\lambda_1}$. Let

$$t_0 = \sup\{t > 0 \mid x_{\lambda_1} > t x_{\lambda_2},\ x_{\lambda_2} > t x_{\lambda_1}\}.$$

Then we have $0 < t_0 < 1$ and

$$x_{\lambda_1} > t_0 x_{\lambda_2}, \qquad x_{\lambda_2} > t_0 x_{\lambda_1}. \tag{2.11}$$

Applying the mixed monotonicity of the operator A, we get

$$\begin{aligned}\lambda_1 x_{\lambda_1} &= A(x_{\lambda_1}, x_{\lambda_1}) \geq A\big(t_0 x_{\lambda_2}, t_0^{-1} x_{\lambda_2}\big)\\ &\geq t_0[1+\eta(x_{\lambda_2}, x_{\lambda_2}, t_0)]A(x_{\lambda_2}, x_{\lambda_2}) = t_0[1+\eta(x_{\lambda_2}, x_{\lambda_2}, t_0)]\lambda_2 x_{\lambda_2},\\ \lambda_2 x_{\lambda_2} &= A(x_{\lambda_2}, x_{\lambda_2}) \geq A\big(t_0 x_{\lambda_1}, t_0^{-1} x_{\lambda_1}\big)\\ &\geq t_0[1+\eta(x_{\lambda_1}, x_{\lambda_1}, t_0)]A(x_{\lambda_1}, x_{\lambda_1}) = t_0[1+\eta(x_{\lambda_1}, x_{\lambda_1}, t_0)]\lambda_1 x_{\lambda_1}.\end{aligned}$$

Furthermore, we get

$$x_{\lambda_1} \geq t_0[1+\eta(x_{\lambda_2}, x_{\lambda_2}, t_0)]\lambda_1^{-1}\lambda_2 x_{\lambda_2}, \qquad x_{\lambda_2} \geq t_0[1+\eta(x_{\lambda_1}, x_{\lambda_1}, t_0)]\lambda_2^{-1}\lambda_1 x_{\lambda_1}.$$

Noting that $t_0[1+\eta(x_{\lambda_2}, x_{\lambda_2}, t_0)]\lambda_1^{-1}\lambda_2 > t_0$, from the definition of t_0, we have

$$t_0[1+\eta(x_{\lambda_1}, x_{\lambda_1}, t_0)]\lambda_2^{-1}\lambda_1 \leq t_0.$$

Let $\eta(x,y,t) = t^{\alpha(t)-1} - 1$. Then $t^{\alpha(t)} = t[1+\eta(x,y,t)]$ for $\alpha(t) \in [0,1)$. Thus we can get

$$x_{\lambda_1} \geq t_0^{\alpha(t_0)}\lambda_1^{-1}\lambda_2 x_{\lambda_2}, \qquad x_{\lambda_2} \geq t_0^{\alpha(t_0)}\lambda_2^{-1}\lambda_1 x_{\lambda_1}.$$

So

$$\lambda_1^{-1}\lambda_2 t_0^{\alpha(t_0)} > t_0, \qquad \lambda_2^{-1}\lambda_1 t_0^{\alpha(t_0)} \leq t_0,$$

which implies that

$$t_0 \geq \left(\frac{\lambda_1}{\lambda_2}\right)^{1/(1-\alpha(t_0))}. \tag{2.12}$$

Then

$$x_{\lambda_1} \geq \lambda_1^{-1}\lambda_2 \left(\frac{\lambda_1}{\lambda_2}\right)^{\alpha(t_0)/(1-\alpha(t_0))} x_{\lambda_2} = \left(\frac{\lambda_2}{\lambda_1}\right)^{(1-2\alpha(t_0))/(1-\alpha(t_0))} x_{\lambda_2}. \tag{2.13}$$

Note that $t[1+\eta(x,y,t)] > t^{1/2}$ implies $\alpha(t_0) < 1/2$. Consequently, we have $(\lambda_2/\lambda_1)^{(1-2\alpha(t_0))/(1-\alpha(t_0))} > 1$. Thus, $x_{\lambda_1} > x_{\lambda_2}$.

(2) Next we prove (R2). Let $t^{\alpha(t)} = t[1+\eta(x,y,t)]$, but $t[1+\eta(x,y,t)] > t^{\beta}$. Then $\alpha(t) < \beta$, for $t \in (0,1)$. From (2.11) and (2.12), we have

$$\begin{aligned}\left(\frac{\lambda_1}{\lambda_2}\right)^{1/(1-\beta)} x_{\lambda_2} &\leq \left(\frac{\lambda_1}{\lambda_2}\right)^{1/(1-\alpha(t_0))} x_{\lambda_2} \leq x_{\lambda_1}\\ &\leq \frac{1}{t_0} x_{\lambda_2} \leq \left(\frac{\lambda_2}{\lambda_1}\right)^{1/(1-\alpha(t_0))} x_{\lambda_2} \leq \left(\frac{\lambda_2}{\lambda_1}\right)^{1/(1-\beta)} x_{\lambda_2},\end{aligned} \tag{2.14}$$

$$\begin{aligned}\left(\frac{\lambda_1}{\lambda_2}\right)^{1/(1-\beta)} x_{\lambda_1} &\leq \left(\frac{\lambda_1}{\lambda_2}\right)^{1/(1-\alpha(t_0))} x_{\lambda_1} \leq x_{\lambda_2}\\ &\leq \frac{1}{t_0} x_{\lambda_1} \leq \left(\frac{\lambda_2}{\lambda_1}\right)^{1/(1-\alpha(t_0))} x_{\lambda_1} \leq \left(\frac{\lambda_2}{\lambda_1}\right)^{1/(1-\beta)} x_{\lambda_1}.\end{aligned} \tag{2.15}$$

Moreover,

$$\theta \leq x_{\lambda_1} - \left(\frac{\lambda_1}{\lambda_2}\right)^{1/(1-\beta)} x_{\lambda_2} \leq \left[\left(\frac{\lambda_2}{\lambda_1}\right)^{1/(1-\beta)} - \left(\frac{\lambda_1}{\lambda_2}\right)^{1/(1-\beta)}\right] x_{\lambda_2}.$$

Then, from the normality of cone P and (2.14), we get

$$\begin{aligned}\|x_{\lambda_1}-x_{\lambda_2}\| &\le \left\|x_{\lambda_1}-\left(\frac{\lambda_1}{\lambda_2}\right)^{1/(1-\beta)}x_{\lambda_2}\right\|+\left\|\left(\frac{\lambda_1}{\lambda_2}\right)^{1/(1-\beta)}x_{\lambda_2}-x_{\lambda_2}\right\| \\ &\le N\left[\left(\frac{\lambda_2}{\lambda_1}\right)^{1/(1-\beta)}-\left(\frac{\lambda_1}{\lambda_2}\right)^{1/(1-\beta)}\right]\|x_{\lambda_2}\|+\left|\left(\frac{\lambda_1}{\lambda_2}\right)^{1/(1-\beta)}-1\right|\|x_{\lambda_2}\|,\end{aligned}$$

where N is the normality constant. Let $\lambda_1\to\lambda_2^-$, then we have $\|x_{\lambda_1}-x_{\lambda_2}\|\to 0$. Similarly, let $\lambda_2\to\lambda_1^+$, from (2.15), we have $\|x_{\lambda_2}-x_{\lambda_1}\|\to 0$. Then the conclusion (R2) holds.

(3) Finally we prove (R3). Let $t^{\alpha(t)}=t[1+\eta(x,y,t)]$, $t\in(0,1)$, $\alpha(t)\in[0,1)$. Then $t[1+\eta(x,y,t)]\ge t^\beta,\beta\in(0,1/2)$ tells us that $\alpha(t)\le\beta<1/2$. Let $\lambda_1=1,\lambda_2=\lambda$ in (2.13), then we have

$$x_1\ge\lambda^{(1-2\alpha(t_0))/(1-\alpha(t_0))}x_\lambda\ge\lambda^{(1-2\beta)/(1-\beta)}x_\lambda,\quad \lambda>1.$$

Thus $\|x_\lambda\|\le N/\lambda^{(1-2\beta)/(1-\beta)}$, for all $\lambda>1$, where N is the normality constant. Let $\lambda\to\infty$, then we get $\|x_\lambda\|\to 0$.

Similarly, if we let $\lambda_1=\lambda,\lambda_2=1$ in (2.13), then we get

$$x_\lambda\ge\lambda^{-(1-2\alpha(t_0))/(1-\alpha(t_0))}x_1\ge\lambda^{(1-2\beta)/(1-\beta)}x_1,\quad 0<\lambda<1.$$

So $\|x_\lambda\|\ge N^{-1}\lambda^{-(1-2\beta)/(1-\beta)}\|x_1\|$, for all $0<\lambda<1$, where N is the normality constant. Let $\lambda\to 0^+$, then we know $\|x_\lambda\|\to\infty$. □

Remark 2.10. For the operator equation $\lambda x=Ax$, where $A(x)$ is an e-concave and increasing operator, we can still discuss its dependency to the parameter and obtain the solution of the nonlinear eigenvalue equation. These conclusions can be obtained by reducing the operator $A(x,x)$ in Theorem 2.9 to $A(x)$.

3. Applications

In this section, we will give an example to demonstrate the application of our main result Theorem 2.2.

Let

$$u(x)=\int_G k(x,y)[f(y,u(y),u(y))+h(y,u(y),u(y))]\,dy, \tag{3.1}$$

where $G\subset R^n$ is a measurable set, $k(x,y)$ is nonnegative and measurable on $G\times G$ and

$$\begin{aligned} f(x,u,v)&=a_0+\sum_{i=1}^m a_i(x)u^{\alpha_i}+a_{m+1}(x)u+\sum_{j=1}^n b_j(x)v^{\beta_j},\\ h(x,u,v)&=\sum_{s=1}^p c_s(x)u^{\gamma_s}+\sum_{l=1}^q d_l(x)v^{\mu_l},\end{aligned}$$

where $0 < \alpha_i < 1$, $-1 < \beta_j < 0$, $0 < \gamma_s < 1$, $-1 < \mu_l < 0$, a_i, b_j, c_s, d_l are nonnegative and measurable on G ($i = 1, \dots, m$, $j = 1 \dots, n$, $s = 1, \dots, p$, $l = 1, \dots, q$). We denote the measure of G by mG, the set of all measurable functions on G by $M(G)$, and

$$M^+(G) = \{u(x) \in M(G) \mid u(x) \text{ is bounded and nonnegative, } u(x) \not\equiv 0\}.$$

Theorem 3.1. *Suppose $0 < mG \le \infty$. Assume that there exist nonnegative measurable functions $\varphi_1(x)$, $\varphi_2(x)$ not identical to zero, and $g(x) \in M^+(G)$ such that*

$$\varphi_2(y)g(x) \le k(x,y) \le \varphi_1(y)g(x), \quad \text{for all } x, y \in G,$$

$$\int_G \varphi_1 f(x, g(x), g(x))\,dx < \infty, \qquad \int_G \varphi_1 h(x, g(x), g(x))\,dx < \infty,$$

and there exists a real number $R > 0$ such that $\sum_{i=0}^{m} a_i(x) \ge R a_{m+1}(x), x \in G$, and $R + \overline{u} > 1$, where $\overline{u} = \sup_{x\in G} u(x)$. Then we have:

(a) *Equation* (3.1) *has exactly one solution $u^*(x)$ in $M^+(G)$.*

(b) *Constructing successively the sequence of functions*

$$\kappa_n = \int_G k(x,y)[f(y, \kappa_{n-1}(y), \kappa_{n-1}(y)) + h(y, \kappa_{n-1}(y), \kappa_{n-1}(y))]dy,$$

for $n = 1, 2, \dots$ and for any initial function $\kappa_0(x) \in M^+(G)$, then $\{\kappa_n(x)\}$ must converge to $u^*(x)$ on $M^+(G)$.

Proof. First, we will show condition (H1) of Theorem 2.2 is satisfied. Let $E = M(G)$, the order of E derived by the cone

$$P = \{u(x) \in E \mid u(x) \ge 0,\ x \in G\}, \quad e = g(x),$$

$$C(u,v) = \int_G k(x,y)\Big(a_0 + \sum_{i=1}^{m} a_i(x)u^{\alpha_i} + \sum_{j=1}^{n} b_j(x)v^{\beta_j}\Big)\,dy, \tag{3.2}$$

$$D(u) = \int_G k(x,y)a_{m+1}(x)u(y)\,dy, \tag{3.3}$$

$$A(u,v) = \int_G k(x,y) f(y, u(y), v(y))\,dy, \quad \text{for all } u, v \in P. \tag{3.4}$$

Then

$$A(u,v) = C(u,v) + D(u),$$

$$C_e = \{u(x) \in E \mid \alpha_u g(x) \le u(x) \le \beta_u g(x),\ \exists\, \beta_u \ge \alpha_u > 0\}.$$

For $\alpha = \max_{1\le i\le m,\, 1\le j\le n}\{\alpha_i, -\beta_j\}$, then

$$C\Big(ru, \frac{1}{r}v\Big) \ge r^{\alpha} C(u,v), \quad \text{for all } u, v \in P^+,\ 0 < r < 1.$$

For any $u(x) \in C_e$, we know

$$\overline{u} = \sup_{x\in G} u(x) \quad \text{and} \quad R + \overline{u} > 1.$$

Then, if $\overline{u} \le 1$, we have

$$a_0 + \sum_{i=1}^{m} a_i(x) u^{\alpha_i} \ge R a_{m+1}(x)\, u(x).$$

If $\overline{u} > 1$, we have

$$a_0 + \sum_{i=1}^{m} a_i(x) u^{\alpha_i} \ge \frac{R}{\overline{u}}\, a_{m+1}(x)\, u(x).$$

So

$$a_0 + \sum_{i=1}^{m} a_i(x) u^{\alpha_i} \ge \frac{R}{R+\overline{u}}\, a_{m+1}(x) u(x).$$

Then combining (3.2) with (3.3), we know that

$$C(u,v) \ge \frac{R}{R+\overline{u}}\, D(u) \triangleq l(u,v)\, D(u), \tag{3.5}$$

From (3.4) and (3.5), we have

$$C(u,v) \ge \frac{A(u,v)}{1+(l(u,v))^{-1}}.$$

Hence,

$$\begin{aligned} A\bigg(ru, \frac{1}{r}\, v\bigg) - rA(u,v) &= C\bigg(ru, \frac{1}{r}\, v\bigg) + D(ru) - rC(u,v) - rD(u) \\ &\ge [r^\alpha - r] C(u,v) \ge \frac{1}{1+(l(u,v))^{-1}}\, [r^\alpha - r]\, A(u,v), \end{aligned}$$

So

$$A\bigg(ru, \frac{1}{r}\, v\bigg) \ge r\bigg(1 + \frac{1}{1+(l(u,v))^{-1}}\, [r^{\alpha-1} - 1]\bigg)\, A(u,v).$$

Let

$$\eta = \frac{1}{1+(l(u,v))^{-1}} (r^{\alpha-1} - 1) \quad \text{with } r \in (0,1) \text{ and } \alpha \in (0,1).$$

Then $\eta(u,v,r)$ is non-increasing in u, and non-decreasing in v, since $l(u,v)$ is non-increasing in u and non-decreasing in v. So the condition (L2) of Theorem 2.2 is satisfied.

For any $u, v \in C_e$, take $\alpha_{u,v} > 0$, such that

$$\alpha_{u,v} g(x) \le u(x) \le \frac{1}{\alpha_{u,v}}\, g(x), \qquad \alpha_{u,v} g(x) \le v(x) \le \frac{1}{\alpha_{u,v}}\, g(x),$$

for $x \in G$. Then

$$A(u,v) \geq g(x)\eta(g,g,\alpha_{u,v}) \int_G \varphi_2(y) f(y,g(y),g(y))\,dy,$$

$$A(u,v) \leq g(x)\alpha_{u,v}\eta\biggl(\alpha_{u,v}g, \frac{1}{\alpha_{u,v}}\,g, \alpha_{u,v}\biggr) \int_G \varphi_1(y) f(y,g(y),g(y))\,dy,$$

Thus we know that $A\colon C_e \times C_e \to C_e$ is a mixed monotone and e-concave-convex operator, and condition (H1) of Theorem 2.2 is satisfied.

Next, we will prove condition (H2) of Theorem 2.2 is satisfied. Let

$$B(u,v) = \int_G k(x,y)h(y,u(y),v(y))\,dy,$$

then

$$\begin{aligned} B\biggl(ru, \frac{1}{r}v\biggr) &= \int_G k(x,y)\biggl(\sum_{s=1}^{p} c_s(x)(ru)^{\gamma_s} + \sum_{l=1}^{q} d_l(x)\biggl(\frac{1}{r}\,v\biggr)^{\mu_l}\biggr)\,dy \\ &= \int_G k(x,y)\biggl(\sum_{s=1}^{p} c_s(x) r^{\gamma_s} u^{\gamma_s} + \sum_{l=1}^{q} d_l(x) \frac{1}{r^{\mu_l}}\,v^{\mu_l}\biggr)\,dy \\ &> r\int_G k(x,y)\biggl(\sum_{s=1}^{p} c_s(x) u^{\gamma_s} + \sum_{l=1}^{q} d_l(x)\,v^{\mu_l}\biggr)\,dy = rB(u,v). \end{aligned}$$

If $u_1 > u_2$, $v_1 < v_2$, it is clear that $B(u_1,v_1) > B(u_2,v_2)$. Then B satisfies the condition (H2) of Theorem 2.2.

Finally, we prove that condition (H3) of Theorem 2.2 is satisfied. Take $x_0, y_0 \in C_e$ and $x_0 \leq y_0$. Let $0 < t_0 < 1$ be such that $t_0^2 x_0 \leq y_0$. Then we have

$$T(x_0,y_0) = A(x_0,y_0) + B(x_0,y_0) \quad \text{and} \quad T(x_0,y_0) \in C_e.$$

So, there exists m such that $mx_0 \leq T(x_0,y_0)$. Let

$$m = \frac{1}{1 + \eta(x_0,y_0,t_0)/c}.$$

Take c the same as in Theorem 2.2. Let $u_0 = t_0 x_0$, $v_0 = y_0/t_0$. Then we have

$$v_0 = t_0 x_0 = \frac{1}{t_0}\,t_0^2 x_0 \leq \frac{1}{t_0}\,y_0 = w_0.$$

So

$$T(u_0,v_0) = T\biggl(t_0 x_0, \frac{1}{t_0}\,y_0\biggr) \geq t_0\biggl(1 + \frac{1}{c}\,\eta(x_0,y_0,t_0)\biggr)T(x_0,y_0) \geq t_0 x_0 = u_0,$$

$$\begin{aligned} T(v_0,u_0) &= T\biggl(\frac{1}{t_0}\,y_0, t_0 x_0\biggr) \\ &\leq \frac{1}{t_0}\biggl(1 + \frac{1}{c}\,\eta\biggl(\frac{1}{t_0}\,y_0, t_0 x_0, t_0\biggr)\biggr)^{-1} T(y_0,x_0) \leq \frac{1}{t_0}\,y_0 = v_0. \end{aligned}$$

Now all conditions of Theorem 2.2 are satisfied, thus we end the proof of Theorem 3.1. □

Remark 3.2. Note that the problem (3.1) can't be solved by theorems in [17], [16], [8], [5], [9], [4].

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