Topological Methods in Nonlinear Analysis
Volume 53, No. 1, 2019, 97–109
DOI: 10.12775/TMNA.2019.002

STABILITY OF MULTIVALUED ATTRACTORS

Miroslav Rypka

Abstract. Stimulated by recent problems in the theory of iterated function systems, we provide a variant of the Banach converse theorem for multivalued maps. In particular, we show that attractors of continuous multivalued maps on metric spaces are stable. Moreover, such attractors in locally compact, complete metric spaces may be obtained by means of the Banach theorem in the hyperspace.

1. Introduction

Multivalued maps and their attractors are studied in relation to dynamical systems, for instance iterated function systems, backward dynamics or differential inclusions. Throughout the whole paper, we consider continuous multivalued maps with compact values which generate continuous operators on hyperspaces, as discussed in the next section.

Our motivation is the following. We would like to state a variant of Jánoš theorem for operators on hyperspaces induced by multivalued maps. Under Jánoš theorem, we understand the results on the converse of Banach theorem developed in [18], [26], [17], [24], [25], [31]. In spite of the metric nature of the Banach theorem, these papers provide several topological conditions on a map to be a contraction.

2010 *Mathematics Subject Classification.* Primary: 37B25, 54H20.

Key words and phrases. Converse of Banach theorem; multivalued map; multivalued attractor; stability.

Supported by the project StatGIS Team No. CZ 1.07/2.3.00/20.0170.

Although the theory of the converses to the Banach theorem seems to be complete, analogical problems in the theory of multivalued maps (for detailed treatment of attractors of multivalued maps, see e.g. [1], [2], [29]) and iterated function systems are still addressed. Since the Hutchinson's seminal work [16] (see also [32]), the metric approach to attractors of IFSs dominated. With only a few exceptions ([20]–[23]), the attractors of IFS were obtained by means of the Banach theorem. Recently, it was pointed out ([5], [10], [11]) that the attractor of IFS is a topological notion and the contractivity of maps in an IFS is only a sufficient condition for the existence of an attractor (for different approaches, see e.g. [6], [7], [27]). The novelty of this fact is caused by the prevailing interest in affine IFSs on Euclidean spaces, for which the existence of an attractor is equivalent to the existence of equivalent metric on original space with respect to which the maps in IFS are contractions [5].

The question whether it holds for any IFS with a point-fibred attractor A was raised by Kameyama in [19] (see also [5] and [6]). *Does there exist a metric on A such that Hutchinson operator $F|_A$ is a contraction and the topology on A induced by this metric is the same as the topology of X restricted to A*?

Similar problem for multivalued maps was stated by Fryszkowski ([17]). *Let X be an arbitrary nonempty set, 2^X be the family of all nonempty subsets of X and $F\colon X \to 2^X$ be a multivalued map. Find necessary conditions and (or) sufficient conditions for the existence of a complete metric d on X such that given $c \in (0,1)$, F would be a Nadler* ([29]) *multivalued c-contraction with respect to d, that is*

$$d_H(F(x), F(y)) \leq cd(x,y) \quad \textit{for all } x, y \in X,$$

where d_H denotes the Hausdorff metric generated by d.

In contrast to Fryszkowski's problem or Kameyama's question, we shift the search for the metric with respect to which a map is contracting on the hyperspace. In particular, we will investigate when continuous multivalued operators with attractors are contractions.

We will proceed in the following way. Next section recalls the basic notions, e.g. multivalued maps, attractors and strict attractors. Main results can be found in Section 3. Theorem 3.1 shows that attractors of continuous multivalued maps on metric spaces are stable fixed points of associated operators on hyperspace. The stability of the attractor is implied by the monotonicity of such operators. If an original space is, in addition, locally compact (e.g. Euclidean), there exists a metric equivalent to the Hausdorff metric with respect to which the multivalued operator is a contraction. This result, stated in Corollary 3.5, may be regarded as a variant of Jánoš theorem for multivalued operators. The same conditions as in Theorem 3.1 imply also the stability of strict attractors. Hence, we express the analogical results for strict attractors in Corollaries 3.3 and 3.7.

Finally, a few examples are provided. Example 3.8 illustrates the relation of our theory to Fryszkowski's problem. It proves that multivalued maps generating contracting operators need not be contractions even on compact metric spaces. Example 3.10 shows that we cannot drop the monotonicity condition. The attractivity of multivalued operators that are not generated by multivalued maps of multivalued operators does not imply the contractivity of the operators even on compact spaces.

2. Notation

Throughout the whole paper, we deal with a metric space (X,d).

Definition 2.1. We denote by $\mathbb{K}(X)$ the space of nonempty compact subsets of X, called the *hyperspace,* endowed usually with the Hausdorff metric d_H defined (cf. e.g. [16])

$$d_H(A,\,B) := \inf\{r > 0 : A \subset O_r(B) \text{ and } B \subset O_r(A)\},$$

where $O_r(A) := \{x \in X : \exists a \in A : d(x,a) < r)\}$ and $A, B \in \mathbb{K}(X)$.

An alternative definition reads

$$\begin{aligned} d_H(A,\,B) &:= \max\Big\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\Big\} \\ &= \max\Big\{\sup_{a\in A}\Big(\inf_{b\in B} d(a,b)\Big), \sup_{b\in B}\Big(\inf_{a\in A} d(a,b)\Big)\Big\}. \end{aligned}$$

Remark 2.2. The Hausdorff metric is induced by d. However, there exist metrics on $\mathbb{K}(X)$ which cannot be induced by any d. Thus, we will denote a general metric on $\mathbb{K}(X)$ by D.

Notice that a hyperspace $(\mathbb{K}(X), d_H)$ inherits most of the features of a metric space (X,d).

Lemma 2.3 (cf. e.g. [3, Table 1]). *Let (X,d) be a metric space. The hyperspace $(\mathbb{K}(X), d_H)$ is locally compact if and only if (X,d) is locally compact.*

We will often treat a neighbourhood of a compact set as a subset of a hyperspace.

Definition 2.4. Let $A \in (\mathbb{K}(X), d_H)$. We will write $\mathcal{N}_r(A) := \{B \in \mathbb{K}(X) : d_H(A,B) < r\}$.

In general, letters such as $\mathcal{B}$, $\mathcal{N}$, $\mathcal{O}$, $\mathcal{U}$ will stand for classes of sets.

Definition 2.5. A map $F\colon X \to \mathbb{K}(X)$ is called *a multivalued map* and the operator $F\colon \mathbb{K}(X) \to \mathbb{K}(X)$, defined by

$$F(A) = \bigcup_{x\in A} F(x),$$

is called *a multivalued operator.*

Throughout the paper, all multivalued maps and operators are continuous with respect to d and d_H.

Remark 2.6. A continuous multivalued map on a metric space generates a continuous multivalued operator (cf. [1], [9]).

Definition 2.7. A multivalued operator $F\colon \mathbb{K}(X) \to \mathbb{K}(X)$ is called *monotone* if $F(A \cup B) = F(A) \cup F(B)$ for all $A, B \in \mathbb{K}(X)$.

Remark 2.8. Any multivalued operator induced by a multivalued map is obviously monotone.

Remark 2.9. The condition for F given in Definition 2.7 is in fact stronger than the monotonicity which is usually understood as follows. For any $A, B \in K(X)$, $A \subseteq B$ implies that $F(A) \subseteq F(B)$.

Definition 2.10. Let (X, d) and (Y, d') be metric spaces. A map $f\colon X \to Y$ is *a contraction* (*contracting*) if, for some $c \in [0, 1)$,

$$d'(f(x), f(y)) \le c d(x, y) \quad \text{for any } x, y \in X.$$

Remark 2.11. We will distinguish contractive maps (see [14]) for which

$$d'(f(x), f(y)) < d(x, y) \quad \text{for any } x, y \in X,\ x \neq y.$$

Multivalued maps and operators are often generated by iterated function systems (IFSs).

Definition 2.12. *An iterated function system* consists of finite number of continuous maps f_i, $i = 1, \dots, N$, on a metric space (X, d).

Any IFS yields a continuous multivalued map $F\colon X \to \mathbb{K}(X)$,

$$F(x) = \bigcup_{i=1}^{N} \{f_i(x)\},$$

and a continuous multivalued operator $F\colon \mathbb{K}(X) \to \mathbb{K}(X)$,

$$F(A) = \bigcup_{x \in A} F(x),$$

called *Hutchinson operator.*

Usually, IFSs of contracting maps f_i on complete metric spaces are treated. They possess an attractor $A^* \in \mathbb{K}(X)$ due to the Banach theorem (cf. [16]). Notice that this theorem gives not only the existence and attractivity but also the speed of convergence necessary for the visualization of attractors.

Let us start the discussion of the notion of attractor with the single-valued case.

DEFINITION 2.13. Let (X,d) be a metric space and $f\colon X \to X$ continuous with a fixed point x^*. The point x^* is called *an attractor* (*attractive*) if there exists an open set $U \subset X$, $x^* \in U$, such that

$$\lim_{n\to\infty} d(f^n(x), x^*) = 0, \quad \text{for all } x \in U.$$

The set

$$B(x^*) = \Big\{x \in X : \lim_{n\to\infty} d(f^n(x), x^*) = 0\Big\}$$

is called *the basin of attraction* of x^*.

REMARK 2.14. The basin of attraction is the maximal neighbourhood U from Definition 2.13 and $f(B(x^*)) \subset B(x^*)$.

Let us also recall the notion of stability of a fixed point.

DEFINITION 2.15. Let (X,d) be a metric space and $f\colon X \to X$ continuous with a fixed point $x^* \in X$. The point x^* is called *stable* if, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $d(f^n(x), x^*) < \varepsilon$ for any $n \in \mathbb{N}$ and $x \in X$ with $d(x, x^*) < \delta$. Fixed point x^* is *asymptotically stable* if it is attractive and stable.

Analogically to Definition 2.13, we define an attractor in a hyperspace.

DEFINITION 2.16. Let (X,d) be a metric space and $F\colon X \to \mathbb{K}(X)$ be a continuous multivalued map. Let $A^* \in \mathbb{K}(X)$ be such that $F(A^*) = A^*$ and an open set $\mathcal{U} \subset \mathbb{K}(X)$ be such that $A^* \in \mathcal{U}$ and $\lim\limits_{n\to\infty} d_H(F^n(B), A^*) = 0$ for all $B \in \mathcal{U}$. Then A^* is called *an attractor* of the multivalued map F.

The previous definition may be formulated also for an IFS.

DEFINITION 2.17. Let $\{X; f_1, \ldots, f_N\}$ be an IFS and F its Hutchinson operator with a fixed point $A^* \in \mathbb{K}(X)$. The set A^* is an attractor of the IFS if there exists an open set $\mathcal{U} \subset \mathbb{K}(X)$ such that $A^* \in \mathcal{U}$ and, for all $B \in \mathcal{U}$,

$$\lim_{n\to\infty} d_H(F^n(B), A^*) = 0.$$

However, this definition has its drawbacks in hyperspaces as can be seen from the following example.

EXAMPLE 2.18. The IFS $\{([-1,1], d_{\mathrm{eucl}}); f(x) = \sqrt[3]{x}\}$ possesses three overlapping attractors

$$\begin{aligned}
A_1 &= \{-1\}, & \mathcal{U}_1 &= \mathbb{K}([-1,0)),\\
A_2 &= \{1\}, & \mathcal{U}_2 &= \mathbb{K}((0,1]),\\
A_3 &= \{-1,1\}, & \mathcal{U}_3 &= \mathbb{K}([-1,1] \setminus \{0\}) \setminus (\mathcal{U}_1 \cup \mathcal{U}_2).
\end{aligned}$$

In order attractors do not overlap, we introduce the notion of a strict attractor (see [13], [10], [11]).

Definition 2.19. A compact set $A^* \subset X$ is a *strict attractor* of a multivalued map $F\colon X \to \mathbb{K}(X)$, if there exists an open set $X \supset U \supset A^*$ such that

$$F^n(S) \to A^* \quad \text{for all } S \in \mathbb{K}(U).$$

The maximal open set U with the above property is called the *basin of attraction* of the attractor A^* (with respect to F) and denoted by $B(A^*)$.

Remark 2.20. The existence of the maximal open set $B(A^*)$ is proven in [11].

Remark 2.21. Strict attractor is a topological invariant ([8, Lemma 2.8]) and it is an attractor.

3. Results

We are ready to prove the stability of multivalued attractors and even contractivity of corresponding operators on basins of attraction.

Theorem 3.1. *Any attractor of continuous multivalued map on a metric space (X,d) is asymptotically stable.*

The proof of the theorem proceeds in two steps. First, we show the stability of A^* in $\mathbb{K}(A^*)$. Then we extend it to the neighbourhood of A^*.

Lemma 3.2. *Let (A^*,d) be a compact metric space. Let $F\colon \mathbb{K}(A^*) \to \mathbb{K}(A^*)$ be induced by continuous multivalued map $F\colon A^* \to \mathbb{K}(A^*)$ such that $F(A^*) = A^*$ and there exists $\varepsilon > 0$ such that*

$$\lim_{n\to\infty} d_H(F^n(\{B\}), A^*) = 0, \quad \textit{for all } B \in \mathcal{N}_\varepsilon(A^*).$$

Then A^* is asymptotically stable.

Proof. We only need to show the stability of A^*. If A^* is a singleton, then it is obviously stable on $\mathbb{K}(A^*)$.

Suppose that A^* is not a singleton, which means $\operatorname{diam}(A^*) > 0$. We will proceed by contradiction. Assume A^* is not stable. Then there exist $\varepsilon > 0$ and a sequence

$$\{B_n\} \in \mathbb{K}(A^*), \qquad d_H(B_n, A^*) < \min\left\{\frac{1}{n}, \varepsilon\right\}$$

such that there exists $k_n \in \mathbb{N}$ such that $d_H(F^{k_n}(B_n), A^*) > \varepsilon$. Observe that $k_n \to \infty$, otherwise the operator F would not be continuous.

Since F is continuous (see Remark 2.6), for any $n \in \mathbb{N}$, the set $\mathcal{B}_n \subset \mathbb{K}(A^*)$ defined by $\mathcal{B}_n = \{B \in \mathcal{N}_\varepsilon(A^*) : d_H(F^{k_n}(B), A^*) > \varepsilon\}$ is open.

In the following part, we will employ the monotonicity of F (see Remark 2.8). Notice that, for any set $C \in \mathbb{K}(A^*)$, $C \subset B \in \mathcal{B}_n$,

$$d_H(F^{k_n}(C), A^*) > \varepsilon \tag{3.1}$$

since $d_H(F^{k_n}(B), A^*) \le d_H(F^{k_n}(C), A^*)$ according to Definition 2.1.

We will show that there exists a set $K \in \mathbb{K}(A^*)$ such that K belongs to the subsequence of $\overline{\mathcal{B}_n}$. For $\mathcal{B} \subseteq \mathbb{K}(A^*)$, denote by $\widetilde{\mathcal{B}}$ the set $\{C \in \mathcal{N}_\varepsilon(A^*); \text{exists } B \in \mathcal{B}\ , B \subset C\}$. The set $\widetilde{\mathcal{B}}$ is open if $\mathcal{B}$ is open. Furthermore, let $(\mathcal{B}' \cap \widetilde{\mathcal{B}})^{-1}$ stand for the set $\{B \in \mathcal{B} : \text{exists } B' \in \mathcal{B}',\ B \subset B'\}$ which is again open for $\mathcal{B}$, $\mathcal{B}'$ open.

Observe that, for any $n_1 \in \mathbb{N}$, there exists $n_2 \in \mathbb{N}$ such that $\mathcal{B}_{n_1n_2} := \big(\mathcal{B}'_{n_2} \cap \widetilde{\mathcal{B}_{n_1}}\big)^{-1}$ is nonempty and open. Since $\mathcal{B}_{n_1}$ is nonempty and open, there exists an open ball $\mathcal{O}_{n_1} \subset \mathcal{B}_{n_1}$ with radius r_{n_1}. For $n_2 \in \mathbb{N}$, $n_2 > 1/r_{n_1}$, consider the set $\mathcal{B}_{n_2}$. The inequality $d_H(B_{n_2}, A^*) < r_{n_1}$ and (3.1) imply that $\big(\mathcal{B}_{n_2} \cap \widetilde{\mathcal{B}_{n_1}}\big)^{-1}$ is nonempty. It is also open, since $\mathcal{B}_{n_1}$ and $\mathcal{B}_{n_2}$ are open. We will simplify the notation using $\mathcal{B}_{n_1n_2}$ instead of $\big(\mathcal{B}_{n_2} \cap \widetilde{\mathcal{B}_{n_1}}\big)^{-1}$.

Again, since $\mathcal{B}_{n_1n_2}$ is open, there exists an open ball $\mathcal{O}_{n_1n2} \subset \mathcal{B}_{n_1n2}$ with radius $r_{n_1n_2}$ and $n_3 \in \mathbb{N}$ such that

$$\mathcal{B}_{n_1n_2n_3} := \big(\mathcal{B}_{n_3} \cap \widetilde{\mathcal{B}_{n_1n_2}}\big)^{-1}$$

is nonempty and open. Repeating this process to infinity, we obtain the sequence

$$\mathcal{B}_{n_1}, \mathcal{B}_{n_1n_2}, \mathcal{B}_{n_1n_2n_3}, \mathcal{B}_{n_1,n_2,n_3n_4}, \ldots \tag{3.2}$$

Observe that the sequence is nested, i.e.

$$\mathcal{B}_{n_1} \supset \mathcal{B}_{n_1n_2} \supset \mathcal{B}_{n_1n_2n_3} \supset \mathcal{B}_{n_1n_2n_3n_4} \supset \ldots$$

and, for any $p \in \mathbb{N}$,

$$d_H(F^{n_p}(B), A^*) > \varepsilon, \quad B \in \mathcal{B}_{n_1n_2\ldots n_p}.$$

Last, consider the sequence of closures

$$\overline{\mathcal{B}_{n_1}}, \overline{\mathcal{B}_{n_1n_2}}, \overline{\mathcal{B}_{n_1n_2n_3}}, \overline{\mathcal{B}_{n_1n_2n_3n_4}}, \ldots \tag{3.3}$$

where $\overline{\mathcal{B}_{n_1\ldots n_k}}$ is compact in $\mathbb{K}(A^*)$ for any k. Since the sequence (3.3) is also nested, the intersection $\overline{\mathcal{B}} = \bigcap\limits_{i=1}^{\infty} \overline{\mathcal{B}_{n_1n_2\ldots n_i}}$ is nonempty. Continuity of F implies

$$d_H(F^{n_i}(K), A^*) \geq \varepsilon, \quad i \in \mathbb{N},\ K \in \overline{\mathcal{B}},$$

which is a contradiction to the attractivity of A^* in $\mathcal{N}_\varepsilon(A^*)$. □

Proof (Continuation of the proof of Theorem 3.1). In the final part of the proof, we will use the asymptotic stability of A^* on a compact subset of $\mathcal{N}_\varepsilon(A^*)$ and the uniform continuity of F on a feasible compact subset of $\mathbb{K}(X)$. We will proceed by contradiction.

Assume that A^* is not stable. Then there exist $\varepsilon > 0$, a sequence of sets $B_n \in \mathcal{U}$, $d_H(B_n, A^*) < 1/n$ and a sequence $i_n \in \mathbb{N}$ such that

$$d_H(F^{i_n}(B_n), A^*) > \varepsilon, \quad \text{for all } n \in \mathbb{N}.$$

For the sake of simplicity, let us denote $F^i(B_n)$ by B_n^i.

Observe that
$$\overline{B} = \bigcup_{n=1}^{\infty} B_n \cup A^*$$
is a compact subset of X as well as
$$\widehat{B} := \bigcup_{i=1}^{\infty} F^i(\overline{B}).$$
The compactness of $\widehat{B}$ implies that for all $\varepsilon > 0$ there exists $m_0 \in \mathbb{N}$ such that
$$d_H\big(F^i\big(\widehat{B}\big), A^*\big) < \varepsilon \quad \text{for all } i \geq i_0.$$
Similarly, from the monotonicity of F, we have
$$\forall\, \varepsilon\ \exists\, i_0 \in \mathbb{N}\ \forall\, i \geq i_0\ \forall\, B_n^i\ \exists\, C_n^i \in \mathbb{K}(A^*) \cap \mathcal{U} : d_H(B_n^{i_n}, C_n^i) < \varepsilon.$$
Consider closed neighbourhood $\mathcal{U} \subset \mathbb{K}(A^*)$ such that $\mathcal{U} \subset \mathcal{N}_\varepsilon(A^*)$.

The asymptotic stability of F implies
$$\forall\, \varepsilon > 0\ \exists\, k_0 \in \mathbb{N} : d_H(F^k(B), A^*) < \frac{\varepsilon}{2}, \quad k \geq k_0,\ \forall\, B \in \mathcal{U}. \tag{3.4}$$
The operator F is uniformly continuous on any compact subset of $\mathbb{K}(X)$, which implies that, for any $k \in \mathbb{N}$ and $\varepsilon > 0$, there exists $\delta_0 > 0$ such that
$$d_H(F^k(B), F^k(B')) < \frac{\varepsilon}{2}, \quad \text{for all } B, B' \in \mathbb{K}(\widehat{B}),\ d_H(B, B') < \delta_0. \tag{3.5}$$
Let k_0 fulfill (3.4) for our chosen ε and δ_0 fulfill (3.5) for $k = k_0$. From the attractivity of A^* and from $F^n\big(\widehat{B}\big) \to A^*$, we get
$$\forall\, \delta > 0\ \exists\, n_0 \in \mathbb{N}\ \forall\, i \in \mathbb{N}\ \forall\, B_n^i\ \exists\, C_n^i \in \mathbb{K}(A^*) : d_H(B_n^i, C_n^i) < \delta. \tag{3.6}$$
Notice that there always exists $C_n^i \in \mathbb{K}(A^*)$ such that
$$d_H(C_n^i, A^*) \leq d_H(B_n^i, A^*). \tag{3.7}$$
Consider a sequence $\{i_n\} \in \mathbb{N}$ such that
$$d_H(B_n^{i_n}, A^*) \geq \varepsilon \tag{3.8}$$
and if $d_H(B_n^i, A^*) \geq \varepsilon$ then $i > i_n$. Observe that $i_n \to \infty$, otherwise F would not be continuous.

Let us investigate the behaviour of F^{k_0} on $B_n^{i_n - k_0}$ for $n \in \mathbb{N}$ such that $i_n > k_0$. From (3.4) and (3.5), we obtain (notice that $d_H(C_n^{i_n - k_0}, A^*) < \varepsilon$ follows from (3.6) and (3.7))
$$d_H(F^{k_0}(B_n^{i_n - k_0}), A^*) \leq d_H(F^{k_0}(B_n^{i_n - k_0}), C_n^{i_n - k_0}) + d_H(F^{k_0}(C_n^{i_n - k_0}), A^*)$$
implying $d_H(B_n^{i_n}, A^*) < \varepsilon$ which is a contradiction to (3.8). Therefore, the attractor A^* is stable. □

Since a strict attractor is an attractor, we obtain:

Corollary 3.3. *Any strict attractor of continuous multivalued map on a metric space is asymptotically stable.*

Adding condition of local compactness on an original space X, a multivalued map with an attractor generates a contraction on the basin of attraction.

Lemma 3.4 (cf. e.g. [31, Theorem 2.1]). *Let (X, d) be a locally compact metric space. Let $f\colon X \to X$ be a continuous map with a fixed point $x^* \in X$ such that $\lim_{n\to\infty} d(f^n(x), x^*) = 0$, for any $x \in X$ and x^* be stable. Then there exists metric d' on X equivalent to d such that f is a contraction on (X, d').*

Theorem 3.1, Lemmas 2.3 and 3.4 imply a corollary.

Corollary 3.5. *Let (X, d) be a locally compact metric space. Let $F\colon X \to \mathbb{K}(X)$ be a ultivalued map with an attractor A^* and an open set $\mathcal{U} \subset \mathbb{K}(X)$ be such that $A^* \in \mathcal{U}$, $F(\mathcal{U}) \subset \mathcal{U}$ and $\lim_{n\to\infty} d_H(F^n(B), A^*) = 0$ for all $B \in \mathcal{U}$. Then there exists metric D on $\mathcal{U}$ equivalent with d_H such that the operator $F\colon \mathbb{K}(X) \to \mathbb{K}(X)$ is a contraction on $\mathcal{U}$ with respect to D.*

Remark 3.6. In order to define precisely the contraction on a neighbourhood $\mathcal{U}$ of an attractor, we need in addition to previous results the forward invariance of $\mathcal{U}$ with respect to F. Such maximal neighbourhood of A^* is its basin of attraction (see Remark 2.14).

Furthermore, for strict attractors and their basins of attraction, we immediately receive the following.

Corollary 3.7. *Let (X, d) be a locally compact metric space. Let $F\colon X \to \mathbb{K}(X)$ be a multivalued map with a strict attractor A^* and basin of attraction $B(A^*)$. There exists metric D on $\mathbb{K}(B(A^*))$ equivalent with d_H such that the operator $F\colon \mathbb{K}(X) \to \mathbb{K}(X)$ is a contraction on $(\mathbb{K}(B(A^*)), D)$.*

The contractivity of operator F provides us with the speed of convergence to the attractor in addition to its existence and uniqueness in the basin of attraction. The previous two corollaries obviously apply to any Euclidean space X. Together, it means that all attractors of IFSs in Euclidean spaces are "Banach" in hyperspace (compare with [7], [30]).

The metric D may be constructed by means of [31] or [18]. However, D need not be induced by any d on X. In other words, multivalued maps inducing contracting operators on hyperspace do not fulfill Fryszkowski's condition in general. We will show it in the following example.

Example 3.8. Consider the IFS $F = \{(X, d), f_1, f_2\}$, $X = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$, where d is ordinary Euclidean metric and $f_1(x, y) := (x, y)$, $f_2(x, y) := (x\cos\alpha - y\sin\alpha, x\sin\alpha + y\cos\alpha)$, where α/π is irrational. Observe that X is a (strict) attractor of F in X and, according to Corollary 3.5, there exists metric D on $\mathbb{K}(X)$ equivalent to d_H such that operator F is a contraction with respect to D.

We will show that there is no metric d' on X equivalent to d such that multivalued map $F\colon X \to \mathbb{K}(X)$ associated to the IFS is a contraction (with respect to d' and d'_H). Let us prove it by contradiction.

Suppose, there exists such metric d'. Then we can find a point $x_0 \in X$ and $\varepsilon > 0$ such that $d'(x, x') \leq d(f_2(x), f_2(x'))$, $x, x' \in N_\varepsilon(x_0)$. Otherwise, f_2 would be contractive with a fixed point according to [14, Theorem 1] and d' would not be equivalent to d.

Since $0 < \alpha < 2\pi$, we can pick $x, x' \in N_\varepsilon(x_0)$ close enough so that

$$d(f_2(x), \{x, x'\}) \geq d(x, x') \quad \text{and} \quad d(f_2(x'), \{x, x'\}) \geq d(x, x').$$

From Definition 2.1, we obtain

$$d'_H(F(x), F(x')) = d'_H(\{x, f_2(x)\}, \{x', f_2(x')\}) = d'(f_2(x), f_2(x')) \geq d'(x, y),$$

which is a contradiction to contractivity of F with respect to d' and d'_H.

Remark 3.9. The attractor X in Example 3.8 is not point-fibred, as defined in [20] (topologically contracting according to [6]). In contrast to [19] and [6], the point-fibredness is not relevant for our results.

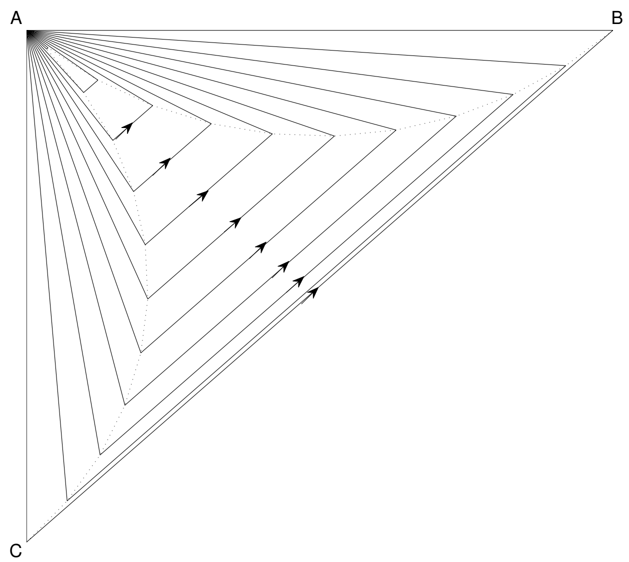

Figure 1. Attracting operator on $\mathbb{K}_C(I)$ which is not a contraction.

In general, an attracting operator on a compact hyperspace is not a contraction. Let us demonstrate that we can not drop the condition that the operator is generated by a multivalued map.

EXAMPLE 3.10. Let us denote by $\mathbb{K}_C(I)$ the set of closed subintervals of $I = [0,1]$, endowed with Euclidean metric. The space $(\mathbb{K}_C(I), d_H)$ is equivalent to a filled triangle $T = ABC$, $A = [0,1]$, $B = [1,1]$, $C = [0,0]$ in $\mathbb{R}^2$ endowed with the maximum metric (cf. e.g. [4]).

Inspired by the examples in [15, Theorem 10] and [31, Theorem 2.1], we will construct an operator on $\mathbb{K}(I)$ which is attractive, but not a contraction. We consider the space T as a union of triangles $T_\alpha, \alpha \in [0,1]$ with empty interiors and the common point A as illustrated in Figure 1. Observe that there is a homotopy $h\colon T \times [0,1] \to T$ such that $T_\alpha = h(T_1, \alpha)$.

Similarly, as discussed in [10], consider the circle X which may be projected to the set of real numbers and infinity on the real line $\mathbb{R}^*$. Let $f^*\colon \mathbb{R}^* \to \mathbb{R}^*$ be such that $f^*(x) = x + 1$, $x \neq \infty$ and $f^*(\infty) = \infty$.

Observe that the corresponding map $f\colon X \to X$ is continuous with respect to the Euclidean metric on the circle. Moreover, each point of X is attracted to the point ∞ (see Figure 2). On the other hand, f is not a contraction since $f(X) = X$ (cf. [31]).

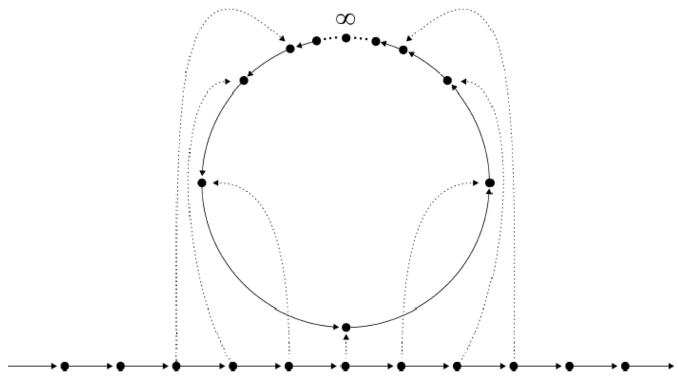

FIGURE 2. Construction of an attracting map which is not a contraction.

We can construct a homeomorphism $g\colon X \to T_1$ such that $A = \infty$. Define a map $G\colon T \to T$ such that $G(t) = h(g(f(g^{-1}(p_T(h^{-1}(t))))), \alpha)$ for $t \in T_\alpha$, $\alpha > 0$ and $G(A) = A$ where $p_T\colon T \times I \to T$ is an ordinary projection.

The map G is obviously continuous (see Figure 2) and may be applied on $\mathbb{K}_C(I)$. Let us extend the map $G\colon \mathbb{K}_C(I) \to \mathbb{K}_C(I)$ to the whole hyperspace $\mathbb{K}(I)$ and define a map $Q\colon \mathbb{K}(I) \to \mathbb{K}(I)$ by $Q(D) = [\min(D), \max(D)]$, which is continuous.

Defining $F\colon \mathbb{K}(I) \to \mathbb{K}(I)$, $F = G \circ Q$, we obtain an operator on $\mathbb{K}(I)$ which has an attractor I. However, the operator F is not a contraction, since $F^n(\mathbb{K}(I)) \nrightarrow \{I\}$ (see [31] and [15, Theorem 10]).

References

[1] J. Andres and J. Fišer, *Metric and topological multivalued fractals*, Intrnat. J. Bifur. Chaos Appl. Sci. Engrg. **14** (2004), no. 4, 1277–1289.

[2] J. Andres, J. Fišer, G. Gabor and K. Leśniak, *Multivalued fractals*, Chaos Solitons Fractals **24** (2005), no. 3, 665–700.

[3] J. Andres and M. Rypka, *Multivalued fractals and hyperfractals*, Internat. J. Bifurc. Chaos Appl. Sci. Engrg. **22** (2012), no. 1, 1250009.

[4] J. Andres and M. Rypka, *Visualization of hyperfractals*, Internat. J. Bifurc. Chaos Appl. Sci. Engrg. **23** (2013), no. 10, 1350167.

[5] R. Atkins, M.F. Barnsley, A. Vince and D.C. Wilson, *A characterization of hyperbolic affine iterated function systems*, Topol. Proc. **36** (2010), 189–211.

[6] T. Banakh, W. Kubiś, N. Novosad, M. Nowak and F. Strobin, *Contractive function systems, their attractors and metrization*, Topol. Methods Nonlinear Anal. **46** (2015), no. 2, 1029–1066.

[7] T. Banakh and M. Nowak, *A* 1*-dimensional Peano continuum which is not an* IFS *attractor*, Proc. Amer. Math. Soc. **141** (2013), no. 3, 931–935.

[8] M.F. Barnsley, B. Harding and M. Rypka, *Measure preserving fractal homeomorphisms*, Fractals, Wavelets and their Applications (C. Bandt, ed.), Springer Proceedings in Mathematics and Statistics, vol. 92, Springer, Berlin, 2014, 79–102.

[9] M.F. Barnsley and K. Leśniak, *On the continuity of the Hutchinson Operator*, Symmetry **7** (2015), 1831–1840.

[10] M.F. Barnsley, K. Leśniak and M. Rypka, *Chaos game for* IFS*s on topological spaces*, J. Math. Anal. Appl. **435** (2016), no. 2, 1458–1466.

[11] M.F. Barnsley, K. Leśniak and M. Rypka, *Basic topological structure of fast basins*, Fractals, doi: 10.1142/S0218348X18500111,

[12] M.F. Barnsley and A. Vince, *The Conley attractors of an iterated function system*, Bull. Austral. Math. Soc. **88** (2013), no. 2, 267–279.

[13] M. F. Barnsley, D.C. Wilson, K. Leśniak, *Some recent progress concerning topology of fractals*, Recent Progress in General Topology III (K.P. Hart, J. van Mill, P. Simon, eds.), Atlantis Press, Amsterdam, 2014, 69–92.

[14] M. Edelstein, *On fixed and periodic points under contractive mappings*, J. London Math. Soc. **1** (1962), no. 1, 74–79.

[15] J.L.G. Guirao, D. Kwietniak, M. Lampart, P. Oprocha and A. Peris, *Chaos on hyperspaces*, Nonlinear Anal. **71** (2009), no. 1, 1–8.

[16] J.E. Hutchinson, *Fractals and self-similarity*, Indiana Univ. Math. J. **30** (1981), 713–747.

[17] J. Jachymski, *A short proof of the converse to the contraction principle and some related results*, Topol. Methods Nonlinear Anal. **15** (2000), 179–186.

[18] L. Janos, *A converse of Banach's contraction theorem*, Proc. Amer. Math. Soc. **18** (1967), no. 2, 287–289.

[19] A. Kameyama, *Distances on topological self-similar sets and the kneading determinants*, J. Math. Kyoto Univ. **40** (2000), no. 4, 604–672.

[20] B. Kieninger, *Iterated function systems on compact Hausdorff spaces*, Shaker (2002).

[21] A. Lasota and J. Myjak, *Semifractals*, Bull. Polish Acad. Sci. Math. **44** (1996), 5–21.

[22] A. Lasota and J. Myjak, *Attractors of multifunctions*, Bull. Polish Acad. Sci. Math. **48** (2000), no. 3, 319–334.

[23] A. Lasota and J. Myjak, *Fractals, multifunctions and Markov operators*, Trends in Mathematics: Fractals in Graz 2001 (P. Grabner, W. Woess, eds.) Birkhäuser, Basel, (2002), 197–210.

[24] S. Leader, *A topological characterization of Banach contractions*, Pacific J. Math **69** (1977), no. 2, 461–466.

[25] S. Leader, *Uniformly contractive fixed points in compact metric spaces*, Proc. Amer. Math. Soc. **86** (1982), no. 1, 153–158.

[26] P. R. Meyers, *A converse to Banach's contraction theorem*, J. Res. Nat. Bur. Standards Sect. B **71**, (1967), 73–76.

[27] A. Mihail, *A topological version of iterated function systems*, An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. **58** (2012), no. 1, 105–120.

[28] J. Myjak and T. Szarek, *Attractors of iterated function systems and Markov operators*, Abstr. Appl. Anal. **2003:8** (2003), 479–502.

[29] S.B. Nadler, *Multi-valued contraction mappings*, Pacific J. Math **30** (1969), 475–488.

[30] M. Nowak and T. Szarek, *The shark teeth is a topological* IFS-*attractor*, Siberian Math. J. **55** (2014), no. 2, 296–300.

[31] V.I. Opoitsev, *A converse to the principle of contracting maps*, Russian Math. Surveys **31** (1976), no. 4, 175–204.

[32] R.F. Williams, *Composition of contractions*, Bol. Soc. Brasil. Mat. **2** (1971), no. 2, 55–59.

Manuscript received September 22, 2017

accepted December 7, 2018

Miroslav Rypka
Deparment of Mathematical Analysis
and Applications of Mathematics
Faculty of Science
Palacký University
17. listopadu 12
771 46 Olomouc, CZECH REPUBLIC

E-mail address: miroslav.rypka01@upol.cz