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A DIFFUSIVE LOGISTIC EQUATION WITH U-SHAPED DENSITY DEPENDENT DISPERSAL ON THE BOUNDARY

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Abstract. We study positive solutions to the steady state reaction diffusion equation:

$$\begin{cases} -\Delta v = \lambda v(1-v), & x \in \Omega_0, \\ \dfrac{\partial v}{\partial \eta} + \gamma\sqrt{\lambda}(v-A)^2 v = 0, & x \in \partial\Omega_0, \end{cases}$$

where Ω_0 is a bounded domain in $\mathbb{R}^n$; $n \geq 1$ with smooth boundary $\partial\Omega_0$, $\partial/\partial\eta$ is the outward normal derivative, $A \in (0,1)$ is a constant, and λ, γ are positive parameters. Such models arise in the study of population dynamics when the population exhibits a U-shaped density dependent dispersal on the boundary of the habitat. We establish existence, multiplicity, and uniqueness results for certain ranges of the parameters λ and γ. We obtain our existence and mulitplicity results via the method of sub-super solutions.

1. Introduction

Let $\Omega_0 = (0,1)$ or be a bounded domain in $\mathbb{R}^n$; $n = 2, 3$ with smooth boundary $\partial\Omega_0$ and $|\Omega_0| = 1$. Let $\Omega = \{\ell x \mid x \in \Omega_0\}$, where ℓ is a positive parameter representing the patch size of Ω. We will consider a population that satisfies

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a logistic growth in the patch Ω. We will also assume that the diffusion rate in Ω is D and that Ω is surrounded by a matrix Ω_M, where the matrix diffusion rate is D_0 and the matrix death rate is S_0, with D, D_0 and $S_0 > 0$ (see Figure 1).

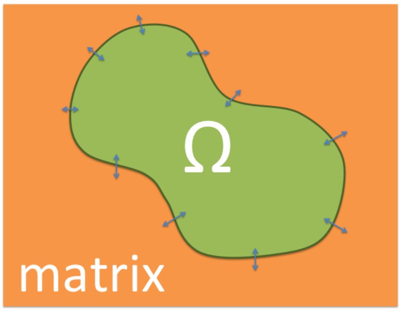


FIGURE 1. Illustration of a patch Ω surrounded by a matrix Ω_M with dispersal across the boundary.

We will further assume that the population exhibits density dependent dispersal (DDD) on the boundary. Denoting by $\alpha(u)$ the probability of the population staying in Ω when it reaches the boundary (here u is the population density), the resulting model is (see [18], [19] and [4]):

$$\begin{cases} u_t = D\Delta u + ru\left(1 - \dfrac{u}{K}\right), & x \in \Omega,\ t > 0, \\ u(0,x) = u_0(x), & x \in \Omega, \\ D\alpha(u)\dfrac{\partial u}{\partial \eta} + \dfrac{\sqrt{S_0 D_0}}{\kappa}[1 - \alpha(u)]u = 0, & x \in \partial\Omega,\ t > 0, \end{cases} \tag{1.1}$$

with corresponding steady state equation:

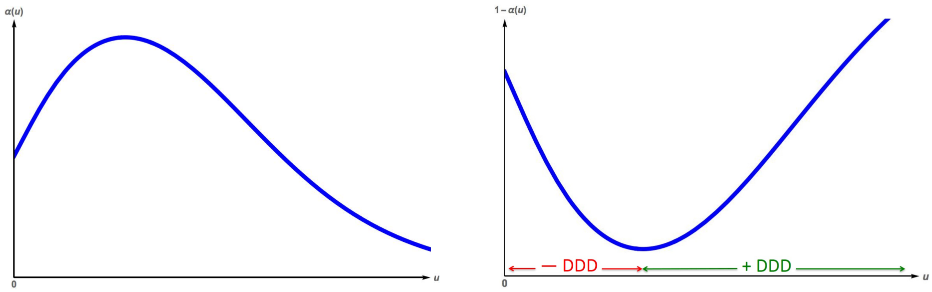
$$\begin{cases} -\Delta u = \dfrac{1}{D} ru\left(1 - \dfrac{u}{K}\right), & x \in \Omega, \\ D\alpha(u)\,\dfrac{\partial u}{\partial \eta} + \dfrac{\sqrt{S_0 D_0}}{\kappa}[1 - \alpha(u)]u = 0, & x \in \partial\Omega \end{cases} \tag{1.2}$$

or equivalently,

$$\begin{cases} -\Delta u = \dfrac{\ell^2}{D}\, ru\left(1 - \dfrac{u}{K}\right), & x \in \Omega_0, \\ \dfrac{D}{\ell}\alpha(u)\,\dfrac{\partial u}{\partial \eta} + S^*[1 - \alpha(u)]u = 0, & x \in \partial\Omega_0, \end{cases} \tag{1.3}$$

where $r > 0$ is the patch intrinsic growth rate, $K > 0$ is the carrying capacity, and $S^* = \sqrt{S_0 D_0}/\kappa$. Here κ is a positive parameter that encapsulates hypotheses regarding the patch/matrix interface (see [4]).

In this paper, we will be interested in the case when the dispersal $[1 - \alpha(u)]$ on the boundary is U-shaped (i.e. negative density dependent dispersal for lower densities and positive density dependent dispersal for higher densities, see Figure 2). In empirical studies several important factors have been found that influence emigration (the first stage of dispersal), particularly conspecific density. The paradigmatic view is that dispersal increases with density; i.e. positive density-dependent dispersal (see [22], [20], [3]). However, alternate forms of density dependent dispersal (DDD) have been reported in the literature ([5]). Negative or U-shaped relationships between conspecific density and dispersal have been observed in a wide range of taxa. In fact, seemingly contradictory evidence for both negative and positive DDD has simultaneously been observed in the same organism for both the Glanville fritillary butterfly, *Melitaea cinxia* (see [16] and [17] for negative and [5] for positive) and the Blue-footed Booby, *Sula nebouxii*, (see [15]). Kim et al. and Enfjall et al. both independently proposed a hypothesis to reconcile these contradictory patterns of DDD by integrating the conspecific hypothesis with the traditional competition hypothesis (see [15] and [5]). They suggested that the relationship between density and dispersal is not linear but U-shaped (see Figure 1). Kim et al. empirically confirmed this U-shaped relationship between density and dispersal in the Blue-footed Booby *Sula nebouxii* (see [15]). To date, U-shaped DDD has received little attention with regard to theoretical population modeling. However, see [1], [2], [6]–[9], [12] and [13] for development and analysis of several negative DDD models.

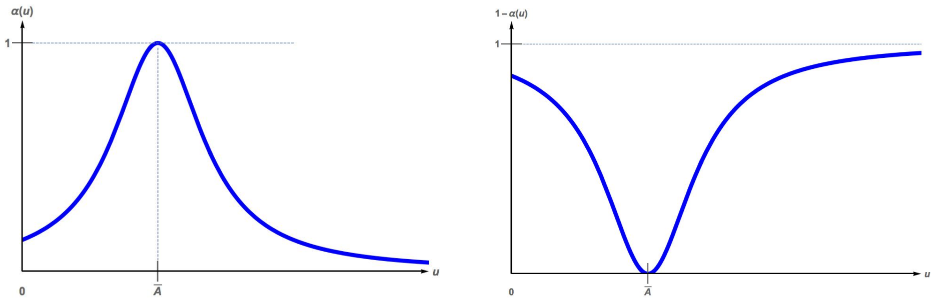


(a) Graph of $\alpha(u)$ (b) Graph of $1-\alpha(u)$

Figure 2. Example of $\alpha(u)$ which gives rise to U-shaped density dependent dispersal, $1-\alpha(u)$.

In particular, we will consider $\alpha(u)$ of the form $\alpha(u) = (m_1)/(m_1 + g(u))$ where $m_1 > 0$ and $g(u) = (u - \overline{A})^2/m_2$ where $m_2 > 0$ and $\overline{A} \in (0, K)$. Applying the change of variables $v = u/K$, $\lambda = r\ell^2/D$, $\gamma = S^*K^2/(\sqrt{rD}m_1m_2)$, and $A = \overline{A}/K$, (1.3) reduces to:

$$
\begin{cases} -\Delta v = \lambda v(1-v) & x \in \Omega_0, \\ \dfrac{\partial v}{\partial \eta} + \gamma\sqrt{\lambda}(v-A)^2 v = 0, & x \in \partial\Omega_0. \end{cases} \tag{1.4}
$$



(a) Graph of $\alpha(u)$ (b) Graph of $1-\alpha(u)$

Figure 3. Specific form of $\alpha(u)$ and its effects on dispersal $[1-\alpha(u)]$.

We now state our results. Let $\lambda_1(\varepsilon,\gamma)>0$ be the principal eigenvalue of the problem:

$$
\begin{cases} -\Delta w = \lambda w, & x\in\Omega_0,\\ \dfrac{\partial w}{\partial\eta} = -\gamma\sqrt{\lambda}\varepsilon w, & x\in\partial\Omega_0, \end{cases} \tag{1.5}
$$

for a given $\varepsilon>0$ (see Appendix and also [10]). We establish:

Theorem 1.1. *Let $\gamma>0$ and*

$$
\Gamma = \{u\in C^2(\Omega_0)\cap C^1(\overline{\Omega}_0) \mid u(x)\in[A,1] \text{ for all } x\in\overline{\Omega}_0\}.
$$

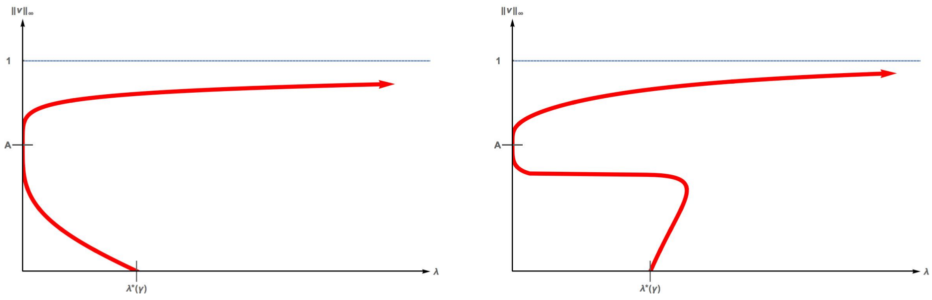
For each $\lambda>0$, (1.4) has a positive solution $v_{1,\lambda}\in\Gamma$ and this solution is unique. Further, for $\lambda\in(0,\lambda_1(A^2,\gamma))$, (1.4) has another positive solution $v_{2,\lambda}\in C^2(\Omega_0)\cap C^1(\overline{\Omega}_0)$ with $v_{2,\lambda}\notin\Gamma$.

Theorem 1.2. *Let $\gamma\gg 1$. There exists $\delta_\gamma>\lambda_1(A^2,\gamma)$ so that for $\lambda=\delta_\gamma$, (1.4) has at least two positive solutions $v_{i,\lambda}\in C^2(\Omega_0)\cap C^1(\overline{\Omega}_0)$ with $v_{i,\lambda}\notin\Gamma$ for $i=2,3$.*

Remark 1.3. Combining Theorem 1.2 with the unique solution $v_{1,\lambda}\in\Gamma$ for each $\lambda>0$, when $\gamma\ll 1$, for $\lambda=\delta_\gamma$, (1.4) has at least three positive solutions.

In the case when $n=1$ and $\Omega_0=(0,1)$, (1.4) was studied in detail in [11] via a quadrature method. In particular, they obtain exact bifurcation curves for positive solutions of the form described in Figure 4. Our results (Theorems 1.1 and 1.2) extend some of the conclusions obtained in the $n=1$ case to the higher dimensional case.

In Section 2, we will recall some important results in the theory of sub and supersolutions and some preliminary results. In Section 3, we will provide the proofs of Theorems 1.1 and 1.2. We also include results for certain eigenvalue problems that we use in our proofs in Appendix (these results also appear in [10]).



(a) Case $\gamma \approx 0$ (b) Case $\gamma \gg 1$

Figure 4. Bifurcation diagrams exhibited in [11] for the one-dimensional case of (1.4).

2. Preliminaries

Consider the boundary value problem:

$$\begin{cases} -\Delta u = f(u), & x \in \Omega_0, \\ \dfrac{\partial u}{\partial \eta} = g(u), & x \in \partial\Omega_0, \end{cases} \tag{2.1}$$

where f and g are in $C^1(\mathbb{R})$. By a solution u of (2.1), we will mean a $u \in C^2(\Omega_0) \cap C^1(\overline{\Omega}_0)$ which satisfies (2.1).

Definition 2.1. A function $u \in C^2(\Omega_0) \cap C^1(\overline{\Omega}_0)$ is called a subsolution (supersolution) of (2.1) if u satisfies

$$\begin{cases} -\Delta u \le (\ge)\, f(u), & x \in \Omega_0, \\ \dfrac{\partial u}{\partial \eta} \le (\ge)\, g(u), & x \in \partial\Omega_0. \end{cases}$$

A subsolution or supersolution which is not a solution is called strict.

We now state some well-known results in the theory of sub-supersolutions.

Lemma 2.2 (see [14, Theorem 1]). *Let $\underline{u}$ and $\overline{u}$ be sub- and supersolutions of* (2.1) *respectively such that $\underline{u} \le \overline{u}$, $x \in \Omega_0$. Then* (2.1) *has a maximal solution $\widehat{u}$ and a minimal solution $\check{u}$ such that $\underline{u} \le \check{u} \le \widehat{u} \le \overline{u}$.*

Lemma 2.3 (see [14, Theorem 2]). *Let $\underline{u}_1$ and $\overline{u}_2$ be sub- and super solutions of* (2.1) *respectively such that $\underline{u}_1 \le \overline{u}_2$, $x \in \Omega_0$. Let $\underline{u}_2$ and $\overline{u}_1$ be strict sub- and supersolutions of* (2.1) *respectively such that $\underline{u}_2, \overline{u}_1 \in [\underline{u}_1, \overline{u}_2]$ and $\underline{u}_2 \not\le \overline{u}_1$. Then* (2.1) *has at least three solutions u_i, $i = 1, 2, 3$ where $u_i \in [\underline{u}_i, \overline{u}_i]$, $i = 1, 2$ and $u_3 \in [\underline{u}_1, \overline{u}_2] \setminus \big([\underline{u}_1, \overline{u}_1] \cup [\underline{u}_2, \overline{u}_2]\big)$.*

Next, we consider positive solutions $z \in C^2(\Omega_0) \cap C^1(\overline{\Omega}_0)$ to the boundary value problem:

$$\begin{cases} -\Delta z = \lambda z(1-z), & x \in \Omega_0, \\ \dfrac{\partial z}{\partial \eta} = -\sqrt{\lambda}\gamma\varepsilon z, & x \in \partial\Omega_0, \end{cases} \tag{2.2}$$

for a given $\varepsilon > 0$.

We recall the following result from [10].

LEMMA 2.4. *There exists a unique positive solution z_λ to (2.2) when $\lambda > \lambda_1(\varepsilon,\gamma)$, and when $\lambda < \lambda_1(\varepsilon,\gamma)$, there is no positive solution to (2.2). Furthermore, $\|z_\lambda\|_\infty \to 0^+$ as $\lambda \to \lambda_1(\varepsilon,\gamma)^+$ and $\|z_\lambda\|_\infty \to 1^-$ as $\lambda \to \infty$.*

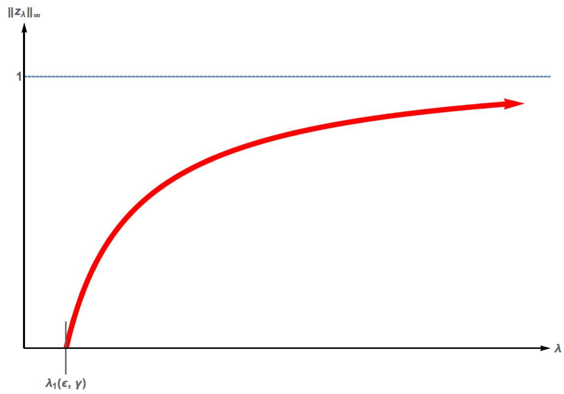


FIGURE 5. Bifurcation curve for (2.2).

3. Proofs of Theorems 1.1 and 1.2

3.1. Proof of Theorem 1.1. Clearly $\overline{v}_2 \equiv 1$ is a supersolution and $\underline{v}_2 \equiv A$ is a strict subsolution for (1.4) for every $\lambda > 0$. Hence, by Lemma 2.2, (1.4) has a maximal positive solution $v_{2,\lambda} \in \Gamma$. Suppose there exist another distinct solution $v_{1,\lambda} \in \Gamma$ to (1.4). Since $v_{2,\lambda}$ is the maximal solution, we have $v_{2,\lambda} \geq v_{1,\lambda}$. Let $v_1 = v_{1,\lambda}$ and $v_2 = v_{2,\lambda}$. Now, by Green's second identity, we have

$$\begin{aligned} \int_{\Omega_0} [(\Delta v_2)v_1 - (\Delta v_1)v_2]\,dx &= \int_{\partial\Omega_0} \left[\left(\frac{\partial v_2}{\partial \eta}\right)v_1 - \left(\frac{\partial v_1}{\partial \eta}\right)v_2\right] ds \\ &= \int_{\partial\Omega_0} \sqrt{\lambda}\gamma v_1 v_2 \left[(v_1 - A)^2 - (v_2 - A)^2\right] ds \\ &= \sqrt{\lambda}\gamma \int_{\partial\Omega_0} v_1 v_2 [v_1 - v_2][v_1 + v_2 - 2A]\,ds \leq 0, \end{aligned}$$

since $v_2 \geq v_1 \geq A$. However,

$$\int_{\Omega_0} [(\Delta v_2)v_1 - (\Delta v_1)v_2]\, dx = \lambda \int_{\Omega_0} v_1 v_2 (v_2 - v_1)\, dx > 0,$$

since we are assuming v_1 and v_2 are distinct. This is a contradiction, and hence $v_1 \equiv v_2$ and hence the solution $v_{2,\lambda} \in \Gamma$ is unique.

Next, consider the case when $\lambda < \lambda_1(A^2, \gamma)$. Let μ_1 be the principal eigenvalue and $\widetilde{\phi} > 0$ be the corresponding eigenfunction such that $\|\widetilde{\phi}\|_\infty = 1$ for the eigenvalue problem:

$$\begin{cases} -\Delta\widetilde{\phi} = \lambda\widetilde{\phi} + \mu\widetilde{\phi}, & x \in \Omega_0, \\ \dfrac{\partial\widetilde{\phi}}{\partial\eta} = -\sqrt{\lambda}\gamma A^2\widetilde{\phi} + \mu\widetilde{\phi}, & x \in \partial\Omega_0. \end{cases} \tag{3.1}$$

Note that $\mu_1 > 0$ when $\lambda < \lambda_1(A^2, \gamma)$, where λ_1^D is the principle eigenvalue of the $-\Delta$ operator with Dirichlet boundary conditions (see Appendix). Let $\overline{v}_1 = \alpha\widetilde{\phi}$ where $\alpha \in (0, A)$ will be chosen later.

Now, since $\mu_1 > 0$,

$$-\Delta\overline{v}_1 = \alpha\Big\{\lambda\widetilde{\phi} + \mu_1\widetilde{\phi}\Big\} \geq \lambda\alpha\widetilde{\phi}(1 - \alpha\widetilde{\phi}) = \lambda\overline{v}_1(1 - \overline{v}_1),$$

for $x \in \Omega_0$. Also, for $x \in \partial\Omega_0$, we have

$$\frac{\partial\overline{v}_1}{\partial\eta} = \alpha\frac{\partial\widetilde{\phi}}{\partial\eta} = \alpha\Big[-\sqrt{\lambda}\gamma A^2\widetilde{\phi} + \mu\widetilde{\phi}\Big] \geq -\sqrt{\lambda}\gamma\Big[\alpha\widetilde{\phi} - A\Big]^2\alpha\widetilde{\phi} = -\sqrt{\lambda}\gamma[\overline{v}_1 - A]^2\overline{v}_1,$$

provided

$$\alpha\widetilde{\phi}\Big\{\mu_1 + \sqrt{\lambda}\gamma\alpha^2\widetilde{\phi}^2 - 2\sqrt{\lambda}\gamma\alpha\widetilde{\phi}A\Big\} > 0 \quad \text{for all } x \in \partial\Omega_0. \tag{3.2}$$

Since $\mu_1 > 0$, (3.2) will clearly hold for α sufficiently small. Hence, $\overline{v}_1 = \alpha\widetilde{\phi}$ with $\alpha \in (0, A)$ and sufficiently small is a strict supersolution to (1.4). Note that $\underline{v}_1 \equiv 0$ is a solution, and hence a subsolution. Thus, by Lemma 2.3, for $\lambda < \lambda_1(A^2, \gamma)$, (1.4) has at least three distinct solutions, $v_{1,\lambda} \in [A, 1]$, $v_{3,\lambda} \in \Big[0, \alpha\widetilde{\phi}\Big]$, and $v_{2,\lambda} \in [0, 1] \setminus \Big(\Big[0, \alpha\widetilde{\phi}\Big] \cup [A, 1]\Big)$. Clearly $v_{1,\lambda}$ and $v_{2,\lambda}$ are positive solutions, and Theorem 1.1 is proved. □

3.2. Proof of Theorem 1.2. Let $\lambda > \lambda_1(A^2, \gamma)$. Clearly $\overline{v}_2 \equiv 1$ is a supersolution and $\underline{v}_2 \equiv A$ is a strict subsolution. Let $\underline{v}_1 = \beta\widetilde{\phi}$ where ϕ is as before (see the proof of Theorem 1.1) and $\beta > 0$ will be chosen later. Now

$$-\Delta\underline{v}_1 = \beta\Big[\lambda\widetilde{\phi} + \mu_1\widetilde{\phi}\Big] \leq \lambda\Big[\beta\widetilde{\phi}(1 - \beta\widetilde{\phi})\Big] = \lambda\underline{v}_1(1 - \underline{v}_!), \quad x \in \Omega_0,$$

provided,

$$\beta\widetilde{\phi}\Big[\mu_1 + \lambda\beta\widetilde{\phi}\Big] \leq 0, \quad x \in \Omega_0, \tag{3.3}$$

holds. Also,

$$\frac{\partial \underline{v}_1}{\partial \eta} = \beta \frac{\partial \widetilde{\phi}}{\partial \eta} = \beta\Big[-\sqrt{\lambda}\gamma A^2\widetilde{\phi}+\mu_1\widetilde{\phi}\Big] \leq -\sqrt{\lambda}\gamma\Big(\beta\widetilde{\phi}-A\Big)^2\beta\widetilde{\phi} = -\sqrt{\lambda}\gamma(\underline{v}_1-A)^2\underline{v}_1,$$

for all $x \in \partial\Omega_0$ provided,

$$\beta\widetilde{\phi}\Big\{\mu_1 + \sqrt{\lambda}\beta^2\widetilde{\phi}^2 - 2\sqrt{\lambda}\gamma\beta\widetilde{\phi}A\Big\} \leq 0, \quad x \in \partial\Omega_0, \tag{3.4}$$

holds. But $\mu_1 < 0$ since $\lambda > \lambda_1(A^2,\gamma)$ (see Appendix). Hence both (3.3) and (3.4) will hold for $\beta > 0$ sufficiently small, and $\underline{v}_1 = \beta\widetilde{\phi}$ will be a small positive subsolution to (1.4).

Next, consider the boundary value problem (2.2) with $\varepsilon = A^2/4$, namely,

$$\begin{cases} -\Delta z = \lambda z(1-z), & x \in \Omega_0, \\ \dfrac{\partial z}{\partial \eta} = -\sqrt{\lambda}\gamma\, \dfrac{A^2}{4}\, z, & x \in \partial\Omega_0. \end{cases} \tag{3.5}$$

By Lemma 2.4, there exists $\delta_\gamma > \lambda_1(A^2/4,\gamma)$ such that the unique positive solution z_λ to (3.5) satisfies $\|z_\lambda\|_\infty = A/3$ when $\lambda = \delta_\gamma$.

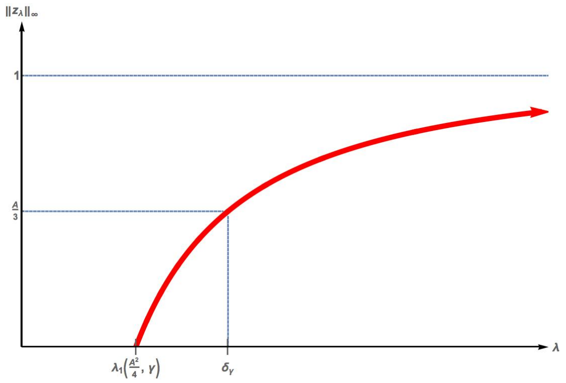


Figure 6. Bifurcation curve for (2.2) showing $A/3$ and δ_γ.

Note that $\lambda_1(A^2/4,\gamma) < \lambda_1(A^2,\gamma)$ and, as $\gamma \to \infty$, $\lambda_1(A^2,\gamma) \to \lambda_1^D$ (see Appendix). However, $\lim_{\gamma\to\infty} \delta_\gamma > \lambda_1^D$ (see Figure 6). For $\lambda = \delta_\gamma$, let $\overline{v}_1 \equiv z_\lambda$, where z_λ is the unique solution of (3.5). Note that $\|\overline{v}_1\|_\infty \leq A/3$. Now, $-\Delta\overline{v}_1 = \lambda\overline{v}_1(1-\overline{v}_1)$, $x \in \Omega_0$ and

$$\frac{\partial \overline{v}_1}{\partial \eta} = -\sqrt{\lambda}\gamma\frac{A^2}{4}\,\overline{v}_1 > -\sqrt{\lambda}\gamma(\overline{v}_1 - A)^2\overline{v}_1, \quad x \in \partial\Omega_0,$$

provided,

$$\frac{A^2}{4} < (A-\overline{v}_1)^2, \quad x \in \partial\Omega_0, \tag{3.6}$$

holds. But clearly (3.6) holds since $\|\overline{v}_1\|_\infty = \|z_\lambda\|_\infty = A/3$ and hence $\overline{v}_1 = z_\lambda$ is a strict supersolution to (1.4). Now, combining with $\underline{v}_1$ (where β is chosen sufficently small so that $\underline{v}_1 \leq \overline{v}_1$), $\underline{v}_2 \equiv A$, and $\overline{v}_2 \equiv 1$, by Lemma 2.3, (1.4) has at least three positive solutions, $v_{1,\lambda} \in \Gamma$, $v_{2,\lambda} \in [\underline{v}_1, \overline{v}_1]$ and $v_{3,\lambda} \in [\underline{v}_1, \overline{v}_2] \setminus \big([\underline{v}_1, \overline{v}_1] \cup [\underline{v}_2, \overline{v}_2]\big)$ for $\lambda = \delta_\gamma$ and Theorem 1.2 is proven. $\square$

Appendix A. Results for eigenvalue problems (1.5) and (3.1)

First we consider the eigenvalue problem

$$\begin{cases} -\Delta z = \lambda z & \text{in } \Omega_0, \\ \dfrac{\partial z}{\partial \eta} = kz & \text{in } \partial\Omega_0, \end{cases} \tag{A.1}$$

for a given $k \in \mathbb{R}$. We recall the following result from [21].

LEMMA A.1. *For each $k \in \mathbb{R}$,* (A.1) *has a principal eigenvalue $\overline{\lambda}_1(k)$ and the eigencurve $\overline{\lambda}_1(k) \subset \mathbb{R}^2$ is Lipschitz continuous, strictly decreasing, and concave. Furthermore, $\overline{\lambda}_1(0) = 0$ and the eigenfunciton associated with any point on $\overline{\lambda}_1(k)$ is strictly positive in Ω_0.*

We now state and prove a result regarding the limiting value of $\overline{\lambda}_1(k)$ as $k \to -\infty$. Figure 7 illustrates Lemma A.2.

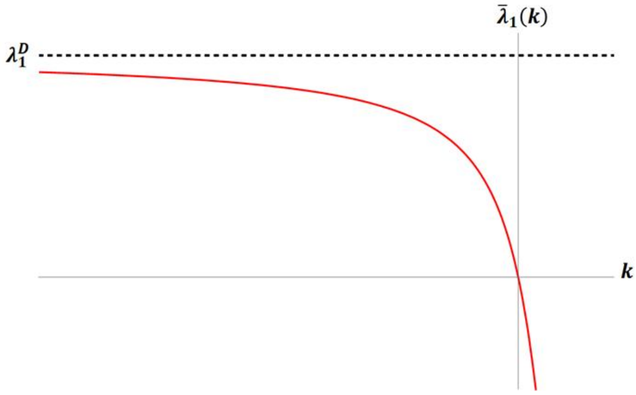


FIGURE 7. Plot of k vs. $\overline{\lambda}_1(k)$. The curve illustrates the fact that $\overline{\lambda}_1(k) \to \lambda_1^D$ as $k \to -\infty$.

LEMMA A.2. *$\overline{\lambda}_1(k) \to \lambda_1^D$ as $k \to -\infty$ where λ_1^D is the principal eigenvalue of the $-\Delta$ operator with Dirichlet boundary conditions.*

PROOF. We note that for any $k \in \mathbb{R}$, we may characterize $\overline{\lambda}_1(k)$ by

$$\overline{\lambda}_1(k) = \min_{u \in H^1(\Omega_0)\setminus\{0\}} \frac{\displaystyle\int_{\Omega_0} |\nabla u|^2\,dx - k \int_{\partial\Omega_0} u^2\,ds}{\displaystyle\int_{\Omega_0} u^2\,dx}. \tag{A.2}$$

Let ϕ_1^D the corresponding eigenfunction to the eigenvalue λ_1^D be chosen such that $\int_{\Omega_0}\phi_1^D = 1$. Testing (A.2) with $u = 1$ and $u = \phi_1^D$ shows that

$$\overline{\lambda}_1(k) \le -k\,\frac{|\partial\Omega_0|}{|\Omega_0|} \quad \text{and} \quad \overline{\lambda}_1(k) \le \lambda_1^D,$$

respectively. Taking a sequence $k_n \to -\infty$ such that the corresponding eigenfunctions u_n, without loss of generality, satisfy $\int_{\Omega_0} u_n^2\,dx = 1$, we observe that

$$\overline{\lambda}_1(k_n) = \int_{\Omega_0}|\nabla u_n|^2\,dx - k_n\int_{\partial\Omega_0} u_n^2\,ds.$$

Since $k_n < 0$, we have $0 = \overline{\lambda}_1(0) < \overline{\lambda}_1(k_n) < \lambda_1^D$. By Lemma A.1,

$$\lim_{k\to-\infty}\overline{\lambda}_1(k) = \overline{\lambda}_1(-\infty) \le \lambda_1^D \quad \text{for some } \overline{\lambda}_1(-\infty)\in\mathbb{R}.$$

Without loss of generality, we may assume that

$$-k_n\int_{\partial\Omega_0} u_n^2\,ds \to \alpha \ge 0,$$

and thus

$$\int_{\partial\Omega_0} u_n^2 \to 0.$$

Since $\{u_n\}$ is bounded in $H^1(\Omega_0)$, we may select a subsequence so that $u_n \rightharpoonup u$ in $H^1(\Omega_0)$, $u_n \to u$ in $L^2(\Omega_0)$ and in $L^2(\partial\Omega_0)$. It follows that

$$\int_{\Omega_0} u^2\,dx = 1 \quad \text{and} \quad \int_{\partial\Omega_0} u^2\,ds = 0,$$

and hence $u \in H_0^1(\Omega_0)$.

By the weak lower semicontinuity of $\int_{\Omega_0}|\nabla u|^2\,dx$, we get that,

$$\int_{\Omega_0}|\nabla u|^2\,dx + \alpha \le \liminf_{n\to\infty}\left(\int_{\Omega_0}|\nabla u_n|\,dx - k_n\int_{\partial\Omega_0} u_n^2\,ds\right) = \overline{\lambda}_1(-\infty) \le \lambda_1^D.$$

But by Poincaré's Inequality, we have

$$\lambda_1^D \le \int_{\Omega_0}|\nabla u|^2\,dx,$$

and hence we must have $\alpha = 0$ and $\overline{\lambda}_1(-\infty) = \lambda_1^D$. Furthermore,

$$\int_{\Omega_0}|\nabla u|^2\,dx = \lambda_1^D,$$

and thus, without loss of generality, $u = \phi_1^D$. Moreover,

$$\lim_{n\to\infty}\int_{\Omega_0}|\nabla u_n|^2\,dx = \int_{\Omega_0}|\nabla u|^2\,dx,$$

and hence $u_n \to u = \phi_1^D$ in $H^1(\Omega)$. $\square$

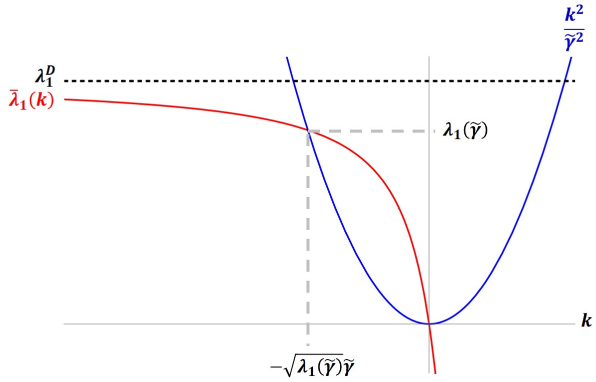


FIGURE 8. Illustration of the existence of $\lambda_1(\widetilde{\gamma})$.

Next, we consider the eigenvalue problem of the form (1.5), namely,

$$\begin{cases} -\Delta w = \lambda w & \text{in } \Omega_0, \\ \dfrac{\partial w}{\partial \eta} + \widetilde{\gamma}\sqrt{\lambda} w = 0, & \text{in } \partial\Omega_0. \end{cases} \tag{A.3}$$

for a given $\widetilde{\gamma} > 0$. It is easy to see that the principal eigenvalue $\lambda_1(\widetilde{\gamma})$ of (A.3) is nothing but the y-coordinate of the intersection of the curves $\overline{\lambda}_1(k)$ and $k^2/\widetilde{\gamma}^2$ (see Figure 8).

It is also straightforward to show that $\lambda_1(\widetilde{\gamma})$ is an increasing function of $\widetilde{\gamma}$, $\lambda_1(\widetilde{\gamma}) \to \lambda_1^D$ as $\widetilde{\gamma} \to \infty$, and $\lambda_1(\widetilde{\gamma}) \to 0$ as $\widetilde{\gamma} \to 0$ (see Figure 9).

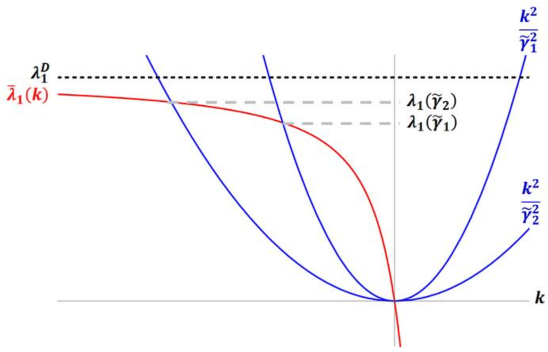


FIGURE 9. The plot illustrates that $\lambda_1(\widetilde{\gamma})$ is an increasing function. Note that $\widetilde{\gamma}_2 > \widetilde{\gamma}_1$.

Now, we consider the eigenvalue problem:

$$\begin{cases} -\Delta \psi - \lambda \psi = \sigma \psi & \text{in } \Omega_0, \\ \dfrac{\partial \psi}{\partial \eta} + \widetilde{\gamma}\sqrt{\lambda}\psi = 0 & \text{in } \partial\Omega_0 \end{cases} \tag{A.4}$$

for a given $\lambda > 0$ and $\widetilde{\gamma} > 0$. Once again, it is easy to see that the principal eigenvalue $\sigma_1(\lambda, \widetilde{\gamma})$ of (A.4) exists and must satisfy

$$\lambda + \sigma_1(\lambda, \widetilde{\gamma}) = \overline{\lambda}_1\big(-\sqrt{\lambda}\widetilde{\gamma}\big) \tag{A.5}$$

(see Figure 10). Furthermore, the following result holds:

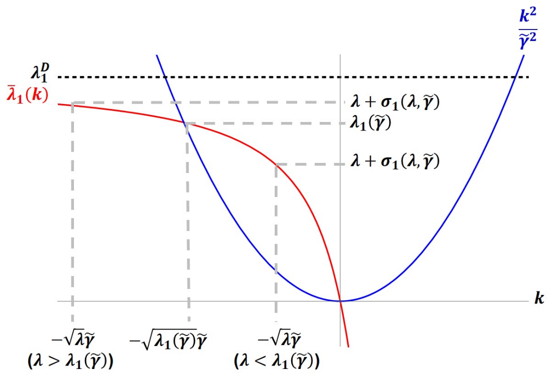


Figure 10. The plot illustrates the existence of $\sigma_1(\lambda, \widetilde{\gamma})$.

Lemma A.3. *If $\lambda < \lambda_1(\widetilde{\gamma})$ then $\sigma_1(\lambda, \widetilde{\gamma}) > 0$ and, if $\lambda = \lambda_1(\widetilde{\gamma})$, then $\sigma_1(\lambda, \widetilde{\gamma}) = 0$. Also, if $\lambda > \lambda_1(\widetilde{\gamma})$, then $\sigma_1(\lambda, \widetilde{\gamma}) < 0$.*

Proof. Note that,

$$\text{if} \quad \lambda < (>)\ \lambda_1(\widetilde{\gamma}), \quad \text{then} \quad -\sqrt{\lambda}\widetilde{\gamma} > (<) -\sqrt{\lambda_1(\widetilde{\gamma})}\widetilde{\gamma}.$$

Hence,

$$\lambda = \frac{\big(-\sqrt{\lambda}\widetilde{\gamma}\big)^2}{\widetilde{\gamma}^2} < (>)\, \overline{\lambda}_1\big(-\sqrt{\lambda}\widetilde{\gamma}\big) \tag{A.6}$$

(see Figure 9) and, by (A.5), we have $\sigma_1(\lambda, \widetilde{\gamma}) > (<)\, 0$ Note that $\lambda = \lambda_1(\widetilde{\gamma})$ implies that we have equality in (A.6), and thus $\sigma_1(\lambda, \widetilde{\gamma}) = 0$. □

Next, for fixed λ and $\widetilde{\gamma}$ we consider the eigenvalue problem,

$$\begin{cases} -\Delta\phi - \lambda\phi = \delta\pi & \text{in } \Omega_0, \\ \dfrac{\partial\phi}{\partial\eta} + \widetilde{\gamma}\sqrt{\lambda}\phi = \delta\phi & \text{in } \partial\Omega_0. \end{cases} \tag{A.7}$$

Notice that by letting $\widetilde{\delta} = \delta - \sqrt{\lambda}\widetilde{\gamma}$ implies that (A.7) becomes

$$\begin{cases} -\Delta\phi = \big(\lambda + \sqrt{\lambda}\widetilde{\gamma} + \widetilde{\delta}\big)\phi & \text{in } \Omega_0, \\ \dfrac{\partial\phi}{\partial\eta} = \widetilde{\delta}\phi, & \text{in } \partial\Omega_0 \end{cases} \tag{A.8}$$

and the principle eigenvalue $\widetilde{\delta}_1(\lambda, \widetilde{\gamma})$ is nothing but the x-coordinate of the intersection of the curves $\overline{\lambda}_1(\widetilde{\delta})$ and $\widetilde{\delta} + \big(\lambda + \sqrt{\lambda}\widetilde{\gamma}\big)$ (see Figure 11). Hence the principal eigenvalue $\delta_1(\lambda, \widetilde{\gamma})$ of (A.7) exists and is given by:

$$\delta_1(\lambda, \widetilde{\gamma}) = \widetilde{\delta}_1(\lambda, \widetilde{\gamma}) + \sqrt{\lambda}\widetilde{\gamma}. \tag{A.9}$$

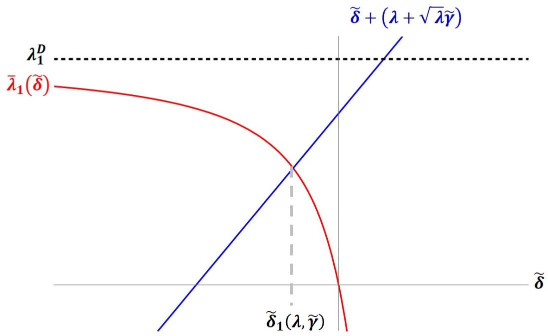


FIGURE 11. The plot illustrates the existence of $\widetilde{\delta}_1(\lambda, \widetilde{\gamma})$.

We next establish a relationship between the signs of $\delta_1(\lambda, \widetilde{\gamma})$ and $\sigma_1(\lambda, \widetilde{\gamma})$ in the following result.

LEMMA A.4. $\operatorname{sign}(\delta_1(\lambda, \widetilde{\gamma})) = \operatorname{sign}(\sigma_1(\lambda, \widetilde{\gamma}))$.

PROOF. Let ϕ_1 and ϕ_2 be corresponding positive eigenfunctions in (A.7) and (A.8). Then by Green's Second Identity, we have that

$$\int_{\Omega_0} \big[(\Delta\phi_1)\phi_2 - (\Delta\phi_2)\phi_1\big]\, dx = \int_{\partial\Omega_0} \left[\frac{\partial \phi_1}{\partial \eta}\phi_2 - \frac{\partial \phi_2}{\partial \eta}\phi_1\right] ds \tag{A.10}$$

which implies

$$\big[\delta_1(\lambda, \widetilde{\gamma}) - \sigma_1(\delta, \widetilde{\gamma})\big] \int_{\Omega_0} \phi_1\phi_2\, dx = -\delta_1(\lambda, \widetilde{\gamma}) \int_{\partial\Omega_0} \phi_2\phi_1\, ds. \tag{A.11}$$

Now, it immediately follows that $\sigma_1(\lambda, \widetilde{\gamma}) = 0$ if and only if $\delta_1(\lambda, \widetilde{\gamma}) = 0$ and if $\delta_1(\lambda, \widetilde{\gamma}) \neq 0$ then we have

$$\frac{\sigma_1(\lambda, \widetilde{\gamma}) - \delta_1(\lambda, \widetilde{\gamma})}{\delta_1(\lambda, \widetilde{\gamma})} > 0. \tag{A.12}$$

Thus, if $\delta_1(\lambda, \widetilde{\gamma}) > 0$ then we must have that $\sigma_1(\lambda, \widetilde{\gamma}) > \delta_1(\lambda, \widetilde{\gamma}) > 0$ and if $\delta_1(\lambda, \widetilde{\gamma}) < 0$ then $\sigma_1(\lambda, \widetilde{\gamma}) < \delta_1(\lambda, \widetilde{\gamma}) < 0$. Hence the result. □

Finally, combining Lemmas A.3 and A.4, the following lemma immediately follows:

LEMMA A.5. *If* $\lambda < \lambda_1(\widetilde{\gamma})$ *then* $\delta_1(\lambda, \widetilde{\gamma}) > 0$. *Also, if* $\lambda > \lambda_1(\widetilde{\gamma})$ *then* $\delta_1(\lambda, \widetilde{\gamma}) < 0$.

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