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A VERSION OF KRASNOSEL'SKII'S COMPRESSION–EXPANSION FIXED POINT THEOREM IN CONES FOR DISCONTINUOUS OPERATORS WITH APPLICATIONS

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Abstract. We introduce a new fixed point theorem of Krasnosel'skiĭ type for discontinuous operators. As an application we use it to study the existence of positive solutions of a second-order differential problem with separated boundary conditions and discontinuous nonlinearities.

1. Introduction

A classical problem [11], [12], [14] is that of the existence of positive solutions for the differential equation

$$u''(t) + g(t)f(u(t)) = 0, \quad 0 < t < 1, \tag{1.1}$$

along with suitable boundary conditions (BCs). This problem arises in the study of radial solutions in $\mathbb{R}^n$, $n \geq 2$, for the partial differential equation (PDE)

$$\Delta v + h(\|x\|)f(v) = 0, \quad x \in \mathbb{R}^n, \ \|x\| \in [R_1, R_2],$$

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with the appropriate boundary conditions, see [5], [11], [12].

Recently, in the paper [10], the authors study the existence of nontrivial radial solutions for a system of PDEs of the previous type. First, they turn the former problem into a system of ordinary differential equations similar to (1.1).

The main novelty in this paper is that we will let f be discontinuous. A usual idea in the literature when a discontinuous differential problem as

$$x'(t) = f(t, x(t)), \quad \text{a.a. } t \in I,$$

under some initial or boundary conditions is studied consists of a regularization of the nonlinearity f which is replaced by a multivalued mapping F (see [3], [4], [7]). Then the aim is to give assumptions about f which ensure that the solutions of the differential inclusion

$$x'(t) \in F(t, x(t)), \quad \text{a.a. } t \in I,$$

are solutions for the first problem. A well-known way of building F is the Krasovsky regularization (see [7]) where the multivalued mapping is given by

$$F(t, x) := \bigcap_{\varepsilon > 0} \overline{\text{co}}\, f(t, \overline{B}_\varepsilon(x)), \quad (t, x) \in I \times \mathbb{R},$$

and $\overline{B}_\varepsilon(x)$ and $\overline{\text{co}}$ are defined as below. Our technique here is similar, but we regularize the fixed point operator associated to the BVP instead of the nonlinear part of the differential equation. Then we look for fixed points of the multivalued operator which we show to be fixed points of the integral operator associated to the BVP when admissible discontinuity curves are considered.

The classical compression–expansion fixed point theorem of Krasnosel'skiĭ (see [2] or [15]) is a well-known tool of nonlinear analysis and it has proven very useful to deduce existence of solutions for nonlinear problems. Here we prove a generalization of that theorem which allows discontinuous operators. The idea is similar to that employed in [6], [13], where Schauder's fixed point theorem was extended. Then we return to problem (1.1) along with Sturm–Liouville BCs and we use our extension of Krasnosel'skiĭ's theorem to get a result about existence of positive solutions when f is not necessarily continuous.

2. Krasnosel'skiĭ's fixed point theorem for discontinuous operators

In the sequel we need the following definitions. A closed and convex subset K of a Banach space $(X, \|\cdot\|)$ is a cone if it satisfies the following conditions:

(i) if $x \in K$, then $\lambda x \in K$ for all $\lambda \geq 0$;
(ii) if $x \in K$ and $-x \in K$, then $x = 0$.

A cone K defines the partial order in X given by $x \preceq y$ if and only if $y - x \in K$. Let U be a relatively open subset of K and let $T\colon \overline{U} \subset K \to K$ be an operator, not necessarily continuous.

DEFINITION 2.1. The closed-convex envelope of an operator $T\colon \overline{U} \to K$ is the multivalued mapping $\mathbb{T}\colon \overline{U} \to 2^K$ given by

$$\mathbb{T}x = \bigcap_{\varepsilon>0} \overline{\mathrm{co}}\, T(\overline{B}_\varepsilon(x) \cap \overline{U}) \quad \text{for every } x \in \overline{U}, \tag{2.1}$$

where $\overline{B}_\varepsilon(x)$ denotes the closed ball centered at x and radius ε, and $\overline{\mathrm{co}}$ means closed convex hull. In other words, we say that $y \in \mathbb{T}x$ if for every $\varepsilon > 0$ and every $\rho > 0$ there exist $m \in \mathbb{N}$ and a finite family of vectors $x_i \in \overline{B}_\varepsilon(x) \cap \overline{U}$ and coefficients $\lambda_i \in [0,1]$, $i = 1,\ldots,m$, such that $\sum \lambda_i = 1$ and

$$\bigg\| y - \sum_{i=1}^{m} \lambda_i\, T x_i \bigg\| < \rho.$$

The previous definition was formulated for open subsets of a cone, but it works for arbitrary nonempty subsets of a Banach space (see [13]). This object is similar to Krasovsky regularization and it is studied for instance in [4].

Closed-convex envelopes (cc-envelopes, for short) need not be upper semicontinuous (usc, for short), see [4, Example 1.2], unless some additional assumptions are imposed on T.

PROPOSITION 2.2. *Let $\mathbb{T}$ be the cc-envelope of an operator $T\colon \overline{U} \to K$. The following properties are satisfied*:

(a) *If T maps bounded sets into relatively compact sets, then $\mathbb{T}$ assumes compact values and it is usc.*
(b) *If $T\overline{U}$ is relatively compact, then $\mathbb{T}\overline{U}$ is relatively compact.*

PROOF. Let $x \in \overline{U}$ be fixed and let us prove that $\mathbb{T}x$ is compact. We know that $\mathbb{T}x$ is closed, so it suffices to show that it is contained in a compact set. To do so, we note that

$$\mathbb{T}x = \bigcap_{\varepsilon>0} \overline{\mathrm{co}}\, T\big(\overline{B}_\varepsilon(x) \cap \overline{U}\big) \subset \overline{\mathrm{co}}\, T\big(\overline{B}_1(x) \cap \overline{U}\big) \subset \overline{\mathrm{co}}\, \overline{T\big(\overline{B}_1(x) \cap \overline{U}\big)},$$

and $\overline{\mathrm{co}}\, \overline{T\big(\overline{B}_1(x) \cap \overline{U}\big)}$ is compact because it is the closed convex hull of a compact subset of a Banach space; see [1, Theorem 5.35]. Hence $\mathbb{T}x$ is compact for every $x \in \overline{U}$, and this property allows us to check that $\mathbb{T}$ is usc by means of sequences, see [1, Theorem 17.20]: let $x_n \to x$ in $\overline{U}$ and let $y_n \in \mathbb{T}x_n$ for all $n \in \mathbb{N}$ be such that $y_n \to y$; we have to prove that $y \in \mathbb{T}x$. Let $\varepsilon > 0$ be fixed and take $N \in \mathbb{N}$ such that $\overline{B}_\varepsilon(x_n) \subset \overline{B}_{2\varepsilon}(x)$ for all $n \geq N$. Then we have $y_n \in \overline{\mathrm{co}}\, T(\overline{B}_\varepsilon(x_n) \cap \overline{U}) \subset \overline{\mathrm{co}}\, T(\overline{B}_{2\varepsilon}(x) \cap \overline{U})$ for all $n \geq N$, which implies that $y \in \overline{\mathrm{co}}\, T(\overline{B}_{2\varepsilon}(x) \cap \overline{U})$. Since $\varepsilon > 0$ was arbitrary, we conclude that $y \in \mathbb{T}x$.

Arguments are similar for the second part of the proposition. For every $x \in \overline{U}$ and $\varepsilon > 0$ we have $\overline{\mathrm{co}}\, T(\overline{B}_\varepsilon(x) \cap \overline{U}) \subset \overline{\mathrm{co}}\, \overline{T\overline{U}}$, and therefore $\mathbb{T}x \subset \overline{\mathrm{co}}\, \overline{T\overline{U}}$ for all $x \in \overline{U}$. Hence, $\overline{\mathbb{T}\overline{U}}$ is compact because it is a closed subset of the compact set $\overline{\mathrm{co}}\, \overline{T\overline{U}}$. $\square$

Now we recall the fixed point theorem mentioned above (see [15, Theorem 13.D]).

Theorem 2.3 (Krasnosel'skiĭ). *Let $r_i \le R$, $i = 1, 2$, be positive numbers with $r_1 \neq r_2$ and let $T\colon \overline{B}(0,R) \cap K \to K$ be a compact mapping. Suppose that*

(a) *$Tx \not\geq x$ for all $x \in K$ with $\|x\| = r_1$,*
(b) *$Tx \not\leq x$ for all $x \in K$ with $\|x\| = r_2$.*

Then T has a fixed point $x \in K$ such that

$$\min\{r_1, r_2\} < \|x\| < \max\{r_1, r_2\}.$$

In this section we introduce a generalization of the previous theorem which is based on the following idea: given a possibly discontinuous operator T, we build its cc-envelope $\mathbb{T}$ and we prove that it has fixed points by means of the version of Krasnosel'skiĭ fixed point theorem for multivalued mappings given by Fitzpatrick–Petryshyn [8]. Then we impose suitable conditions on T which, roughly speaking, guarantee that fixed points of $\mathbb{T}$ are fixed point of T too. For completeness, we recall [8, Theorem 3.2].

Theorem 2.4. *Let X be a Fréchet space with a cone $K \subset X$. Let d be a metric on X and let $r_1, r_2 \in (0, \infty)$, $r = \max\{r_1, r_2\}$ and $F\colon \overline{B}(0,r) \cap K \to 2^K$ usc and condensing. Suppose there exists a continuous seminorm p such that $(I - F)(\overline{B}(0, r_1) \cap K)$ is p-bounded. Moreover, suppose that F satisfies:*

(a) *there is some $w \in K$ with $p(w) \neq 0$ and such that $x \notin F(x) + tw$ for any $t > 0$ and $x \in \partial_K B(0, r_1)$;*
(b) *$\lambda x \notin F(x)$ for any $\lambda > 1$ and $x \in \partial_K B(0, r_2)$.*

Then F has a fixed point x_0 with $\min\{r_1, r_2\} \le d(x_0, 0) \le \max\{r_1, r_2\}$.

We are already in a position to introduce and prove two extensions of Krasnosel'skiĭ fixed point theorem for discontinuous single-valued operators which we will use to derive new existence results on positive solutions for discontinuous boundary value problems. While our next two results are not particularly surprising corollaries of Theorem 2.4, they have a much more general applicability range than its classical versions (say, Theorem 2.3), as we shall show in our final example.

Proposition 2.5. *Let $r_i \le R$, $i = 1, 2$, with $r_1 \neq r_2$, be positive numbers and $T\colon \overline{B}(0,R) \cap K \to K$ a mapping such that $T(\overline{B}(0,R) \cap K)$ is relatively compact and*

$$\{x\} \cap \mathbb{T}x \subset \{Tx\} \quad \text{for all } x \in \overline{B}(0,R) \cap K, \tag{2.2}$$

where $\mathbb{T}$ is the cc-envelope of T as defined in (2.1). Suppose that

(a) *$\lambda x \notin \mathbb{T}x$ for all $x \in K$ with $\|x\| = r_1$ and all $\lambda \ge 1$,*

(b) *there exists* $w \in K$ *with* $\|w\| \neq 0$ *such that* $x \notin \mathbb{T}x + \lambda w$ *for all* $\lambda \geq 0$ *and all* $x \in K$ *with* $\|x\| = r_2$.

Then T *has a fixed point* $x \in K$ *such that*

$$\min\{r_1, r_2\} < \|x\| < \max\{r_1, r_2\}. \tag{2.3}$$

Proof. Notice that the multivalued mapping $\mathbb{T}$ fulfills all the conditions in Theorem 2.4, so there exists a point $x \in \mathbb{T}x$ with (2.3). Moreover we deduce from (2.2) that $x = Tx$ because $\{x\} \cap \mathbb{T}x = \{x\}$. □

A second result leans on compression–expansion type conditions.

Proposition 2.6. *Let* $r_i \leq R$, $i = 1, 2$, *with* $r_1 \neq r_2$, *be positive numbers and* $T \colon \overline{B}(0, R) \cap K \to K$ *a mapping such that* $T(\overline{B}(0, R) \cap K)$ *is relatively compact and fulfils condition* (2.2). *Let* $\mathbb{T}$ *be the cc-envelope of* T *and suppose that*

(a) $y \not\succeq x$ *for all* $y \in \mathbb{T}x$ *and all* $x \in K$ *with* $\|x\| = r_1$,
(b) $y \not\preceq x$ *for all* $y \in \mathbb{T}x$ *and all* $x \in K$ *with* $\|x\| = r_2$.

Then T *has a fixed point* $x \in K$ *satisfying* (2.3).

Proof. It suffices to show that all the conditions in Proposition 2.5 are satisfied. First, we show that condition (a) implies condition (a) in Proposition 2.5. Let $x \in K$ be such that $\|x\| = r_1$ and let $\lambda \geq 1$; we have to prove that $\lambda x \notin \mathbb{T}x$. Reasoning by contradiction, we assume that $y = \lambda x \in \mathbb{T}x$. Then we have

$$y - x = (\lambda - 1)x \in K \quad (\text{because } \lambda - 1 \geq 0),$$

and this implies that $y \succeq x$, a contradiction with condition (a). Now for condition (b) in Proposition 2.5. Once again we use a contradiction argument: we assume that for every $w \in K$ such that $\|w\| \neq 0$ we can find $x \in \partial_K B(0, r_2)$ and $\lambda \geq 0$ such that $x \in \mathbb{T}x + \lambda w$, i.e., there exists $y \in \mathbb{T}x$ such that $x = y + \lambda w$. Hence, $x - y = \lambda w \in K$, a contradiction with (b). □

Remark 2.7. Condition (2.2) is weaker than continuity, since if T is continuous then $\mathbb{T}x = \{Tx\}$, so (2.2) is trivially satisfied. In addition, it is not difficult to find discontinuous mappings that verify this condition as we show in our next section. Notice that condition (2.2) can be written simply as $\operatorname{Fix} \mathbb{T} \subseteq \operatorname{Fix} T$, where $\operatorname{Fix} S$ denotes the set of all fixed points of a given operator S.

Neither Proposition 2.5 nor 2.6 remain true if we replace $\mathbb{T}$ by T in the assumptions, as we show in the following example.

Example 2.8. In $X = \mathbb{R}^2$ we consider the cone $K = \mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2 : x, y \geq 0\}$. Let $0 < r < R$ and define a mapping $T \colon K \to K$ in polar coordinates as

$$T(\rho,\theta) = \begin{cases} (0,0) & \text{if } \rho \neq r, \\ (r,\pi/2) & \text{if } \theta \in [0,\pi/4),\ \rho = r, \\ (r,0) & \text{if } \theta \in [\pi/4,\pi/2],\ \rho = r. \end{cases}$$

Note that $\mathbb{T}x = \{Tx\} = \{(0,0)\}$ for all $x \in K$ such that $\|x\| \neq r$ because T is continuous at those points. For points $x = (r,\theta)$, with $\theta \in [0,\pi/2]$, we have three possibilities: if $\theta \in [0,\pi/4)$, then $\mathbb{T}x$ is the segment with endpoints $(0,0)$ and $(r,\pi/2)$; if $\theta \in (\pi/4,\pi/2]$, then $\mathbb{T}x$ is the segment with endpoints $(0,0)$ and $(r,0)$; finally, $\mathbb{T}(r,\pi/4)$ is the triangle with vertices $(0,0)$, $(r,0)$ and $(r,\pi/2)$. Therefore,

$$\{x\} \cap \mathbb{T}x \subset \{Tx\} \quad \text{for all } x \in K.$$

Moreover, conditions (a) and (b) in Proposition 2.6 are satisfied if we replace $\mathbb{T}$ by T (and we take $r_1 = R$ and $r_2 = r$). However, T has no fixed point in $\overline{B}(0,R) \setminus B(0,r)$.

3. Application to Sturm–Liouville problems

We consider the following generalization of equation (1.1) with separated BCs:

$$\begin{aligned} u''(t) + g(t)f(t,u(t)) &= 0, \quad 0 < t < 1, \\ \alpha u(0) - \beta u'(0) &= 0, \\ \gamma u(1) + \delta u'(1) &= 0, \end{aligned} \tag{3.1}$$

where $\alpha,\beta,\gamma,\delta \geq 0$ and $\Gamma := \gamma\beta+\alpha\gamma+\alpha\delta > 0$. A usual approach to this problem consists in turning it into a fixed point problem with the integral operator

$$Tu(t) := \int_0^1 G(t,s)g(s)f(s,u(s))\,ds,$$

where G is the Green's function associated to the differential problem. Motivated by this situation, we study existence of fixed points of Hammerstein integral operators

$$Tu(t) := \int_0^1 k(t,s)g(s)f(s,u(s))\,ds, \tag{3.2}$$

defined in a suitable space. Here we consider $\mathcal{C}([0,1])$, endowed with the usual supremum norm $\|u\| = \max_{t\in[0,1]} |u(t)|$. Fixed points of T will be looked for in the cone

$$K = \Big\{u \in \mathcal{C}([0,1]) : u \geq 0, \min_{t\in[a,b]} u(t) \geq c\|u\|\Big\},$$

where $[a,b] \subset [0,1]$ and $c \in (0,1]$. This cone was introduced by Guo and it was intensively employed in recent years, for example, see [9, 11, 14].

We suppose that the terms of the Hammerstein equation (3.2) satisfy the following hypotheses:

(H1) $f\colon [0,1]\times[0,\infty)\to[0,\infty)$ is such that:
  (a) compositions $f(\,\cdot\,,u(\,\cdot\,))$ are measurable whenever $u\in\mathcal{C}([0,1])$; and
  (b) for each $r>0$ there exists $R>0$ such that $f(t,u)\le R$ for almost all $t\in[0,1]$ and all $u\in[0,r]$.

(H2) g is measurable and $g(s)\ge 0$ almost everywhere.

(H3) $k\colon [0,1]\times[0,1]\to[0,\infty)$ is continuous.

(H4) There exists a measurable function $\Phi\colon [0,1]\to[0,\infty)$ satisfying

$$\Phi g\in L^1(0,1)\quad\text{and}\quad\int_a^b\Phi(s)g(s)\,ds>0,$$

and a constant $c\in(0,1]$ such that

$$\begin{aligned}k(t,s)&\le\Phi(s)&&\text{for all } t,s\in[0,1],\\ c\Phi(s)&\le k(t,s)&&\text{for all } t\in[a,b],\ s\in[0,1].\end{aligned}$$

Remark 3.1. Conditions (H1)–(H4) are similar to those requested in [11] with the exception that we do not require f to be continuous. In addition, our assumptions are more general than those in [12] or [14] where the authors require $g\in L^1(0,1)$ and $\Phi\in\mathcal{C}([0,1])$.

Lemma 3.2. *If conditions* (H1)–(H4) *are satisfied, then the operator* $T\colon K\to K$ *introduced in* (3.2) *is well-defined and maps bounded sets into relatively compact sets.*

Proof. The operator T maps K into K. Indeed, we have

$$\|Tu\|=\max_{t\in[0,1]}\bigg\{\int_0^1 k(t,s)g(s)f(s,u(s))\,ds\bigg\}\le\int_0^1\Phi(s)g(s)f(s,u(s))\,ds.$$

Moreover,

$$\min_{t\in[a,b]}\{Tu(t)\}\ge c\int_0^1\Phi(s)g(s)f(s,u(s))\,ds.$$

Hence, $Tu\in K$ for every $u\in K$.

Now we prove that if $B\subset K$ is an arbitrary nonempty bounded set, then TB is relatively compact. Let $r>0$ be such that $u\in B$ implies $0\le u(t)\le r$ for all $t\in[0,1]$, and let $R>0$ be the constant associated to $r>0$ by condition (H1) (b). Given $u\in B$, we have

$$\int_0^1 k(t,s)g(s)f(s,u(s))\,ds\le R\int_0^1\Phi(s)g(s)\,ds<\infty,$$

so TB is uniformly bounded. To see that TB is equicontinuous, it suffices to show that for every $\tau\in[0,1]$ and $t_n\to\tau$, we have

$$\lim_{t_n\to\tau}\int_0^1|k(t_n,s)g(s)f(s,u(s))-k(\tau,s)g(s)f(s,u(s))|\,ds=0\tag{3.3}$$

uniformly in $u \in B$. To prove it, we note that for every $u \in B$,

$$\big|k(t_n,s)g(s)f(s,u(s)) - k(\tau,s)g(s)f(s,u(s))\big| \le Rg(s)|k(t_n,s) - k(\tau,s)|, \tag{3.4}$$

which tends to zero as n tends to infinity for a.a. $s \in [0,1]$ because k is continuous in $[0,1]$. Moreover,

$$R\,g(s)|k(t_n,s) - k(\tau,s)| \le 2R\Phi(s)g(s) \quad \text{for all } n \in \mathbb{N},$$

and $2R\Phi g \in L^1(0,1)$, by (H4), so the dominated convergence theorem and (3.4) yield (3.3). □

Moreover, suppose that the discontinuities of f allow the operator T to satisfy the condition

$$\{u\} \cap \mathbb{T}u \subset \{Tu\} \quad \text{for all } u \in K \cap \mathbb{T}K, \tag{3.5}$$

where $\mathbb{T}$ is the multivalued mapping associated to T defined in (2.1). Examples of this type of nonlinearities f can be looked up in [6, 13].

Lemma 3.3. *Suppose that condition* (3.5) *holds and*

(I^1_ρ) *there exist $\rho > 0$ and $\varepsilon > 0$ such that $f^{\rho,\varepsilon} < m$, where*

$$f^{\rho,\varepsilon} := \sup_{0\le t\le 1,\, 0\le u\le \rho+\varepsilon} \left\{\frac{f(t,u)}{\rho}\right\} \quad \textit{and} \quad \frac{1}{m} := \sup_{t\in[0,1]} \int_0^1 k(t,s)g(s)\,ds.$$

Then $\lambda u \notin \mathbb{T}u$ for all $u \in \partial_K B(0,\rho)$ and all $\lambda \ge 1$.

Proof. Suppose that there exist $\lambda \ge 1$ and $u \in \partial_K B(0,\rho)$ such that $\lambda u = Tv$ for some $v \in \overline{B}_\varepsilon(u) \cap K$, i.e.,

$$\lambda u(t) = \int_0^1 k(t,s)g(s)f(s,v(s))\,ds.$$

Taking the supremum for $t \in [0,1]$, we get

$$\begin{aligned} \lambda\rho &\le \sup_{t\in[0,1]} \int_0^1 k(t,s)g(s)f(s,v(s))\,ds \\ &\qquad\le \rho f^{\rho,\varepsilon} \sup_{t\in[0,1]} \int_0^1 k(t,s)g(s)\,ds \le \rho f^{\rho,\varepsilon}\frac{1}{m} < \rho, \end{aligned} \tag{3.6}$$

a contradiction. Given $m \in \mathbb{N}$, it can be similarly proven that $\lambda u \ne \sum_{i=1}^{m} \lambda_i T v_i$ for any $v_i \in \overline{B}_\varepsilon(u) \cap K$ and $\lambda_i \in [0,1]$ with $\sum_{i=1}^{m} \lambda_i = 1$. Hence, $\lambda u \notin \operatorname{co}(T(\overline{B}_\varepsilon(u) \cap K))$.

To see $\lambda u \notin \overline{\operatorname{co}}(T(\overline{B}_\varepsilon(u) \cap K))$ we consider two cases: $\lambda = 1$ and $\lambda > 1$. If $\lambda = 1$, we have $u \notin \mathbb{T}u$ because $u \ne Tu$ and $\{u\} \cap \mathbb{T}u \subset \{Tu\}$. If $\lambda > 1$, we obtain from (3.6) that $\lambda\rho \le \rho$, which in this case brings a contradiction too. □

In the sequel we denote

$$V_\rho = \Big\{u \in K : \min_{a\le t\le b} u(t) < \rho\Big\}.$$

In addition, it is trivial to see that $B(0,\rho)\cap K \subset V_\rho \subset B(0,\rho/c)\cap K$, and V_ρ is a relatively open subset of K (since the minimum function is continuous).

Lemma 3.4. *Suppose that condition* (3.5) *holds and*

(I^0_ρ) *there exist $\rho > 0$ and $\varepsilon > 0$ such that $f_{\rho,\varepsilon} > M(a,b)$, where*

$$f_{\rho,\varepsilon} := \inf_{a\le t\le b,\, c(\rho-\varepsilon)\le u\le \rho/c+\varepsilon} \bigg\{\frac{f(t,u)}{\rho}\bigg\}$$

and

$$\frac{1}{M(a,b)} := \inf_{t\in[a,b]} \int_a^b k(t,s)g(s)\,ds.$$

Then $u \notin \mathbb{T}u + \lambda e$ for all $u \in \partial V_\rho$, all $\lambda \ge 0$ and $e(t)\equiv 1$.

Proof. Suppose there exist $u \in \partial V_\rho$ and $\lambda \ge 0$ such that $u = Tv + \lambda e$ for some $v \in \overline{B}_\varepsilon(u)\cap K$. Then

$$u(t) = \int_0^1 k(t,s)g(s)f(s,v(s))\,ds + \lambda.$$

Notice that $\|v\| \le \|u\|+\varepsilon \le \rho/c+\varepsilon$ and $\min\limits_{t\in[a,b]} v(t) \ge c\|v\| \ge c(\|u\|-\varepsilon) \ge c(\rho-\varepsilon)$. Therefore, for $t\in[a,b]$,

$$\begin{aligned} u(t) &= \int_0^1 k(t,s)g(s)f(s,v(s))\,ds + \lambda \\ &\ge \int_a^b k(t,s)g(s)f(s,v(s))\,ds + \lambda \ge \rho f_{\rho,\varepsilon}\int_a^b k(t,s)g(s)\,ds + \lambda. \end{aligned}$$

Taking the infimum in $[a,b]$ we have

$$\rho \ge \rho f_{\rho,\varepsilon} \inf_{t\in[a,b]} \int_a^b k(t,s)g(s)\,ds + \lambda > \rho + \lambda,$$

a contradiction because $\lambda \ge 0$.

Given $m\in\mathbb{N}$, it is similarly checked that $u \ne \sum\limits_{i=1}^m \lambda_i T v_i + \lambda e$ for any $v_i \in \overline{B}_\varepsilon(u)$ and $\lambda_i \in [0,1]$, $i = 1,\dots,n$, with $\sum\limits_{i=1}^m \lambda_i = 1$. Hence, $u \notin \operatorname{co}(T(\overline{B}_\varepsilon(u)\cap K)) + \lambda e$. If we consider two cases: $\lambda = 0$ and $\lambda > 0$, and we work in a similar way as in the previous lemma, we obtain that $u \notin \mathbb{T}u + \lambda e$. □

Theorem 3.5. *Under the hypothesis* (H1)–(H4) *and* (3.5)*, the Hammerstein integral operator* (3.2) *has a nonzero fixed point in K if either of the following conditions holds*:

(a) *There exist $\rho_1,\rho_2 \in (0,\infty)$ with $\rho_1/c < \rho_2$ such that* ($\mathrm{I}^0_{\rho_1}$) *and* ($\mathrm{I}^1_{\rho_2}$) *hold.*
(b) *There exist $\rho_1,\rho_2 \in (0,\infty)$ with $\rho_1 < \rho_2$ such that* ($\mathrm{I}^1_{\rho_1}$) *and* ($\mathrm{I}^0_{\rho_2}$) *hold.*

Proof. It is an immediate consequence of the generalization of Krasnosel'-skiĭ's Proposition 2.5 together with both lemmas above: 3.3 and 3.4. □

Remark 3.6. Multiplicity results can be obtained combining previous conditions (see [11]).

Now we return to the differential BVP (3.1). We will say that u is a solution of that problem if $u \in W^{2,1}([0,1])$ (i.e. if $u \in \mathcal{C}^1([0,1])$ and $u' \in \mathrm{AC}([0,1])$, where $\mathrm{AC}([0,1])$ denotes the space of absolutely continuous functions defined in $[0,1]$) and if it satisfies (3.1).

The problem (3.1) was widely studied in looking for positive solutions [5], [11]. However, the novelty here is to let the function f be discontinuous. In [5], the authors consider the problem with $g(t)f(t,u) = h(t,u(t))$ where h is continuous and they use a norm compression–expansion theorem in order to guarantee the existence of solutions. On the other hand, in [11], Lan considers f autonomous and continuous and weaker conditions about g, he even replaces the integrability hypothesis by measurability, but it is necessary that $\int_0^1 \Phi(s)g(s)\,ds < \infty$. Here, as f can be discontinuous, we will require $g \in L^1(0,1)$.

We can write the differential problem (3.1) as

$$u(t) = \int_0^1 G(t,s)g(s)f(s,u(s))\,ds =: Tu(t),$$

where G is the associated Green function, which in this case [11] is given by

$$G(t,s) = \frac{1}{\Gamma}\begin{cases} (\gamma+\delta-\gamma t)(\beta+\alpha s) & \text{if } 0 \le s \le t \le 1, \\ (\beta+\alpha t)(\gamma+\delta-\gamma s) & \text{if } 0 \le t < s \le 1, \end{cases} \tag{3.7}$$

and is nonnegative. As $G(t,s) \le G(s,s)$ for all $s,t \in [0,1]$, it is possible to choose

$$\Phi(s) = G(s,s) = \frac{1}{\Gamma}(\gamma+\delta-\gamma s)(\beta+\alpha s).$$

Moreover, we can choose a, b and c in the following way [11]:

- (C1) $a, b \in [0,1]$ so that $-\beta/\alpha < a < b < 1+\delta/\gamma$, where we consider $\beta/\alpha = \infty$ if $\alpha = 0$ and $\delta/\gamma = \infty$ if $\gamma = 0$.
- (C2) $c = \min\{(\gamma+\delta-\gamma b)/(\gamma+\delta), (\beta+\alpha a)/(\alpha+\beta)\}$.

These choices guarantee that $c\Phi(s) \le G(t,s)$ for $t \in [a,b]$ and $s \in [0,1]$. We shall work, as before, in the cone

$$K = \Big\{u \in \mathcal{C}[0,1] \,:\, u \ge 0,\ \min_{t\in[a,b]} u(t) \ge c\|u\|\Big\}.$$

We allow $f\colon [0,1]\times[0,\infty) \to [0,\infty)$ to have discontinuities over the graphs of some admissible curves. This type of curves is similar to Filippov's discontinuity sets, see [7].

DEFINITION 3.7. We say that $\gamma\colon [r,s] \subset I = [0,1] \to [0,\infty)$, $\gamma \in W^{2,1}([r,s])$, is an admissible discontinuity curve for the differential equation $u'' = -g(t)f(t,u)$ if one of the following conditions holds:

(a) $\gamma''(t) = -g(t)f(t,\gamma(t))$ for a.e. $t \in [r,s]$ (then we say γ is viable for the differential equation);

(b) there exist $\varepsilon > 0$ and $\psi \in L^1(r,s)$, $\psi(t) > 0$ for almost every $t \in [r,s]$, such that either

$$\gamma''(t) + \psi(t) < -g(t)f(t,y) \text{ for a.e. } t \in I, \text{ all } y \in [\gamma(t)-\varepsilon, \gamma(t)+\varepsilon], \text{ or} \tag{3.8}$$

$$\gamma''(t) - \psi(t) > -g(t)f(t,y) \text{ for a.e. } t \in I, \text{ all } y \in [\gamma(t)-\varepsilon, \gamma(t)+\varepsilon]. \tag{3.9}$$

In this case we say that γ is inviable.

Working with admissible discontinuity curves involves some technicalities gathered in the next lemma and its subsequent corollaries whose proofs will be omitted because they can be found in [13].

LEMMA 3.8 ([13, Lemma 4.1]). *Let $a,b \in \mathbb{R}$, $a < b$, and let $g,h \in L^1(a,b)$, $g \geq 0$ almost everywhere and $h > 0$ almost everywhere in (a,b). For every measurable set $J \subset (a,b)$ with $m(J) > 0$ there is a measurable set $J_0 \subset J$ with $m(J \setminus J_0) = 0$ such that for every $\tau_0 \in J_0$ we have*

$$\lim_{t\to\tau_0^+} \frac{\displaystyle\int_{[\tau_0,t]\setminus J} g(s)\,ds}{\displaystyle\int_{\tau_0}^{t} h(s)\,ds} = 0 = \lim_{t\to\tau_0^-} \frac{\displaystyle\int_{[t,\tau_0]\setminus J} g(s)\,ds}{\displaystyle\int_{t}^{\tau_0} h(s)\,ds}.$$

COROLLARY 3.9 ([13, Corollary 4.2]). *Let $a,b \in \mathbb{R}$, $a < b$, and let $h \in L^1(a,b)$ be such that $h > 0$ almost everywhere in (a,b). For every measurable set $J \subset (a,b)$ with $m(J) > 0$ there is a measurable set $J_0 \subset J$ with $m(J \setminus J_0) = 0$ such that for all $\tau_0 \in J_0$ we have*

$$\lim_{t\to\tau_0^+} \frac{\displaystyle\int_{[\tau_0,t]\cap J} h(s)\,ds}{\displaystyle\int_{\tau_0}^{t} h(s)\,ds} = 1 = \lim_{t\to\tau_0^-} \frac{\displaystyle\int_{[t,\tau_0]\cap J} h(s)\,ds}{\displaystyle\int_{t}^{\tau_0} h(s)\,ds}.$$

COROLLARY 3.10 ([13, Corollary 4.3]). *Let $a,b \in \mathbb{R}$, $a < b$, and let functions $f, f_n\colon [a,b] \to \mathbb{R}$ be absolutely continuous s on $[a,b]$, $n \in \mathbb{N}$, and such that $f_n \to f$ uniformly on $[a,b]$ and suppose that for a measurable set $A \subset [a,b]$ with $m(A) > 0$ we have*

$$\lim_{n\to\infty} f_n'(t) = g(t) \quad \text{for a.a. } t \in A.$$

If there exists $M \in L^1(a,b)$ such that $|f'(t)| \leq M(t)$ almost everywhere in $[a,b]$ and also $|f_n'(t)| \leq M(t)$ almost everywhere in $[a,b]$, $n \in \mathbb{N}$, then $f'(t) = g(t)$ for almost all $t \in A$.

We shall also need the following result.

Lemma 3.11. *If $M \in L^1(0,1)$, $M \ge 0$ almost everywhere, then the set*

$$Q = \left\{ u \in \mathcal{C}^1([0,1]) : |u'(t) - u'(s)| \le \int_s^t M(r)\,dr \text{ whenever } 0 \le s \le t \le 1 \right\},$$

is closed in $\mathcal{C}([0,1])$ with the maximum norm topology. Moreover, if $u_n \in Q$ for all $n \in \mathbb{N}$ and $u_n \to u$ uniformly in $[0,1]$, then there exists a subsequence $\{u_{n_k}\}$ that tends to u in the $\mathcal{C}^1$ norm.

Proof. Let $\{u_n\}$ be a sequence of elements of Q which converges uniformly on $[0,1]$ to some function $u \in \mathcal{C}([0,1])$; we have to show that $u \in Q$ and a subsequence $\{u_{n_k}\}$ tends to u in the $\mathcal{C}^1$ norm.

Since each u_n is continuously differentiable, the Mean Value Theorem guarantees the existence of some $t_n \in (0,1)$ such that

$$u_n'(t_n) = u_n(1) - u_n(0).$$

This implies the existence of some $K > 0$ such that $|u_n'(t_n)| \le K$ for all $n \in \mathbb{N}$, because $\{u_n\}$ is uniformly bounded in $[0,1]$. Hence, for every $n \in \mathbb{N}$ and every $t \in [0,1]$, we have

$$|u_n'(t)| \le |u_n'(t) - u_n'(t_n)| + |u_n'(t_n)| \le \int_0^1 M(s)\,ds + K,$$

so $\{u_n\}$ is bounded in the $\mathcal{C}^1$ norm. Moreover, the definition of Q implies that the sequence $\{u_n'\}$ is equicontinuous in $[0,1]$, so the Ascoli–Arzelá Theorem ensures that some subsequence of $\{u_n\}$, say $\{u_{n_k}\}$, converges in the $\mathcal{C}^1$ norm to some $v \in \mathcal{C}^1([0,1])$. As a result, $u = v$, so u is continuously differentiable in $[0,1]$ and $\{u_{n_k}\}$ tends to u in the $\mathcal{C}^1$ norm. In particular, $\{u_{n_k}'\}$ tends to u' uniformly in $[0,1]$. Moreover, for $s,t \in [0,1]$, $s \le t$, and all $k \in \mathbb{N}$, we have

$$|u_{n_k}'(t) - u_{n_k}'(s)| \le \int_s^t M(r)\,dr,$$

and going to the limit as k tends to infinity we deduce that

$$|u'(t) - u'(s)| \le \int_s^t M(r)\,dr. \qquad \square$$

We are now ready for the proof of the main result in this section.

Theorem 3.12. *Suppose that f and g satisfy the following hypothesis:*

(i) $f\colon [0,1] \times [0,\infty) \to [0,\infty)$ *is such that:*
  - *compositions $f(\,\cdot\,, u(\,\cdot\,))$ are measurable whenever $u \in \mathcal{C}([0,1])$; and*
  - *for each $r > 0$ there exists $R > 0$ such that $f(t,u) \le R$ for almost all $t \in [0,1]$ and all $u \in [0,r]$.*

(ii) *There exist admissible discontinuity curves* $\gamma_n\colon I_n = [a_n, b_n] \to [0,\infty)$, $n \in \mathbb{N}$, *such that for almost all* $t \in I$ *the function* $u \mapsto f(t,u)$ *is continuous on* $[0,\infty) \setminus \bigcup_{\{n:t\in I_n\}} \{\gamma_n(t)\}$.

(iii) $g \in L^1(0,1)$ *and* $g(s) \geq 0$ *almost everywhere with* $\int_a^b g(s)\,ds > 0$, *where* a *and* b *are given in* (C1).

Moreover, assume that one of the following conditions holds:

(a) *There exist* $\rho_1, \rho_2 \in (0,\infty)$ *with* $\rho_1/c < \rho_2$ *such that* $(\mathrm{I}^0_{\rho_1})$ *and* $(\mathrm{I}^1_{\rho_2})$ *hold.*

(b) *There exist* $\rho_1, \rho_2 \in (0,\infty)$ *with* $\rho_1 < \rho_2$ *such that* $(\mathrm{I}^1_{\rho_1})$ *and* $(\mathrm{I}^0_{\rho_2})$ *hold.*

Then the differential problem with separated BCs (3.1) *has a positive solution* $u \in W^{2,1}([0,1])$.

Proof. The operator $T\colon K \to K$ given by

$$Tu(t) = \int_0^1 G(t,s)g(s)f(s,u(s))\,ds$$

is well defined and it maps bounded sets into relatively compact ones, as a consequence of Lemma 3.2. In addition, as G is the Green function associated to a second-order homogeneous differential problem, $Tu \in W^{2,1}([0,1])$ for all $u \in K$. On the other hand, given $u \in B(0,\rho_2/c) \cap K = K_2$, we have $g(t)f(t,u(t)) \in L^1[0,1]$, and there exists $M(t) \in L^1[0,1]$ such that

$$h(t,u) := g(t)f(t,u) \leq M(t) \quad \text{for a.e. } t \in [0,1] \text{ and all } u \in K_2. \tag{3.10}$$

We consider the set

$$Q = \left\{ u \in \mathcal{C}^1([0,1]) \,:\, |u'(t) - u'(s)| \leq \int_s^t M(r)\,dr \text{ for } s \leq t \right\}, \tag{3.11}$$

which is closed in $(\mathcal{C}([0,1]), \|\cdot\|_\infty)$ by virtue of Lemma 3.11. Hence, since $T\,K_2 \subset Q$ and Q is a closed and convex subset of $\mathcal{C}([0,1])$, we have $\mathbb{T}K_2 \subset Q$.

Now we will prove that

$$\{u\} \cap \mathbb{T}u \subset \{Tu\} \quad \text{for all } u \in K_2 \cap \mathbb{T}K_2. \tag{3.12}$$

To do so, we fix an arbitrary function $u \in K_2 \cap Q$ and consider three different cases.

Case 1. $m(\{t \in I_n : u(t) = \gamma_n(t)\}) = 0$ *for all* $n \in \mathbb{N}$. Let us prove that then T is continuous at u. The assumption implies that for almost all $t \in I$ the mapping $h(t,\,\cdot\,)$ is continuous at $u(t)$. Hence if $u_k \to u$ in $K_2 \cap Q$ then

$$h(t,u_k(t)) \to h(t,u(t)) \quad \text{for a.a. } t \in I,$$

which, along with (3.10), yields $Tu_k \to Tu$ in $\mathcal{C}(I)$.

Case 2. $m(\{t \in I_n : u(t) = \gamma_n(t)\}) > 0$ *for some* $n \in \mathbb{N}$ *such that* γ_n *is inviable.* In this case we can prove that $u \notin \mathbb{T}u$. First, we fix some notation.

Let us assume that for some $n \in \mathbb{N}$ we have $m(\{t \in I_n : u(t) = \gamma_n(t)\}) > 0$ and there exist $\varepsilon > 0$ and $\psi \in L^1(I_n)$, $\psi(t) > 0$ for almost all $t \in I_n$, such that (3.9) holds with γ replaced by γ_n. (The proof is similar if we assume (3.8) instead of (3.9), so we omit it.)

We denote $J = \{t \in I_n : u(t) = \gamma_n(t)\}$, and we deduce from Lemma 3.8 that there is a measurable set $J_0 \subset J$ with $m(J_0) = m(J) > 0$ such that for all $\tau_0 \in J_0$,

$$\lim_{t\to\tau_0^+} \frac{2\displaystyle\int_{[\tau_0,t]\setminus J} M(s)\,ds}{\dfrac{1}{4}\displaystyle\int_{\tau_0}^{t}\psi(s)\,ds} = 0 = \lim_{t\to\tau_0^-} \frac{2\displaystyle\int_{[t,\tau_0]\setminus J} M(s)\,ds}{\dfrac{1}{4}\displaystyle\int_{t}^{\tau_0}\psi(s)\,ds}. \tag{3.13}$$

By Corollary 3.9 there exists $J_1 \subset J_0$ with $m(J_0 \setminus J_1) = 0$ such that for all $\tau_0 \in J_1$,

$$\lim_{t\to\tau_0^+} \frac{\displaystyle\int_{[\tau_0,t]\cap J_0} \psi(s)\,ds}{\displaystyle\int_{\tau_0}^{t}\psi(s)\,ds} = 1 = \lim_{t\to\tau_0^-} \frac{\displaystyle\int_{[t,\tau_0]\cap J_0} \psi(s)\,ds}{\displaystyle\int_{t}^{\tau_0}\psi(s)\,ds}. \tag{3.14}$$

Let us now fix a point $\tau_0 \in J_1$. From (3.13) and (3.14) we deduce that there exist $t_- < \widetilde{t}_- < \tau_0$ and $t_+ > \widetilde{t}_+ > \tau_0$, $t_\pm$ sufficiently close to τ_0, so that the following inequalities are satisfied for all $t \in [\widetilde{t}_+, t_+]$:

$$2\int_{[\tau_0,t]\setminus J} M(s)\,ds < \frac{1}{4}\int_{\tau_0}^{t}\psi(s)\,ds, \tag{3.15}$$

$$\int_{[\tau_0,t]\cap J} \psi(s)\,ds \geq \int_{[\tau_0,t]\cap J_0} \psi(s)\,ds > \frac{1}{2}\int_{\tau_0}^{t}\psi(s)\,ds, \tag{3.16}$$

and for all $t \in [t_-, \widetilde{t}_-]$:

$$2\int_{[t,\tau_0]\setminus J} M(s)\,ds < \frac{1}{4}\int_{t}^{\tau_0}\psi(s)\,ds, \tag{3.17}$$

$$\int_{[t,\tau_0]\cap J} \psi(s)\,ds > \frac{1}{2}\int_{t}^{\tau_0}\psi(s)\,ds. \tag{3.18}$$

Finally, we define a positive number

$$\widetilde{\rho} = \min\left\{\frac{1}{4}\int_{\widetilde{t}_-}^{\tau_0}\psi(s)\,ds,\ \frac{1}{4}\int_{\tau_0}^{\widetilde{t}_+}\psi(s)\,ds\right\}, \tag{3.19}$$

and we are now in a position to prove that $u \notin \mathbb{T}u$. It suffices to prove the following claim:

Claim. *Let $\varepsilon > 0$ be given by our assumptions over γ_n and let*

$$\rho = \frac{\widetilde{\rho}}{2}\min\{\widetilde{t}_- - t_-, t_+ - \widetilde{t}_+\}$$

where $\widetilde{\rho}$ is as in (3.19). *For every finite family $u_i \in \overline{B}_\varepsilon(u) \cap K$ and $\lambda_i \in [0,1]$, $i = 1, \dots, m$, with $\sum_i \lambda_i = 1$, we have $\left\| u - \sum_i \lambda_i T u_i \right\| \geq \rho$.*

Let u_i and λ_i be as in the Claim and, for simplicity, denote $y = \sum_i \lambda_i T u_i$. For almost all $t \in J = \{t \in I_n : u(t) = \gamma_n(t)\}$ we have

$$y''(t) = \sum_{i=1}^{m} \lambda_i (T u_i)''(t) = -\sum_{i=1}^{m} \lambda_i h(t, u_i(t)). \tag{3.20}$$

On the other hand, for every $i \in \{1, \dots, m\}$ and every $t \in J$ we have

$$|u_i(t) - \gamma_n(t)| = |u_i(t) - u(t)| < \varepsilon,$$

and then the assumptions on γ_n ensure that for almost all $t \in J$,

$$y''(t) = -\sum_{i=1}^{m} \lambda_i h(t, u_i(t)) < \sum_{i=1}^{m} \lambda_i (\gamma_n''(t) - \psi(t)) = u''(t) - \psi(t). \tag{3.21}$$

Now for $t \in [t_-, \widetilde{t}_-]$ we compute

$$\begin{aligned} y'(\tau_0) - y'(t) &= \int_t^{\tau_0} y''(s)\, ds = \int_{[t,\tau_0] \cap J} y''(s)\, ds + \int_{[t,\tau_0] \setminus J} y''(s)\, ds \\ &< \int_{[t,\tau_0] \cap J} u''(s)\, ds - \int_{[t,\tau_0] \cap J} \psi(s)\, ds + \int_{[t,\tau_0] \setminus J} M(s)\, ds \end{aligned}$$

(by (3.21), (3.20) and (3.10))

$$\begin{aligned} &= u'(\tau_0) - u'(t) - \int_{[t,\tau_0] \setminus J} u''(s)\, ds \\ &\quad - \int_{[t,\tau_0] \cap J} \psi(s)\, ds + \int_{[t,\tau_0] \setminus J} M(s)\, ds \\ &\leq u'(\tau_0) - u'(t) - \int_{[t,\tau_0] \cap J} \psi(s)\, ds + 2 \int_{[t,\tau_0] \setminus J} M(s)\, ds \\ &< u'(\tau_0) - u'(t) - \frac{1}{4} \int_t^{\tau_0} \psi(s)\, ds \end{aligned}$$

(by (3.17) and (3.18)), hence $y'(t) - u'(t) \geq \widetilde{\rho}$ provided that $y'(\tau_0) \geq u'(\tau_0)$. Therefore, by integration we obtain

$$\begin{aligned} y(\widetilde{t}_-) - u(\widetilde{t}_-) = y(t_-) - u(t_-) + \int_{t_-}^{\widetilde{t}_-} (y'(t) - u'(t))\, dt \\ \geq y(t_-) - u(t_-) + \widetilde{\rho}(\widetilde{t}_- - t_-), \end{aligned}$$

so if $y(t_-) - u(t_-) \leq -\rho$, then $\|y - u\| \geq \rho$. Otherwise, if $y(t_-) - u(t_-) > -\rho$, then we have $y(\widetilde{t}_-) - u(\widetilde{t}_-) > \rho$ and thus $\|y - u\| \geq \rho$ too.

Similar computations in the interval $[\widetilde{t}_+, t_+]$ instead of $[t_-, \widetilde{t}_-]$ show that if $y'(\tau_0) \leq u'(\tau_0)$ then we have $u'(t) - y'(t) \geq \widetilde{\rho}$ for all $t \in [\widetilde{t}_+, t_+]$ and this also implies $\|y - u\| \geq \rho$. The claim is proven.

Case 3. $m(\{t \in I_n : u(t) = \gamma_n(t)\}) > 0$ *only for some of those* $n \in \mathbb{N}$ *such that* γ_n *is viable.* Let us prove that in this case the relation $u \in \mathbb{T}u$ implies $u = Tu$. Let us consider the subsequence of all viable admissible discontinuity curves in the conditions of Case 3, which we denote again by $\{\gamma_n\}_{n\in\mathbb{N}}$ to avoid overloading notation. We have $m(J_n) > 0$ for all $n \in \mathbb{N}$, where

$$J_n = \{t \in I_n : u(t) = \gamma_n(t)\}.$$

For each $n \in \mathbb{N}$ and for almost all $t \in J_n$ we have

$$u''(t) = \gamma_n''(t) = -h(t, \gamma_n(t)) = -h(t, u(t)),$$

and therefore $u''(t) = -h(t, u(t))$ almost everywhere in $J = \bigcup_{n\in\mathbb{N}} J_n$.

Now we assume that $u \in \mathbb{T}u$ and we prove that it implies $u''(t) = -h(t, u(t))$ almost everywhere in $I \setminus J$, thus showing that $u = Tu$. Since $u \in \mathbb{T}u$ then for each $k \in \mathbb{N}$ we can guarantee that we can find functions $u_{k,i} \in \overline{B}_{1/k}(u) \cap K_2$ and coefficients $\lambda_{k,i} \in [0,1]$, $i = 1, \ldots, m(k)$, such that $\sum_i \lambda_{k,i} = 1$ and

$$\left\| u - \sum_{i=1}^{m(k)} \lambda_{k,i} T u_{k,i} \right\| < \frac{1}{k}.$$

Let us denote $y_k = \sum_{i=1}^{m(k)} \lambda_{k,i} T u_{k,i}$, and notice that $y_k \to u$ uniformly in I and $\|u_{k,i} - u\| \leq 1/k$ for all $k \in \mathbb{N}$ and all $i \in \{1, \ldots, m(k)\}$. For every $k \in \mathbb{N}$ we have $y_k \in Q$ as defined in (3.11), and therefore Lemma 3.11 guarantees that $u \in Q$ and, up to a subsequence, $y_k \to u$ in the $\mathcal{C}^1$ topology.

For almost all $t \in I \setminus J$ we have that $h(t, \cdot\,)$ is continuous at $u(t)$, so for any $\varepsilon > 0$ there is some $k_0 = k_0(t) \in \mathbb{N}$ such that for all $k \in \mathbb{N}$, $k \geq k_0$, we have

$$|h(t, u_{k,i}(t)) - h(t, u(t))| < \varepsilon \quad \text{for all } i \in \{1, \ldots, m(k)\},$$

and therefore

$$|y_k''(t) + h(t, u(t))| \leq \sum_{i=1}^{m(k)} \lambda_{k,i} |h(t, u_{k,i}(t)) - h(t, u(t))| < \varepsilon.$$

Hence $y_k''(t) \to -h(t, u(t))$ for almost all $t \in I \setminus J$, and then Corollary 3.10 guarantees that $u''(t) = -h(t, u(t))$ for almost all $t \in I \setminus J$. Therefore the proof of condition (3.12) is over and we conclude by means of Theorem 3.5. □

Remark 3.13. The differential problem (3.1) contains Dirichlet and Robin problems, so the previous result generalizes the existence results given in [12], because here we allow f to be discontinuous.

Finally, we illustrate the applicability of our result with a discontinuous modification of an example due to Infante [9].

EXAMPLE 3.14. Let $\lambda > 0$ and consider the following second-order differential problem

$$
\begin{aligned}
u''(t) + g(t)f(t,u(t)) = 0 \quad 0 < t < 1,\\
u(0) = u(1) = 0,
\end{aligned}
\tag{3.22}
$$

where $g \equiv 1$ and f is given by

$$f(t,u) = \lambda u^2(\cos^2(\lfloor 1/(t^2 - u)\rfloor) + 1) \quad \text{if } u \neq t^2,$$

($\lfloor x\rfloor$ denotes the integer part of x) and $f(t,t^2) = 2\lambda t^4$. For each $n \in \mathbb{Z}\setminus\{0\}$ the curves

$$\gamma_n(t) = t^2 - n^{-1} \quad \text{for } t \in [n^{-1/2}, 1] \quad \text{and} \quad \gamma_0(t) = t^2 \quad \text{for } t \in [0,1],$$

are admissible discontinuity curves. We will show that they are inviable. On the one hand, for every $n \in \mathbb{Z}$ we have $\gamma_n''(t)=2$. On the other hand, $g(t)f(t,u(t))\geq 0$ for almost every $t \in I$ and all $u \in [0,\infty)$. Hence if we take $\psi \equiv 1$, then for every $n \in \mathbb{Z}$,

$$\gamma_n''(t) - \psi(t) > 0 \geq -g(t)f(t,u(t)) \quad \text{for a.e. } t \in I \text{ and all } u \in [0,\infty),$$

and (3.9) holds. For almost all $t \in I$ the function $u \mapsto f(t,u)$ is continuous on $[0,\infty)\setminus \bigcup_{\{n:t\in I_n\}} \{\gamma_n(t)\}$.

We can take $a = 1/4$ and $b = 3/4$ in (C1), so $c = 1/4$. In addition it is easy to check that $m = 8$ and $M(1/4,3/4) = 16$ as defined in Lemmas 3.3 and 3.4, respectively. Moreover, we have the following bounds for f:

$$\lambda u^2 \leq f(t,u) \leq 2\lambda u^2 \quad \text{for a.a. } t \in I \text{ and all } u \in [0,\infty).$$

Since

$$f^{\rho_1,\varepsilon} := \sup_{0\leq t\leq 1,\, 0\leq u\leq \rho_1+\varepsilon} \left\{\frac{f(t,u)}{\rho_1}\right\} \leq \frac{2\lambda(\rho_1+\varepsilon)^2}{\rho_1},$$

it is sufficient to choose $\rho_1 < 4/\lambda$ for ensuring that $f^{\rho_1,\varepsilon} < m$ and so $(I^1_{\rho_1})$ is satisfied. Now if we take $\rho_0 = 1$ then

$$f_{\rho_0,\varepsilon} := \inf_{1/4\leq t\leq 3/4,\, 1/4-\varepsilon/4\leq u\leq 4+\varepsilon} \{f(t,u)\} \geq \lambda\,(1/4-\varepsilon)^2 > \lambda/25$$

if ε is small enough and $f_{\rho_0,\varepsilon} > M(1/4,3/4) = 16$ provided $\lambda > 400$. Hence Theorem 3.12 guarantees that the differential problem (3.22) has a positive solution for every $\lambda > 400$.

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