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INFINITELY MANY SOLUTIONS FOR A CLASS OF QUASILINEAR EQUATION WITH A COMBINATION OF CONVEX AND CONCAVE TERMS

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Abstract. We consider the following quasilinear elliptic equation with convex and concave nonlinearities:

$$-\Delta_p u - (\Delta_p u^2)u + V(x)|u|^{p-2}u = \lambda K(x)|u|^{q-2}u + \mu g(x,u), \quad \text{in } \mathbb{R}^N,$$

where $2 \leq p < N$, $1 < q < p$, $\lambda, \mu \in \mathbb{R}$, V and K are potential functions, and $g \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ is a continuous function. Under some suitable conditions on V, K and g, the existence of infinitely many solutions is established.

1. Introduction

In this paper, we study the following quasilinear Schrödinger equation:

$$(1.1) \quad -\Delta_p u - (\Delta_p u^2)u + V(x)|u|^{p-2}u = \lambda K(x)|u|^{q-2}u + \mu g(x,u), \quad \text{in } \mathbb{R}^N,$$

where $-\Delta_p u = -\mathrm{div}(|\nabla u|^{p-2}\nabla u)$, $2 \leq p < N$, $1 < q < p$, $\lambda, \mu \in \mathbb{R}$ are two parameters. In order to deal with the concave term we make the following assumptions on the potentials V and K:

(V$_1$) $V \in C(\mathbb{R}^N, \mathbb{R})$ and $\inf\limits_{x \in \mathbb{R}^N} V(x) \geq V_0 > 0$;

(V$_2$) $\int_{\mathbb{R}^N} V(x)^{-1/(p-1)}\, dx < +\infty$;

(K$_0$) $K \in L^\infty(\mathbb{R}^N)$, $K(x) \geq 0$, $K(x) \not\equiv 0$;

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(K_1) $K \in L^{2p^*/(2p^*-q)}(\mathbb{R}^N)$, where $p^* = Np/(N-p)$.

Also, we pose the following assumptions on g:

(g_0) $g \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ and $\lim_{|s|\to 0} g(x,s)/|s|^{p-2}s = 0$ uniformly for $x \in \mathbb{R}^N$.

(g_1) There exist $c > 0$ and $p < r < 2p^*$ such that $|g(x,u)| \leq c(1 + |u|^{r-1})$ for all $x \in \mathbb{R}^N$ and $u \in \mathbb{R}$.

(g_2) There exists $2p < \theta < 2p^*$ such that $0 < \theta G(x,u) \leq ug(x,u)$ for all $x \in \mathbb{R}^N$ and $u \in \mathbb{R}\backslash\{0\}$.

(g_3) $g(x,u)$ is odd in u.

Remark 1.1. The assumption $p \geq 2$ is a consequence of the choice of the work space $\mathcal{E}_f$ which requires $|f(t)|^p$ to verify the convexity property and the Δ_2 condition, see Proposition 2.5.

The quasilinear Schrödinger equation of type (1.1) has served for modeling of several physical phenomena. It is related to the existence of standing wave solutions for quasilinear Schrödinger equation of the form

$$iz_t = -\Delta z + W(x)z - f(|z|^2)z - \kappa\Delta h(|z|^2)h'(|z|^2)z, \quad x \in \mathbb{R}^N, \tag{1.2}$$

where W is a given potential, κ is a real constant, f and h are real functions. For instance, in the case $h(s) = s$, it corresponds to the superfluid film equations in plasma physics, see Kurihara [16]. In the case $h(s) = (1+s)^{1/2}$, it models the self-channeling of a high-power ultra short laser in the matter, see [8]. Equation (1.2) also appears in plasma physics and fluid mechanics, see [16] and [17], in theory of Heisenberg ferromagnets and magnons, see [15], [32]. Considering the case $h(s) = s$, $\kappa = 1$ and setting $z(x,t) = \exp(-iwt)u(x)$, $w \in \mathbb{R}$, it is easy to obtain the corresponding equation

$$-\Delta u - (\Delta u^2)u + V(x)u = g(u), \quad x \in \mathbb{R}^N, \tag{1.3}$$

where $V(x) = W(x) - w$, $g(u) = f(|u|^2)u$. Because one of the main difficulties of problem (1.3) is that there is no suitable work space on which the energy functional is of class C^1, the standard critical point theory cannot be applied directly. The existence and multiplicity of solutions to the problems like (1.3) have been considered by many authors in the recent years. To the best of our knowledge, there are some powerful methods developed, such as, the minimizing method [18], [26], the Nehari manifold method [6], [21], the method of change of variables which was independently applied in [20] and [12], the method of nonsmooth critical point theory [22], [23], the perturbation method [25], [19]. By the change of variables, the quasilinear equation (1.3) reduces to a semilinear one, so the usual methods for semilinear Schrödinger equations can be adopted. This method has become the fundamental trick for studying quasilinear problem (1.3). For the recent progress in this regard, we refer the interested readers to [1]–[3], [9] and references therein.

In [13], the authors proposed the following question:

Question 1.2. Does problem (1.3) have infinitely many solutions when nonlinearities are symmetric in the sense of being odd in u and involve a combination of concave and convex terms?

The main purpose of this paper is to treat the above problem. For this, we consider a more general quasilinear problem (1.1). When $p = 2$ it is reduced to problem (1.3). Our main strategy is as follows: we will develop the Orlicz–Sobolev framework for problem (1.1), where we deduce some new phenomenon in the abstract Orlicz–Sobolev space, compared with the results in [14], [13] this will allow us more easily to verify the mountain pass geometric conditions and (PS) condition.

In the past decades, nonlinear elliptic problems involving concave and convex terms have attracted intensive interest. For example, for semilinear and quasilinear problems, we refer the readers to Ambrosetti, Brezis, Cerami [5], Ambrosetti, Azorero, Peral [4], Bartsch and Willem [7] and references therein. In [7], Bartsch and Willem proved the existence of infinity many solutions for the semilinear problem in an open bounded domain with Dirichlet boundary conditions

$$\begin{cases} -\Delta u = \mu|u|^{q-2}u + \lambda|u|^{p-2}u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where $1 < q < 2 < r < 2^*$, $\lambda, \mu \in \mathbb{R}$. In [33], the author considered the following problem:

$$\begin{cases} -\Delta u - \lambda g(x)u = k(x)|u|^{q-2}u - h(x)|u|^{p-2}u, & x \in \mathbb{R}^N, \\ u > 0, & x \in \mathbb{R}^N, \end{cases} \tag{1.4}$$

where $N \geq 3$ and $1 < q < 2 < p \leq 2^* = 2N/(N-2)$. With integrability and sign conditions on g, k and h, by using the Fountain theorem and dual Fountain theorem, infinitely many solutions for problem (2.1) were obtained. After that, some results in [33] were generalized to an equation of p-Laplacian type in [24]

$$\begin{cases} -\Delta_p u - \lambda g(x)|u|^{p-2}u = k(x)|u|^{q-2}u - h(x)|u|^{s-2}u, & x \in \mathbb{R}^N, \\ u > 0, & x \in \mathbb{R}^N, \end{cases} \tag{1.5}$$

where $N \geq 3$ and $1 < q < p < s \leq p^*$. The authors in [24] introduced a new space as their framework for the study of problem (1.5) and used the Clark theorem to establish the existence of infinitely many solutions for problem (1.5). This idea was successfully applied to Schrödinger–Poisson systems, see [31].

For the concave-convex type problem for equation (1.1), the author in [29] obtained infinitely many solutions via the Fountain theorem. The same method can be applied to generalize the results of [7] to quasilinear case in dimension one

$$-(|u'|^{p-2}u')' - (|(u^2)'|^{p-2}(u^2)')'u + V(x)|u|^{p-2}u = \lambda|u|^{q-2}u + \mu|u|^{r-2}u, \quad \text{in } \mathbb{R},$$

where λ, μ are real parameters. For the higher dimensional case, there are few papers to tackle this problem. For instance, in [13], the authors considered equation (1.1) in the case $p = 2$ and obtained the existence of one nontrivial solution and two nontrivial solutions, respectively, in [37], the authors considered equation (1.1) on a bounded open domain in the case $p = 2$ and proved the existence of infinitely many solutions by using the perturbed method which was developed in [25].

Now let us state our main results.

THEOREM 1.3. *Assume that* (V_1)–(V_2) *and* (g_0)–(g_3) *hold. Then for every* $\mu > 0$ *and* $\lambda \le 0$, *problem* (1.1) *has infinitely many solutions for* $1 < q < p$.

THEOREM 1.4. *Assume that* (V_1)–(V_2), (g_0)–(g_1) *and* (g_3) *hold. Then for every* $\mu < 0$ *and* $\lambda \ge 0$, *problem* (1.1) *possesses infinitely many solutions for* $1 < q < p/2$ *with* $p > 2$.

REMARK 1.5. Our main results complement some of the results in [29], in the sense that we are considering a higher dimensional case, while they are new even in the semilinear case because we will answer the question posed in [13]. However, in our framework, there is a gap in the case $2 \le p < 4$. We aim to solve it in the forthcoming paper by using perturbed methods which were developed in [25] and [19]. In fact, in [36], by using the perturbed methods, the authors proved the existence of infinitely many small energy solutions for modified Kirchhoff-type equation via the Clark theorem, where the potential V satisfies the coercive hypothesis $\lim_{|x|\to\infty} V(x) = +\infty$, the nonlinearity possesses the concave and convex structure. However, there are some non-coercive functions verifying assumption (V_2), see [13].

This paper is organized as follows. Section 2 contains some preliminary results and we state abstract theorems which will be used in the sequel. In Section 3, we present proofs of the main results.

2. Preliminaries

As usual, for $1 \le s \le +\infty$, we let

$$\|u\|_s = \left(\int_{\mathbb{R}^N} |u(x)|^s \, dx \right)^{1/s}, \quad u \in L^s(\mathbb{R}^N).$$

We denote by C, C_i $(i = 0, 1, \ldots)$ various positive constants throughout this paper. Let $W^{1,p}(\mathbb{R}^N)$ denote the Sobolev space endowed with the norm

$$\|u\|_{W^{1,p}} = \left(\int_{\mathbb{R}^N} (|\nabla u|^p + |u|^p)\, dx \right)^{1/p}.$$

We define the following function space:

$$X = \left\{ u \in W^{1,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|u|^p\, dx < \infty \right\}$$

which is a reflexive and separable Banach space endowed with the norm

$$\|u\|_X = \left(\int_{\mathbb{R}^N} (|\nabla u|^p + V(x)|u|^p)\, dx \right)^{1/p}.$$

Due to (V_1) and (V_2), we can establish the following compactness lemma.

LEMMA 2.1. *If conditions* (V_1) *and* (V_2) *hold, the embedding* $X \hookrightarrow L^s(\mathbb{R}^N)$ *is continuous for* $1 \le s \le p^*$ *and compact for* $1 \le s < p^*$.

PROOF. Let $u \in X$, by (V_2), we have

$$\begin{aligned} \int_{\mathbb{R}^N} |u|\, dx &\le \left(\int_{\mathbb{R}^N} V(x)^{-1/(p-1)}\, dx \right)^{(p-1)/p} \left(\int_{\mathbb{R}^N} V(x)|u|^p\, dx \right)^{1/p} \\ &\le \left(\int_{\mathbb{R}^N} V(x)^{-1/(p-1)}\, dx \right)^{(p-1)/p} \|u\|_X. \end{aligned}$$

By (V_1), we get

$$\begin{aligned} \int_{\mathbb{R}^N} (|\nabla u|^p + |u|^p)\, dx &\le \int_{\mathbb{R}^N} \left(|\nabla u|^p + \frac{V(x)}{V_0}|u|^p \right) dx \\ &\le \max\left\{ 1, \frac{1}{V_0} \right\} \int_{\mathbb{R}^N} (|\nabla u|^p + V(x)|u|^p)\, dx \end{aligned}$$

which means that $X \hookrightarrow W^{1,p}(\mathbb{R}^N)$. Hence, $X \hookrightarrow L^s(\mathbb{R}^N)$ for $p \le s \le p^*$. Therefore, by an interpolation argument, we conclude that $X \hookrightarrow L^s(\mathbb{R}^N)$ for $1 \le s \le p^*$.

Let $\{u_n\}$ be a bounded sequence in X, that is $\|u_n\| \le C$ ($C > 0$ is a positive constant). Up to a subsequence, we may assume that $u_n \rightharpoonup u_0$ in X. From (V_2), it follows that for a given $\varepsilon > 0$, there exists $R > 0$ such that

$$\int_{|x| \ge R} V(x)^{-1/(p-1)}\, dx < \left(\frac{\varepsilon}{2(C + \|u_0\|_X)} \right)^{p/(p-1)}.$$

Thus, we have

$$\int_{|x| \ge R} |u_n - u_0|\, dx \le \left(\int_{\mathbb{R}^N} V(x)^{-1/(p-1)}\, dx \right)^{(p-1)/p} \|u_n - u_0\|_X \le \frac{\varepsilon}{2}.$$

By the local embedding compactness, we see that $u_n \to u_0$ in $L^1_{\rm loc}(\mathbb{R}^N)$. It follows that there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$\int_{|x|<R} |u_n - u_0|\, dx \leq \frac{\varepsilon}{2}.$$

Thus, $u_n \to u_0$ in $L^1(\mathbb{R}^N)$. By a standard interpolation argument, we conclude that $u_n \to u_0$ in $L^s(\mathbb{R}^N)$ for $1 \leq s < p^*$. $\square$

Let us make the change of variables $v = f^{-1}(u)$, where f is defined by

$$f'(t) = \frac{1}{[1 + 2^{p-1}|f(t)|^p]^{1/p}}, \quad t \in [0, \infty),$$

and

$$f(-t) = -f(t), \quad t \in (-\infty, 0].$$

After the change of variables, equation (1.1) reduces to the following one:

$$\begin{aligned} (2.1) \qquad -\Delta_p v + V(x)|f(v)|^{p-2} f(v) f'(v) \\ = \lambda K(x)|f(v)|^{q-2} f(v) f'(v) + g(x, f(v)) f'(v). \end{aligned}$$

Lemma 2.2. *The function f and its derivative satisfy the following properties*:

- (f_1) *f is uniquely defined, C^2 and invertible*;
- (f_2) $|f'(t)| \leq 1$ *for all* $t \in \mathbb{R}$;
- (f_3) $|f(t)| \leq |t|$ *for all* $t \in \mathbb{R}$;
- (f_4) $f(t)/t \to 1$ *as* $t \to 0$;
- (f_5) $f(t)/\sqrt{t} \to a > 0$ *as* $t \to +\infty$;
- (f_6) $f(t)/2 \leq t f'(t) \leq f(t)$ *for all* $t > 0$;
- (f_7) $f^2(t)/2 \leq t f'(t) f(t) \leq f^2(t)$ *for all* $t \in \mathbb{R}$;
- (f_8) $|f(t)| \leq 2^{1/(2p)}|t|^{1/2}$ *for all* $t \in \mathbb{R}$;
- (f_9) *there exists a positive constant C such that*

$$|f(t)| \geq \begin{cases} C|t| & \text{if } |t| \leq 1, \\ C|t|^{1/2} & \text{if } |t| \geq 1; \end{cases}$$

- (f_{10})

$$\begin{cases} f^2(st) \leq s f^2(t) & \text{if } 0 \leq s \leq 1,\ t \in \mathbb{R}, \\ f^2(st) \leq s^2 f^2(t) & \text{if } s \geq 1,\ t \in \mathbb{R}; \end{cases}$$

- (f_{11}) $|f(t) f'(t)| \leq 1/2^{(p-1)/p}$;
- (f_{12}) $|t| \leq C_1 f(t) + C_2 f^2(t)$ *for all* $t \in \mathbb{R}$;
- (f_{13}) $|f(t)|^{p-2} f(t) f'(t)$ *is a non-decreasing function in* $t \in \mathbb{R}$.

PROOF. We only need to verify that (f_{10}) holds, the other properties can be found in Lemma 2.1 of [3] and [30]. Since

$$\begin{aligned}(f^2)'' = 2(f')^2 + 2ff'' &= \frac{2}{[1+2^{p-1}|f(t)|^p]^{2/p}} - \frac{2^p|f(t)|^p}{[1+2^{p-1}|f(t)|^p]^{2/p+1}} \\ &= \frac{2}{[1+2^{p-1}|f(t)|^p]^{2/p+1}} > 0, \quad t > 0,\end{aligned}$$

$f^2(t)$ is strictly convex and $f^2(st) \le sf^2(t)$ for all $t \in \mathbb{R}$ and $s \in [0,1]$. On the other hand, setting $h(s) = f(st) - sf(t)$, $s \ge 1$, from (f_6) we have $h'(s) = tf'(st) - f(t) \le tf'(t) - f(t) \le 0$. Note that $h(1) = 0$, hence $h(s) \le 0$, for all $s \ge 1$, that is, $f(st) \le sf(t)$ and the conclusion follows. □

REMARK 2.3. (f_{10}) implies that

$$\begin{aligned}&|f(st)|^p \le s^{p/2}|f(t)|^p \quad \text{and} \quad |f(s^{-1}t)|^p \ge \frac{1}{s^{p/2}}\,|f(t)|^p, \quad \text{for all } s \le 1,\\ &|f(st)|^p \le s^{p}|f(t)|^p \quad \text{and} \quad |f(s^{-1}t)|^p \ge \frac{1}{s^{p}}\,|f(t)|^p, \quad \text{for all } s \ge 1.\end{aligned}$$

We only need to verify

$$\begin{aligned}|f(s^{-1}t)|^p &\ge \frac{1}{s^{p/2}}\,|f(t)|^p, \quad \text{for all } s \le 1,\\ |f(s^{-1}t)|^p &\ge \frac{1}{s^{p}}\,|f(t)|^p, \quad \text{for all } s \ge 1.\end{aligned}$$

Indeed, for every $0 < s \le 1$, by (f_{10}), we have

$$f^2(t) = f^2(ss^{-1}t) \le sf^2(s^{-1}t) \ \Rightarrow\ |f(s^{-1}t)|^p \ge \frac{1}{s^{p/2}}\,|f(t)|^p.$$

Similarly, we obtain

$$f^2(t) = f^2(ss^{-1}t) \le s^2f^2(s^{-1}t) \ \Rightarrow\ |f(s^{-1}t)|^p \ge \frac{1}{s^{p}}\,|f(t)|^p, \quad \text{for all } s \ge 1.$$

REMARK 2.4. Actually, we have more accurate estimates than in Remark 2.9. Since

$$(|f(t)|^p)'' = \frac{p(p-1)|f(t)|^{p-2} + p(p-2)2^{p-1}|f(t)|^{2p-2}}{(1+2^{p-1}|f(t)|^p)^{2/p+1}} \ge 0,$$

$|f(t)|^p$ is convex for $p \ge 2$ and $|f(st)|^p \le s|f(t)|^p$, for all $t \in \mathbb{R}$ and $s \in [0,1]$.

Now we introduce a set

$$\mathcal{E}_f = \bigg\{v\colon \mathbb{R}^N \to \mathbb{R} \text{ measurable} : \int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx < \infty\bigg\}.$$

PROPOSITION 2.5.

(a) *$\mathcal{E}_f$ is a linear space.*

(b) *We define* $|\cdot|_f$ *on* $\mathcal{E}_f$ *as follows:*

$$|v|_f = \inf_{\xi>0} \xi\bigg(1 + \int_{\mathbb{R}^N} V(x)|f(\xi^{-1}v)|^p\,dx\bigg),$$

then it holds that

$$|v|_f \le 2\max\bigg\{\bigg(\int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx\bigg)^{1/p}, \bigg(\int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx\bigg)^{2/p}\bigg\} \tag{2.2}$$

and

$$|v|_f \ge \begin{cases} c_p \min\bigg\{\bigg(\displaystyle\int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx\bigg)^{1/p}, \\ \qquad\qquad \bigg(\displaystyle\int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx\bigg)^{2/p}\bigg\}, & \text{if } p>2, \\ \min\bigg\{\bigg(\displaystyle\int_{\mathbb{R}^N} V(x)|f(v)|^2\,dx\bigg)^{1/2}, \\ \qquad\qquad \displaystyle\int_{\mathbb{R}^N} V(x)|f(v)|^2\,dx\bigg\}, & \text{if } p=2, \end{cases} \tag{2.3}$$

where $c_p = \min\{1, ((p-2)/2)^{2/p}\}$.

(c) $(\mathcal{E}_f, |\cdot|_f)$ *is a Banach space.*

(d) $\mathcal{E}_f$ *is separable and reflexive when* $p>2$.

Proof. (a) It is easy to see that $\mathcal{E}_f$ is a nonempty set. Since $|f(t)|^p$ $(p\ge 2)$ is a convex function and satisfies the Δ_2 condition (i.e. $|f(2t)|^p \le C|f(t)|^p$), $\mathcal{E}_f$ is a linear space. Indeed, $0\in E_f$. Let $v\in\mathcal{E}_f$, $\alpha\in\mathbb{R}\setminus\{0\}$ and $k\in\mathbb{N}$ be such that $|\alpha|/2^k\in(0,1)$. By the convexity of f^p, we have

$$\begin{aligned}\int_{\mathbb{R}^N} V(x)|f(\alpha v)|^p\,dx &= \int_{\mathbb{R}^N} V(x)\bigg|f\bigg(2^k\frac{|\alpha|}{2^k}v\bigg)\bigg|^p\,dx \\ &\le C^k\int_{\mathbb{R}^N} V(x)\bigg|f\bigg(\frac{|\alpha|}{2^k}v\bigg)\bigg|^p\,dx \le \frac{C^k|\alpha|}{2^k}\int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx,\end{aligned}$$

which means that $\alpha v\in\mathcal{E}_f$. Let $u,v\in\mathcal{E}_f$, using the convexity of f^p, we have

$$\int_{\mathbb{R}^N} V(x)|f(u+v)|^p\,dx \le \frac{1}{2}\int_{\mathbb{R}^N} V(x)|f(2u)|^p\,dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|f(2v)|^p\,dx,$$

which implies that $u+v\in\mathcal{E}_f$. Thus $\mathcal{E}_f$ is a linear space.

(b) *Case 1.* If $\int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx<1$, let $\xi = \big(\int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx\big)^{1/p}$. By the definition of $|v|_f$ and Remark 2.9, we have

$$\begin{aligned}|v|_f &\le \bigg(\int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx\bigg)^{1/p} + \bigg(\int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx\bigg)^{1/p} \\ &\quad\cdot\int_{\mathbb{R}^N} V(x)\bigg|f\bigg(v\Big/\bigg(\int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx\bigg)^{1/p}\bigg)\bigg|^p\,dx \\ &\le \bigg(\int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx\bigg)^{1/p}\end{aligned}$$

$$
\begin{aligned}
&+\Big(\int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx\Big)^{-1+1/p}\int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx\\
&=2\Big(\int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx\Big)^{1/p}.
\end{aligned}
$$

Case 2. If $\int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx \geq 1$, let $\xi = \big(\int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx\big)^{2/p}$. By Remark 2.9, we have

$$
\begin{aligned}
|v|_f \leq{}&\Big(\int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx\Big)^{2/p}+\Big(\int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx\Big)^{2/p}\\
&\cdot\int_{\mathbb{R}^N} V(x)\bigg|f\bigg(v\Big/\Big(\int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx\Big)^{2/p}\bigg)\bigg|^p\,dx\\
\leq{}&\Big(\int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx\Big)^{2/p}+\Big(\int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx\Big)^{2/p}\\
={}&2\Big(\int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx\Big)^{2/p}.
\end{aligned}
$$

On the other hand, by Remark 2.9, for every $\xi \geq 1$, we deduce that

$$
\begin{aligned}
&\xi+\xi\int_{\mathbb{R}^N} V(x)|f(\xi^{-1}v)|^p\,dx \geq \xi+\frac{1}{\xi^{p-1}}\int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx\\
&\geq \frac{p}{p-1}\,(p-1)^{1/p}\Big(\int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx\Big)^{1/p}\geq\Big(\int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx\Big)^{1/p}.
\end{aligned}
$$

For every $\xi \leq 1$, by Remark 2.9, we get

$$
\begin{aligned}
\xi+\xi\int_{\mathbb{R}^N} V(x)|f(\xi^{-1}v)|^p\,dx &\geq \xi+\frac{1}{\xi^{p/2-1}}\int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx\\
&\geq\begin{cases}\dfrac{p}{p-2}\Big(\dfrac{p-2}{2}\Big)^{2/p}\Big(\displaystyle\int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx\Big)^{2/p} & \text{if } p>2,\\ \displaystyle\int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx & \text{if } p=2,\end{cases}\\
&\geq\begin{cases}\Big(\dfrac{p-2}{2}\Big)^{2/p}\Big(\displaystyle\int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx\Big)^{2/p} & \text{if } p>2,\\ \displaystyle\int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx & \text{if } p=2.\end{cases}
\end{aligned}
$$

Hence, (2.3) is proved.

(c) First, we will show that $|\cdot|_f$ is a norm on $\mathcal{E}_f$.

1. Obviously, $v=0$ implies that $|v|_f=0$. On the other hand, we assume that $|v|_f=0$, we infer that $v=0$. By (2.3), we see that

$$
\int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx=0
$$

which implies that $v = 0$ almost everywhere in $\mathbb{R}^N$ and so we conclude that $v = 0$ in $\mathcal{E}_f$.

2. For every $\lambda \in \mathbb{R} \setminus \{0\}$ ($\lambda = 0$ obviously holds true). By the definition of $|v|_f$ and since f^2 is an even function, we have

$$\begin{aligned}
|\lambda v|_f &= \inf_{\xi>0} \xi\biggl(1 + \int_{\mathbb{R}^N} V(x)|f(\xi^{-1}\lambda v)|^p \,dx\biggr) \\
&= \inf_{\xi>0} \xi\biggl(1 + \int_{\mathbb{R}^N} V(x)\biggl|f\biggl(\frac{1}{\xi|\lambda|^{-1}} v\biggr)\biggr|^p \,dx\biggr) \\
&= |\lambda| \inf_{\xi>0} |\lambda|^{-1}\xi\biggl(1 + \int_{\mathbb{R}^N} V(x)\biggl|f\biggl(\frac{1}{\xi|\lambda|^{-1}} v\biggr)\biggr|^p \,dx\biggr) \\
&= |\lambda| \inf_{\xi>0} \xi\biggl(1 + \int_{\mathbb{R}^N} V(x)|f(\xi^{-1}v)|^p \,dx\biggr).
\end{aligned}$$

That is, $|\lambda v|_f = |\lambda||v|_f$, for any $\lambda \in \mathbb{R}$, and $v \in \mathcal{E}_f$.

3. We verify the triangle inequality. Let $u, v \in \mathcal{E}_f$, using the convexity of $|f|^p$, we get

$$\begin{aligned}
&\int_{\mathbb{R}^N} V(x)|f(\xi^{-1}(u+v))|^p \,dx \\
&\qquad\le \frac{1}{2}\int_{\mathbb{R}^N} V(x)|f(2\xi^{-1}u)|^p \,dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|f(2\xi^{-1}v)|^p \,dx.
\end{aligned}$$

Hence

$$\begin{aligned}
&\xi\biggl(1 + \int_{\mathbb{R}^N} V(x)|f(\xi^{-1}(u+v))|^p \,dx\biggr) \\
&\quad\le \frac{\xi}{2}\biggl(1 + \int_{\mathbb{R}^N} V(x)|f(2\xi^{-1}u)|^p \,dx\biggr) + \frac{\xi}{2}\biggl(1 + \int_{\mathbb{R}^N} V(x)|f(2\xi^{-1}v)|^p \,dx\biggr)
\end{aligned}$$

which implies that

$$|u+v|_f \le |u|_f + |v|_f, \quad \text{for all } u, v \in \mathcal{E}_f.$$

The next task is to show that $(\mathcal{E}_f, |\cdot|_f)$ is a Banach space. Let $\{v_n\}$ be a Cauchy sequence in $\mathcal{E}_f$, that is, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|v_{n+l} - v_n|_f < \varepsilon, \quad \text{for all } n \ge N \text{ and } l \in \mathbb{N}.$$

Let $\varepsilon = 1/2^{i+2}$, $i = 1, 2, \ldots$, we can find a subsequence $\{v_{n_i}\}$ of $\{v_n\}$ such that

$$|v_{n_{i+1}} - v_{n_i}|_f < \frac{1}{2^{i+2}}, \quad i = 1, 2, \ldots$$

From the definition of $|\cdot|_f$, it follows that there exist $\xi_i \in (0,1)$, $i = 1, 2, \ldots$, such that

$$\xi_i\biggl[1 + \int_{\mathbb{R}^N} V(x)|f(\xi_i^{-1}(v_{n_{i+1}} - v_{n_i}))|^p \,dx\biggr] < \frac{1}{2^{i+1}}, \quad i = 1, 2, \ldots, \tag{2.4}$$

which implies that

$$\sum_{i=1}^{\infty} \xi_i \leq \sum_{i=1}^{\infty} \frac{1}{2^{i+1}} = \frac{1}{2}.$$

For every $x \in \mathbb{R}^N$, we define

$$h_k(x) = \sum_{i=1}^{k} [\xi_i V(x) |f(\xi_i^{-1}(v_{n_{i+1}}(x) - v_{n_i}(x)))|^p] + \frac{1}{2} V(x) |f(2v_{n_1}(x))|^p,$$
$$h(x) = \sum_{i=1}^{\infty} [\xi_i V(x) |f(\xi_i^{-1}(v_{n_{i+1}}(x) - v_{n_i}(x)))|^p] + \frac{1}{2} V(x) |f(2v_{n_1}(x))|^p.$$

Obviously, $\lim_{k\to\infty} h_k(x) = h(x)$ for all $x \in \mathbb{R}^N$. Thus, by Fatou's lemma, we get

$$\begin{aligned} \int_{\mathbb{R}^N} h(x)\,dx &\leq \liminf_{k\to\infty} \int_{\mathbb{R}^N} h_k(x)\,dx \\ &\leq \liminf_{k\to\infty} \int_{\mathbb{R}^N} \sum_{i=1}^{k} [\xi_i V(x) |f(\xi_i^{-1}(v_{n_{i+1}}(x) - v_{n_i}(x)))|^p]\,dx \\ &\quad + \int_{\mathbb{R}^N} \frac{1}{2} V(x) |f(2v_{n_1}(x))|^p\,dx \\ &\leq \sum_{i=1}^{\infty} \frac{1}{2^{i+1}} + C < \infty. \end{aligned}$$

This shows that $h \in L^1(\mathbb{R}^N)$. Thus, $h(x) < +\infty$ almost everywhere in $\mathbb{R}^N$. If $x_0 \in \mathbb{R}^N$ is such that

$$\sum_{i=1}^{\infty} [\xi_i V(x_0) |f(\xi_i^{-1}(v_{n_{i+1}}(x_0) - v_{n_i}(x_0)))|^p] < \infty, \tag{2.5}$$

then $\{v_{n_k}(x_0)\}$ is a Cauchy sequence in $\mathbb{R}$. In fact, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|v - 0| = |f^{-1}(f(v)) - 0| < \varepsilon, \quad \text{for all } |f(v)| = |f(v) - f(0)| < \delta. \tag{2.6}$$

From (2.5), it follows that for the above $\delta > 0$, there exists $N \in \mathbb{N}$ such that

$$\sum_{i=k+1}^{k+l} [\xi_i V(x_0) |f(\xi_i^{-1}(v_{n_{i+1}}(x_0) - v_{n_i}(x_0)))|^p] < V(x_0)\delta^p, \tag{2.7}$$

for all $k > N$ and $l \in \mathbb{N}$.

By the convexity of $|f|^p$ and $f(0)=0$, we see that

$$\begin{aligned}
&V(x_0)|f(v_{k+l}(x_0)-v_{k+1}(x_0))|^p\\
&\quad=V(x_0)|f(v_{k+l}(x_0)-v_{k+l-1}(x_0)+\ldots+v_{k+2}(x_0)-v_{k+1}(x_0))|^p\\
&\quad=V(x_0)|f(\xi_{k+l}\xi_{k+l}^{-1}(v_{k+l}(x_0)-v_{k+l-1}(x_0))+\ldots\\
&\qquad+\xi_{k+1}\xi_{k+1}^{-1}(v_{k+2}(x_0)-v_{k+1}(x_0))+(1-\xi_{k+1}-\ldots-\xi_{k+l})0)|^p\\
&\quad\le\sum_{i=k+1}^{k+l}[\xi_i V(x_0)|f(\xi_i^{-1}(v_{n_{i+1}}(x_0)-v_{n_i}(x_0)))|^p].
\end{aligned}$$

Hence, by (2.7), we obtain

$$|f(v_{n_{k+l}}(x_0)-v_{n_{k+1}}(x_0))|<\delta,\quad\text{for all } k>N \text{ and } l\in\mathbb{N}.$$

From (2.6), it follows that

$$|v_{n_{k+l}}(x_0)-v_{n_{k+1}}(x_0)|<\varepsilon,\quad\text{for all } k>N \text{ and } l\in\mathbb{N}.$$

That is, $\{v_{n_k}(x_0)\}$ is a Cauchy sequence in $\mathbb{R}$. Hence there exists $v(x_0)$ such that $\lim\limits_{k\to\infty}v_{n_k}(x_0)=v(x_0)$. Therefore, $v_{n_k}(x)\to v(x)$ for almost every $x\in\mathbb{R}^N$. Using the convexity of $|f|^p$ and $f(0)=0$, we get

$$\begin{aligned}
&V(x)|f(v_{n_k}(x))|^p\\
&\quad=V(x)|f(v_{n_k}(x)-v_{n_{k-1}}(x)+\ldots+v_{n_2}(x)-v_{n_1}(x)+v_{n_1}(x))|^p\\
&\quad=V(x)\bigg|f\bigg(\xi_{k-1}\xi_{k-1}^{-1}(v_{n_k}(x)-v_{n_{k-1}}(x))+\ldots\\
&\qquad+\xi_1\xi_1^{-1}(v_{n_2}(x)-v_{n_1}(x))+\frac{1}{2}\,2v_{n_1}(x)+\bigg(\frac{1}{2}-\xi_1-\ldots-\xi_{k-1}\bigg)\bigg)\bigg|^p\\
&\quad\le\sum_{i=1}^{k-1}[V(x)\xi_i|f(v_{n_{i+1}}(x)-v_{n_i}(x))|^p]+\frac{1}{2}\,V(x)|f(2v_{n_1}(x))|^p\le h(x).
\end{aligned}$$

From the Lebesgue Dominated Convergence theorem, it follows that

$$\lim_{k\to\infty}\int_{\mathbb{R}^N}V(x)|f(v_{n_k})|^p\,dx=\int_{\mathbb{R}^N}V(x)|f(v)|^p\,dx.$$

By (2.2), we see that $v\in\mathcal{E}_f$. It remains to show that $v_n\to v$ in $\mathcal{E}_f$.

Since $\{v_n\}$ is a Cauchy sequence, for every $\varepsilon>0$, there exist $N\in\mathbb{N}$ and $\xi_0\in(0,+\infty)$ such that

$$\xi_0\bigg(1+\int_{\mathbb{R}^N}V(x)|f\xi_0^{-1}(v_{n+l}(x)-v_n(x)))|^p\,dx\bigg)<\varepsilon,\quad\text{for all } l\in\mathbb{N}. \tag{2.8}$$

Replacing v_{n_1} and $1/2^{i+1}$ by v_n and $\xi_0/2^{i+1}$ in (2.4) (clearly $\xi_i \leq \xi_0/2^{i+1}$, $i = 1, 2, \ldots$), by the convexity of $|f|^p$ and $f(0) = 0$, we have

$$\xi_0 V(x)|f(\xi_0^{-1}(v_{n_k}(x) - v_n(x)))|^p \leq \xi_0 \sum_{i=1}^{k-1} \frac{\xi_i}{\xi_0} V(x)|f(\xi_i^{-1}(v_{n_{i+1}}(x) - v_{n_i}(x)))|^p \leq h(x)$$

for almost every $x \in \mathbb{R}^N$. By the Lebesgue Dominated Convergence theorem, we obtain

$$\xi_0 \int_{\mathbb{R}^N} V(x)|f(\xi_0^{-1}(v(x) - v_n(x)))|^p \, dx = \lim_{k\to\infty} \xi_0 \int_{\mathbb{R}^N} V(x)|f(\xi_0^{-1}(v_{n_k}(x) - v_n(x)))|^p \, dx.$$

Replacing v_{n+l} by v_{n_k} in (2.8), we get

$$|v_n - v|_f = \inf_{\xi>0} \xi\bigg(1 + \int_{\mathbb{R}^N} V(x)|f(\xi^{-1}(v(x) - v_n(x)))|^p \, dx\bigg) \leq \xi_0 \int_{\mathbb{R}^N} V(x)|f(\xi_0^{-1}(v(x) - v_n(x)))|^p \, dx \leq \varepsilon.$$

Consequently, $v_n \to v$ in $\mathcal{E}_f$.

(d) If $p > 2$, the function $|f(t)|^p$ is a convex N-function in view of the fact

$$|f(t)|^p = 0 \iff t = 0, \quad \text{and} \quad \lim_{t\to\infty} \frac{|f(t)|^p}{t} = +\infty, \quad \lim_{t\to 0} \frac{|f(t)|^p}{t} = 0,$$

where we have used (f_4) and (f_9) of Lemma 2.2. From (f_{10}) it follows that $|f(t)|^p$ satisfies the Δ_2 condition. By Theorem 1.10 in [35, p. 64], we see that $\mathcal{E}_f$ is separable. By Remark 2.9, $|f(t)|^p$ satisfies the ∇_2 condition. From Corollary 12 in [28, p. 113], space $\mathcal{E}_f$ is reflexive. □

Remark 2.6. Observe that when $p = 2$, the function $|f(t)|^p$ is not an N-function, but a Young function. So $\mathcal{E}_f$ might not be separable and reflexive.

Let $\mathcal{D}^{1,p}(\mathbb{R}^N)$ ($N \geq 3$) be the usual Banach space defined as

$$\mathcal{D}^{1,p}(\mathbb{R}^N) = \bigg\{v \in L^{p^*}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |\nabla v|^p \, dx < +\infty\bigg\}$$

which is equipped with the norm

$$\|v\|_{\mathcal{D}^{1,p}} = \bigg(\int_{\mathbb{R}^N} |\nabla v|^p \, dx\bigg)^{1/p} \quad \text{and} \quad S_p = \inf_{v\in\mathcal{D}^{1,p}(\mathbb{R}^N)\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla v|^p \, dx}{\bigg(\displaystyle\int_{\mathbb{R}^N} |v|^{p^*} dx\bigg)^{p/p^*}}.$$

We introduce a vector space

$$E = \bigg\{v \in \mathcal{E}_f : \int_{\mathbb{R}^N} |\nabla v|^p \, dx < +\infty\bigg\} = \mathcal{E}_f \cap \mathcal{D}^{1,p}(\mathbb{R}^N)$$

equipped with the norm

$$\|v\| = \|\nabla v\|_{L^p} + |v|_f.$$

From Proposition 2.5, it is easy to verify that $\|\cdot\|$ is a norm on E. Moreover, we can prove the following conclusion.

Proposition 2.7.

(a) *E is a Banach space with respect to the norm $\|\cdot\|$.*
(b) *X is embedded continuously into E.*
(c) *When $p > 2$, E is separable and reflexive.*

Proof. (a) It is standard to verify that E is a complete space with the norm $\|\cdot\|$, we refer the readers to the proof of Proposition 1.1 in [10].

(b) By the definition of $|v|_f$, we get

$$\begin{aligned} |v|_f &= \inf_{\xi>0} \xi\bigg(1 + \int_{\mathbb{R}^N} V(x)|f(\xi^{-1}v)|^p\,dx\bigg) \\ &\le \inf_{\xi>0}\bigg(\xi + \frac{1}{\xi^{p-1}}\int_{\mathbb{R}^N} V(x)|v|^p\,dx\bigg) = C\bigg(\int_{\mathbb{R}^N} V(x)|v|^p\,dx\bigg)^{1/p}. \end{aligned}$$

Therefore, it is easy to check that

$$\|v\| = \|\nabla v\|_{L^p} + |v|_f \le C\|v\|_X,$$

which means that X can be regarded as a subspace of E, thus X is continuously embedded into E.

(c) If $p > 2$, by Proposition 2.5 and properties of $\mathcal{D}^{1,p}(\mathbb{R}^N)$, it is easy to conclude the desired. □

Motivated by the ideas of Lemma 2.1 in [14], we prove a precise estimate for the norm $\|\cdot\|$ in the Banach space E.

Lemma 2.8. *There exist two constants $C_1, C_2 > 0$ depending only on p such that for any $v \in E$, it holds that*

$$\begin{aligned} C_1 \min\{\|v\|^{p/2}, \|v\|^p\} &\le \int_{\mathbb{R}^N} (|\nabla v|^p + V(x)|f(v)|^p)\,dx \\ &\le C_2 \max\{\|v\|^{p/2}, \|v\|^p\}. \end{aligned} \tag{2.9}$$

Proof. We first prove that

$$C_1 \min\{\|v\|^{p/2}, \|v\|^p\} \le \int_{\mathbb{R}^N} (|\nabla v|^p + V(x)|f(v)|^p)\,dx.$$

Case 1. If $\int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx < 1$, by (2.2), using the inequality

$$(x^{1/p} + y^{1/p}) \le 2^{1-1/p}(x+y)^{1/p} \quad \text{for } x, y \ge 0,$$

we get

$$\|v\| = \|\nabla v\|_{L^p} + |v|_f \le \|\nabla v\|_{L^p} + 2\bigg(\int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx\bigg)^{1/p}$$
$$\le 2^{(2p-1)/p}\bigg(\|\nabla v\|_{L^p}^p + \int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx\bigg)^{1/p}.$$

Case 2. If $\int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx \ge 1$, by (2.2), using the inequality

$$(x^{1/p} + y^{1/p}) \le 2^{1-1/p}(x+y)^{1/p} \quad \text{for } x, y \ge 0,$$

we get

$$\begin{aligned}\|v\|^{p/2} &= (\|\nabla v\|_{L^p} + |v|_f)^{p/2} \le \bigg(\|\nabla v\|_{L^p} + 2\bigg(\int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx\bigg)^{2/p}\bigg)^{p/2}\\
&\le \bigg[2\bigg(\|\nabla v\|_{L^p} + \bigg(\int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx\bigg)^{1/p}\bigg)^2\bigg]^{p/2}\\
&\le \bigg[4\bigg(\|\nabla v\|_{L^p}^2 + \bigg(\int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx\bigg)^{2/p}\bigg)\bigg]^{p/2}\\
&\le 4^{p/2}2^{p/2-1}\bigg(\|\nabla v\|_{L^p}^p + \int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx\bigg).\end{aligned}$$

Therefore, we see that

$$C_1 \min\{\|v\|^{p/2}, \|v\|^p\} \le \int_{\mathbb{R}^N} (|\nabla v|^p + V(x)|f(v)|^p)\,dx, \quad C_1 = \frac{1}{2^{2p-1}}.$$

On the other hand, from (2.3), we note that the case $p = 2$ is similar to $p > 2$, so we consider only the case $p > 2$.

Case 1. If $\big(\int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx\big)^{1/p} \ge 1$, by (2.3), we obtain

$$\begin{aligned}\|v\| &= \|\nabla v\|_{L^p} + |v|_f\\
&\ge \|\nabla v\|_{L^p} + \min\bigg\{1, \bigg(\frac{p-2}{2}\bigg)^{2/p}\bigg\}\bigg(\int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx\bigg)^{1/p}\\
&\ge \min\bigg\{1, \bigg(\frac{p-2}{2}\bigg)^{2/p}\bigg\}\bigg(\|\nabla v\|_{L^p}^p + \int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx\bigg)^{1/p},\end{aligned}$$

that is,

$$\|\nabla v\|_{L^p}^p + \int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx \le \bigg(\frac{1}{\min\{1, ((p-2)/2)^{2/p}\}}\bigg)^p \|v\|^p. \tag{2.10}$$

Case 2. If $\big(\int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx\big)^{1/p} \le 1$, by (2.3), we have

$$\begin{aligned}\|v\| &= \|\nabla v\|_{L^p} + |v|_f \qquad (2.11)\\
&\ge \|\nabla v\|_{L^p} + \min\bigg\{1, \bigg(\frac{p-2}{2}\bigg)^{2/p}\bigg\}\bigg(\int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx\bigg)^{2/p}.\end{aligned}$$

When $\|\nabla v\|_{L^p} \le 1$, by (2.11) and the inequality $a^p + b^p \le (a+b)^p$ for $a, b \ge 0$, we deduce that

$$\begin{aligned}\|v\|^{p/2} &\ge \bigg(\|\nabla v\|_{L^p} + \min\bigg\{1, \bigg(\frac{p-2}{2}\bigg)^{2/p}\bigg\}\bigg(\int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx\bigg)^{2/p}\bigg)^{p/2} \\ &\ge \bigg(\|\nabla v\|_{L^p}^2 + \min\bigg\{1, \bigg(\frac{p-2}{2}\bigg)^{2/p}\bigg\}\bigg(\int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx\bigg)^{2/p}\bigg)^{p/2} \\ &\ge \bigg(\min\bigg\{1, \bigg(\frac{p-2}{2}\bigg)^{2/p}\bigg\}\bigg)^{p/2}\bigg(\|\nabla v\|_{L^p}^p + \int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx\bigg).\end{aligned}$$

That is,

$$\|\nabla v\|_{L^p}^p + \int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx \le \frac{1}{(\min\{1, ((p-2)/2)^{2/p}\})^{p/2}}\,\|v\|^{p/2}. \tag{2.12}$$

When $\|\nabla v\|_{L^p} \ge 1$, by (2.11) and the inequality $x^p + x^{p/2}y^{p/2} \le (x+y)^p$ for $x, y \ge 0$, we have

$$\begin{aligned}\|v\|^{p} &\ge \bigg(\|\nabla v\|_{L^p} + \min\bigg\{1, \bigg(\frac{p-2}{2}\bigg)^{2/p}\bigg\}\bigg(\int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx\bigg)^{2/p}\bigg)^{p} \\ &\ge \bigg(\min\bigg\{1, \bigg(\frac{p-2}{2}\bigg)^{2/p}\bigg\}\bigg)^{p/2}\bigg(\|\nabla v\|_{L^p}^p + \|\nabla v\|_{L^p}^{p/2}\int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx\bigg) \\ &\ge \bigg(\min\bigg\{1, \bigg(\frac{p-2}{2}\bigg)^{2/p}\bigg\}\bigg)^{p/2}\bigg(\|\nabla v\|_{L^p}^p + \int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx\bigg).\end{aligned}$$

That is,

$$\|\nabla v\|_{L^p}^p + \int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx \le \frac{1}{(\min\{1, ((p-2)/2)^{2/p}\})^{p/2}}\,\|v\|^{p}. \tag{2.13}$$

Therefore, it follows from (2.10), (2.12) and (2.13) that

$$\|\nabla v\|_{L^p}^p + \int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx \le C_2 \max\{\|v\|^{p/2}, \|v\|^p\},$$

where $C_2 > 0$ is a constant depending only on p. $\square$

Remark 2.9. From (f_2) of Lemma 2.2, it is easy to check that

$$\|f(v)\|_X^p = \int_{\mathbb{R}^N} (|\nabla f(v)|^p + V(x)|f(v)|^p)\,dx \le \|\nabla v\|_{L^p}^p + \int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx.$$

Corollary 2.10. *Let $(V_1), (V_2)$ hold, then there holds*

$$\|v\|_{W^{1,p}}^p \le C\bigg(\int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx + \int_{\mathbb{R}^N} |\nabla v|^p\,dx\bigg) \le C\max\{\|v\|^{p/2}, \|v\|^p\}.$$

PROOF. By hypotheses (V_1) and (V_2), using Hölder's and Young's inequalities, we have

$$\begin{aligned}
\int_{\mathbb{R}^N} |v|^p\,dx &= \int_{|v|\le 1} |v|^p\,dx + \int_{|v|>1} |v|^p\,dx \\
&= \frac{1}{V_0}\int_{|v|\le 1} V(x)|v|^p\,dx \\
&\quad + \int_{|v|>1} V(x)^{-(p^*-p)/(p(p^*-1))} V(x)^{(p^*-p)/(p(p^*-1))} \\
&\qquad\cdot |v|^{p^*-p/(p^*-1)} |v|^{(p^*(p-1))/(p^*-1)}\,dx \\
&\le \frac{1}{CV_0}\int_{|v|\le 1} V(x)|f(v)|^p\,dx + \bigg(\int_{\mathbb{R}^N} V(x)^{-1/(p-1)}\,dx\bigg)^{(p-1)(p^*-p)/(p(p^*-1))} \\
&\quad \cdot \bigg(\int_{|v|>1} V(x)|v|^p\,dx\bigg)^{(p^*-p)/(p(p^*-1))} \bigg(\int_{\mathbb{R}^N} |v|^{p^*}\,dx\bigg)^{(p-1)/(p^*-1)} \\
&\le \frac{1}{CV_0}\int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx \\
&\quad + C\bigg(\int_{|v|>1} V(x)|v|^{2p}\,dx\bigg)^{(p^*-p)/(p(p^*-1))} \bigg(\int_{\mathbb{R}^N} |\nabla v|^p\,dx\bigg)^{p^*(p-1)/(p(p^*-1))} \\
&\le \frac{1}{CV_0}\int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx \\
&\quad + C\bigg(\int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx\bigg)^{(p^*-p)/(p(p^*-1))} \bigg(\int_{\mathbb{R}^N} |\nabla v|^p\,dx\bigg)^{p^*(p-1)/(p(p^*-1))} \\
&\le C\bigg(\int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx + \int_{\mathbb{R}^N} |\nabla v|^p\,dx\bigg).
\end{aligned}$$

LEMMA 2.11.

(a) *There exists a positive constant C such that for all $v \in E$,*

$$\frac{\displaystyle\int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx}{\bigg[1 + \bigg(\displaystyle\int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx\bigg)^{(p-1)/p}\bigg]} \le C(\|v\| + \|v\|^{p/2}).$$

(b) *If $v_n \to v$ in E, then*

$$\int_{\mathbb{R}^N} V(x)\big||f(v_n)|^p - |f(v)|^p\big|\,dx \to 0$$

and

$$\int_{\mathbb{R}^N} V(x)|f(v_n) - f(v)|^p\,dx \to 0.$$

(c) *If $v_n \to v$ almost everywhere, $\int_{\mathbb{R}^N} V(x)|f(v_n)|^p\,dx \to \int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx$ and $\int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx < +\infty$, then $|v_n - v|_f \to 0$ as $n \to \infty$.*

(d) *If* (V_1) *and* (V_2) *hold, the map* $v \mapsto f(v)$ *from* E *into* $L^s(\mathbb{R}^N)$ *is continuous for* $1 \le s \le 2p^*$ *and compact for* $1 \le s < 2p^*$.

Proof. (a) Let us define, for $v \in E$ and $\xi > 0$,

$$A_\xi = \{x \in \mathbb{R}^N : \xi^{-1}|v| \le 1\}.$$

From properties (f_3) and (f_9) in Lemma 2.2, we have

$$\begin{aligned}\int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx &= \int_{A_\xi} V(x)|f(v)|^p\,dx + \int_{A_\xi^c} V(x)|f(v)|^p\,dx \\ &\le \int_{A_\xi} V(x)|f(v)|^{p-1}|v|\,dx + C\int_{A_\xi^c} V(x)|v|^{p/2}\,dx.\end{aligned}$$

By Hölder's inequality and (f_9) in Lemma 2.2, we have

$$\begin{aligned}\int_{A_\xi} V(x)|f(v)|^{p-1}|v|\,dx &\le \bigg(\int_{A_\xi} V(x)|f(v)|^p\,dx\bigg)^{(p-1)/p}\bigg(\int_{A_\xi} V(x)|v|^p\,dx\bigg)^{1/p} \\ &\le \bigg(\int_{A_\xi} V(x)|f(v)|^p\,dx\bigg)^{(p-1)/p} C\xi\bigg(\int_{A_\xi} V(x)|f(\xi^{-1}v)|^p\,dx\bigg)^{1/p} \\ &\le \bigg(\int_{A_\xi} V(x)|f(v)|^p\,dx\bigg)^{(p-1)/p} C\xi\bigg[1+\int_{A_\xi} V(x)|f(\xi^{-1}v)|^p\,dx\bigg],\end{aligned}$$

where we used the inequality $s^{1/p} \le 1+s$ for all $s \ge 0$.

By (f_9) in Lemma 2.2, we get

$$\begin{aligned}\int_{A_\xi^c} V(x)|v|^{p/2}\,dx &= \xi\int_{A_\xi^c} V(x)|v|^{p/2-1}|\xi^{-1}v|\,dx \\ &\le \xi\bigg(\int_{A_\xi^c} V(x)|v|^{p/2}\,dx\bigg)^{(p-2)/p}\bigg(\int_{A_\xi^c} V(x)|\xi^{-1}v|^{p/2}\,dx\bigg)^{2/p} \\ &\le \bigg(\int_{A_\xi^c} V(x)|v|^{p/2}\,dx\bigg)^{(p-2)/p} C\xi\bigg(\int_{A_\xi^c} V(x)|f(\xi^{-1}v)|^p\,dx\bigg)^{2/p},\end{aligned}$$

which leads to

$$\int_{A_\xi^c} V(x)|v|^{p/2}\,dx \le C^{p/2}\xi^{p/2}\bigg[1+\int_{A_\xi^c} V(x)|f(\xi^{-1}v)|^p\,dx\bigg]^{p/2}.$$

Thus, conclusion (a) follows.

(b) Assume that $v_n \to v$ in E, then $v_n \to v$ in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ and $|v_n - v|_f \to 0$. From (2.3), it follows that

$$\int_{\mathbb{R}^N} V(x)|f(v_n - v)|^p\,dx \to 0,$$

which implies that, up to a subsequence, there exists $k \in L^1(\mathbb{R}^N)$ such that

$$V(x)|f(v_n - v)|^p \le k(x) \quad \text{a.e. in } \mathbb{R}^N. \tag{2.14}$$

Moreover, we can obtain that there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}^N} V(x)|f(v_n)|^p \, dx \le C.$$

Since $v_n \to v$ in $\mathcal{D}^{1,p}(\mathbb{R}^N)$, it is enough to verify that $v_n \to v$ almost everywhere in $\mathbb{R}^N$. By Fatou's lemma, we see that

$$\int_{\mathbb{R}^N} V(x)|f(v)|^p \, dx \le \liminf_{n\to\infty} \int_{\mathbb{R}^N} V(x)|f(v_n)|^p \, dx \le C. \tag{2.15}$$

By (f_{10}), (2.14) and (2.15), using the convexity of $|f|^p$, we have

$$\begin{aligned} V(x)|f(v_n)|^p &\le \frac{V(x)}{2}|f(2(v_n - v))|^p + \frac{V(x)}{2}|f(2v)|^p \\ &\le 2^{p-1}\big(V(x)|f((v_n - v))|^p + V(x)|f(v)|^p\big) \\ &\le 2^{p-1}\big(k(x) + V(x)|f(v)|^p\big) \end{aligned}$$

and

$$\begin{aligned} V(x)|f(v_n) - f(v)|^p &\le 2^{p-1}\big(V(x)|f(v_n)|^p + V(x)|f(v)|^p\big) \\ &\le 2^{p-1}\bigg(\frac{V(x)}{2}|f(2(v_n - v))|^p + \frac{V(x)}{2}|f(2v)|^p + V(x)|f(v)|^p\bigg) \\ &\le 2^{p-1}\big(2^{p-1}V(x)|f(v_n - v)|^p + 2^{p-1}V(x)|f(v)|^p + V(x)|f(v)|^p\big) \\ &\le 2^{2p-1}(k(x) + V(x)|f(v)|^p). \end{aligned}$$

By the Lebesgue Dominated Convergence theorem, the proof of (b) is completed.

(c) In virtue of (2.2), it is sufficient to prove that

$$\int_{\mathbb{R}^N} V(x)|f(v_n - v)|^p \, dx \to 0.$$

Indeed, since $\int_{\mathbb{R}^N} V(x)(|f(v_n)|^p - |f(v)|^p) \, dx \to 0$, up to a subsequence, there exists $k_1 \in L^1(\mathbb{R}^N)$ such that

$$V(x)(|f(v_n)|^p - |f(v)|^p) \le k_1(x) \quad \text{a.e. in } \mathbb{R}^N. \tag{2.16}$$

Using the convexity and evenness of $|f|^p$, by (f_{10}) and (2.16), we have

$$\begin{aligned} V(x)|f(v_n - v)|^p &\le \frac{V(x)}{2}|f(2v_n)|^p + \frac{V(x)}{2}|f(2v)|^p \\ &\le 2^{p-1}(V(x)|f(v_n)|^p + V(x)|f(v)|^p) \le 2^p(k_1(x) + V(x)|f(v)|^p). \end{aligned}$$

Since $k_1(x) + V(x)|f(v)|^p \in L^1(\mathbb{R}^N)$, due to $v_n \to v$ almost everywhere in $\mathbb{R}^N$ and the Lebesgue Dominated Convergence theorem, we can get the conclusion.

(d) From Remark 2.9 and Lemma 2.1, it follows that

$$\begin{aligned} C\|f(v)\|_{L^1}^p \le \|f(v)\|_X^p &= \int_{\mathbb{R}^N} |\nabla f(v)|^p\,dx + \int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx \\ &= \int_{\mathbb{R}^N} |\nabla v|^p|f'(v)|^p\,dx + \int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx \\ &\le \int_{\mathbb{R}^N} |\nabla v|^p\,dx + \int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx \le C_2 \max\{\|v\|^{p/2}, \|v\|^p\}, \end{aligned}$$

for all $v \in E$, which implies that $f \in L^p(\mathbb{R}^N)$. On the other hand, by (f_2), we see that

$$\begin{aligned} \bigg(\int_{\mathbb{R}^N} |f^2(v)|^{p^*}\,dx\bigg)^{1/p^*} &\le C\bigg(\int_{\mathbb{R}^N} |\nabla f^2(v)|^p\,dx\bigg)^{1/p} \\ &\le C\bigg(\int_{\mathbb{R}^N} |\nabla v|^p\,dx\bigg)^{1/p} \\ &\le C\bigg(\int_{\mathbb{R}^N} |\nabla v|^p\,dx + \int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx\bigg)^{1/p} \\ &\le C\max\{\|v\|^{1/2}, \|v\|\}, \end{aligned} \tag{2.17}$$

for all $v \in E$, which implies that $f \in L^{2p^*}(\mathbb{R}^N)$. By a standard interpolation argument, we obtain that $f \in L^s(\mathbb{R}^N)$ for $s \in [1, 2p^*]$.

Let $\{v_n\}$ be a sequence in E such that $v_n \to v$ as $n \to \infty$. We will show that the map f is continuous. That is, we need to show that $f(v_n) \to f(v)$ in $L^s(\mathbb{R}^N)$ for $s \in [1, 2p^*]$. In fact, since $v_n \to v$ in E,

$$v_{nx_i} \to v_{x_i} \quad \text{in } L^p(\mathbb{R}^N) \text{ for } i = 1, \ldots, N,$$

and from conclusion (a), we see that

$$\int_{\mathbb{R}^N} V(x)|f(v_n) - f(v)|^p\,dx \to 0. \tag{2.18}$$

Thus, up to a subsequence, there exists $h_i \in L^p(\mathbb{R}^N)$ such that

$$|v_{nx_i}(x)| \le h_i(x) \quad \text{a.e. in } \mathbb{R}^N, \text{ for } i = 1, \ldots, N, \tag{2.19}$$

and

$$v_n(x) \to v(x), \quad v_{nx_i}(x) \to v_{x_i}(x) \quad \text{a.e. in } \mathbb{R}^N, \text{ for } i = 1, \ldots, N. \tag{2.20}$$

Hence, by (f_1), (f_2), (2.19) and (2.20), we have

$$\left|\frac{\partial f(v_n)}{\partial x_i}\right| = |f'(v_n)v_{nx_i}| \le |v_{nx_i}(x)| \le h_i(x)$$

and

$$\frac{\partial f(v_n)}{\partial x_i} = f'(v_n)v_{nx_i} \to f'(v)v_{x_i} = \frac{\partial f(v)}{\partial x_i},$$

almost everywhere in $\mathbb{R}^N$ for $i = 1, 2, \ldots, N$. Therefore, by the Lebesgue Dominated Convergence theorem, we conclude that $f(v_n) \to f(v)$ in $\mathcal{D}^{1,p}(\mathbb{R}^N)$. Thus, by (2.18), we see that $f(v_n) \to f(v)$ in X, which implies that

$$f(v_n) \to f(v) \quad \text{in } L^s(\mathbb{R}^N), \text{ for } 1 \le s \le p^*. \tag{2.21}$$

Next, we will show that $f(v_n) \to f(v)$ in $L^{2p^*}(\mathbb{R}^N)$. From (2.17), we see that

$$f^2(v_n - v) \to 0 \quad \text{in } L^{p^*}(\mathbb{R}^N).$$

Thus, up to a subsequence, we conclude that $v_n \to v$ almost everywhere in $\mathbb{R}^N$ and there exists $\widehat{h} \in L^{p^*}(\mathbb{R}^N)$ such that

$$f^2(v_n - v) \le \widehat{h}(x) \quad \text{a.e. in } \mathbb{R}^N.$$

Using the convexity of f^2 and (f_{10}), we deduce that

$$|f(v_n)|^{2p^*} \le 2^{p^*-1}\big(\widehat{h}^{p^*} + |f|^{2p^*}\big) \in L^1(\mathbb{R}^N)$$

which together with the Lebesgue Dominated Convergence theorem implies that

$$f(v_n) \to f(v) \quad \text{in } L^{2p^*}(\mathbb{R}^N).$$

Combining the above with (2.21), by an interpolation argument, the part one of conclusion (c) follows. Next, we will show that the compactness is true. Let $\{v_n\} \subset E$ be a bounded sequence. By Corollary 2.10, we see that $\{v_n\}$ is bounded in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ and $\int_{\mathbb{R}^N} V(x)|f(v_n)|^p\,dx$ is bounded. Using Remark 2.9 and (2.17), we have that $\{f(v_n)\}$ is bounded in X and $L^{2p^*}(\mathbb{R}^N)$. By Corollary 2.8, we have that $\{v_n\}$ is bounded in $W^{1,p}(\mathbb{R}^N)$. Hence, up to a subsequence, there exists $v \in W^{1,p}(\mathbb{R}^N)$ such that $v_n \to v$ almost everywhere in $\mathbb{R}^N$. On the other hand, it follows from Lemma 2.1, that up to a subsequence, there is $w \in X$ such that $f(v_n) \to w$ in $L^p(\mathbb{R}^N)$. By the uniqueness of the limit, we see that $w = f(v)$ almost everywhere in $\mathbb{R}^N$. By Fatou's lemma, we see that $f \in L^{2p^*}(\mathbb{R}^N)$. Thus, using the interpolation inequality, we have that

$$\|f(v_n) - f(v)\|_s \le \|f(v_n) - f(v)\|_p^{\theta}\,\|f(v_n) - f(v)\|_{2p^*}^{1-\theta},$$

where $s \in [1, 2p^*)$ and $\theta \in (0, 1]$ such that $\theta/p + (1-\theta)/2p^* = 1/s$. Hence, the proof is completed. □

Corollary 2.12. *Let $\{v_n\} \subset E$ be a bounded sequence. If* (V_1) *and* (V_2) *hold, then there exists $v \in W^{1,p}(\mathbb{R}^N)$ such that $v_n \to v$ in $L^s(\mathbb{R}^N)$ for $1 \le s < p^*$.*

Proof. We only show that the part one of the conclusion holds, the other one is similar so we omit it. By Lemma 2.11, up to a subsequence, there exists $v \in W^{1,p}(\mathbb{R}^N)$ such that $f(v_n) \to f(v)$ in $L^s(\mathbb{R}^N)$ for $1 \le s < 2p^*$ and $v_n \to v$ almost everywhere in $\mathbb{R}^N$. Then, up to a subsequence, there exists $\widehat{k} \in L^s(\mathbb{R}^N)$ such that

$$|f(v_n) - f(v)| \le \widehat{k}(x) \quad \text{a.e. in } \mathbb{R}^N.$$

Using the convexity of f^2 and (f_{10}), we deduce that

$$|f(v_n - v)|^s \le 2^s\big(\widehat{k}^s + |f(v)|^s\big) \in L^1(\mathbb{R}^N)$$

which together with the Lebesgue Dominated Convergence theorem implies that

$$f(v_n - v) \to 0 \quad \text{in } L^s(\mathbb{R}^N), \text{ for } 1 \le s < 2p^*.$$

Using (f_{12}) of Lemma 2.2, we have that

$$\int_{\mathbb{R}^N} |v_n - v|^s\,dx \le C\bigg(\int_{\mathbb{R}^N} |f(v_n - v)|^s\,dx + \int_{\mathbb{R}^N} |f(v_n - v)|^{2s}\,dx\bigg)$$

which implies that $v_n \to v$ in $L^s(\mathbb{R}^N)$ for $s \in [1, p^*)$. □

To prove our main results, we also need to introduce the following Sobolev space. For a nonnegative measurable function ω and a real number $q > 1$, denote by $L^q_\omega(\mathbb{R}^N)$ the weighted Lebesgue space of all measurable functions u which satisfy $\int_{\mathbb{R}^N} \omega(x)|v|^q\,dx < \infty$, with the associated seminorm

$$|u|_{q,\omega} = \bigg(\int_{\mathbb{R}^N} \omega(x)|v|^q\,dx\bigg)^{1/q}.$$

We introduce two sets

$$\Omega = \{x \in \mathbb{R}^N : K(x) = 0\}, \qquad \widetilde{E} = \{v \in E : v(x) = 0 \text{ a.e. } x \in \Omega\},$$

then $\widetilde{E}$ is an infinitely dimensional Banach space with the norm of $\|\cdot\|$. Therefore, the seminorm

$$|v|_{q,K} = \bigg(\int_{\mathbb{R}^N} K(x)|v|^q\,dx\bigg)^{1/q}$$

is a norm on $\widetilde{E}$. Indeed, by (f_8), (f_9), Hölder's and Sobolev's inequalities, for every $v \in E$, we deduce that

$$\begin{aligned}
\int_{\mathbb{R}^N} K(x)|v|^q\,dx &\le \int_{\{|v|\le 1\}} K(x)|v|^q\,dx + \int_{\{|v|\ge 1\}} K(x)|v|^q\,dx \\
&\le \int_{\{|v|\le 1\}} K(x)|v|^{q/2}\,dx + \|K\|_\infty \int_{\{|v|\ge 1\}} |v|^{p^*}dx \\
&\le \int_{\mathbb{R}^N} K(x)|f(v)|^{q/2}\,dx + \|K\|_\infty S_p^{p^*/p}\bigg(\int_{\mathbb{R}^N} |\nabla v|^p\,dx\bigg)^{p^*/p} \\
&\le \|K\|_{L^{(2p^*-q)/(2p^*)}}\bigg(\int_{\mathbb{R}^N} |f(v)|^{2p^*}dx\bigg)^{q/(2p^*)} \\
&\quad + \|K\|_\infty S_p^{p^*/p}\bigg(\int_{\mathbb{R}^N} |\nabla v|^p\,dx\bigg)^{p^*/p} \\
&\le \|K\|_{L^{(2p^*-q)/(2p^*)}}\bigg(\int_{\mathbb{R}^N} |v|^{p^*}dx\bigg)^{q/(2p^*)} + \|K\|_\infty S_p^{p^*/p}\bigg(\int_{\mathbb{R}^N} |\nabla v|^p\,dx\bigg)^{p^*/p} \\
&\le \|K\|_{L^{(2p^*-q)/(2p^*)}} S_p^{q/(2p)}\bigg(\int_{\mathbb{R}^N} |\nabla v|^p\,dx\bigg)^{q/(2p)}
\end{aligned}$$

$$+ \|K\|_\infty S_p^{p^*/p}\bigg(\int_{\mathbb{R}^N} |\nabla v|^p\,dx\bigg)^{p^*/p}$$

which implies that $|v|_{q,K}$ is well defined. It remains to verify that $v \in \widetilde{E}$ and $|v|_{q,K} = 0$ implies that $v = 0$ almost everywhere in $\mathbb{R}^N$. Since

$$0 = \int_{\mathbb{R}^N} K(x)|v|^q\,dx = \int_{\{K(x)>0\}} K(x)|v|^q\,dx,$$

this implies that $v(x) = 0$ almost everywhere on $\{x \in \mathbb{R}^N : K(x) > 0\}$. From $v \in \widetilde{E}$, it follows that $v(x) = 0$ almost everywhere in $\mathbb{R}^N$.

2.1. Variational framework. The energy functional $I\colon E \to \mathbb{R}$ corresponding to problem (2.1) is given by

$$I_{\lambda,\mu}(v) = \frac{1}{p}\int_{\mathbb{R}^N} (|\nabla v|^p + V(x)|f(v)|^p)\,dx \\ - \frac{\lambda}{q}\int_{\mathbb{R}^N} K(x)|f(v)|^q\,dx - \mu\int_{\mathbb{R}^N} G(x,f(v))\,dx,$$

where $G(x,t) = \int_0^t g(x,s)\,ds$. Under assumptions (V_1)–(V_2), (K_0)–(K_1) and (g_0)–(g_2), we infer the following properties of the functional $I_{\lambda,\mu}$:

Proposition 2.13.

(a) *$I_{\lambda,\mu}$ is well defined in E,*
(b) *$I_{\lambda,\mu}$ is continuous in E,*
(c) *$I_{\lambda,\mu} \in C^1(E,\mathbb{R})$ and for any $v,\varphi \in E$, there holds:*

$$\langle I'_{\lambda,\mu}(v),\varphi\rangle = \int_{\mathbb{R}^N} |\nabla v|^{p-2}\nabla v\nabla\varphi\,dx \\ + \int_{\mathbb{R}^N} V(x)|f(v)|^{p-2}f(v)f'(v)\varphi\,dx - \langle\Psi'(v),\varphi\rangle,$$

where

$$\langle\Psi'(v),\varphi\rangle = \lambda\int_{\mathbb{R}^N} K(x)|f(v)|^{q-2}f(v)f'(v)\varphi\,dx + \mu\int_{\mathbb{R}^N} g(x,f(v))f'(v)\varphi\,dx.$$

Proof. The verification of (a) and (b) is trivial. For conclusion (c), we need to show that:

(i) $I_{\lambda,\mu}$ is Gateaux-differentiable on E and

$$\langle DI_{\lambda,\mu}(v),\varphi\rangle = \int_{\mathbb{R}^N} |\nabla v|^{p-2}\nabla v\nabla\varphi\,dx \\ + \int_{\mathbb{R}^N} V(x)|f(v)|^{p-2}f(v)f'(v)\varphi\,dx - \langle D\Psi(v),\varphi\rangle$$

and

$$\langle D\Psi(v),\varphi\rangle = \lambda\int_{\mathbb{R}^N} K(x)|f(v)|^{q-2}f(v)f'(v)\varphi\,dx + \mu\int_{\mathbb{R}^N} g(x,f(v))f'(v)\varphi\,dx,$$

where $DI_{\lambda,\mu}(v)$ and $D\Psi(v)$ are the Gateaux-derivatives of $I_{\lambda,\mu}$ and Ψ at v, respectively.

(ii) For $v \in E$, $DI_{\lambda,\mu}(v) \in E'$, and if $v_n \to v$ in E, then

$$\|DI_{\lambda,\mu}(v_n) - DI_{\lambda,\mu}(v)\|_{E'} = \sup_{\varphi \in E} \langle DI_{\lambda,\mu}(v_n) - DI_{\lambda,\mu}(v), \varphi\rangle \to 0.$$

If (i) and (ii) are verified, we conclude that $I_{\lambda,\mu}$ is Frechet differentiable on E and so $DI_{\lambda,\mu}(v) = I'_{\lambda,\mu}(v)$. Consequently, $I_{\lambda,\mu} \in C^1(E,\mathbb{R})$.

The proofs of (i) and (ii) are similar to Proposition 2.5 in [13]. For the readers' convenience, we only need to verify that there hold

$$\lim_{t\to 0} \int_{\mathbb{R}^N} \frac{V(x)[|f(v+t\varphi)|^p - |f(v)|^p]}{t}\, dx = \int_{\mathbb{R}^N} V(x)|f(v)|^{p-2} f(v) f'(v)\varphi\, dx, \quad \varphi \in E, \tag{2.22}$$

$$\int_{\mathbb{R}^N} V(x)|f(v)|^{p-2} f(v) f'(v)\varphi_n\, dx \to 0, \quad \text{as } \varphi_n \to 0 \text{ in } E \tag{2.23}$$

and, for every $\varphi \in E$, there holds

$$\langle DI_{\lambda,\mu}(v_n) - DI_{\lambda,\mu}(v), \varphi\rangle \to 0 \quad \text{as } v_n \to v \text{ in } E. \tag{2.24}$$

Verification of (2.22). By the mean value theorem, we have

$$\lim_{t\to 0} \frac{1}{p} \int_{\mathbb{R}^N} \frac{V(x)[|f(v+t\varphi)|^p - |f(v)|^p]}{t}\, dx = \lim_{t\to 0} \int_{\mathbb{R}^N} V(x)|f(\xi)|^{p-2} f(\xi) f'(\xi)\varphi\, dx,$$

where $\min\{v(x), v(x)+t\varphi(x)\} \le \xi(x) \le \max\{v(x), v(x)+t\varphi(x)\}$.

Using (f_2), (f_{11}), (f_{12}) and (f_{13}) of Lemma 2.2, we infer that for $|t| \le 1$, there holds:

$$\begin{aligned}
&|V(x)|f(\xi)|^{p-2} f(\xi) f'(\xi)\varphi| \\
&\quad \le C_1 V(x)||f(\xi)|^{p-2} f(\xi) f'(\xi)| f(|\varphi|) + C_2 V(x)||f(\xi)|^{p-2} f(\xi) f'(\xi)| f^2(|\varphi|) \\
&\quad \le C_1 V(x)|f(|v|+|\varphi|)|^{p-2} f(|v|+|\varphi|) f'(|v|+|\varphi|) f(|\varphi|) \\
&\quad + C_2 V(x)|f(|v|+|\varphi|)|^{p-2} f(|v|+|\varphi|) f'(|v|+|\varphi|) f^2(|\varphi|) \\
&\quad \le C_1 V(x)|f(|v|+|\varphi|)|^{p-1} f(|\varphi|) + \frac{C_2 V(x)}{2^{(p-1)/p}} |f(|v|+|\varphi|)|^{p-2} f^2(|\varphi|),
\end{aligned}$$

where the right hand side belongs to $L^1(\mathbb{R}^N)$. Since

$$V(x)|f(\xi)|^{p-2} f(\xi) f'(\xi)\varphi \to V(x)|f(v)|^{p-2} f(v) f'(v)\varphi \quad \text{for a.e. } x \in \mathbb{R}^N,$$

as $t \to 0$, by the Lebesgue Dominated Convergence theorem, we conclude that (2.22) is true.

Verification of (2.23). From $\varphi_n \to 0$, we see that

$$\int_{\mathbb{R}^N} V(x)|f(\varphi_n)|^p\,dx \to 0.$$

From (f_2), (f_{12}) of Lemma 2.2, by Hölder's inequality, we deduce that

$$\begin{aligned}
&\left|\int_{\mathbb{R}^N} V(x)|f(v)|^{p-2}f(v)f'(v)\varphi_n\,dx\right| \\
&\quad\le C_1\int_{\mathbb{R}^N} V(x)|f(v)|^{p-2}|f(v)f'(v)|f(\varphi_n)\,dx \\
&\qquad + C_2\int_{\mathbb{R}^N} V(x)|f(v)|^{p-2}|f(v)f'(v)|f^2(\varphi_n)\,dx \\
&\quad\le C_1\int_{\mathbb{R}^N} V(x)|f(v)|^{p-1}f(\varphi_n)\,dx + C_2\int_{\mathbb{R}^N} V(x)|f(v)|^{p-2}f^2(\varphi_n)\,dx \\
&\quad\le C_1\biggl(\int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx\biggr)^{(p-1)/p}\biggl(\int_{\mathbb{R}^N} V(x)|f(\varphi_n)|^p\,dx\biggr)^{1/p} \\
&\qquad + C_2\biggl(\int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx\biggr)^{(p-2)/p}\biggl(\int_{\mathbb{R}^N} V(x)|f(\varphi_n)|^p\,dx\biggr)^{2/p},
\end{aligned}$$

which implies that

$$\int_{\mathbb{R}^N} V(x)|f(v)|^{p-2}f(v)f'(v)\varphi_n\,dx \to 0.$$

Verification of (2.24). Let $v_n \to v$ in E, then $v_n \to v$ in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ and $f(v_n) \to f(v)$ in $L^s(\mathbb{R}^N)$ for $s \in [1, 2p^*]$ and $v_n \to v$ almost everywhere in $\mathbb{R}^N$. By Hölder's inequality, we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^N} (|\nabla v_n|^{p-2}\nabla v_n - |\nabla v|^{p-2}\nabla v)\nabla\varphi\,dx\right| \\
&\quad\le \biggl(\int_{\mathbb{R}^N} ||\nabla v_n|^{p-2}\nabla v_n - |\nabla v|^{p-2}\nabla v|^{p/(p-1)}\,dx\biggr)^{(p-1)/p}\biggl(\int_{\mathbb{R}^N} |\nabla\varphi|^p\,dx\biggr)^{1/p} \\
&\quad\le (p-1)\biggl(\int_{\mathbb{R}^N} (|\nabla v_n| + |\nabla v|)^{p-2}|\nabla v_n - \nabla v|\,dx\biggr)^{(p-1)/p}\biggl(\int_{\mathbb{R}^N} |\nabla\varphi|^p\,dx\biggr)^{1/p} \\
&\quad\le C\biggl(\int_{\mathbb{R}^N} (|\nabla v_n|^{p-2} + |\nabla v|^{p-2})|\nabla v_n - \nabla v|\,dx\biggr)^{(p-1)/p}\biggl(\int_{\mathbb{R}^N} |\nabla\varphi|^p\,dx\biggr)^{1/p} \\
&\quad\le C\biggl(\biggl(\int_{\mathbb{R}^N} |\nabla v_n|^p\,dx\biggr)^{(p-2)/p} + \biggl(\int_{\mathbb{R}^N} |\nabla v|^p\,dx\biggr)^{(p-2)/p}\biggr) \\
&\qquad\cdot\biggl(\int_{\mathbb{R}^N} |\nabla v_n - \nabla v|^p\,dx\biggr)^{1/p}\biggl(\int_{\mathbb{R}^N} |\nabla\varphi|^p\,dx\biggr)^{1/p} \to 0
\end{aligned}$$

as $n \to \infty$.

By Lemma 2.11, we have that

$$\int_{\mathbb{R}^N} V(x)|f(v_n) - f(v)|^p\,dx \to 0.$$

Then, up to a subsequence, there exists $\widetilde{h} \in L^1(\mathbb{R}^N)$ such that $V(x)|f(v_n) - f(v)|^p \le \widetilde{h}$ almost everywhere in $\mathbb{R}^N$. Thus, by (f_2), (f_{11}) of Lemma 2.2, using Young's inequality, we have

$$\begin{aligned}
&V(x)(|f(v_n)|^{p-2}f(v_n)f'(v_n) - |f(v)|^{p-2}f(v)f'(v))\varphi \\
&\le V(x)(|f(v_n)|^{p-2}|f(v_n)f'(v_n)| + |f(v)|^{p-2}|f(v)f'(v)|)|\varphi| \\
&\le V(x)(|f(v_n)|^{p-2}|f(v_n)f'(v_n)| + |f(v)|^{p-2}|f(v)f'(v)|)(C_1 f(|\varphi|) + C_2 f^2(|\varphi|)) \\
&\le C(V(x)|f(v_n)|^p + V(x)|f(\varphi)|^p + V(x)|f(v)|^p) \\
&\le C(\widetilde{h}(x) + V(x)|f(\varphi)|^p + V(x)|f(v)|^p) \in L^1(\mathbb{R}^N).
\end{aligned}$$

Therefore, if follows from the Lebesgue Dominated Convergence theorem that

$$\left| \int_{\mathbb{R}^N} V(x)(|f(v_n)|^{p-2}f(v_n)f'(v_n) - |f(v)|^{p-2}f(v)f'(v))\varphi\,dx \right| \to 0. \tag{2.25}$$

Similarly to the proof of (2.25), we can deduce that $\langle D\Psi(v_n) - D\Psi(v), \varphi\rangle \to 0$, for any $\varphi \in E$. $\square$

In order to establish our main results, we need the symmetric mountain pass theorem in [27] and Clark theorem in [11] which are stated as follows.

Theorem 2.14 ([27]). *Let E be an infinite dimensional Banach space. We assume that the functional $I \in C^1(E, \mathbb{R})$ satisfies the following conditions*:

(a) $I(0) = 0$ *and* $I(u) = I(-u)$,
(b) *I satisfies the* (PS) *condition*,
(c) *there exist $\rho, \alpha > 0$ such that $I(u) \ge \alpha$ for $u \in E$ and $\|u\| = \rho$*,
(d) *for any finite dimensional subspace of $\widetilde{E} \subset E$, $\{u \in \widetilde{E} : I(u) \ge 0\}$ is a bounded set.*

Then I possesses an unbounded from above sequence of critical values.

Theorem 2.15 ([11]). *Let E be a Banach space. We assume that the functional $I \in C^1(E, \mathbb{R})$ satisfies the following conditions*:

(a) $I(0) = 0$ *and* $I(u) = I(-u)$,
(b) *I is bounded from below and satisfies the* (PS) *condition*,
(c) *for any $k \in \mathbb{N}$, there exist k-dimensional subspace E_k and $\rho_k > 0$ such that $\sup_{E_k \cap S_{\rho_k}} I < 0$, where $S_{\rho_k} = \{u \in E : \|u\| = \rho_k\}$.*

Then I possesses a sequence of critical values $c_k < 0$ such that $c_k \to 0$ as $k \to \infty$.

3. Main results and proofs

In this section, we will apply the symmetric mountain pass Theorem 2.14 and Clark Theorem 2.15 to prove our main results: Theorems 1.3 and 1.4. First we verify all the conditions of Theorem 2.14.

Lemma 3.1. *Assume that* (g_0)–(g_1) *hold, then for every* $\mu > 0$, *there exist* $\rho_\mu, \alpha_\mu > 0$ *such that for any* $\lambda \in (-\infty, q\beta_\mu\rho_\mu^{(p-q)/2}/2C_3\|K\|_\infty]$, $I_{\lambda,\mu}(v) \geq \alpha_\mu$, *for every* $v \in E$ *with* $\|v\| = \rho_\mu$.

Proof. By (g_0) and (g_1), for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|g(x,t)| \leq \frac{\varepsilon}{p}\,|t|^{p-1} + \frac{C_\varepsilon}{r}\,|t|^{r-1}, \qquad |G(x,t)| \leq \varepsilon|t|^p + C_\varepsilon|t|^r, \tag{3.1}$$

for all $x \in \mathbb{R}^N$, $t \in \mathbb{R}$. For any $v \in E$ with $\|v\| \leq 1$, by (3.1), Lemma 2.8 and (c) of Lemma 2.11, we have

$$\begin{aligned} I_{\lambda,\mu}(v) &= \frac{1}{p}\int_{\mathbb{R}^N} (|\nabla v|^p + V(x)|f(v)|^p)\,dx \\ &\quad - \frac{\lambda}{q}\int_{\mathbb{R}^N} K(x)|f(v)|^q\,dx - \mu\int_{\mathbb{R}^N} G(x,f(v))\,dx \\ &\geq \frac{C_1}{p}\min\{\|v\|^{p/2}, \|v\|^p\} - \frac{\lambda}{q}\|K\|_\infty \int_{\mathbb{R}^N} |f(v)|^q\,dx \\ &\quad - \varepsilon\mu\int_{\mathbb{R}^N} |f(v)|^p\,dx - C_\varepsilon\mu\int_{\mathbb{R}^N} |f(v)|^r\,dx \\ &\geq \frac{C_1}{p}\|v\|^p - \frac{\lambda}{q}\|K\|_\infty C_3\max\{\|v\|^{q/2}, \|v\|^q\} \\ &\quad - \varepsilon\mu C_4\max\{\|v\|^{p/2}, \|v\|^p\} - C_5\mu\max\{\|v\|^{r/2}, \|v\|^r\} \\ &= \|v\|^{p/2}\bigg(\frac{C_1}{p}\,\|v\|^{p/2} - \widetilde{C_4}\varepsilon - \widetilde{C_5}\|v\|^{(r-p)/2}\bigg) - \frac{\lambda}{q}\,\|K\|_\infty C_3\|v\|^{q/2}, \end{aligned} \tag{3.2}$$

where $C_1 = 1/2^{2p-1}$, $C_2, C_3, C_4 > 0$ and $\widetilde{C_5} \geq 1$ (can be chosen).

Let $\xi(t) = C_1 t^{p/2}/p - \widetilde{C_4}\varepsilon - \widetilde{C_5}t^{(r-p)/2}$, $t \geq 0$. Obviously, $C_1/p < 1$, $\xi(0) < 0$, $\xi(t) < 0$ for any $t \geq 1$ and by computation, we see that $\xi'(t_\mu) = 0$, where $t_\mu = (C_1/(\widetilde{C_5}(r-p)))^{2/(r-2p)}$. Thus, we have

$$\begin{aligned} \xi(t_\mu) &= \frac{C_1}{p}\bigg(\frac{C_1}{\widetilde{C_5}(r-p)}\bigg)^{p/(r-2p)} - \widetilde{C_4}\varepsilon - \widetilde{C_5}\bigg(\frac{C_1}{C_5(r-p)}\bigg)^{(r-p)/(r-2p)} \\ &= \frac{C_1^{(r-p)/(r-2p)}}{\widetilde{C_5}^{\,p/(r-2p)}}\,\frac{r-2p}{p(r-p)^{(r-p)/(r-2p)}} - C_4\varepsilon \end{aligned}$$

which implies that choosing

$$0 < \varepsilon < \frac{C_1^{(r-p)/(r-2p)}}{\widetilde{C_4}\widetilde{C_5}^{\,p/(r-2p)}}\,\frac{r-2p}{p(r-p)^{(r-p)/(r-2p)}},$$

there holds that $\xi(t) > 0$ for t belonging to some neighbourhoods of t_μ. Hence, there exists $0 < \rho_\mu < 1$ such that

$$\beta_\mu = \frac{C_1}{p}\,\rho_\mu^{p/2} - \widetilde{C_4}\varepsilon - \widetilde{C_5}\rho_\mu^{p/2} > 0.$$

Therefore, by (3.2), we get

$$I_{\lambda,\mu}(v) \geq \beta_\mu \rho_\mu^{p/2} - \frac{\lambda}{q} \|K\|_\infty C_3 \rho_\mu^{q/2} \quad \text{for all } \|v\| = \rho_\mu.$$

Taking $\alpha_\mu = \beta_\mu \rho_\mu^{p/2}/2$, then for any $\lambda \in (-\infty, q\beta_\mu \rho_\mu^{(p-q)/2}/(2\|K\|_\infty C_3)]$, we have that

$$I_{\lambda,\mu}(v) \geq \alpha_\mu \quad \text{for all } \|v\| = \rho_\mu$$

and the proof is completed. □

Lemma 3.2. *Assume that* (g_0)–(g_2) *hold, then for any* $\mu > 0$ *and* $\lambda \leq 0$, $\{u \in \widetilde{E} : I_{\lambda,\mu} \geq 0\}$ *is bounded, where* $\widetilde{E}$ *is a finite dimensional subspace of* E.

Proof. By (g_2), there exists $C_6 > 0$ such that

$$G(x,t) \geq C_6 |t|^\theta, \quad \text{for all } x \in \mathbb{R}^N \text{ and } u \in \mathbb{R}. \tag{3.3}$$

Observe that it is sufficient to show that for any $v \in E$ with $\|v\| > 1$, the conclusion holds true. Hence, let $v \in E$ with $\|v\| > 1$, by (3.3), Lemmas 2.2 (f_3), 2.8 and 2.11 (c), we deduce that

$$\begin{aligned}
0 \leq I_{\lambda,\mu}(v) &\leq \frac{1}{p} \int_{\mathbb{R}^N} (|\nabla v|^p + V(x)|f(v)|^p)\, dx \\
&\quad + |\lambda| \|K\|_\infty \int_{\mathbb{R}^N} |f(v)|^q \, dx - C_6 \int_{\mathbb{R}^N} |v|^\theta \, dx \\
&\leq \frac{C_2}{p} \max\{\|v\|^p, \|v\|^{p/2}\} \\
&\quad + |\lambda| \|K\|_\infty C_7 \max\{\|v\|^{q/2}, \|v\|^q\} - C_8 \|v\|^{\theta/2} \\
&= \frac{C_2}{p} \|v\|^p + |\lambda| \|K\|_\infty C_7 \|v\|^q - C_8 \|v\|^{\theta/2} \leq C_9 \|v\|^p - C_8 \|v\|^{\theta/2}
\end{aligned}$$

which implies that $\{v \in \widetilde{E} : I_{\lambda,\mu}(v) \geq 0\}$ is bounded in E. □

Lemma 3.3. *Suppose that* (g_0)–(g_2) *hold, then* $I_{\lambda,\mu}(v)$ *satisfies the* (PS)-*condition in* E *if* p, q, μ, λ *verify one of the following conditions*:

(a) $\mu > 0$, $\lambda \leq 0$, $p > q$;
(b) $\mu > 0$, $\lambda > 0$, $p > 2q$.

Proof. Let $\{v_n\} \subset E$ be a (PS)-sequence verifying

$$I_{\lambda,\mu}(v_n) \text{ is bounded and } I'_{\lambda,\mu}(v_n) \to 0. \tag{3.4}$$

We will show that $\{v_n\}$ is bounded in E. By computation, we deduce that

$$\nabla\bigg(\frac{f(v_n)}{f'(v_n)}\bigg) = \bigg(1 + \frac{2^{p-1}|f(v_n)|^p}{1 + 2^{p-1}|f(v_n)|^p}\bigg) \nabla v_n \tag{3.5}$$

and thus

$$\bigg\| \nabla \frac{f(v_n)}{f'(v_n)} \bigg\|_p \leq 2 \|\nabla v_n\|_p.$$

By (f_6) of Lemma 2.2, there holds $1 \le f(t)/(tf'(t)) \le 2$ for any $t \neq 0$. Hence, we can use (f_{10}) to get

$$\left|f\left(\xi^{-1}\frac{f(t)}{f'(t)}\right)\right|^p = \left|f\left(\xi^{-1}t\,\frac{f(t)}{tf'(t)}\right)\right|^p \le \left(\frac{f(t)}{tf'(t)}\right)^p |f(\xi^{-1}t)|^p \le 2^p|f(\xi^{-1}t)|^p,$$

for any $\xi > 0$, $t \in \mathbb{R}$. Thus, using the above estimate, we have

$$\left|\frac{f(v_n)}{f'(v_n)}\right|_f = \inf_{\xi>0}\xi\left(1 + \int_{\mathbb{R}^N} V(x)\left|f\left(\xi^{-1}\frac{f(v_n)}{f'(v_n)}\right)\right|^p dx\right) \le 2^p|v_n|_f.$$

Accordingly, we have proved that

$$\left\|\frac{f(v_n)}{f'(v_n)}\right\| \le 2^p\|v_n\|. \tag{3.6}$$

By (g_2), and (c) of Lemma 2.11, we have

$$\begin{aligned}
I_{\lambda,\mu}(v_n) - \frac{1}{\theta}\left\langle I'_{\lambda,\mu}(v_n), \frac{f(v_n)}{f'(v_n)}\right\rangle \qquad\qquad &\\
= \left(\frac{1}{p} - \frac{1}{\theta}\right)\int_{\mathbb{R}^N} (|\nabla v_n|^p + V(x)|f(v_n)|^p)\,dx &\\
- \frac{1}{\theta}\int_{\mathbb{R}^N} \frac{2^{p-1}|f(v_n)|^p}{1 + 2^{p-1}|f(v_n)|^p}\,|\nabla v_n|^p\,dx &\\
+ \lambda\left(\frac{1}{\theta} - \frac{1}{q}\right)\int_{\mathbb{R}^N} K(x)|f(v_n)|^q\,dx &\\
- \mu\int_{\mathbb{R}^N}\left(G(x, f(v_n)) - \frac{1}{\theta}\,g(x, f(v_n))\right)dx &\\
\ge \left(\frac{1}{p} - \frac{2}{\theta}\right)\int_{\mathbb{R}^N} (|\nabla v_n|^p + V(x)|f(v_n)|^p)\,dx &\\
+ \lambda\left(\frac{1}{\theta} - \frac{1}{q}\right)\int_{\mathbb{R}^N} K(x)|f(v_n)|^q\,dx.&
\end{aligned} \tag{3.7}$$

If $\lambda \le 0$, it follows from (3.4), (3.7) and Lemma 2.8 that

$$\begin{aligned}
M + \varepsilon_n\|v_n\| &\ge \left(\frac{1}{p} - \frac{2}{\theta}\right)\int_{\mathbb{R}^N} (|\nabla v_n|^p + V(x)|f(v_n)|^p)\,dx\\
&\ge \left(\frac{1}{p} - \frac{2}{\theta}\right)\min\{\|v_n\|^p, \|v_n\|^{p/2}\},
\end{aligned}$$

where $M > 0$ is a positive constant, $0 < \varepsilon_n < 1$ and $\varepsilon_n \to 0$ as $n \to \infty$. The above inequality implies that $\{v_n\}$ is bounded in E.

If $\lambda > 0$, by (3.4), (3.7), Lemma 2.8 and (c) of Lemma 2.11, using (3.5), we have

$$\left(\frac{1}{p} - \frac{2}{\theta}\right)C_1\min\{\|v_n\|^p, \|v_n\|^{p/2}\} - C_9\max\{\|v_n\|^q, \|v_n\|^{q/2}\} \le \|v_n\| + M,$$

where $C_9 > 0$ is a constant. Clearly, $\{v_n\}$ is bounded in E for $p > 2q$.

Next, we will show that $\{v_n\}$ possesses a convergent subsequence in E. In fact, by Lemma 2.11 and Corollary 2.12, up to a subsequence, still denoted by v_n, there exists $v \in W^{1,p}(\mathbb{R}^N)$ such that $v_n \to v$ in $L^s(\mathbb{R}^N)$ for $s \in [1, p^*)$ and $f(v_n) \to f(v)$ in $L^s(\mathbb{R}^N)$ for $s \in [1, 2p^*)$.

Using the convexity of $\int_{\mathbb{R}^N} |\nabla v|^p + V(x)|f(v)|^p\,dx$ and (3.4), we have that

$$\begin{aligned}
\frac{1}{p}\int_{\mathbb{R}^N} &(|\nabla v|^p + V(x)|f(v)|^p)\,dx - \frac{1}{p}\int_{\mathbb{R}^N}(|\nabla v_n|^p + V(x)|f(v_n)|^p)\,dx \\
&\geq \frac{1}{p}\int_{\mathbb{R}^N}(|\nabla v_n|^{p-2}\nabla v_n(\nabla v - \nabla v_n) + V(x)|f(v)|^{p-2}f(v_n)f'(v_n)(v - v_n))\,dx \\
&= \lambda \int_{\mathbb{R}^N} K(x)|f(v_n)|^{q-2}f(v_n)f'(v_n)(v - v_n)\,dx \\
&\quad + \mu\int_{\mathbb{R}^N} g(x, f(v_n))f'(v_n)(v - v_n)\,dx + \langle I'_\lambda(v_n), v - v_n\rangle.
\end{aligned}$$

Since

$$\int_{\mathbb{R}^N} K(x)|f(v_n)|^{q-2}f(v_n)f'(v_n)(v - v_n)\,dx \leq \|K\|_\infty\|f(v_n)\|_q\|v_n - v\|_q \to 0$$

as $n \to \infty$ and

$$\int_{\mathbb{R}^N} g(x, f(v_n))f'(v_n)(v - v_n)\,dx \leq C_{10}(\|v_n - v\| + \|f(v_n)\|_r^{r-1}\|v_n - v\|_r) \to 0$$

as $n \to \infty$, so we get

$$\begin{aligned}
\int_{\mathbb{R}^N} |\nabla v|^p + V(x)|f(v)|^p\,dx &\geq \liminf_{n\to\infty}\int_{\mathbb{R}^N} |\nabla v_n|^p + V(x)|f(v_n)|^p\,dx \\
&\geq \liminf_{n\to\infty}\int_{\mathbb{R}^N} |\nabla v_n|^p\,dx + \liminf_{n\to\infty}\int_{\mathbb{R}^N} V(x)|f(v_n)|^p\,dx
\end{aligned}$$

By the semicontinuity of seminorm and Fatou's Lemma, we have

$$\begin{aligned}
\int_{\mathbb{R}^N} |\nabla v|^p\,dx &\leq \liminf_{n\to\infty}\int_{\mathbb{R}^N} |\nabla v_n|^p\,dx, \\
\int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx &\leq \liminf_{n\to\infty}\int_{\mathbb{R}^N} V(x)|f(v_n)|^p\,dx.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\int_{\mathbb{R}^N} |\nabla v|^p\,dx &= \lim_{n\to\infty}\int_{\mathbb{R}^N} |\nabla v_n|^p\,dx, \\
\int_{\mathbb{R}^N} V(x)|f(v)|^p\,dx &= \lim_{n\to\infty}\int_{\mathbb{R}^N} V(x)|f(v_n)|^p\,dx.
\end{aligned}$$

By the Radon theorem and (b) of Lemma 2.11, we conclude that

$$\|\nabla v_n - \nabla v\|_p \to 0, \quad |v_n - v|_f \to 0 \quad \text{as } n \to \infty$$

which implies that $\|v_n - v\| \to 0$ as $n \to \infty$. $\square$

PROOF OF THEOREM 1.3. By hypotheses (g_0) and (g_3), obviously, $I_{\lambda,\mu}$ is an even functional and $I_{\lambda,\mu}(0) = 0$. From Lemmas 3.1, 3.2 and 3.3, it follows that all the conditions of Theorem 2.14 are verified. Hence, there exists a sequence of unbounded critical values, which implies that there exists $\{v_n\} \subset E$ such that $I_{\lambda,\mu}(v_n) \to +\infty$ as $n \to \infty$ and the proof is completed. □

Second, we shall use the Clark theorem 2.15 to give the proof of Theorem 1.4.

PROOF OF THEOREM 1.4. By (g_0) and (g_3), we see that $I_{\lambda,\mu}$ is an even functional and $I_{\lambda,\mu}(0) = 0$. Using (g_2), Lemma 2.8, Sobolev's inequality and $\mu \le 0$, we have that

$$\begin{aligned}
I_{\lambda,\mu}(v) &= \frac{1}{p}\int_{\mathbb{R}^N} (|\nabla v|^p + V(x)|f(v)|^p)\,dx \\
&\quad - \frac{\lambda}{q}\int_{\mathbb{R}^N} K(x)|f(v)|^q\,dx - \mu\int_{\mathbb{R}^N} G(x,f(v))\,dx \\
&\ge \frac{1}{p}\int_{\mathbb{R}^N} (|\nabla v|^p + V(x)|f(v)|^p)\,dx \\
&\quad - \frac{\lambda}{q}\,\|K\|_{L^{2p^*/(2p^*-q)}}\bigg(\int_{\mathbb{R}^N} |f(v)|^{2p^*}\,dx\bigg)^{q/(2p^*)} \\
&\ge C_1\min\{\|v\|^{p/2},\|v\|^p\} - C_{11}\|v\|^{q/2}.
\end{aligned}$$

Owing to $q < p$, we conclude that $I_{\lambda,\mu}$ is bounded from below on E and any (PS) sequence is bounded in E. Similarly to the proof of Lemma 3.3, $I_{\lambda,\mu}$ satisfies the (PS) condition.

By (f_9) of Lemma 2.2, we can verify that

$$|t|^q \le \frac{|f(t)|^q}{C^q} + \frac{|f(t)|^{2p}}{C^{2p}}, \quad \text{for all } t \in \mathbb{R}. \tag{3.8}$$

In fact, using (f_9) in Lemma 2.2, if $|t| \le 1$, we get that $|t|^q \le |f(t)|^q/C^q$; if $|t| \ge 1$, it follows that $|t|^q \le |t|^p \le |f(t)|^{2p}/C^{2p}$. Thus (3.8) is proved.

For $\Omega = \{x \in \mathbb{R}^N : K(x) = 0\}$, we denote by $|\Omega|$ the Lebesgue measure.

Case 1. Assume that $|\Omega| = 0$, that is $K(x) > 0$ almost everywhere in $\mathbb{R}^N$. By (3.1), (3.8) and Lemma 2.8, we deduce that

$$\begin{aligned}
I_{\lambda,\mu}(v) &\le \int_{\mathbb{R}^N} (|\nabla v|^p + V(x)|f(v)|^p)\,dx - \frac{\lambda}{q}\int_{\mathbb{R}^N} K(x)|f(v)|^q\,dx \\
&\quad + |\mu|\varepsilon\int_{\mathbb{R}^N} |f(v)|^p\,dx + |\mu|C_\varepsilon\int_{\mathbb{R}^N} |f(v)|^{2r}\,dx \\
&\le \int_{\mathbb{R}^N}\bigg(|\nabla v|^p + \bigg(1+\frac{|\mu|}{V_0}\,\varepsilon\bigg)V(x)|f(v)|^p\bigg)\,dx - \frac{\lambda}{q}\,C^q\int_{\mathbb{R}^N} K(x)|v|^q\,dx \\
&\quad + C_{12}\int_{\mathbb{R}^N} K(x)|f(v)|^{2p}\,dx + C_{13}\max\{\|v\|^r,\|v\|^{2r}\} \\
&\le C_{14}\max\{\|v\|^{p/2},\|v\|^p\} - C_{12}\int_{\mathbb{R}^N} K(x)|v|^q\,dx
\end{aligned}$$

$$+ C_{15} \max\{\|v\|^p, \|v\|^{2p}\} + C_{13} \max\{\|v\|^r, \|v\|^{2r}\},$$

where $C_{12}, C_{13}, C_{14}, C_{15} > 0$ are constants. For each $k \in \mathbb{N}$, we can choose a k-dimensional subspace E_k of E, and $|\cdot|_{q,K}$ is a norm of E_k. Then for $v \in E_k$ with $\|v\| \leq 1$, using the fact that all norms on finite dimensional space are equaivalent and $1 < q < p/2 < r$, there exists $\rho_k > 0$ small enough, such that

$$I_{\lambda,\mu}(v) \leq C_{14}\|v\|^{p/2} + C_{15}\|v\|^p + C_{13}\|v\|^r - C_{16}\|v\|^q < 0, \quad \text{if } \|v\| = \rho_k.$$

Case 2. If $|\Omega| > 0$, by the argument in Section 2, the seminorm $|\cdot|_{q,K}$ is a norm on the $\widetilde{E}$. Given $k \in \mathbb{N}$, let $\widetilde{E}_k$ be a k-dimensional space of $\widetilde{E}$, then for $v \in \widetilde{E}_k$, using the fact that all norms on finite dimensional space are equivalent and $1 < q < p/2 < r$, there exists $\rho_k > 0$ small enough, we have that

$$\begin{aligned} I_{\lambda,\mu}(v) &\leq C_{14} \max\{\|v\|^{p/2}, \|v\|^p\} - C_{12} \int_{\mathbb{R}^N} K(x)|v|^q \, dx \\ &\quad + C_{15} \max\{\|v\|^p, \|v\|^{2p}\} + C_{13} \max\{\|v\|^r, \|v\|^{2r}\} \\ &\leq C_{14}\|v\|^{p/2} + C_{15}\|v\|^p + C_{13}\|v\|^r - C_{17}\|v\|^q < 0 \end{aligned}$$

for all $\|v\| = \rho_k$.

Finally, using Theorem 2.15, $I_{\lambda,\mu}$ possesses an sequence of critical values $c_k < 0$ verifying $c_k \to 0$ as $k \to \infty$. Therefore, there exists $\{v_k\} \subset E$ such that $I_{\lambda,\mu}(v_k) = c_k \to 0$ as $k \to \infty$. □

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