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MULTIPLE NODAL SOLUTIONS FOR SEMILINEAR ROBIN PROBLEMS WITH INDEFINITE LINEAR PART AND CONCAVE TERMS

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Abstract. We consider a semilinear Robin problem driven by Laplacian plus an indefinite and unbounded potential. The reaction function contains a concave term and a perturbation of arbitrary growth. Using a variant of the symmetric mountain pass theorem, we show the existence of smooth nodal solutions which converge to zero in $C^1(\overline{\Omega})$. If the coefficient of the concave term is sign changing, then again we produce a sequence of smooth solutions converging to zero in $C^1(\overline{\Omega})$, but we cannot claim that they are nodal.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following semilinear Robin problem:

$$
(1.1)\qquad \begin{cases} -\Delta u(z) + \xi(z)u(z) = \vartheta(z)|u(z)|^{q-2}u(z) + f(z,u(z)) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} + \beta(z)u = 0 & \text{on } \partial\Omega, \\ & \text{for } 1 < q < 2. \end{cases}
$$

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In this problem the potential function $\xi \in L^s(\Omega)$, with $s > N$, and it is sign changing, so the linear part of the problem is indefinite. In the reaction part of the problem (the right hand side), there is a "concave"' (that is, strictly sublinear) term, which is the term $\vartheta(z)|u(z)|^{q-2}u(z)$ $(1 < q < 2)$ with the weight $\vartheta \in L^\infty(\Omega)$, $\vartheta(z) > 0$ for almost all $z \in \Omega$. In the last part of the paper we allow ϑ to be sign changing. There is also a perturbation term f which is assumed to be a Carathéodory function (that is, for all $x \in \mathbb{R}$, $z \mapsto f(z,x)$ is measurable and for almost all $z \in \Omega$, $x \mapsto f(z,x)$ is continuous). A special feature of our work is that we do not impose any growth condition on $f(z,\cdot)$. We only impose conditions near zero and we assume that $f(z,\cdot)$ is odd and the whole reaction function minus the potential term (that is, the function $x \mapsto \vartheta(z)|x|^{q-2}x + f(z,x) - \xi(z)x$) exhibits a kind of oscillatory behaviour near zero. Suitable truncations make the behaviour of $f(z,\cdot)$ near $\pm\infty$ irrelevant. Using a variant of the symmetric mountain pass theorem, which is due to Heinz [3] and Kajikiya [5], we produce a whole sequence of distinct smooth (that is, they belong in $C^1(\overline{\Omega})$) nodal (that is, sign changing) solutions, which converge to zero in $C^1(\overline{\Omega})$. Finally we see what happens when the weight function ϑ is sign changing. In this case, again we produce a sequence of smooth solutions in $C^1(\overline{\Omega})$ converging to zero, but we no longer claim that these solutions are nodal.

Infiniteness of the set of solutions for indefinite semilinear Dirichlet equations was established by Yu, Yongqing [16], Zhang, Liu [17], Qin, Tang, Zhang [13], Zhang, Tang, Zhang [18]. In all these works, the reaction term is superlinear but with subcritical polynomial growth in the x variable. However, nodality of solutions is not shown. We also mention the related works of Wang [15], Qian [11] and Qian, Li [12]. In [15] the problem is Dirichlet with zero potential (that is, $\xi \equiv 0$) and the reaction function f_0 is continuous on $\Omega \times \mathbb{R}$ and no growth condition is imposed on $f_0(z,\cdot)$. In [15], the author produces infinitely many distinct solutions but also does not show that they are nodal. Infinitely many nodal solutions were produced by Qian [11] for Neumann problems with a coercive differential operator of the form

$$u \to -\Delta u + au, \quad \text{for all } u \in H^1(\Omega) \text{ with } 0 < a < +\infty.$$

In [11], the reaction function $f_0(z,x)$ is assumed to be continuous on $\Omega \times \mathbb{R}$ and superlinear in $x \in \mathbb{R}$, but with subcritical polynomial growth. For Robin problems with zero potential term (that is, $\xi \equiv 0$), there is the work of Qian, Li [12], where the reaction function $f_0(z,x)$ is continuous on $\overline{\Omega}\times\mathbb{R}$ and superlinear in $x \in \mathbb{R}$, but again with subcritical polynomial growth. Qian, Li [12] produce a whole sequence of distinct solutions but they again do not show that these solutions are nodal.

So, the above survey of the relevant literature reveals that only Wang [15] deals with a problem where the reaction function $f_0(z,x)$ is of arbitrary growth in $x \in \mathbb{R}$. But in [15], the problem is Dirichlet with zero potential (so the differential operator is coercive). Also, the solutions produced are not necessarily nodal. Infinitely many nodal solutions are produced only by Qian [11] for Neumann equations with a coercive differential operator and a reaction term $f_0(z,x)$ continuous on $\Omega \times \mathbb{R}$ and of subcritical polynomial growth in $x \in \mathbb{R}$. So, we see that our work here is more general that all the aforementioned papers.

Finally, we mention that multiple nodal solutions (but not infinitely many) for problems with indefinite linear part, were obtained by Papageorgiou, Papalini [7] (Dirichlet problems) and by Papageorgiou, Radulescu [9] (Robin problems).

2. Mathematical background

Let X be a Banach space and X^* its topological dual. By $\langle \cdot\,,\cdot \rangle$ we denote the duality brackets for the pair (X^*, X). Given $\varphi \in C^1(X, \mathbb{R})$, we say that φ satisfies the Palais–Smale condition (the "PS-condition" for short), if the following compactness-type condition holds:

- "Every sequence $\{u_n\}_{n\geq 1} \subseteq X$ such that $\{\varphi(u_n)\}_{n\geq 1} \subseteq \mathbb{R}$ is bounded and $\varphi'(u_n) \to 0$ in X^* as $n \to +\infty$ admits a strongly convergent subsequence".

In the analysis of problem (1.1), we will use the following spaces:

- The Sobolev space $H^1(\Omega)$.
- The Banach space $C^1(\overline{\Omega})$.
- The Lebesgue space $L^r(\partial\Omega)$, $r \in [1, +\infty]$.

We know that $H^1(\Omega)$ is a Hilbert space with inner product

$$(u, v) = \int_\Omega uv\,dz + \int_\Omega (\nabla u, \nabla v)_{\mathbb{R}^N}\,dz \quad \text{for all } u, v \in H^1(\Omega).$$

The corresponding norm is

$$\|u\| = [\|u\|_2^2 + \|\nabla u\|_2^2]^{1/2} \quad \text{for all } u \in H^1(\Omega).$$

The space $C^1(\overline{\Omega})$ is an ordered Banach space with positive (order) cone given by

$$C_+ = \{u \in C^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior which contains

$$D_+ = \{u \in C^1(\overline{\Omega}) : u(z) > 0 \text{ for all } z \in \overline{\Omega}\}.$$

On $\partial\Omega$ we consider the $(N-1)$-dimensional Hausdorff (surface) measure σ. Using this measure, we can define in the usual way the "boundary"' Lebesgue spaces $L^r(\partial\Omega)$ $(1 \leq r \leq +\infty)$. From the theory of Sobolev spaces, we know that there is a unique continuous linear map $\gamma_0\colon H^1(\Omega)\to L^2(\partial\Omega)$, known as the "trace map", such that

$$\gamma_0(u)=u|_{\partial\Omega}\quad\text{for all } u\in H^1(\Omega)\cap C(\overline{\Omega}).$$

So, the trace map assigns boundary values to all Sobolev functions. This map is compact into $L^r(\partial\Omega)$ with $r\in[1,(2N-2)/(N-2))$ if $N\geq 3$ and into $L^r(\partial\Omega)$ for all $r\geq 1$ if $N=1,2$. Moreover, we have

$$\ker\gamma_0=H^1_0(\Omega)\quad\text{and}\quad \operatorname{im}\gamma_0=H^{1/2,2}(\partial\Omega).$$

In the sequel for the sake of notational simplicity, we drop the use of the trace map γ_0. All restrictions of the Sobolev functions on $\partial\Omega$ are understood in the sense of traces.

Consider the following linear eigenvalue problem:

$$\begin{cases}-\Delta u(z)+\xi(z)u(z)=\widehat{\lambda}u(z) & \text{in } \Omega,\\ \dfrac{\partial u}{\partial n}+\beta(z)u=0 & \text{on } \partial\Omega.\end{cases}\tag{2.1}$$

We make the following hypotheses concerning the data of (2.1):

- $\xi\in L^{N/2}(\Omega)$ if $N\geq 3$, $\xi\in L^r(\Omega)$ with $r>1$ if $N=2$, $\xi\in L^1(\Omega)$ if $N=1$.
- $\beta\in W^{1,\infty}(\partial\Omega)$ with $\beta(z)\geq 0$ for all $z\in\partial\Omega$.

Consider the C^1-functional $\gamma\colon H^1(\Omega)\to\mathbb{R}$ defined by

$$\gamma(u)=\|\nabla u\|_2^2+\int_\Omega \xi(z)u^2\,dz+\int_{\partial\Omega}\beta(z)u^2\,d\sigma\quad\text{for all } u\in H^1(\Omega).$$

From D'Aguì, Marano, Papageorgiou [1], we know that we can find $\mu>0$ such that

$$\gamma(u)+\mu\|u\|_2^2\geq c_0\|u\|^2,\quad\text{for all } u\in H^1(\Omega),\ \text{some } c_0>0.\tag{2.2}$$

Using (2.2) and the spectral theorem for compact self-adjoint operators, we show that problem (2.1) admits a whole sequence $\{\widehat{\lambda}_k\}_{k\in\mathbb{N}}$ of distinct eigenvalues such that $\widehat{\lambda}_k\to+\infty$. By $E(\widehat{\lambda}_k)$ we denote the corresponding eigenspace. We know that each $E(\widehat{\lambda}_k)$, $k\in\mathbb{N}$, is finite dimensional and we have the orthogonal direct sum decomposition $H^1(\Omega)=\overline{\bigoplus_{k\geq 1}E(\widehat{\lambda}_k)}$. Moreover, $\dim E(\widehat{\lambda}_1)=1$ (that is, $\widehat{\lambda}_1$ is simple) and

$$\widehat{\lambda}_1=\inf\bigg[\frac{\gamma(u)}{\|u\|_2^2}: u\in H^1(\Omega),\ u\neq 0\bigg].\tag{2.3}$$

The infimum in (2.3) is realized on $E(\widehat{\lambda}_1)$. Evidently, the elements of $E(\widehat{\lambda}_1)$ have fixed sign and by $\widehat{u}_1$ we denote the positive L^2-normalized (that is, $\|\widehat{u}_1\|_2=1$) eigenfunction. If $\xi\in L^s$ with $s>N$, then using the regularity theory of Wang [14], we have $\widehat{u}_1\in C_+\setminus\{0\}$. Moreover, the Harnack inequality (see Pucci, Serrin [10, p. 163]) implies that $\widehat{u}_1(z)>0$ for all $z\in\Omega$. Finally if $\xi^+\in L^\infty(\Omega)$, then by Hopf's boundary point lemma (see Pucci, Serrin [10, p. 120]), we have $\widehat{u}_1 \in D_+$.

To produce a whole sequence of distinct nodal solutions, we will use a variant of the classical symmetric mountain pass theorem (see, for example, Gasiński–Papageorgiou [2], p. 688), which is a particular case of a more general result due to Heinz [3] and Kajikiya [5].

Theorem 2.1. *If X is a Banach space, $\varphi \in C^1(X,\mathbb{R})$ satisfies the* PS-*condition, it is even, bounded below, $\varphi(0)=0$ and for every $n \in \mathbb{N}$ there exist a nontrivial finite dimensional subspace Y_n of X and $\rho_n > 0$ such that*

$$\sup[\varphi(u) : u \in Y_n \cap \partial B_{\rho_n}] < 0, \quad \text{where } \partial B_{\rho_n} = \{u \in X : \|u\| = \rho_n\},$$

then there exists a sequence $\{u_n\}_{n\geq 1} \subseteq X$ such that

- *$\varphi'(u_n) = 0$ for all $n \in \mathbb{N}$ (that is, u_n is a critical point of φ),*
- *$\varphi(u_n) < 0$ for all $n \in \mathbb{N}$ and $u_n \to 0$ in X.*

We conclude this section by introducing some notation which will be used in what follows.

By $A \in \mathcal{L}(H^1(\Omega), H^1(\Omega)^*)$ we denote the linear operator defined by

$$\langle A(u), h\rangle = \int_\Omega (\nabla u, \nabla h)_{\mathbb{R}^N}\, dz \quad \text{for all } u,h \in H^1(\Omega).$$

For $x \in \mathbb{R}$ we set $x^\pm = \max\{\pm x, 0\}$. Then for $u \in H^1(\Omega)$ we can define

$$u^\pm(\,\cdot\,) = u(\cdot)^\pm.$$

We know that $u^\pm \in H^1(\Omega)$, $|u| = u^+ + u^-$, $u = u^+ - u^-$.

3. Nodal solutions

Our hypotheses on the data of problem (1.1) are the following:

H(ξ) $\xi \in L^s(\Omega)$ with $s > N$ and $\xi^+ \in L^\infty(\Omega)$.

H(ϑ) $\vartheta \in L^s(\Omega)$ with $s > N$, $\vartheta(z) > 0$ for almost all $z \in \Omega$.

H(β) $\beta \in W^{1,\infty}(\partial\Omega)$, $\beta(z) \geq 0$ for all $z \in \partial\Omega$.

Remark 3.1. If $\beta \equiv 0$, then we recover the Neumann problem.

H(f) Let $e_+ \in H^1(\Omega) \cap C(\overline{\Omega})$ with $0 < \eta_+ \leq e_+(z)$ for all $z \in \Omega$, $\widehat{\eta} = \|e_+\|_\infty$ and $f\colon \Omega \times [-\widehat{\eta}, \widehat{\eta}] \to \mathbb{R}$ is a Carathéodory function such that

(i)
- for almost all $z \in \Omega$, $f(z,\cdot\,)|_{[-\widehat{\eta},\widehat{\eta}]}$ is odd,
- $\vartheta(z)e_+(z)^{q-1} + f(z, e_+(z)) - \xi(z)e_+(z) \leq 0$ for almost all $z \in \Omega$, and $0 \leq A(e_+)$ in $H^1(\Omega)^*$,
- there exists $a_{\widehat{\eta}} \in L^s(\Omega)_+$ with $s > N$ such that $|f(z,x)| \leq a_{\widehat{\eta}}(z)$ for almost all $z \in \Omega$, all $|x| \leq \widehat{\eta}$,

(ii) there exist functions $\widehat{c}_0, \widehat{c}_1 \in L^s(\Omega)$ with $s > N$, and $r > 2$ such that

$$-\widehat{c}_0(z) \le \liminf_{x\to 0} \frac{f(z,x)}{|x|^{r-2}x} \le \limsup_{x\to 0} \frac{f(z,x)}{|x|^{r-2}x} \le \widehat{c}_1(z)$$

uniformly for almost all $z \in \Omega$.

Remark 3.2. Evidently all the hypotheses on $f(z,\,\cdot\,)$ concern the interval $[-\widehat{\eta}, \widehat{\eta}]$. The behaviour of $f(z,\,\cdot\,)$ beyond $[-\widehat{\eta}, \widehat{\eta}]$ is irrelevant. If

$$k(z,x) = \vartheta(z)|x|^{q-2}x + f(z,x) - \xi(z)x,$$

then this is a Carathéodory function which is odd in x and it satisfies

$$k(z, e_+(z)) \le 0 \le k(z, -e_+(z)), \quad \text{for a.a. } z \in \Omega.$$

This fact together with

$$A(-e_+) \le 0 \le A(e_+) \quad \text{in } H^1(\Omega)^* \quad (\text{see H}(f)\,(\text{i}))$$

allow us to treat e_+ (resp. $-e_+$) as an upper (resp. lower) solution for the problem and work with suitable truncations and perturbations of the reaction term. Then we obtain extremal constant sign solutions $v_* \le 0 \le u_*$. This is important since we can focus on the order interval $[v_*, u_*]$ and apply Theorem 2.1 (the Heinz–Kajikiya result) to generate a sequence of nodal solutions. If we can find $\tau > 0$ such that

$$(3.1)\qquad \begin{aligned} &f(z,\,\cdot\,)|_{[-\tau,\tau]} \text{ is odd} && \text{for a.a. } z \in \Omega,\\ &\vartheta(z)\tau^{q-1} + f(z,\tau) - \xi(z)\tau \le 0 && \text{for a.a. } z \in \Omega,\\ &|f(z,x)| \le a_\tau(z) && \text{for a.a. } z \in \Omega,\\ & && \text{all } |x| \le \tau \text{ with } a_\tau \in L^s(\Omega), \end{aligned}$$

then hypothesis H(f) (i) is satisfied.

Hypotheses H(f) imply that we can find $c_1 \in L^s(\Omega)_+$ and $r > 2$ such that

$$(3.2)\quad \vartheta(z)|x|^q + f(z,x)x \ge \vartheta(z)|x|^q - c_1(z)|x|^r \quad \text{for a.a. } z \in \Omega, \text{ all } x \in [-\tau, \tau].$$

Based on this unilateral growth estimate for the reaction function, we introduce the following Carathéodory function:

$$(3.3)\qquad g(z,x) = \begin{cases} -\vartheta(z)e_+(z)^{q-1} + c_1(z)e_+(z)^{r-1} & \text{if } x < -e_+(z),\\ \vartheta(z)|x|^{q-2}x - c_1(z)|x|^{r-2}x & \text{if } -e_+(z) \le x \le e_+(z),\\ \vartheta(z)e_+(z)^{q-1} - c_1(z)e_+(z)^{r-1} & \text{if } e_+(z) < x. \end{cases}$$

Using g, we consider the following auxiliary Robin problem:

$$(3.4)\qquad \begin{cases} -\Delta u(z) + \xi(z)u(z) = g(z,u(z)) & \text{in } \Omega,\\ \dfrac{\partial u}{\partial n} + \beta(z)u = 0 & \text{on } \partial\Omega. \end{cases}$$

Proposition 3.3. *If hypotheses* $\mathrm{H}(\xi), \mathrm{H}(\vartheta), \mathrm{H}(\beta), \mathrm{H}(f)$ *hold, then problem has a unique positive solution* $\overline{u} \in D_+$ *and* $\overline{v} = -\overline{u} \in -D_+$ *is the unique negative solution of* (3.4).

Proof. With $\mu > 0$ as in (2.2), we introduce the following Carathéodory function:

$$\widehat{g}(z,x) = \begin{cases} g(z,-e_+(z)) - \mu e_+(z) & \text{if } x < -e_+(z), \\ g(z,x) + \mu x & \text{if } -e_+(z) \le x \le e_+(z), \\ g(z,e_+(z)) + \mu e_+(z) & \text{if } e_+(z) < x, \end{cases} \tag{3.5}$$

we set $\widehat{G}(z,x) = \int_0^x \widehat{g}(z,s)\,ds$ and consider the C^1-functional $\widehat{\psi}_+ \colon H^1(\Omega) \to \mathbb{R}$ defined by

$$\widehat{\psi}_+(u) = \frac{1}{2}\,\gamma(u) + \frac{\mu}{2}\,\|u\|_2^2 - \int_\Omega \widehat{G}(z,u^+)\,dz \quad \text{for all } u \in H^1(\Omega).$$

From (2.2) and (3.5) it follows that

$$\widehat{\psi}_+(u) \ge \frac{c_0}{2}\,\|u\|^2 - c_2 \quad \text{for some } c_2 > 0, \text{ all } u \in H^1(\Omega),$$

implies $\widehat{\psi}_+$ is coercive.

Moreover, the Sobolev embedding theorem and the compactness of the trace map imply that $\widehat{\psi}_+$ is sequentially weakly lower semicontinuous. So, by the Weierstrass–Tonelli theorem, we can find $\overline{u} \in H^1(\Omega)$ such that

$$\widehat{\psi}_+(\overline{u}) = \inf\big[\widehat{\psi}_+(u) : u \in H^1(\Omega)\big]. \tag{3.6}$$

Let $t \in (0,1)$ be small such that

$$t\widehat{u}_1(z) \in (0,\eta_+] \quad \text{for all } z \in \overline{\Omega} \text{ (recall that } \widehat{u}_1 \in D_+).$$

Using Hölder's inequality, we have for some $\widehat{c}_1$

$$\begin{aligned} \widehat{\psi}_+(t\widehat{u}_1) &= \frac{1}{2}\,\gamma(t\widehat{u}_1) - \frac{1}{q}\int_\Omega \vartheta(z)(t\widehat{u}_1)^q\,dz + \frac{\widehat{c}_1}{r}\,\|t\widehat{u}_1\|_r^r \quad \text{(see (3.5) and (3.3))} \\ &= \frac{t^2}{2}\,\widehat{\lambda}_1 + \frac{\widehat{c}_1 t^r}{r}\,\|\widehat{u}_1\|_r^r - \frac{t^q}{q}\int_\Omega \vartheta(z)\widehat{u}_1^q\,dz \qquad \text{(recall that } \|\widehat{u}_1\|_2 = 1). \end{aligned}$$

Since $q < 2 < r$ and using hypothesis $\mathrm{H}(\vartheta)$, we see that by choosing $t \in (0,1)$ even smaller if necessary, we have

$$\begin{aligned} \widehat{\psi}_+(t\widehat{u}_1) < 0 \ &\Rightarrow \widehat{\psi}_+(\overline{u}) < 0 = \widehat{\psi}_+(0) \quad \text{(see (3.6))}, \\ &\Rightarrow \overline{u} \ne 0. \end{aligned}$$

From (3.6) we have $\widehat{\psi}'_+(\overline{u}) = 0$, which, for all $h \in H^1(\Omega)$, implies

$$\langle A(\overline{u}), h\rangle + \int_\Omega (\xi(z) + \mu)\overline{u}h\,dz + \int_{\partial\Omega} \beta(z)\overline{u}h\,d\sigma = \int_\Omega \widehat{g}(z,\overline{u}^{\,+})h\,dz. \tag{3.7}$$

In (3.7) first we choose $h = -\overline{u}^- \in H^1(\Omega)$. We obtain

$$\begin{aligned}\gamma(\overline{u}^-) + \mu\|\overline{u}^-\|_2^2 = 0 &\Rightarrow c_0\|\overline{u}^-\|^2 \le 0 \qquad \text{(see (2.2))},\\ &\Rightarrow \overline{u} \ge 0, \quad \overline{u} \ne 0.\end{aligned}$$

Also in (3.7) we choose $h = (\overline{u} - e_+)^+ \in H^1(\Omega)$. Then

$$\begin{aligned}
&\|\nabla(\overline{u} - e_+)^+\|_2^2 + \int_\Omega (\xi(z)+\mu)\overline{u}(\overline{u}-e_+)^+\,dz + \int_{\partial\Omega} \beta(z)\overline{u}(\overline{u}-e_+)^+\,d\sigma\\
&\quad= \int_\Omega \widehat{g}(z,\overline{u})(\overline{u}-e_+)^+\,dz\\
&\quad= \int_\Omega [g(z,e_+) + \mu e_+](\overline{u}-e_+)^+\,dz \qquad \text{(see (3.5))}\\
&\quad= \int_\Omega [\vartheta(z)e_+^{q-1} - c_1(z)e_+^{r-1} + \mu e_+](\overline{u}-e_+)^+\,dz \qquad \text{(see (3.3))}\\
&\quad\le \int_\Omega [\vartheta(z)e_+^{q-1} + f(z,e_+) + \mu e_+](\overline{u}-e_+)^+\,dz \qquad \text{(see (3.2))}\\
&\quad\le \int_\Omega [\xi(z)+\mu]e_+(\overline{u}-e_+)^+dz \qquad \text{(see hypothesis H}(f)\text{ (i))}\\
&\quad\quad\Rightarrow\ \gamma((\overline{u}-e_+)^+) + \mu\|(\overline{u}-e_+)^+\|_2^2 \le 0 \qquad \text{(see hypothesis H}(\beta))\\
&\quad\quad\Rightarrow\ c_0\|(\overline{u}-e_+)^+\|^2 \le 0 \qquad \text{(see (2.2))}\\
&\quad\quad\Rightarrow\ \overline{u} \le e_+.
\end{aligned}$$

So, we have proved that

$$\overline{u} \in [0,e_+] = \{u \in H^1(\Omega) : 0 \le u(z) \le e_+(z) \text{ for a.a. } z \in \Omega\}, \quad \overline{u} \ne 0. \tag{3.8}$$

From (3.3), (3.5), (3.7) and (3.8) it follows that

$$\begin{cases} -\Delta\overline{u}(z) + \xi(z)\overline{u}(z) = \vartheta(z)\overline{u}(z)^{q-1} - c_1(z)\overline{u}(z)^{r-1} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for a.a. } z \in \Omega \\ \dfrac{\partial\overline{u}}{\partial n} + \beta(z)\overline{u} = 0 \qquad\qquad\qquad\qquad \text{on } \partial\Omega \end{cases} \tag{3.9}$$

(see Papageorgiou–Radulescu [8]), which implies $\overline{u} \in H^1(\Omega)$ is a positive solution of (3.4). Let

$$a(z) = \begin{cases} 0 & \text{if } \overline{u}(z) \le \dfrac{\widehat{\eta}}{2},\\ \dfrac{\vartheta(z)}{\overline{u}(z)^{2-q}} - c_1(z)\overline{u}(z)^{r-2} - \xi(z) & \text{if } \dfrac{\widehat{\eta}}{2} < \overline{u}(z), \end{cases}$$

and

$$b(z) = \begin{cases} \vartheta(z)\overline{u}(z)^{q-1} - c_1(z)\overline{u}(z)^{r-1} - \xi(z)\overline{u}(z) & \text{if } \overline{u}(z) \le \dfrac{\widehat{\eta}}{2},\\ 0 & \text{if } \dfrac{\widehat{\eta}}{2} < \overline{u}(z). \end{cases}$$

Evidently $a, b \in L^s(\Omega)$ (see hypotheses $\mathrm{H}(\xi), \mathrm{H}(\vartheta)$) and from (3.9) we have

$$\begin{cases} -\Delta\overline{u}(z) = a(z)\overline{u}(z) + b(z) & \text{for a.a. } z \in \Omega, \\ \dfrac{\partial \overline{u}}{\partial n} + \beta(z)\overline{u} = 0 & \text{on } \partial\Omega. \end{cases}$$

Then, from Wang [14] (see Lemmas 5.1 and 5.2), we have $\overline{u} \in C_+\backslash\{0\}$. Moreover, from (3.9) it follows that

$$\Delta\overline{u}(z) \le (c_1 + \xi^+(z))\overline{u}(z) \le (c_1 + \|\xi^+\|_\infty)\overline{u}(z) \quad \text{for a.a. } z \in \Omega$$

(see hypotheses $\mathrm{H}(\xi), \mathrm{H}(\vartheta)$), implies $\overline{u} \in D_+$ (by the strong maximum principle).

Next we show that this positive solution is in fact unique. So, suppose that $\widetilde{u} \in H^1(\Omega)$ is another positive solution of (3.4). As above, we show that $\widetilde{u} \in [0, \tau] \cap D_+$. We have, for all $h \in H^1(\Omega)$,

$$\langle A(\overline{u}), h\rangle + \int_\Omega \xi(z)\overline{u}h\,dz + \int_{\partial\Omega} \beta(z)\overline{u}h\,d\sigma = \int_\Omega [\vartheta(z)\overline{u}^{q-1} - c_1(z)\overline{u}^{r-1}]h\,dz, \tag{3.10}$$

$$\langle A(\widetilde{u}), h\rangle + \int_\Omega \xi(z)\widetilde{u}h\,dz + \int_{\partial\Omega} \beta(z)\widetilde{u}h\,d\sigma = \int_\Omega [\vartheta(z)\widetilde{u}^{q-1} - c_1(z)\widetilde{u}^{r-1}]h\,dz. \tag{3.11}$$

In (3.10) we choose $h = \widetilde{u} \in H^1(\Omega)$ and in (3.11) we choose $h = \overline{u} \in H^1(\Omega)$. We obtain

$$\int_\Omega (\nabla\overline{u}), \nabla\widetilde{u})_{\mathbb{R}^N}\,dz + \int_\Omega \xi(z)\overline{u}\widetilde{u}\,dz + \int_{\partial\Omega} \beta(z)\overline{u}\widetilde{u}\,d\sigma = \int_\Omega [\vartheta(z)\overline{u}^{q-1} - c_1(z)\overline{u}^{r-1}]\widetilde{u}\,dz, \tag{3.12}$$

$$\int_\Omega (\nabla\widetilde{u}, \nabla\overline{u}))_{\mathbb{R}^N}\,dz + \int_\Omega \xi(z)\widetilde{u}\overline{u}\,dz + \int_{\partial\Omega} \beta(z)\widetilde{u}\overline{u}\,d\sigma = \int_\Omega [\vartheta(z)\widetilde{u}^{q-1} - c_1(z)\widetilde{u}^{r-1}]\overline{u}\,dz. \tag{3.13}$$

We subtract (3.13) from (3.12) and have

$$\begin{aligned} &\int_\Omega \left[\vartheta(z)\left(\frac{1}{\overline{u}^{2-q}} - \frac{1}{\widetilde{u}^{2-q}}\right) - c_1(z)\left(\overline{u}^{r-2} - \widetilde{u}^{r-2}\right)\right]\overline{u}\widetilde{u}\,dz = 0 \\ &\qquad \Rightarrow \overline{u} = \widetilde{u} \qquad \text{(recall that } q < 2 < r) \\ &\qquad \Rightarrow \overline{u} \in D_+ \quad \text{is the unique positive solution of problem (3.4).} \end{aligned}$$

Note that problem (3.4) is odd. So, we infer that $\overline{v} = -\overline{u} \in [-e_+, 0] \cap (-D_+)$ is the unique negative solution of problem (3.4). □

We will use the solutions from Proposition 3.3 to produce a lower (resp. upper) bound for the positive (resp. negative) solutions of problem (1.1). So, let

$$\begin{aligned} S_+ &= \{u \in H^1(\Omega) : u \text{ is a positive solution of problem (1.1) in } [0,\widehat{\eta}]\},\\ S_- &= \{u \in H^1(\Omega) : u \text{ is a negative solution of problem (1.1) in } [-\widehat{\eta},0]\}. \end{aligned}$$

Proposition 3.4. *If hypotheses* $\mathrm{H}(\xi), \mathrm{H}(\vartheta), \mathrm{H}(\beta), \mathrm{H}(f)$ *hold, then* $\overline{u} \le u$ *for all* $u \in S_+$ *and* $v \le \overline{v}$ *for all* $v \in S_-$.

Proof. Let $u \in S_+$. We introduce the following Carathéodory function:

$$\widetilde{g}_+(z,x) = \begin{cases} 0 & \text{if } x < 0,\\ \widehat{g}(z,x) & \text{if } 0 \le x \le u(z),\\ \widehat{g}(z,u(z)) & \text{if } u(z) < x \end{cases} \tag{3.14}$$

with $\widehat{g}(z,x)$ from (3.5). We set

$$\widetilde{G}_+(z,x) = \int_0^x \widetilde{g}_+(z,s)\,ds$$

and consider the C^1-functional $\widetilde{\psi}_+ \colon H^1(\Omega) \to \mathbb{R}$ defined by

$$\widetilde{\psi}_+(u) = \frac{1}{2}\,\gamma(u) + \frac{\mu}{2}\,\|u\|_2^2 - \int_\Omega \widetilde{G}_+(z,u)\,dz \quad \text{for all } u \in H^1(\Omega).$$

From (2.2), (3.5) and (3.14) it follows that $\widetilde{\psi}_+$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\widetilde{u} \in H^1(\Omega)$ such that

$$\widetilde{\psi}_+(\widetilde{u}) = \inf\big[\widetilde{\psi}_+(u) : u \in H^1(\Omega)\big]. \tag{3.15}$$

As before (see the proof of Proposition 3.3), since $q < 2 < r$, we have

$$\widetilde{\psi}_+(\widetilde{u}) < 0 = \widetilde{\psi}_+(0) \;\Rightarrow\; \widetilde{u} \neq 0.$$

From (3.15) we have

$$\begin{aligned} \widetilde{\psi}'_+(\widetilde{u}) &= 0\\ &\Rightarrow \langle A(\widetilde{u}), h\rangle + \int_\Omega (\xi(z)+\mu)\widetilde{u}h\,dz + \int_{\partial\Omega} \beta(z)\widetilde{u}h\,d\sigma\\ &= \int_\Omega \widetilde{g}_+(z,\widetilde{u})h\,dz, \quad \text{for all } h \in H^1(\Omega). \end{aligned} \tag{3.16}$$

In (3.16) first we choose $h = -\widetilde{u}^- \in H^1(\Omega)$ and obtain

$$\begin{aligned} \gamma(\widetilde{u}^-) + \mu\|\widetilde{u}^-\|_2^2 &= 0 && \text{(see (3.14))},\\ \Rightarrow c_0\|\widetilde{u}^-\|^2 &\le 0 && \text{(see (2.2))},\\ \Rightarrow \widetilde{u} \ge 0, &\quad \widetilde{u} \neq 0. \end{aligned}$$

Next in (3.16) we choose $h = (\widetilde{u} - u)^+ \in H^1(\Omega)$. Then

$$\begin{aligned}
&\langle A(\widetilde{u}), (\widetilde{u}-u)^+\rangle + \int_\Omega (\xi(z)+\mu)\widetilde{u}(\widetilde{u}-u)^+\,dz + \int_{\partial\Omega}\beta(z)\widetilde{u}(\widetilde{u}-u)^+\,d\sigma \\
&\quad= \int_\Omega \widehat{g}(z,u)(\widetilde{u}-u)^+\,dz && \text{(see (3.14))} \\
&\quad= \int_\Omega [\vartheta(z)u^{q-1} - c_1(z)u^{r-1} + \mu u](\widetilde{u}-u)^+\,dz \\
&\qquad\qquad \text{(see (3.3), (3.5) and recall that } u \in [0,\tau]) \\
&\quad\le \int_\Omega [\vartheta(z)u^{q-1} + f(z,u) + \mu u](\widetilde{u}-u)^+\,dz && \text{(see (3.2))} \\
&\quad= \langle A(u), (\widetilde{u}-u)^+\rangle + \int_\Omega (\xi(z)+\mu)u(\widetilde{u}-u)^+\,dz \\
&\qquad + \int_{\partial\Omega}\beta(z)u(\widetilde{u}-u)^+\,d\sigma && \text{(since } u \in S_+), \\
&\qquad \Rightarrow\ \gamma((\widetilde{u}-u)^+) + \mu\|(\widetilde{u}-u)^+\|_2^2 \le 0, \\
&\qquad \Rightarrow\ c_0\|(\widetilde{u}-u)^+\|^2 \le 0 && \text{(see (2.2))} \\
&\qquad \Rightarrow\ \widetilde{u} \le u.
\end{aligned}$$

So, we have proved that

$$\widetilde{u} \in [0,u] = \{v \in H^1(\Omega) : 0 \le v(z) \le u(z) \text{ for a.a. } z \in \Omega\}, \quad \widetilde{u} \ne 0. \tag{3.17}$$

From (3.3), (3.5), (3.14) and (3.17) it follows that

$$\begin{cases} -\Delta\widetilde{u}(z) + \xi(z)\widetilde{u}(z) = \vartheta(z)\widetilde{u}(z)^{q-1} - c_1(z)\widetilde{u}(z)^{r-1} & \text{for a.a. } z \in \Omega, \\ \dfrac{\partial\widetilde{u}}{\partial n} + \beta(z)\widetilde{u} = 0 & \text{on } \partial\Omega,\ \widetilde{u} \ne 0 \end{cases}$$

(see Papageorgiou–Radulescu [8]), which implies $\widetilde{u} = \overline{u} \in D_+$ (see Proposition 3.3) and hence $\overline{u} \le u$ for all $u \in S_+$ (see (3.17)).

In a similar fashion we show that $\overline{v} \le v$ for all $v \in S_-$. □

Using these bounds we can produce extremal constant sign solutions for problem (1.1), that is, a smallest element in S_+ and a biggest element in S_-.

Proposition 3.5. *If hypotheses* $\mathrm{H}(\xi), \mathrm{H}(\vartheta), \mathrm{H}(\beta), \mathrm{H}(f)$ *hold, then there exists* $u_* \in S_+ \subseteq [0, e_+] \cap D_+$ *such that* $u_* \le u$ *for all* $u \in S_+$ *and there exists* $v_* \in S_- \subseteq [-e_+, 0] \cap (-D_+)$ *such that* $v \le v_*$ *for all* $v \in S_-$.

Proof. Invoking Lemma 3.10 of Hu and Papageorgiou [4, p. 178], we can find $\{u_n\}_{n\ge1} \subseteq S_+$ such that

$$\inf S_+ = \inf_{n\ge1} u_n.$$

We have

$$\tag{3.18} \begin{aligned}\langle A(u_n), h\rangle + \int_\Omega \xi(z) u_n h\, dz + \int_{\partial\Omega} \beta(z) u_n h\, d\sigma \\ = \int_\Omega [\vartheta(z) u_n^{q-1} + f(z, u_n)] h\, dz \quad \text{for all } h \in H^1(\Omega), \text{ all } n \in \mathbb{N}.\end{aligned}$$

Since $u_n(z) \in [0, \tau]$ for all $z \in \overline{\Omega}$, all $n \in \mathbb{N}$, from (3.18) it follows that

$$\{u_n\}_{n\geq 1} \subseteq H^1(\Omega) \text{ is bounded.}$$

(Just choose $h = u_n \in H^1(\Omega)$ in (3.18) and use hypotheses $\mathrm{H}(\xi), \mathrm{H}(\vartheta), \mathrm{H}(\beta)$ and $\mathrm{H}(f)$ (i).) So, we may assume that

$$\tag{3.19} u_n \xrightarrow{\text{w}} u_* \quad \text{in } H^1(\Omega) \quad \text{and} \quad u_n \to u_* \quad \text{in } L^r(\Omega) \text{ and in } L^2(\partial\Omega).$$

If in (3.18) we pass to the limit as $n \to +\infty$ and use (3.19), then

$$\tag{3.20} \begin{aligned}\langle A(u_*), h\rangle + \int_\Omega \xi(z) u_* h\, dz + \int_{\partial\Omega} \beta(z) u_* h\, d\sigma \\ = \int_\Omega [\vartheta(z) u_*^{q-1} - c_1(z) u_*^{r-1}] h\, dz \quad \text{for all } h \in H^1(\Omega).\end{aligned}$$

From Proposition 3.4 we know that $\overline{u} \leq u_n$ for all $n \in \mathbb{N}$, which implies

$$\tag{3.21} \overline{u} \leq u_* \quad (\text{see (3.19)}).$$

From (3.20) and (3.21) it follows that

$$u_* \in S_+ \subseteq [0, e_+] \cap D_+ \quad \text{and} \quad u_* = \inf S_+.$$

Similarly, we produce

$$v_* \in S_- \subseteq [-e_+, 0] \cap (-D_+) \quad \text{and} \quad v_* = \sup S_-. \qquad \square$$

Now we are ready to produce a whole sequence of distinct nodal solutions for problem (1.1).

Theorem 3.6. *If hypotheses* $\mathrm{H}(\xi)$, $\mathrm{H}(\vartheta)$, $\mathrm{H}(\beta)$, $\mathrm{H}(f)$ *hold, then there exists a whole sequence* $\{u_n\}_{n\geq 1} \subseteq C^1(\overline{\Omega})$ *of distinct nodal solutions for problem* (1.1) *such that* $u_n \to 0$ *in* $C^1(\overline{\Omega})$.

Proof. Let $u_* \in D_+$ and $v_* \in -D_+$ be the two extremal constant sign solutions of problem (1.1) produced in Proposition 3.5. We introduce the following Carathéodory function:

$$\tag{3.22} \widehat{f}(z, x) = \begin{cases} \vartheta(z)|v_*(z)|^{q-2} v_*(z) + f(z, v_*(z)) + \mu v_*(z) & \text{if } x < v_*(z), \\ \vartheta(z)|x|^{q-2} x + f(z, x) + \mu x & \text{if } v_*(z) \leq x \leq u_*(z), \\ \vartheta(z) u_*(z)^{q-1} + f(z, u_*(z)) + \mu u_*(z) & \text{if } u_*(z) < x. \end{cases}$$

We set $\widehat{F}(z,x) = \int_0^x \widehat{f}(z,s)\,ds$ and consider the C^1-functional $\widehat{\varphi}\colon H^1(\Omega) \to \mathbb{R}$ defined by

$$\widehat{\varphi}(u) = \frac{1}{2}\,\gamma(u) + \frac{\mu}{2}\,\|u\|_2^2 - \int_\Omega \widehat{F}(z,u)\,dz \quad \text{for all } u \in H^1(\Omega).$$

Claim. $K_{\widehat{\varphi}} = \{u \in H^1(\Omega) : \widehat{\varphi}'(u) = 0\} \subseteq [v_*, u_*] \cap C^1(\overline{\Omega})$.

Let $u \in K_{\widehat{\varphi}}$. Then, for all $h \in H^1(\Omega)$, $\widehat{\varphi}'(u) = 0$ implies

$$\langle A(u), h\rangle + \int_\Omega (\xi(z)+\mu)uh\,dz + \int_{\partial\Omega} \beta(z)uh\,d\sigma = \int_\Omega \widehat{f}(z,u)h\,dz. \tag{3.23}$$

In (3.23) we choose $h = (u-u_*)^+ \in H^1(\Omega)$. Then

$$\begin{aligned}
\langle A(u),&(u-u_*)^+\rangle + \int_\Omega (\xi(z)+\mu)u(u-u_*)^+\,dz + \int_{\partial\Omega} \beta(z)u(u-u_*)^+\,d\sigma \\
&= \int_\Omega \widehat{f}(z,u)(u-u_*)^+\,dz \\
&= \int_\Omega [\vartheta(z)u_*^{q-1} + f(z,u_*) + \mu u_*](u-u_*)^+\,dz \qquad \text{(see (3.22))} \\
&= \langle A(u_*), (u-u_*)^+\rangle + \int_\Omega (\xi(z)+\mu)u_*(u-u_*)^+\,dz \\
&\quad + \int_{\partial\Omega} \beta(z)u_*(u-u_*)^+\,d\sigma \qquad \text{(since } u_* \in S_+\text{)}, \\
&\quad \Rightarrow\ \gamma((u-u_*)^+) + \mu\|(u-u_*)^+\|_2^2 \le 0, \\
&\quad \Rightarrow\ c_0\|(u-u_*)^+\|^2 \le 0 \qquad \text{(see (2.2))}, \\
&\quad \Rightarrow\ u \le u_*.
\end{aligned}$$

Similarly, if in (3.23) we choose $h = (v_* - u)^+ \in H^1(\Omega)$, then we show that $v_* \le u$. Therefore, we have proved that

$$u \in [v_*, u_*] = \{v \in H^1(\Omega) : v_*(z) \le v(z) \le u_*(z) \text{ for a.a. } z \in \Omega\}.$$

Moreover, using the regularity theory of Wang [14], we conclude that

$$K_{\widehat{\varphi}} \subseteq [v_*, u_*] \cap C^1(\overline{\Omega}).$$

This proves the claim.

On account of this claim, we infer that the elements of $K_{\widehat{\varphi}} \setminus \{0, u_*, v_*\}$ are all nodal solutions of problem (1.1). Let $m_0 = \min\big\{\min_{\overline{\Omega}} u_*, \min_{\overline{\Omega}}(-v_*)\big\} > 0$ (recall that $u_* \in D_+$ and $v_* \in -D_+$). Hypothesis H(f) (ii) implies that we can find $c_3 \in L^s(\Omega)$ and $\delta \in (0, m_0)$ such that

$$\begin{aligned}
f(z,x)x &\ge -c_3(z)|x|^r \quad \text{for a.a. } z \in \Omega,\ \text{all } |x| \le \delta, \\
\Rightarrow\ F(z,x) &\ge -\frac{c_3(z)}{r}\,|x|^r \quad \text{for a.a. } z \in \Omega,\ \text{all } |x| \le \delta.
\end{aligned} \tag{3.24}$$

Let Y_m $(m \in \mathbb{N})$ be an m-dimensional subspace of $H^1(\Omega)$. All norms on Y_m are equivalent. So, we can find $\rho_m \in (0,1)$ such that

$$u \in Y_m \quad \text{and} \quad \|u\| \le \rho_m \quad \Rightarrow \quad |u(z)| \le \delta \quad \text{for a.a. } z \in \Omega. \tag{3.25}$$

Therefore for $u \in Y_m$ with $\|u\| \le \rho_m$ we have

$$\begin{aligned}\widehat{\varphi}(u) &= \frac{1}{2}\gamma(u) - \frac{1}{q}\int_\Omega \vartheta(z)u^2\,dz - \int_\Omega F(z,u)\,dz \\ &\qquad\qquad\qquad\qquad\qquad \text{(see (3.22) and (3.25))} \\ &\le \frac{1}{2}\gamma(u) - \frac{1}{q}\int_\Omega \vartheta(z)u^2\,dz + \frac{\widehat{c}_3}{r}\|u\|^r \quad \text{for some } \widehat{c}_3 > 0 \text{ (see (3.24))} \\ &\le \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\int_\Omega \xi^+(z)u^2\,dz \\ &\quad + \frac{1}{2}\int_{\partial\Omega}\beta(z)u^2\,d\sigma + \frac{\widehat{c}_3}{r}\|u\|_r^r - \frac{1}{q}\int_\Omega \vartheta(z)u^2\,dz \\ &\le c_4\|u\|^2 - \frac{1}{q}\int_\Omega \vartheta(z)|u|^q\,dz \qquad \text{for some } c_4 > 0\end{aligned} \tag{3.26}$$

(see hypotheses $H(\xi)$, $H(\beta)$ and recall $2 < r$, $\rho_m < 1$).

Consider the functional $k\colon H^1(\Omega) \to \mathbb{R}$ defined by

$$k(u) = c_4\|u\|^2 - \frac{1}{q}\int_\Omega \vartheta(z)|u|^q\,dz \quad \text{for all } u \in H^1(\Omega).$$

Suppose that $\{u_n\}_{n\ge1} \subseteq H^1(\Omega)$ is a sequence such that

$$u_n \in Y_m \quad \text{for all } n \in \mathbb{N} \quad \text{and} \quad u_n \to 0 \quad \text{in } H^1(\Omega). \tag{3.27}$$

Let $y_n = u_n/\|u_n\|$ for all $n \in \mathbb{N}$. We have $\|y_n\| = 1$ for all $n \in \mathbb{N}$ and

$$\frac{k(u_n)}{\|u_n\|^q} \le c_4\|u_n\|^{2-q} - \frac{1}{q}\int_\Omega \vartheta(z)|y_n|^q\,dz \quad \text{for all } n \in \mathbb{N}. \tag{3.28}$$

We have $\{y_n\}_{n\ge1} \subseteq Y_m$ and $\|y_n\| = 1$ for all $n \in \mathbb{N}$. The finite dimensionality of Y_m implies that at least for a subsequence we have

$$y_n \to y \quad \text{in } H^1(\Omega) \quad \text{and} \quad \|y\| = 1. \tag{3.29}$$

Passing to the limit as $n \to +\infty$ in (3.28) and using (3.27) and (3.29), we obtain

$$\limsup_{n\to+\infty} \frac{k(u_n)}{\|u_n\|^q} \le -\frac{1}{q}\int_\Omega \vartheta(z)|y|^q\,dz < 0$$

(recall $q < 2$ and see hypothesis $H(\vartheta)$ and (3.29)). This fact in conjunction with (3.26), imply that if we take $\rho_m \in (0,1)$ even smaller if needed, then

$$\sup\big[\widehat{\varphi}(u) : u \in Y_m,\ \|u\| = \rho_m\big] < 0.$$

Clearly, $\widehat{\varphi}$ is coercive (see (2.2) and (3.22)). So, it satisfies the PS-condition (see Marano and Papageorgiou [6]) and it is bounded below. Moreover, $\widehat{\varphi}$ is even (see hypothesis H(f) (i)) and $\widehat{\varphi}(0) = 0$. So, we can apply Theorem 2.1 and find a sequence $\{u_n\}_{n\geq 1} \subseteq H^1(\Omega)$ such that

$$\widehat{\varphi}'(u_n) = 0, \quad \widehat{\varphi}(u_n) < 0 \quad \text{for all } n \in \mathbb{N} \text{ and } u_n \to 0.$$

On account of Claim, we have $\{u_n\}_{n\geq 1} \subseteq [v_*, u_*] \cap C^1(\overline{\Omega})$ for all $n \in \mathbb{N}$.

The extremality of u_*, v_* implies that $\{u_n\}_{n\geq 1}$ are smooth nodal solutions of problem (1.1). Moreover, from Wang [14], we know that we can find $\alpha \in (0,1)$ and $c_5 > 0$ such that

$$u_n \in C^{1,\alpha}(\overline{\Omega}) \quad \text{and} \quad \|u_n\|_{C^{1,\alpha}(\overline{\Omega})} \leq c_5 \quad \text{for all } n \in \mathbb{N}.$$

The compact embedding of $C^{1,\alpha}(\overline{\Omega})$ into $C^1(\overline{\Omega})$ implies that $u_n \to u$ in $C^1(\overline{\Omega})$.□

Remark 3.7. As the referee kindly pointed out the second condition in hypothesis H(f) (i) can be dropped. In fact, if this condition is not present, using a cut-off technique we can always transform the problem to an equivalent one in which the perturbation term f satisfies that second condition in H(f) (i). In any case we need to go through the previous argument to establish that the sequence of solutions we produce are nodal. So, essentially there is no loss of generality in assuming from the beginning that condition.

If the coefficient ϑ of the concave term is indefinite (that is, sign changing), then we can still have a sequence of distinct nontrivial smooth solutions converging to zero in $C^1(\overline{\Omega})$. However, we can not claim that these solutions are nodal.

Now the condition on the coefficient ϑ is the following:

H(ϑ)′ $\vartheta \in L^\infty(\Omega)$ and there exist $U \subset \Omega$ open such that $\vartheta(z) > 0$ for almost all $z \in U$.

Theorem 3.8. *If hypotheses* $\mathrm{H}(\xi), \mathrm{H}(\vartheta)', \mathrm{H}(\beta), \mathrm{H}(f)$ *hold, then there exists a whole sequence* $\{u_n\}_{n\geq 1} \subseteq C^1(\overline{\Omega})$ *of distinct nontrivial solutions of problem* (1.1) *such that* $u_n \to 0$ *in* $C^1(\overline{\Omega})$.

Proof. Let $k\colon \Omega \times \mathbb{R} \to \mathbb{R}$ be the Carathéodory function defined by

$$k(z,x) = \begin{cases} -\vartheta(z) e_+(z)^{q-1} - f(z, e_+(z)) - \mu e_+(z) & \\ \qquad\qquad\qquad\qquad \text{if } x < -e_+(z), \\ \vartheta(z)|x|^{q-2}x + f(z,x) + \mu x & \text{if } -e_+(z) \leq x \leq e_+(z), \\ \vartheta(z) e_+(z)^{q-1} + f(z, e_+(z)) + \mu e_+(z) & \\ \qquad\qquad\qquad\qquad \text{if } e_+(z) < x. \end{cases} \tag{3.30}$$

Here $\mu > 0$ is as in (2.2). We set $K(z,x) = \int_0^x k(z,s)\,ds$ and consider the C^1-functional $\varphi_0\colon H^1(\Omega) \to \mathbb{R}$ defined by

$$\varphi_0(u) = \frac{1}{2}\gamma(u) + \frac{\mu}{2}\|u\|_2^2 - \int_\Omega K(z,u)\,dz \quad \text{for all } u \in H^1(\Omega).$$

From (2.2) and (3.30) it is clear that φ_0 is coercive. So, φ_0 is bounded below and satisfies the PS-condition. Also, φ_0 is even and $\varphi_0(0) = 0$.

Let $m \in \mathbb{N}$ and choose $\{w_i\}_{i=1}^m \subseteq C_c^1(U)$ linearly independent. We set

$$Y_m = \operatorname{span}\{w_i\}_{i=1}^m.$$

Then as before (see the proof of Theorem 3.6), using hypothesis H(f) (ii), we have

$$\varphi_0(u) \leq c_6\|u\|^2 - \frac{1}{q}\int_U \vartheta(z)|u|^q\,dz$$

for some $c_6 > 0$ and all $u \in Y_m$ with $\|u\|_m \leq \rho_m \in (0,1)$.

By hypothesis H$(\vartheta)'$, $\vartheta|_U > 0$ and so as in the proof of Theorem 3.6, by choosing $\rho_m \in (0,1)$ even smaller if necessary, we have

$$\sup[\varphi_0(u) : u \in Y_m,\ \|u\| = \rho_m] < 0. \tag{3.31}$$

Claim. $K_{\varphi_0} = \{u \in H^1(\Omega) : \varphi_0'(u) = 0\} \subseteq [-e_+, e_+] \cap C^1(\overline{\Omega})$.

Let $u \in K_{\varphi_0}$. Then $\varphi_0'(u) = 0$, which implies

$$\langle A(u), h\rangle + \int_\Omega (\xi(z) + \mu)uh\,dz + \int_{\partial\Omega} \beta(z)uh\,d\sigma = \int_\Omega k(z,u)h\,dz \tag{3.32}$$

for all $h \in H^1(\Omega)$. In (3.32) we choose $h = (u - e_+)^+ \in H^1(\Omega)$. Then

$$\begin{aligned}
\langle A(u),&(u-e_+)^+\rangle + \int_\Omega (\xi(z)+\mu)u(u-e_+)^+\,dz + \int_{\partial\Omega}\beta(z)u(u-e_+)^+\,d\sigma \\
&= \int_\Omega k(z,u)(u-e_+)^+\,dz \\
&= \int_\Omega [\vartheta(z)e_+^{q-1} + f(z,e_+) + \mu e_+](u-e_+)^+\,dz \quad \text{(see (3.30))} \\
&\leq \langle A(e_+),(u-e_+)^+\rangle \\
&\quad + \int_\Omega (\xi(z)+\mu)e_+(u-e_+)^+\,dz + \int_{\partial\Omega}\beta(z)e_+(u-e_+)^+\,d\sigma
\end{aligned}$$

(see hypotheses H(β), H(f) (i)) which implies

$$\begin{aligned}
\gamma((u-e_+)^+) + \mu\|(u-e_+)^+\|_2^2 &\leq 0, \\
&\Rightarrow c_0\|(u-e_+)^+\|^2 \leq 0 \quad \text{(see (2.2))}, \\
&\Rightarrow u \leq e_+.
\end{aligned}$$

Similarly, choosing $h = (-e_+ - u)^+ \in H^1(\Omega)$ in (3.32), we show that $-e_+ \leq u$. Moreover, from the regularity theory of Wang [14], we have that $u \in C^1(\overline{\Omega})$. Therefore $K_{\varphi_0} \subseteq [-\tau, \tau] \cap C^1(\overline{\Omega})$ and this proves the claim.

Then (3.31), Claim and the properties of φ_0 mentioned in the beginning of the proof, permit the use of Theorem 2.1. So, we can find a sequence $\{u_n\}_{n\geq 1} \subseteq H^1(\Omega)$ such that

$$u_n \in K_{\varphi_0}, \quad \varphi_0(u_n) < 0 \quad \text{for all } n \in \mathbb{N}, \quad u_n \to 0 \quad \text{in } H^1(\Omega). \tag{3.33}$$

From Claim and (3.30) we have that

$$\{u_n\}_{n\geq 1} \subseteq C^1(\overline{\Omega}) \ \text{ are solutions of } (1.1).$$

Moreover, (3.33) and the results of Wang [14] (Lemmas 5.1 and 5.2) imply that $u_n \to 0$ in $C^1(\overline{\Omega})$. □

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