

**MULTIPLICITY OF SOLUTIONS  
FOR POLYHARMONIC DIRICHLET PROBLEMS  
WITH EXPONENTIAL NONLINEARITIES  
AND BROKEN SYMMETRY**

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ABSTRACT. We prove the existence of infinitely many solutions to a class of non-symmetric Dirichlet problems with exponential nonlinearities. Here the domain  $\Omega \in \mathbb{R}^{2l}$  where  $2l$  is also the order of the equation. Considered are the problem with no symmetry requirements, the radial problem on an annulus, and the radial problem on a ball with a Hardy potential term of critical Hardy exponent. These generalize results obtained by Sugimura [31].

**1. Introduction**

This paper is concerned with the multiplicity of solutions to three polyharmonic Dirichlet problems. Common to all three is that the order the equations also equals the dimension of the domain. In the notation below,  $d = 2l$  where  $d$  is the dimension, and  $l$  is the power of the Laplacian. Related to this fact, they all concern odd nonlinearities of exponential growth. They also include perturbations which are not odd. First we seek weak solutions to

$$(P) \quad \begin{cases} (-\Delta)^l u = g(x, u) + \varphi(x) & \text{in } \Omega, \\ \left( \frac{\partial}{\partial \nu} \right)^j u \Big|_{\partial \Omega} = 0 & \text{for } j = 0, \dots, l-1, \end{cases}$$

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2010 *Mathematics Subject Classification.* 35J35.

*Key words and phrases.* Polyharmonic Dirichlet problems; perturbation from symmetry; critical point theory; variational methods.

where the domain  $\Omega \Subset \mathbb{R}^{2l}$  has a smooth boundary. Here  $\varphi \in L^2(\Omega)$ , and  $g(x, u)$  is an odd-in- $u$ , exponential nonlinearity satisfying conditions (g1)–(g5) given below. A typical example to keep in mind would be  $g(x, u) = ue^{|u|^\alpha}$ , with  $0 < \alpha < 1$ . Next in the case that  $\Omega = A_{R_0}^R := \{x \in \mathbb{R}^{2l} : R_0 < |x| < R\}$  is an annulus, with  $R_0 > 0$  and  $R < +\infty$ , we seek weak, radial solutions to

$$(R) \quad \begin{cases} (-\Delta)^l u = 2ue^{u^2} + \varphi(x, u) & \text{in } \Omega, \\ \left(\frac{\partial}{\partial \nu}\right)^j u \Big|_{\partial\Omega} = 0 & \text{for } j = 0, \dots, l-1, \end{cases}$$

where  $\varphi(x, u) = \varphi(|x|, u)$  is not odd in  $u$ . Lastly we seek weak, radial solutions to

$$(H) \quad \begin{cases} (-\Delta)^l u + \bar{b}|x|^{-2l}u = g(x, u) + \varphi(x) & \text{in } B_R(0), \\ \left(\frac{\partial}{\partial \nu}\right)^j u \Big|_{\partial B_R(0)} = 0, & \text{for } j = 0, \dots, l-1, \end{cases}$$

where  $B_R(0) \subset \mathbb{R}^{2l}$  denotes the ball of radius  $R$  centered at the origin, and  $\bar{b} > 0$  is a constant. Here  $\varphi(x) = \varphi(|x|)$ ,  $g(x, u) = g(|x|, u)$ , and  $g$  satisfies conditions (g1)–(g5) given below. We note these are not the most general possible forms such equations can take for the methods to be applicable. We have chosen them for simplicity and because they are instructive.

Many papers have been written on the existence and multiplicity of solutions for second order, nonlinear, elliptic problems, primarily by means of variational methods. The archetype has been the Dirichlet problem

$$\begin{cases} -\Delta u = u|u|^{p-2} + \varphi(x) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases}$$

In its simplest form, the open domain  $\Omega \subset \mathbb{R}^d$  is bounded, with a smooth boundary, and the perturbation  $\varphi(x) \in L^2(\Omega)$ . The exponent of the nonlinearity is such that if  $d \geq 3$ , then  $2 < p < 2d/(d-2)$ , while if  $d = 2$ , then  $2 < p < \infty$ . These restrictions on  $p$  come when one makes use of the Sobolev embeddings  $W_0^{1,2}(\Omega) \hookrightarrow L^p(\Omega)$  and their compactness. If  $\varphi \equiv 0$ , the above equation possesses a  $\mathbb{Z}_2$ -symmetry with respect to the group of reflections in Sobolev space. In this case the Symmetric Mountain Pass Theorem of Ambrosetti and Rabinowitz guarantees the existence of an unbounded sequence of critical values of the functional associated with the variational formulation of the problem. Such methods can be applied to a great variety of nonlinear problems invariant under compact groups of symmetries (see [3], [24], [30] and references therein). However, this brings up the question of what exactly happens to this multitude of critical values when the symmetry of the problem is broken by some non-equivariant perturbation. There are no satisfactory general answers to this question yet. Methods to deal with this problem in certain cases appeared first in the papers [4], [5] and [29]. The general variational principle employed in these works was

later formulated by Rabinowitz in [23] (see also [24]). Roughly speaking, the idea is to estimate the spacing between consecutive symmetric mountain pass levels of the unperturbed functional, and then compare this spacing to the effect of the perturbation. Whenever the perturbation is not sufficient to eliminate this spacing, then the variational principle formulated by Rabinowitz guarantees the existence of a critical value to the perturbed functional. The first methods for estimating the spacing (or more practically the growth rate) of the symmetric mountain pass levels were based on the Weyl asymptotics for the Dirichlet eigenvalues of the Laplacian. A more refined approach, that could be more tailored for a specific problem, came in the papers of Bahri–Lions [6], and Tanaka [32]. Based on Morse theory, these works make use of an estimate for the number of non-positive eigenvalues of Schrödinger operators further described below. However, even for linear perturbations of the functional (like the one coming from  $\varphi$ ), the value of  $p$  needs to be further restricted to  $2 < p < (2d - 2)/(d - 2)$ . To improve the range of  $p$  with these methods one must weaken the perturbation. It is still a central open question of exactly how necessary this trade-off is.

Many sorts of perturbations other than a non-homogenous term  $\varphi$  are of interest. A natural one is to consider the problem of an unperturbed equation, itself formally invariant, but with a non-homogenous boundary condition  $u|_{\partial\Omega} = u_0 \neq 0$ , which destroys the evenness of the problem. This time however, the perturbation is of much higher order, directly entering into the nonlinearity. To deal with such complications Bolle [9] developed a new approach to perturbation theory of minimax levels. Similar in spirit to the earlier approach, but considerably more streamlined, the new approach considers the perturbed functional  $I$  as the endpoint of a continuous path of functionals  $I_\theta$ ,  $\theta \in [0, 1]$ , which starts at the unperturbed functional, denoted  $I_0$ . Bolle’s general theorem explains quantitatively how far apart two consecutive mountain pass levels of the unperturbed functional need to be for a critical level to persist to  $\theta = 1$ . Roughly speaking, it is not the size of the perturbation at general points that determines this, but the size of  $\frac{\partial}{\partial\theta} I_\theta(u)$  at the critical points  $u$  of  $I_\theta$ . This can certainly be helpful because these  $u$  satisfy the corresponding Euler–Lagrange equation. Furthermore, it becomes clearer how the size of the perturbation, as a functional in  $u$ , enters into the problem. This makes it easier to consider perturbations other than simple non-homogenous terms like  $\varphi$ . This approach is further developed and applied in a number of problems in [10].

In the two dimensional case the proper Sobolev embedding is into an Orlicz space given by an exponential  $N$ -function, see [2]. The maximal growth rate of the nonlinearity for which a variational treatment of the problem is possible is like  $e^{Ku^2}$ . This is related to the optimality of the Moser–Trudinger inequality

[22], [34], [19]. A typical problem now is

$$\begin{cases} -\Delta u = g(x, u) + \varphi(x) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

where  $g(x, -t) = -g(x, t)$  is of exponential growth. To guarantee the convergence of general Palais–Smale sequences in  $H_0^1(\Omega)$ ,  $g$  is taken subcritical, which in this case means that  $g(x, t)$  is of order of growth strictly below any positive power of  $e^{t^2}$ . That is

$$\lim_{t \rightarrow \infty} \frac{|g(x, t)|}{e^{\beta t^2}} = 0 \quad \text{for all } \beta > 0.$$

In [31], Sugimura proved that the perturbed symmetric problem above has an infinite number of solutions if the nonlinear term  $g(x, t)$  has growth like  $e^{|t|^\alpha}$ , where  $0 < \alpha < 1/2$ . One of the key points in this paper comes when applying the Morse index approach of [6] and [32]. At that stage one typically applies estimates for the number of non-positive eigenvalues (a.k.a. “bound states”) of Schrödinger operators. Previous results for the problem involving the power nonlinearity  $u|u|^{p-2}$  had made use of a famous estimate from mathematical physics known as the CLR inequality, discovered independently by Rozenblum, Lieb, and Cwikel [26], [27], [18], [13]. To get his result Sugimura proved a 2-dimensional version of this estimate. Later, using the foundation laid by Sugimura, Tarsi [33] streamlined these results by using Bolle’s approach.

One advantage of Rozenblum’s proof of the CLR estimate is that it automatically applies to higher order Schrödinger operators (which is the original form in which Rozenblum stated his result), where the Laplacian is replaced by the poly-Laplacian. Using this fact, it was proved in [16] that the problem

$$\begin{cases} (-\Delta)^l u = u|u|^{p-2} + \varphi(x) & \text{in } \Omega, \\ \left( \frac{\partial}{\partial \nu} \right)^j u \Big|_{\partial\Omega} = \phi_j & \text{for } j = 0, \dots, l-1, \end{cases}$$

where  $\Omega \Subset \mathbb{R}^d$ ,  $d > 2l$ , has infinitely many solutions for  $p$  suitably restricted.

This paper is concerned with appropriate eigenvalue estimates, and their application to problems (P), (R), and (H), which are polyharmonic, in the critical Sobolev dimension, and with exponential nonlinearities. As it is common, stronger results are proved in radially symmetric settings. For simplicity in problems (P) and (H) we consider perturbations that, in the equation, do not depend on  $u$ . However, more general perturbations can be considered as in [33].

In problems (P) and (H), we make the following assumptions on the symmetric term  $g$ :

(g1)  $g \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ .

(g2) Given any constant  $\sigma > 0$ , there exists a constant  $A_\sigma > 0$  such that

$$|g(x, t)| \leq A_\sigma e^{\sigma t^2} \quad \text{for all } (x, t) \in \overline{\Omega} \times \mathbb{R}.$$

(g3) There are constants  $\mu > 0$  and  $r_0 \geq 0$  such that

$$0 < G(x, t) \ln G(x, t) \leq \mu t g(x, t) \quad \text{for } x \in \bar{\Omega}, |t| \geq r_0.$$

(g4)  $g(x, -t) = -g(x, t)$  for  $(x, t) \in \bar{\Omega} \times \mathbb{R}$ .

(g5) There exist  $0 < \alpha_1 \leq \alpha_2 < 1$ , and  $A_1, A_2, B_1$  such that

$$A_1 e^{|t|^{\alpha_1}} - B_1 \leq G(x, t) \leq A_2 e^{|t|^{\alpha_2}} \quad \text{for } (x, t) \in \bar{\Omega} \times \mathbb{R}.$$

Our first result is

**THEOREM 1.1.** *Suppose that  $g$  satisfies conditions (g1)–(g5). Then problem (P) has an unbounded sequence of solutions if  $2/\alpha_2 - 2 > 1/\alpha_1$ .*

As a prototypical example we may take  $g(x, u) := ue^{|u|^\alpha}$ . Then the above theorem asserts that for  $0 < \alpha < 1/2$  problem (P) has an unbounded sequence of solutions. If we impose radial assumptions on the problem we can improve the result. In problem (R), let

$$\Phi(x, u) := \int_0^u \varphi(x, t) dt.$$

**THEOREM 1.2.** *Suppose that there exists  $\beta \in (0, 1)$  such that  $|\Phi(x, t)| + |\varphi(x, t)t| \leq C(t^2 e^{t^2})^\beta$  for sufficiently large values of  $|t|$ . If  $2l > 1/(1 - \beta)$  then problem (R) has an unbounded sequence of radial solutions.*

**THEOREM 1.3.** *Suppose that  $\Omega = B_R(0)$ , the open ball centered at the origin with a finite radius  $R$ . Suppose that, in addition to conditions (g1)–(g5), we have  $g(x, u) = g(|x|, u)$  and  $\varphi(x) = \varphi(|x|)$ . Then if  $2/\alpha_2 - 1 > 1/\alpha_1$ , problem (H) has an unbounded sequence of radial solutions.*

These theorems are proven using upper estimates on the number of non-positive eigenvalues (a.k.a. bound states) of polyharmonic Schrödinger operators. When no radial symmetry is imposed, the corresponding spectral inequality takes the following form:

**PROPOSITION 1.4.** *Let  $\mathcal{B}(t) := (|t| + 1) \ln(|t| + 1) - |t|$  be an  $N$ -function, and  $L_{\mathcal{B}}(\Omega)$  be the corresponding Orlicz space. Let  $V \in L_{\mathcal{B}}(\Omega)$ , and on  $L^2(\Omega)$  consider the unbounded linear operator  $(-\Delta)^l - V(x)$ . Denote by  $N_-((-\Delta)^l - V(x))$  the number of its non-positive eigenvalues. Then there exists a constant  $C = C(l, \Omega)$  such that*

$$N_-((-\Delta)^l - V(x)) \leq C \|V\|_{L_{\mathcal{B}}(\Omega)}.$$

Simplified for our purposes, this result is due to Solomyak [28]. We will not use Proposition 1.4 but rather a corollary of it. As is well known in the theory of Orlicz spaces

$$\|V\|_{L_{\mathcal{B}}(\Omega)} \leq \int_{\Omega} \mathcal{B}(V(x)) dx + 1.$$

See (9.12) in [15].

COROLLARY 1.5.

$$N_-((-\Delta)^l - V(x)) \leq C \int_{\Omega} \mathcal{B}(V(x)) dx + C.$$

This generalizes the eigenvalue estimate originally obtained by Sugimura [31]. It is unknown to the author if the finer estimate in Proposition 1.4 can yield stronger results in this case. To answer such a question a much finer analysis of the critical sequence given by Tanaka's Theorem below (or a possible alternative of it) seems necessary. In the radial setting on an annulus the main eigenvalue estimate we prove is the following

PROPOSITION 1.6. *Let  $\Omega$  be an annulus of outer radius  $R < +\infty$  and inner radius  $R_0 > 0$ . Let  $V(x) = V(|x|)$ . On the space  $L_r^2(\Omega)$ , of radially symmetric, square integrable functions, consider the unbounded linear operator  $(-\Delta)^l - V(x)$ . Denote by  $N_-((-\Delta)^l - V(x))$  the number of its non-positive eigenvalues. There exists a constant  $C = C(l, R_0, R)$  such that*

$$[N_-((-\Delta)^l - V(x))]^{2l} \leq C \int_{\Omega} V^+(x) \left[1 + \log \left(\frac{R}{|x|}\right)\right]^{2i} dx$$

where  $i = 1/2$  when  $l = 1$ , and  $i = 1$  when  $l > 2$ .

## 2. Preliminaries: general outline of the perturbation method

**The symmetric mountain pass sequence and the variational principle of Bolle.** Here we recall the perturbation method of Bolle. See [9] and [10]. Let  $E$  be a Hilbert space, and let  $e_k$  be a basis for  $E$ . Decompose  $E$  as

$$(2.1) \quad E = \overline{\bigcup_{k=0}^{\infty} E_k}, \quad \text{where } E_k = E_{k-1} \oplus \mathbb{R}e_k$$

with  $E_0 = \{0\}$ . For a given increasing sequence of real numbers  $R_k > 0$  set

$$\Gamma_k := \{g \in C(E_k, E) : g \text{ is odd and } g(u) = u \text{ if } u \in E_k, \|u\| \geq R_k\}.$$

The precise values of  $R_k$  that we will use will be defined later. For an even functional  $I_0(u)$  on  $E$  set

$$c_k := \inf_{g \in \Gamma_k} \sup_{u \in g(E_k)} I_0(u).$$

Under typical assumptions on  $I_0(u)$ , the Symmetric Mountain Pass Theorem of Ambrosetti and Rabinowitz shows that, for appropriate values of  $R_k$  the minimax values  $c_k$  are an unbounded sequence of critical values of  $I_0(u)$ .

Consider two continuous functions  $f_1, f_2: [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$  which are Lipschitz continuous with respect to the second variable, and  $f_1 \leq f_2$ . Define the associated flows

$$\begin{cases} \psi_i(0, s) = s, \\ \frac{\partial}{\partial \theta} \psi_i(\theta, s) = f_i(\theta, \psi_i(\theta, s)), \end{cases}$$

$\psi_1$  and  $\psi_2$  are continuous in both variables, and non-decreasing in  $s$ . Moreover, by the comparison theorem for ODEs, since  $f_1 \leq f_2$ , we have  $\psi_1 \leq \psi_2$ . Also, denote

$$\overline{f_i}(s) := \sup_{\theta \in [0, 1]} |f_i(\theta, s)|.$$

We will apply the main theorem of [10]:

**THEOREM 2.1** (Bolle, Ghoussoub, Tehrani [10]). *Let  $E$  be a Hilbert space and  $I: [0, 1] \times E \rightarrow \mathbb{R}$  be a  $C^1$  functional satisfying the following conditions:*

- (H1)  *$I$  satisfies the following analogue of the Palais–Smale condition: For a sequence  $\{(\theta_n, u_n)\}_{n \in \mathbb{N}}$  in  $[0, 1] \times E$  such that  $\|I'_{\theta_n}(u_n)\|_{E^*} \rightarrow 0$  and  $|I_{\theta_n}(u_n)| \leq C$  there is a subsequence of it strongly converging in  $[0, 1] \times E$ .*
- (H2) *For all  $b > 0$  there exists a positive constant  $C(b)$  such that*

$$|I_\theta(u)| \leq b \quad \Rightarrow \quad \left| \frac{\partial}{\partial \theta} I_\theta(u) \right| \leq C(b)(\|I'_\theta(u)\| + 1)(\|u\| + 1).$$

- (H3) *There exist two continuous functions  $f_1, f_2: [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ , with  $f_1 \leq f_2$ , that are Lipschitz continuous relative to the second variable, and such that, for all critical points  $u$  of  $I_\theta$*

$$f_1(\theta, I_\theta(u)) \leq \frac{\partial}{\partial \theta} I_\theta(u) \leq f_2(\theta, I_\theta(u)).$$

- (H4)  *$I_0$  is even and for any finite dimensional subspace  $W$  of  $E$  we have*

$$\sup_{\theta \in [0, 1]} I_\theta(y) \rightarrow -\infty \quad \text{as } \|y\| \rightarrow \infty \text{ for } y \in W.$$

*Then there is  $K > 0$  such that for every  $n$  only one of the two possibilities below holds:*

- (a) *either  $I_1$  has a critical level  $\bar{c}_n$  with  $\psi_2(1, c_n) < \psi_1(1, c_{n+1}) \leq \bar{c}_n$ , or*
- (b)  *$c_{n+1} - c_n \leq K(\overline{f_1}(c_{n+1}) + \overline{f_2}(c_n) + 1)$ .*

The values  $R_k$  that we will use are defined as follows: by hypothesis (H4) in the theorem above we can find  $R_k > 0$  such that  $\sup_{\theta \in [0, 1]} I_\theta(u) < 0$  for all  $u \in E_k$  with  $\|u\| \geq R_k$ .

The theorem above shows that, for each  $n$ , two outcomes are possible. If the second outcome holds for all sufficiently large  $n$  then one can derive an upper bound on the sequence  $c_n$ . In applications, one shows that the first possibility holds for infinitely many  $n$  by obtaining a lower bound for  $c_n$  that contradicts

this alleged upper bound. Because  $c_n \rightarrow +\infty$  and because in typical applications  $\psi_2(1, s) \rightarrow +\infty$  as  $s \rightarrow +\infty$  we get that  $\bar{c}_n \rightarrow +\infty$ . We also note that the above theorem was originally proven for  $I$  in  $C^2(E)$ , by using the gradient flow of  $I$ . However, as typical, for the proof it suffices to use only a pseudo-gradient flow in the sense of Palais. This requires that  $I$  is only in  $C^1(E)$ . See [12].

### 3. Problem (P), the case of an unrestricted domain

**3.1. The variational setup.** Let  $\Omega \Subset \mathbb{R}^d$  be a smooth domain, and where  $d = 2l$ . In the Hilbert space  $L^2(\Omega)$  we consider the dense subspace

$H_0^l(\Omega) :=$  the completion of  $C_0^\infty(\Omega)$  in the following norm

$$(3.1) \quad \|u\|_{H_0^l(\Omega)} = \begin{cases} \|\Delta^k u\|_{L^2(\Omega)} & \text{if } l=2k, \\ \|\nabla(\Delta^k u)\|_{L^2(\Omega)} & \text{if } l = 2k + 1. \end{cases}$$

Typically on the space  $H_0^l(\Omega)$  we use the norm  $\|D^l u\|_{L^2(\Omega)}$ , after taking into account Poincaré's inequality for the lower order terms. However, this norm is equivalent to  $\|u\|_{H_0^l(\Omega)}$  on  $C_0^\infty(\Omega)$  by integration by parts. When convenient, we shall denote the norm  $\|u\|_{H_0^l(\Omega)}$  by  $\|u\|$ . As shorthand, on  $H_0^l(\Omega)$ , define the  $l$ th power of the gradient as

$$(3.2) \quad \nabla^l u = \begin{cases} \Delta^k u & \text{if } l = 2k, \\ \nabla(\Delta^k u) & \text{if } l = 2k + 1. \end{cases}$$

The variational setup for our problem is as follows. On the space  $H_0^l(\Omega)$  we consider the functional

$$(3.3) \quad I_1(u) := \frac{1}{2} \int_{\Omega} |\nabla^l u|^2 dx - \int_{\Omega} G(x, u) dx - \int_{\Omega} \varphi u dx.$$

This is the functional whose critical points correspond to generalized solutions of the boundary value problem

$$(P) \quad \begin{cases} (-\Delta)^l u = g(x, u) + \varphi(x) & \text{for } x \in \Omega, \\ \left(\frac{\partial}{\partial \nu}\right)^j u \Big|_{\partial\Omega} = 0 & \text{for } j = 0, \dots, l-1, \end{cases}$$

where  $G(x, u) = \int_0^u g(x, s) ds$ , in the space  $H_0^l(\Omega)$ . See [14].

For  $I_1(u)$  to be well defined on all of  $H_0^l(\Omega)$ , and for such a variational treatment to be viable, we must restrict the growth rate with respect to  $u$  of the nonlinearity  $G$ . The maximal growth for which a variational treatment



is allowed comes from the optimality of Adam's generalization of the Moser–Trudinger inequality. Namely, on the space  $W_0^{l,d/l}(\Omega)$ ,  $1 \leq l < d$ , Adams showed

$$(3.4) \quad \sup_{\substack{u \in W_0^{l,d/l}(\Omega) \\ \|\nabla^l u\|_{L^{d/l}} \leq 1}} \int_{\Omega} e^{\beta|u|^{d/(d-l)}} dx \begin{cases} \leq C|\Omega| & \text{if } \beta \leq \beta(d,l), \\ = +\infty & \text{if } \beta > \beta(d,l), \end{cases}$$

where  $\beta(d,l)$  is given explicitly. See [1]. In our case,  $d = 2l$  so the exponent  $d/(d-l)$  equals 2. In this case,  $\beta(2l,l) = l!(4\pi)^l$ . For a variational treatment to be possible in  $H_0^l(\Omega)$ ,  $G(x,u)$  cannot grow faster than  $e^{Ku^2}$  for all  $K$ .

Conditions (g1)–(g5) imply that  $I_1(u)$  is a  $C^1$  functional on  $H_0^l(\Omega)$ . For the corresponding path of functionals we simply consider

$$(3.5) \quad I_{\theta}(u) := \frac{1}{2} \int_{\Omega} |\nabla^l u|^2 dx - \int_{\Omega} G(x,u) dx - \theta \int_{\Omega} \varphi u dx$$

where  $\theta \in [0, 1]$ , and for which  $I_0$  is even.

**3.2. Bolle's requirements.** (H1)  $I$  satisfies the following analogue of the Palais–Smale condition: For a sequence  $\{(\theta_n, u_n)\}_{n \in \mathbb{N}}$  in  $[0, 1] \times E$  such that  $\|I'_{\theta_n}(u_n)\|_{E^*} \rightarrow 0$  and  $|I_{\theta_n}(u_n)| \leq C$  there is a subsequence converging strongly in  $[0, 1] \times E$ .

PROOF. Let  $\{(\theta_n, u_n)\}_{n \in \mathbb{N}}$  be such a sequence. Then, after taking a subsequence, we can find constants  $C_0$  and  $\theta_0$  such that

$$(3.6) \quad I_{\theta_n}(u_n) = \frac{1}{2} \int_{\Omega} |\nabla^l u_n|^2 dx - \int_{\Omega} G(x, u_n) dx - \theta_n \int_{\Omega} \varphi u_n dx \rightarrow C_0$$

where  $\theta_n \rightarrow \theta_0$ , and

$$(3.7) \quad \left| \int_{\Omega} \{\nabla^l u_n \cdot \nabla^l v - g(x, u_n)v - \theta_n \varphi(x)v\} dx \right| \leq \varepsilon_n \|v\|_{H_0^l(\Omega)}$$

for all  $v \in H_0^l(\Omega)$ , where  $\varepsilon_n \rightarrow 0$  as  $n \rightarrow \infty$ . Choosing  $v = u_n$  in (3.7) we get, for  $\bar{\mu} > 2$ ,

$$(3.7') \quad \frac{\bar{\mu}}{2} \int_{\Omega} |\nabla^l u_n|^2 dx - \bar{\mu} \int_{\Omega} G(x, u_n) dx - \bar{\mu} \theta_n \int_{\Omega} \varphi(x) u_n dx \leq C_1$$

and

$$(3.8) \quad -\|u_n\|_{H_0^l(\Omega)}^2 + \int_{\Omega} g(x, u_n) u_n dx + \theta_n \int_{\Omega} \varphi(x) u_n dx \leq \varepsilon_n \|u_n\|_{H_0^l(\Omega)}.$$

Adding (3.7') and (3.8) gives

$$(3.9) \quad \left(\frac{\bar{\mu}}{2} - 1\right) \|u_n\|^2 + \int_{\Omega} (g(x, u_n) u_n - \bar{\mu} G(x, u_n)) dx \\ + (1 - \bar{\mu}) \theta_n \int_{\Omega} \varphi(x) u_n dx \leq C' + \varepsilon_n \|u_n\|.$$

By assumption (g3)

$$\int_{\Omega} (g(x, u_n)u_n - \bar{\mu}G(x, u_n)) dx \geq -C_2.$$

Hence (3.9) gives

$$\left(\frac{\bar{\mu}}{2} - 1\right) \|u_n\|^2 \leq C_3 + \varepsilon_n \|u_n\| + C_4 \int |\varphi(x)u_n| dx \leq C_3 + \varepsilon_n \|u_n\| + C_5 \|u_n\|_{L^2(\Omega)}.$$

By the generalized Poincaré inequality, if  $u \in H_0^1(\Omega)$  then there exists a constant  $C_6 > 0$  such that

$$\|u\|_{L^2(\Omega)} \leq C_6 \|u\|_{H_0^1(\Omega)}.$$

So we get

$$\left(\frac{\bar{\mu}}{2} - 1\right) \|u_n\|^2 \leq C_3 + C_7 \|u_n\|.$$

Thus

$$(3.10) \quad \|u_n\| \leq K.$$

Having proven the boundedness of Palais–Smale sequences, we will show that they are pre-compact by proving that  $I'_\theta(u)(\cdot)$  has the form  $L(u)(\cdot) + \kappa(u)(\cdot)$  where  $L: H_0^1(\Omega) \rightarrow H^{-l}(\Omega)$  is an isomorphism, and  $\kappa: H_0^1(\Omega) \rightarrow H^{-l}(\Omega)$  is compact. Although this is not entirely necessary, and a shorter proof which does not rely explicitly on this fact is possible. However, the fact that  $I'_\theta(u)$  has this form will be needed later to apply Tanaka’s Theorem, hence we give the proof now. First

$$(3.11) \quad I'_{\theta_0}(u)(\cdot) = \langle u, \cdot \rangle_{H_0^1(\Omega)} - \langle g(x, u), \cdot \rangle_{L^2(\Omega)} - \theta_0 \langle \varphi(x), \cdot \rangle_{L^2(\Omega)}.$$

Clearly  $L: H_0^1(\Omega) \rightarrow H^{-l}(\Omega): u \mapsto \langle u, \cdot \rangle_{H_0^1(\Omega)}$  is the Riesz map, hence a Hilbert space isomorphism. Clearly the map  $K_1: H_0^1(\Omega) \rightarrow H^{-l}(\Omega): u \mapsto \langle \varphi(x), \cdot \rangle_{L^2(\Omega)}$  is compact because it is a constant map.

To show that  $K_2: H_0^1(\Omega) \rightarrow H^{-l}(\Omega): u \mapsto \langle g(x, u), \cdot \rangle_{L^2(\Omega)}$  is compact it suffices to show that if  $\{u_n\} \subset H_0^1(\Omega)$  is bounded then, up to a subsequence,  $g(x, u_n)$  converges in  $L^2(\Omega)$ . Without loss of generality we may assume, after taking a subsequence, that

- $\|u_n\| \leq K$ ,
- $u_n \rightharpoonup u$  weakly in  $H_0^1(\Omega)$ ,
- $u_n \rightarrow u$  strongly in  $L^p(\Omega)$ ,  $p \geq 1$ ,
- $u_n(x) \rightarrow u(x)$  almost everywhere in  $\Omega$ .

Now since  $g$  has subcritical growth in  $u$  by (g2), we can find  $C_K > 0$  such that

$$(3.12) \quad |g(x, t)| \leq C_K \exp\left(\frac{\beta(2l, l)}{2K^2} t^2\right)$$

where  $\beta(2l, l) = l!(4\pi)^l$  is the optimal constant in Adam's inequality. We apply Adam's inequality:

$$\begin{aligned} \|g(x, u_n)\|_{L^2}^2 &\leq C_K^2 \int_{\Omega} \exp\left(\frac{\beta(2l, l)}{K^2} |u_n|^2\right) dx \\ &\leq C_K^2 \int_{\Omega} \exp\left(\frac{\beta(2l, l)}{\|u_n\|^2} |u_n|^2\right) dx \leq C'_K. \end{aligned}$$

Similarly we have

$$\int_{\Omega} |g(x, u_n)|^2 |u_n| dx \leq C''_K.$$

To obtain the required result we use the following lemma:

LEMMA 3.1. *Let  $\{u_n\}$  be a convergent sequence of functions in  $L^2(\Omega)$ , with  $u_n(x) \rightarrow u(x)$  almost everywhere. Assume that  $g(x, u_n)$  and  $g(x, u)$  are also in  $L^2(\Omega)$  with  $g(x, t)$  continuous in  $t$ , uniformly in  $x$ . If*

$$\int_{\Omega} |g(x, u_n)|^2 |u_n| dx \leq C_8$$

then  $g(x, u_n)$  converges in  $L^2(\Omega)$  to  $g(x, u)$ .

PROOF. Note that since  $g(x, t)$  is continuous in  $t$  and  $u_n(x) \rightarrow u(x)$  almost everywhere, then  $g(x, u_n(x)) \rightarrow g(x, u(x))$  almost everywhere. We have that

$$\begin{aligned} |g(x, u_n(x)) - g(x, u(x))|^2 &\leq [ |g(x, u_n(x))| + |g(x, u(x))| ]^2 \\ &\leq 2|g(x, u_n(x))|^2 + 2|g(x, u(x))|^2. \end{aligned}$$

If we assume  $\|g(x, u_n)\|_{L^2(\Omega)} \rightarrow \|g(x, u)\|_{L^2(\Omega)}$  then

$$\int_{\Omega} (2|g(x, u_n(x))|^2 + 2|g(x, u(x))|^2) dx \rightarrow 4 \int_{\Omega} |g(x, u(x))|^2 dx.$$

Also  $|g(x, u_n(x)) - g(x, u(x))| \rightarrow 0$  almost everywhere. So we can apply the generalized Lebesgue Dominated Convergence Theorem and get that

$$\int |g(x, u_n(x)) - g(x, u(x))|^2 dx \rightarrow 0,$$

which is the required result. So it suffices to prove that

$$\int |g(x, u_n)|^2 \rightarrow \int |g(x, u)|^2 dx.$$

Let  $f(x, t) := g(x, t)^2$ . Since  $f(x, u(x)) \in L^1(\Omega)$ , it follows that for a given  $\varepsilon > 0$  there is  $\delta > 0$  such that

$$(3.13) \quad \int_A f(x, u(x)) dx \leq \varepsilon \quad \text{if } |A| \leq \delta$$

for all measurable subsets  $A \subseteq \Omega$ . Next using the fact that  $u \in L^1(\Omega)$  we find  $M_1 > 0$  such that

$$(3.14) \quad |\{x \in \Omega : |u(x)| \geq M_1\}| \leq \delta.$$

Let  $M := \max\{M_1, C_8/\varepsilon\}$ . We write

$$(3.15) \quad \left| \int f(x, u_n(x)) dx - \int f(x, u(x)) dx \right| \leq I_1 + I_2 + I_3$$

and estimate each integral separately:

$$I_1 \equiv \int_{|u_n(x)| \geq M} f(x, u_n(x)) dx = \int_{|u_n(x)| \geq M} \frac{|g(x, u_n(x))|^2}{|u_n(x)|} |u_n(x)| dx \leq \frac{C_8}{M} \leq \varepsilon.$$

By the choices we have made above (3.13) and (3.14) imply that

$$I_3 \equiv \int_{|u(x)| \geq M} f(x, u(x)) dx \leq \varepsilon.$$

Next we claim that

$$I_2 \equiv \left| \int_{|u_n(x)| < M} f(x, u_n(x)) dx - \int_{|u(x)| < M} f(x, u(x)) dx \right| \rightarrow 0$$

as  $n \rightarrow \infty$ . Indeed,  $h_n(x) := f(x, u_n(x))\chi_{|u_n| < M} - f(x, u(x))\chi_{|u| < M}$  tends to 0 almost everywhere in  $\Omega$ . Moreover,  $|h_n(x)| \leq |f(x, u(x))|$  if  $|u_n(x)| \geq M$  and  $|h_n(x)| \leq C + f(x, u(x))$  if  $|u_n(x)| < M$ . So  $I_2 \rightarrow 0$  as  $n \rightarrow \infty$  by the Lebesgue Dominated Convergence Theorem.  $\square$

Thus  $I'$  has the stated form and (H1) is satisfied.

(H2) Here

$$\frac{\partial}{\partial \theta} I_\theta(u) = - \int_{\Omega} \varphi(x) u(x) dx$$

is bounded in absolute value by  $\|\varphi\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)}$ . By the generalized Poincaré inequality this is bounded by  $C_\varphi \|u\|$ .

(H3) Determining  $f_1, f_2$ .

LEMMA 3.2. *There exists a constant  $C_9 > 0$  such that if  $u \in H_0^1(\Omega)$  is a critical point of  $I_\theta$ , then*

$$(3.16) \quad \left| \frac{\partial}{\partial \theta} I_\theta(u) \right| \leq C_9 [\ln(|I_\theta(u)| + 1)]^{1/\alpha_1} + C_9.$$

PROOF. From (H2) above we have that

$$\left| \frac{\partial}{\partial \theta} I_\theta(u) \right| \leq C_{10} \|u\|_{L^2(\Omega)}.$$

So it suffices to estimate  $\|u\|_{L^2(\Omega)}$ . Assume  $I'_\theta(u) = 0$ . Then

$$\begin{aligned}
 (3.17) \quad I_\theta(u) &= I_\theta(u) - \frac{1}{2} \langle I'_\theta(u), u \rangle \\
 &= \frac{1}{2} \int_\Omega |\nabla^l u|^2 dx - \int_\Omega G(x, u) dx - \theta \int_\Omega \varphi u dx \\
 &\quad - \frac{1}{2} \int_\Omega |\nabla^l u|^2 dx + \int_\Omega \frac{1}{2} g(x, u) u dx + \frac{\theta}{2} \int_\Omega \varphi u dx \\
 &= \int_\Omega \left( \frac{1}{2} g(x, u) u - G(x, u) \right) dx - \frac{\theta}{2} \int_\Omega \varphi u dx.
 \end{aligned}$$

We now apply condition (g3). Take  $\varepsilon > 0$  such that  $\varepsilon < 1/(2\mu)$ . When  $|u(x)| > r_0$  we bound

$$\begin{aligned}
 \frac{1}{2} g(x, u) u - G(x, u) &\geq \frac{1}{2\mu} G(x, u) \ln[G(x, u)] - G(x, u) \\
 &\geq \left( \frac{1}{2\mu} - \varepsilon \right) G(x, u) \ln[G(x, u)] - C_{11}.
 \end{aligned}$$

When  $|u(x)| \leq r_0$  the expression  $g(x, u)u/2 - G(x, u)$  is bounded by a constant since  $g$  and  $G$  are continuous. Since  $\Omega$  is of finite measure we get from the above and equation (3.17)

$$I_\theta(u) \geq \left( \frac{1}{2\mu} - \varepsilon \right) \int_{|u(x)| \geq r_0} G(x, u) \ln[G(x, u)] dx - C_{12} \|u\|_{L^2(\Omega)} - C_{13}.$$

Now applying the growth condition (g5) and again the fact that  $\Omega$  is of finite measure

$$(3.18) \quad I_\theta(u) \geq C_{14} \int_\Omega |u|^{\alpha_1} e^{|u|^{\alpha_1}} dx - C_{12} \|u\|_{L^2(\Omega)} - C_{13}.$$

Now observe that for  $\alpha, \beta > 0$  there exists a constant  $t_0 = t_0(\alpha, \beta)$  such that the function  $t^\beta e^{t^\alpha}$  is convex for  $t \geq t_0$ . We take  $\alpha = \beta = \alpha_1/2$ , and apply Jensen's inequality

$$\begin{aligned}
 (3.19) \quad C_{15} \int_{|u| \geq t_0^{1/2}} |u|^{\alpha_1} e^{|u|^{\alpha_1}} dx \\
 \geq \left\{ \frac{1}{|\Omega|} \int_{|u| \geq t_0^{1/2}} |u|^2 dx \right\}^{\alpha_1/2} \exp \left[ \left\{ \frac{1}{|\Omega|} \int_{|u| \geq t_0^{1/2}} |u|^2 dx \right\}^{\alpha_1/2} \right].
 \end{aligned}$$

Also note that

$$(3.20) \quad \|u\|_{L^2(\Omega)}^{\alpha_1} \leq C_{16} + \left\{ \int_{|u| \geq t_0^{1/2}} |u|^2 dx \right\}^{\alpha_1/2}.$$

Hence we proceed as follows:

$$\begin{aligned}
& \|u\|_{L^2(\Omega)}^{\alpha_1} \exp \left[ \left( \frac{1}{|\Omega|} \right)^{\alpha_1/2} \|u\|_{L^2(\Omega)}^{\alpha_1} \right] \\
& \leq \left\{ C_{16} + \left( \int_{|u| \geq t_0^{1/2}} |u|^2 dx \right)^{\alpha_1/2} \right\} \\
& \quad \cdot \exp \left[ \frac{C_{16}}{|\Omega|^{\alpha_1/2}} + \left\{ \frac{1}{|\Omega|} \int_{|u| \geq t_0^{1/2}} |u|^2 dx \right\}^{\alpha_1/2} \right] \\
& \leq C_{17} + C_{18} \left\{ \int_{|u| \geq t_0^{1/2}} |u|^2 dx \right\}^{\alpha_1/2} \exp \left[ \left\{ \frac{1}{|\Omega|} \int_{|u| \geq t_0^{1/2}} |u|^2 dx \right\}^{\alpha_1/2} \right]
\end{aligned}$$

(here apply (3.19))

$$\begin{aligned}
& \leq C_{19} + C_{20} \int_{|u| \geq t_0^{1/2}} |u|^{\alpha_1} e^{|u|^{\alpha_1}} dx \\
(3.21) \quad & \leq C_{19} + C_{20} \int_{\Omega} |u|^{\alpha_1} e^{|u|^{\alpha_1}} dx.
\end{aligned}$$

Inequality (3.21) now implies

$$(3.22) \quad \|u\|_{L^2(\Omega)} \leq C_{21} \left\{ \ln \left( \int_{\Omega} |u|^{\alpha_1} e^{|u|^{\alpha_1}} dx + 1 \right) \right\}^{1/\alpha_1} + C_{22}.$$

So (3.22) and (3.18) give

$$\begin{aligned}
(3.23) \quad I_{\theta}(u) & \geq C_{23} \int_{\Omega} |u|^{\alpha_1} e^{|u|^{\alpha_1}} dx \\
& \quad - C'_{23} \left\{ \ln \left( \int_{\Omega} |u|^{\alpha_1} e^{|u|^{\alpha_1}} dx + 1 \right) \right\}^{1/\alpha_1} - C_{24} \\
& \geq C_{25} \int_{\Omega} |u|^{\alpha_1} e^{|u|^{\alpha_1}} dx - C_{26}.
\end{aligned}$$

So, by (3.21) and (3.23),

$$I_{\theta}(u) \geq C_{27} \|u\|_{L^2(\Omega)}^{\alpha_1} \exp \left[ \left( \frac{1}{|\Omega|} \right)^{\alpha_1/2} \|u\|_{L^2(\Omega)}^{\alpha_1} \right] - C_{28},$$

that is,

$$\|u\|_{L^2(\Omega)} \leq C_{29} [\ln(|I_{\theta}(u)| + 1)]^{1/\alpha_1} + C_{30},$$

which proves the lemma.  $\square$

Thus we take

$$(3.24) \quad f_i(\theta, t) = f_i(t) = (-1)^i C_9 \{ [\ln(|t| + 1)]^{1/\alpha_1} + 1 \}$$

(H4) This condition is easily satisfied by assumption (g5), which shows that  $G(x, u)$  is super-quadratic (uniformly in  $x$ ) and tends to  $+\infty$  as  $|u| \rightarrow \infty$ .

**3.3. The alleged upper bound.** As noted earlier, we will operate under the assumption that alternative (b) of Theorem 1.2 holds for sufficiently large  $n$ . That is, for  $n > n_0$ , it will be assumed that

$$(3.25) \quad c_{n+1} - c_n \leq K[(\ln(c_{n+1}))^{1/\alpha_1} + (\ln(c_n))^{1/\alpha_1} + 1].$$

We will show this implies that  $c_n \leq An[\ln(n)]^{1/\alpha_1}$  for sufficiently large  $n$ , and for some constant  $A$  to be chosen appropriately.

Let  $\gamma := 1/\alpha_1$  and let  $b_n := An[\ln(n)]^\gamma$ . First we can choose  $A > 0$  so large that  $c_{n_0} < b_{n_0}$  where  $n_0$  is large and fixed. In particular,  $n_0$  is taken to be at least large enough so that

$$K \frac{(\ln t)^{\gamma-1}}{t} < 1$$

for all  $t \geq b_{n_0}$ . For  $n > n_0$

$$\begin{aligned} b_{n+1} - b_n &= A(n+1)[\ln(n+1)]^\gamma - An[\ln(n)]^\gamma \\ &= A[(\ln(n+\theta))^\gamma + \gamma(\ln(n+\theta))^{\gamma-1}], \end{aligned}$$

for some  $\theta \in [0, 1]$ , by the Mean Value Theorem. Hence

$$(3.26) \quad b_{n+1} - b_n \geq \frac{A}{2} [\ln(n)]^\gamma.$$

Now, from the definition of  $b_n$ , we compute

$$(3.27) \quad K[(\ln(b_{n+1}))^\gamma + (\ln(b_n))^\gamma + 1] \leq C_\gamma K(\ln(n))^\gamma + C_\gamma K \ln(A)$$

for  $n > n_0$  sufficiently large.

So we take  $A \gg 2C_\gamma K$ . Then (3.26) and (3.27) combine to give

$$b_{n+1} - b_n > K[(\ln(b_{n+1}))^\gamma + (\ln(b_n))^\gamma + 1].$$

This is the reverse of the inequality satisfied by  $c_n$ . We already have  $b_{n_0} \geq c_{n_0}$ . Assume that  $b_i > c_i$  for  $i = n_0, \dots, n$ . We will show that  $b_{n+1} \geq c_{n+1}$ :

$$\begin{aligned} (3.28) \quad b_{n+1} - c_{n+1} &= b_{n+1} - b_n - (c_{n+1} - c_n) + (b_n - c_n) \\ &\geq b_{n+1} - b_n - (c_{n+1} - c_n) \\ &\geq K[(\ln(b_{n+1}))^\gamma + (\ln(b_n))^\gamma + 1] - K[(\ln(c_{n+1}))^\gamma + (\ln(c_n))^\gamma + 1] \\ &= K(\ln(b_{n+1}))^\gamma - K(\ln(c_{n+1}))^\gamma + [K(\ln(b_n))^\gamma - K(\ln(c_n))^\gamma] \\ &\geq K(\ln(b_{n+1}))^\gamma - K(\ln(c_{n+1}))^\gamma. \end{aligned}$$

Assume that  $b_{n+1} < c_{n+1}$ . Then

$$\begin{aligned} (3.29) \quad K(\ln(b_{n+1}))^\gamma - K(\ln(c_{n+1}))^\gamma &= -K\gamma \int_{b_{n+1}}^{c_{n+1}} \frac{(\ln(t))^{\gamma-1}}{t} dt \\ &> b_{n+1} - c_{n+1}, \end{aligned}$$

where we have used the fact that  $-K\gamma(\ln(t))^{\gamma-1}/t > -1$  for  $t > b_{n+1} \geq b_{n_0}$ , when  $n_0$  is taken to be sufficiently large. This contradicts (3.28), and so  $c_{n+1} \leq b_{n+1}$ . Thus, by induction,  $c_n \leq b_n$  for all  $n > n_0$ , i.e.

$$(3.30) \quad c_n \leq An[\ln(n)]^{1/\alpha_1} \quad \text{for } n > n_0,$$

when assuming alternative (b) of Theorem 2.1.  $\square$

#### 4. Tanaka's Theorem and its requirements

The reference is Theorem B in [32] (see also [6]). The idea is that associated to each minimax value of the symmetric functional there is a sequence of critical points which are at a lower energy level, but which have a large augmented Morse index. For the moment, we are concerned with  $I_0(u)$

$$I_0(u) = \frac{1}{2} \int_{\Omega} |\nabla^l u|^2 dx - \int_{\Omega} G(x, u) dx.$$

We already know from the proof of the Palais–Smale condition that  $I'_0$  has the form of a compact perturbation of a Hilbert space isomorphism. Actually we will apply Tanaka's Theorem to a slightly smoother functional:

$$(4.1) \quad J(u) := \frac{1}{2} \int_{\Omega} |\nabla^l u|^2 dx - \int_{\Omega} H(u) dx,$$

where  $H(t) = a \exp[(t^2 + 1)^b]$  for  $b = \alpha_2/2$ . By assumption (g5), we can choose  $a > 0$  so that  $G(x, t) \leq H(t)$  for  $(x, t) \in \bar{\Omega} \times \mathbb{R}$ . Thus  $I_0(u) \geq J(u)$ .

$J(u)$  has a nonlinearity of subcritical and super-quadratic growth, and so all compactness properties of  $I_0(u)$  also hold for  $J(u)$ . In particular,  $J'$  has the form  $L + \kappa$  where  $L: H_0^l(\Omega) \rightarrow H^{-l}(\Omega)$  is an isomorphism and  $\kappa: H_0^l(\Omega) \rightarrow H^{-l}(\Omega)$  is compact. In addition we have the following compactness conditions needed in the application of Tanaka's Theorem:

Let  $\{E_j\}$  be the decomposition in equation (2.1).

(PS) $_m$  If for some  $M > 0$ ,  $\{u_j\}$  satisfies:  $u_j \in E_m$ ,  $J(u_j) \leq M$  for all  $j$  and  $\|(J|_{E_m})'(u_j)\|_{E'_m} \rightarrow 0$  as  $j \rightarrow \infty$ , then  $\{u_j\}$  is precompact.

(PS) $_*$  If for some  $M > 0$ ,  $\{u_j\}$  satisfies:  $u_j \in E_j$ ,  $J(u_j) \leq M$  for all  $j$  and  $\|(J|_{E_j})'(u_j)\|_{E'_j} \rightarrow 0$  as  $j \rightarrow \infty$ , then  $\{u_j\}$  is precompact.

These conditions follow from the fact that  $J'$  is a compact perturbation of the Riesz representation map, and because such sequences are bounded. (Recall that in the proof of the Palais–Smale condition we only needed that  $I_0(u_n) \leq C$ . See equation (3.7').)

**Applying Tanaka's Theorem: The lower bound.** The goal is to obtain a lower bound for  $c_n$  that will contradict (3.30). For  $J$  define the symmetric minimax levels

$$(4.2) \quad \beta_n := \inf_{g \in \Gamma_n} \sup_{u \in g(E_n)} J(u).$$



Since  $J(u) \leq I_0(u)$  by construction, we have  $\beta_n \leq c_n$ . So it will suffice to obtain a good lower bound on  $\beta_n$ . By [32], Theorem B, there exists a sequence  $u_n$  such that

- (i)  $J(u_n) \leq \beta_n$ ,
- (ii)  $J'(u_n) = 0$ ,
- (iii)  $n \leq \text{index}_0 J''(u_n)$ ,

where the extended Morse index  $\text{index}_0 J''(u)$  is the dimension of the maximal, negative semidefinite subspace corresponding to the form  $J''(u)$ . For simplicity we simply denote  $u_n$  as  $u$ , holding  $n$  fixed for the time being. Now

$$J''(u)(v, w) = \langle v, w \rangle_{H_0^1(\Omega)} - \int_{\Omega} H''(u)vw \, dx.$$

One basis for the maximal negative semidefinite subspace of this bilinear form is the set of eigenfunctions of  $(-\Delta)^l - H''(u)$  with non-positive eigenvalues. So  $\text{index}_0 J''(u) =$  number of non-positive eigenvalues of  $(-\Delta)^l - H''(u)$  on  $L^2(\Omega)$ .

By applying Corollary 1.5, we get

$$(4.3) \quad \text{index}_0 J''(u) \leq C_{31} \int_{\Omega} \mathcal{B}(H''(u(x))) \, dx + C_{31}.$$

So, by Tanaka's Theorem,

$$n \leq C_{32} \int_{\Omega} \mathcal{B}(H''(u(x))) \, dx + C_{32},$$

where we take  $n$  sufficiently large. Since  $\Omega$  is of finite measure, the exact form of  $H''(u)$  is not important, only that it behaves like  $(|u| + 1)^{2\alpha_2 - 2} e^{(u^2 + 1)^b}$  for  $|u|$  large. So that, for some  $C_{33} > 0$ ,

$$\mathcal{B}(H''(u(x))) \leq C_{33}(|u| + 1)^{3\alpha_2 - 2} e^{(u^2 + 1)^b}.$$

So

$$\frac{n}{C_{34}} \leq \int_{\Omega} (|u| + 1)^{3\alpha_2 - 2} e^{(u^2 + 1)^b} \, dx.$$

Since  $u = u_n$  is a critical point of  $J$

$$(4.4) \quad \begin{aligned} J(u) &= A \int_{\Omega} [bu^2(u^2 + 1)^{b-1} - 1] e^{(u^2 + 1)^b} \, dx \\ &\geq C_{35} \int_{\Omega} (|u| + 1)^{\alpha_2} e^{(u^2 + 1)^b} \, dx - C_{36}. \end{aligned}$$

Let  $\tau = (|u| + 1)^{\alpha_2} e^{(u^2 + 1)^b}$  and, for some  $\gamma$ , consider

$$(4.5) \quad \lambda(\tau) := \frac{\tau}{[\ln(\tau)]^\gamma} \geq C_{37}(|u| + 1)^{\alpha_2 - \alpha_2 \gamma} e^{(u^2 + 1)^b}.$$

We let  $\alpha_2 - \alpha_2 \gamma = 3\alpha_2 - 2$ , so  $\gamma = 2/\alpha_2 - 2$ . Note that  $\gamma > 0$  since  $\alpha_2 < 1$ . From (4.5) and the bound on  $n$  we have

$$(4.6) \quad n \leq C_{39} \int_{\Omega} \lambda[(|u| + 1)^{\alpha_2} e^{(u^2 + 1)^b}] \, dx + C_{40}.$$

For large values of  $\tau$ ,  $\lambda''(\tau) < 0$ . So by the eventual concavity of  $\lambda$ , (4.6) gives

$$(4.7) \quad n \leq C_{41} \lambda \left[ \int_{\Omega} (|u| + 1)^{\alpha_2} e^{(u^2+1)^b} dx \right] + C_{42}.$$

Now apply inequality (4.4) by using the sublinearity of  $\kappa$ , inequalities (4.4) and (4.7) give

$$n \leq C_{43} \cdot \lambda[J(u_n)] + C_{44}$$

where we have included the subscript on  $u$ . Using the fact that  $J(u_n) \leq \beta_n$ , that  $\lambda$  is eventually increasing, and that  $\beta_n \rightarrow +\infty$  we have

$$(4.8) \quad n \leq C_{43} \cdot \lambda[\beta_n] + C_{44}.$$

Let  $\theta(\tau) := \tau[\ln(\tau)]^\gamma$ , which is increasing and subexponential. Apply  $\theta(\cdot)$  to both sides of (4.8)

$$(4.9) \quad \theta(n) \leq C_{45} \cdot \theta \circ \lambda[\beta_n] + C_{46}.$$

Now, for large  $\tau$ ,

$$\theta(\lambda(\tau)) = \tau \left[ 1 - \gamma \frac{\ln \ln(\tau)}{\ln(\tau)} \right]^\gamma \leq \tau.$$

So that, for large  $n$ ,  $C_{47}\theta(n) \leq \beta_n$ , i.e.

$$Cn[\ln(n)]^\gamma \leq \beta_n \leq cn, \quad \gamma = 2/\alpha_2 - 2.$$

If  $2/\alpha_2 - 2 > 1/\alpha_1$ , as in the hypothesis of Theorem 1.1, this contradicts (3.30) and so proves Theorem 1.1.  $\square$

## 5. Proof of the eigenvalue bound

We essentially follow the argument in [28]. A general outline of the method is as follows. The initial step is to begin with a group of related embedding inequalities. These are then used to prove a theorem on piecewise-polynomial approximation. This approximation theorem then leads to an eigenvalue estimate for a compact operator on an appropriate Hilbert space. Finally one relates the eigenvalues of this operator to the non-positive eigenvalues of the Schrödinger operator of interest via the Birman–Schwinger principle.

Let  $Q = (0, 1)^d$  be the unit cube in  $\mathbb{R}^d$  and let  $u \in H^1(Q) = W^{1,2}(Q)$ . Let  $\square \subset \mathbb{R}^d$  be a parallelepiped with edges parallel to those of  $Q$ , and denote

$$\mathcal{P}(l, d) = \text{vector space of all polynomials of degree } < l \text{ in } \square,$$

$$m(l, d) := \dim_{\mathbb{R}} \mathcal{P}(l, d).$$

That is, we regard  $\mathcal{P}(l, d)$  as the subspace of  $L^2(\mathbb{R}^d)$  consisting of functions supported in  $\square$ , and which are polynomials of degree less than  $l$  in  $\square$ . We let  $\mathcal{P}_{l,\square}$  be the corresponding orthonormal projection onto  $\mathcal{P}(l, d)$ . That is,  $\mathcal{P}_{l,\square}$  is the  $L^2$ -orthogonal projection of  $L^2(\mathbb{R}^d)$  onto  $\mathcal{P}(l, d)$ .

Furthermore, let  $\Xi$  be a finite covering of  $Q$  by parallelepipeds  $\square$ . To any such covering and any  $l > 0$  we associate an operator of piecewise-polynomial approximation in  $L^2(\mathbb{R}^d)$ : For  $\Xi = \{\square_j\}$ ,  $1 \leq j \leq \text{card}(\Xi)$ , and with  $\chi_j$  the characteristic function of the set  $\square_j \setminus \bigcup_{i < j} \square_i$ , we denote

$$(5.1) \quad K_{\Xi, l} = \sum_j \chi_j \mathcal{P}_{\square_j, l}.$$

Note that  $\text{rank}(K_{\Xi, l}) \leq m(l, d) \cdot \text{card}(\Xi)$ .

In this section we will need to recall the theory of Orlicz spaces (see [2], [15] and [25]). Let  $\mathcal{B}, \mathcal{A}$  be a pair of mutually complementary  $N$ -functions, and  $L_{\mathcal{B}}(\omega), L_{\mathcal{A}}(\omega)$  be the corresponding Orlicz spaces on a set  $\omega \subset \mathbb{R}^d$  of finite Lebesgue measure. We are primarily interested in the pair

$$\mathcal{A}(t) = e^{|t|} - 1 - |t| \quad \text{and} \quad \mathcal{B}(t) = (|t| + 1) \ln(|t| + 1) - |t|.$$

Then Solomyak's main theorem is

**THEOREM 5.1** (Solomyak [28]). *Let  $Q = (0, 1)^d$ ,  $V \in L_{\mathcal{B}}(Q)$ ,  $V \geq 0$ . Then, for any  $n \in \mathbb{N}$ , there exists a covering  $\Xi = \Xi(V, n)$  of  $Q$  by parallelepipeds  $\square \subset Q$  such that*

$$(5.2) \quad \text{card}(\Xi) \leq C_1 n$$

and, for any  $u \in H^l(Q)$ ,  $2l = d$ , we have

$$(5.3) \quad \int_Q V |u - K_{\Xi, l} u|^2 dx \leq C_2 n^{-1} \|V\|_{\mathcal{B}, Q} \int_Q |\nabla^l u|^2 dx,$$

where  $C_1, C_2$  depend only on  $d$ .

Defined on  $H_0^l(\Omega)$ , we consider the quadratic form

$$a(u, v) := \int_{\Omega} \nabla^l u \cdot \nabla^l v dx, \quad \text{with} \quad a(u) := a(u, u) = \int_{\Omega} |\nabla^l u|^2 dx.$$

As an unbounded quadratic form on  $L^2(\Omega)$ ,  $a(u)$  is symmetric and positive. It is also closed in  $L^2(\Omega)$ . To see this note that if  $u_n \rightarrow u$  in  $L^2(\Omega)$  and  $a(u_n - u_n) \rightarrow 0$  then  $\{u_n\}$  is Cauchy in  $H_0^l(\Omega)$ , and hence converges to some  $\bar{u}$  in that space. By the generalized Poincaré inequality,  $u_n \rightarrow \bar{u}$  in  $L^2(\Omega)$ . Thus  $u = \bar{u}$  almost everywhere and so  $u$  is (representable by) an element of  $H_0^l(\Omega)$  and  $\lim_{n \rightarrow \infty} a(u_n - u) = 0$ . On the space  $L^2(\Omega)$  the unbounded operator  $(-\Delta)^l$  is defined as the self-adjoint Friedrichs operator associated to the form  $a(u)$ . That is,

$$D((-\Delta)^l) := \left\{ u \in H_0^l(\Omega) : \text{the linear functional } v \mapsto \int_{\Omega} \nabla^l u \cdot \nabla^l v dx \right. \\ \left. \text{is } L^2\text{-continuous, where } v \in H_0^l(\Omega) \right\}$$

and

$$\langle (-\Delta)^l f, g \rangle_{L^2} = a(f, g) \quad \text{for } f \in D((-\Delta)^l) \text{ and } g \in H_0^l(\Omega).$$

We can do this since  $a$  is closed. See for example Section 5.5 in [35]. Also, on  $L^2(\Omega)$  we consider the form

$$(5.4) \quad a_V(u) := a(u) - \int_{\Omega} V(x)|u|^2 dx,$$

where  $V \in L_{\mathcal{B}}$ ,  $V \geq 0$ . Here the form domain is  $H_0^l(\Omega) \cap L^2(\Omega, V dx)$ . As we will see later in the proof of Theorem 5.2

$$\|u\|_{L^2(\Omega, V dx)}^2 \leq C_{\Omega} \cdot a(u)$$

for  $u \in H_0^l(\Omega)$ . So the domain of  $a_V$  is really just  $d(a_V) = H_0^l(\Omega)$  and  $H_0^l(\Omega)$  embeds into  $L^2(\Omega, V dx)$ . As a matter of fact for any  $\varepsilon > 0$  there exists a constant  $C(\varepsilon)$  such that

$$(5.5) \quad \|u\|_{L^2(\Omega, V dx)}^2 \leq \varepsilon a(u) + C(\varepsilon) \|u\|_{L^2(\Omega)}^2.$$

That is, the quadratic form

$$(5.6) \quad \int_{\Omega} V(x)|u|^2 dx$$

has “zero bound” relative to the form  $a(u)$  in  $L^2(\Omega)$ . This is a consequence of Theorem 5.2 below. By that theorem the operator corresponding to the quadratic form (5.6) in the Hilbert space  $(H_0^l(\Omega), a(\cdot, \cdot))$  is compact. A well-known result then implies the zero-boundedness mentioned.

The bound (5.5) implies that  $a_V(u)$  is lower semi-bounded in  $L^2(\Omega)$ . It is also closed in  $L^2(\Omega)$ . That is, for  $\{u_n\}$  with  $u_n \rightarrow u$  in  $L^2(\Omega)$  and  $a_V(u_n - u_m) \rightarrow 0$  then  $a(u_n - u_m) \rightarrow 0$ . So  $\{u_n\}$  is Cauchy in  $H_0^l(\Omega)$ , and hence in  $L^2(\Omega)$ . Therefore  $u \in H_0^l(\Omega)$ . So as before we can define the associated self-adjoint Friedrichs operator on  $L^2(\Omega, V dx)$ :

$$A_V(u) := (-\Delta)^l u - \alpha V(x)u$$

whose domain is a subset of  $D((-\Delta)^l)$ .

Suppose  $A$  is a self-adjoint operator on a Hilbert space and that the spectrum of  $A$  less than or equal to  $\lambda \in \mathbb{R}$  is discrete. Then define  $N(\lambda, A)$  to be the number of eigenvalues of  $A$  less than or equal to  $\lambda$ , counted according to their multiplicity. For a compact, non-negative, symmetric operator  $T$  denote by

$$n(\lambda, T) = N(-\lambda, -T)$$

the number of eigenvalues of  $T$  greater than or equal to  $\lambda$ . We now consider the quadratic functional

$$b_V(u) := \int_{\Omega} V(x)|u(x)|^2 dx$$

where  $V \in L_{\mathcal{B}}$ . If  $b_V$  is bounded on  $(H_0^l(\Omega), \|\cdot\|_{H_0^l(\Omega)})$ , then it generates a bounded, self-adjoint, non-negative operator on  $H_0^l(\Omega)$  — say  $T_V$ . By definition

$$u = T_V f \Leftrightarrow u \in H_0^l(\Omega); \quad \int_{\Omega} \nabla^l u \cdot \nabla^l w \, dx = \int_{\Omega} V f w \, dx, \quad \text{for all } w \in H_0^l(\Omega).$$

**THEOREM 5.2.** *Let  $\Omega \subset \mathbb{R}^{2l}$  be a bounded region with smooth boundary, and  $V \in L_{\mathcal{B}}(\Omega)$ . Then the operator  $T_V$  is well defined and compact on  $H_0^l(\Omega)$ , and there exists a constant  $C_3 = C_3(\Omega)$  such that, for any  $\lambda > 0$ ,*

$$(5.7) \quad n(\lambda; T_V) \leq C_3 \|V\|_{\mathcal{B}, \Omega} \lambda^{-1}.$$

**PROOF.** Let  $Q \subseteq \mathbb{R}^{2l}$  be a cube such that  $\bar{\Omega} \subseteq Q$ . We can regard  $Q$  as a unit cube, after rescaling. Let  $W$  be the function on  $Q$  equal to  $V$  on  $\Omega$  and  $W = 0$  on  $Q \setminus \Omega$ . Note that by the Hölder inequality for Orlicz spaces, we write

$$(5.8) \quad \int_{\Omega} V \cdot |u|^2 \, dx \leq \|V\|_{\mathcal{B}, \Omega} \cdot \|u^2\|_{\mathcal{A}, \Omega}$$

where the  $N$ -function  $\mathcal{A}(t) := e^t - 1 - t$  is the Young function conjugate to  $\mathcal{B}(t)$ . Now by the well-known Sobolev–Orlicz embedding, see Theorem 8.25 in [2], we have

**PROPOSITION 5.3.** *There exists a constant  $C$  such that, for every  $u \in H^l(\Omega)$ ,  $2l = d$ ,*

$$(5.9) \quad \|u^2\|_{\mathcal{A}, \Omega} \leq C \|u\|_{W^{l,2}(\Omega)}^2.$$

Since  $u \in H_0^l(\Omega)$  we can use the  $H_0^l(\Omega)$  norm. So we get

$$(5.10) \quad \int_{\Omega} V \cdot |u|^2 \, dx \leq C_d \|V\|_{\mathcal{B}, \Omega} \cdot \|\nabla^l u\|_{L^2(\Omega)}^2.$$

Thus  $b_V$  is bounded as a quadratic form on  $H_0^l(\Omega)$  (and on  $\Pi \circ H_0^l(\Omega) \subseteq H_0^l(Q)$ , where  $\Pi$  is the natural extension operator). So  $T_V$  is bounded on  $H_0^l(\Omega)$  and

$$(5.11) \quad n(\lambda; T_V) = 0 \quad \text{for } \lambda > C_d \|V\|_{\mathcal{B}, \Omega}.$$

Now fix  $\lambda \in (0, \lambda_0]$ , where  $\lambda_0 = C_2 C_l \|V\|_{\mathcal{B}, \Omega}$ , for  $C_2$  comes from Theorem 5.1, and  $C_l$  comes from

$$\|u\|_{W^{l,2}(\Omega)}^2 \leq C_l \|u\|_{H_0^l(\Omega)}^2$$

for  $u \in H_0^l(\Omega)$ . Let  $n$  be the minimal integer such that  $n\lambda > \lambda_0$ . For this  $n$  and the function  $W$ , let  $\Xi$  be the covering of  $Q$  constructed in Theorem 5.1 and  $K := K_{\Xi, l}$  the corresponding operator (5.1).

For the subspace  $\mathcal{F} := \ker(K \circ \Pi)$  of  $H_0^l(\Omega)$

$$\text{codim } \mathcal{F} \leq \text{rank } K \leq m(l, d) C_1 \cdot n.$$

For  $u \in H_0^l(\Omega)$  denote by  $U := \Pi(u)$ . Then, by Theorem 5.1, the following inequality holds:

$$\begin{aligned} \int_{\Omega} V u^2 dx &= \int_Q W |U - K(U)|^2 dx \leq C_2 n^{-1} \|W\|_{\mathcal{B}, Q} \int_Q |D^l u|^2 dx \\ &\leq C_l C_2 n^{-1} \|W\|_{\mathcal{B}, Q} \|u\|_{H_0^l}^2 \leq \lambda \|u\|_{H_0^l}^2. \end{aligned}$$

This is enough to show that  $T_V$  is compact. Indeed let  $T_V|_{\mathcal{F}}$  denote the linear operator which is defined as  $T_V$  on  $\mathcal{F}$  and 0 on  $\mathcal{F}^\perp$ . Then  $T_V = T_V|_{\mathcal{F}} + T_V|_{\mathcal{F}^\perp}$  and  $\mathcal{F}^\perp$  is finite dimensional. The above shows that  $\|T_V|_{\mathcal{F}}\| \leq \lambda$ . So taking  $\lambda \rightarrow 0$ , we see that  $T_V$  is the limit of  $T_V|_{\mathcal{F}^\perp}$  in the uniform norm. Therefore  $T_V$  is compact and its spectrum consists of eigenvalues. If  $u_j$  is an eigenvector with eigenvalue  $\lambda_j \geq \lambda$  then

$$(5.12) \quad \lambda \leq \lambda_j = \frac{\langle T_V u_j, u_j \rangle_{H_0^l}}{\langle u_j, u_j \rangle_{H_0^l}} = \frac{\int_{\Omega} V |u_j|^2 dx}{\int_{\Omega} |\nabla^l u_j|^2 dx}.$$

So  $u_j \notin \mathcal{F}$ . Since eigenvectors are orthogonal

$$(5.13) \quad \begin{aligned} n(\lambda; T_V) &\leq \text{codim } \mathcal{F} \leq m(l, d) C_1 n \\ &< m(l, d) C_1 \left( \frac{\lambda_0}{\lambda} + 1 \right) \leq 2m(l, d) C_1 \frac{\lambda_0}{\lambda}, \quad \lambda \leq \lambda_0. \end{aligned}$$

The required estimate (5.7), with  $C_3 = 2m(l, d) C_1 \max\{C_l C_2, C_d\}$ , where  $C_d$  is given in (5.10) and  $C_l$  is given in the inequality after (5.11), is a consequence of (5.11) and (5.13):

- (i) If  $C_l C_2 \geq C_d$  then (5.13) gives the result.
- (ii) If  $\lambda \leq \lambda_0 = C_l C_2 \|V\|_{\mathcal{B}, \Omega}$ , again (5.13) gives the result.
- (iii) If  $C_l C_2 \|V\|_{\mathcal{B}, \Omega} < \lambda \leq C_d \|V\|_{\mathcal{B}, \Omega}$  then (5.13) is applied to  $\tilde{\lambda} = \lambda C_l C_2 / C_d$  and that gives the result.  $\square$

**Proof of Proposition 1.4. The Briman–Schwinger principle.** The reference here is Section 1 in [8]. Let  $a(u)$  be a positive, symmetric, and closed quadratic form in a Hilbert space  $\mathcal{H}$  with domain  $D(a) \subset \mathcal{H}$ . Let  $b(u)$  be another non-negative, symmetric quadratic form such that

$$(5.14) \quad b(u) \leq C \cdot a(u), \quad u \in D(a).$$

Consider the space  $\tilde{D}(a)$ , the completion of  $D(a)$  in the inner product given by  $a(\cdot, \cdot)$ . By (5.14),  $b$  can be extended to all of  $\tilde{D}(a)$ . The extended form defines on  $\tilde{D}$  a bounded, self-adjoint, non-negative operator, which we denote by

$$B: (\tilde{D}(a), a(\cdot, \cdot)) \rightarrow (\tilde{D}(a), a(\cdot, \cdot)).$$

PROPOSITION 5.4 (Birman–Schwinger Principle). *Suppose (5.14) is satisfied and the operator  $B$  is compact as an operator from  $(\tilde{D}(a), a(\cdot, \cdot))$  to itself. Then, for any  $\alpha > 0$ , the quadratic form*

$$a_\alpha(u) := a(u) - \alpha b(u), \quad u \in D(a),$$

*is semi-bounded from below and closed in  $\mathcal{H}$ . As usual, this implies that there is a corresponding self-adjoint Friedrichs operator  $A_{\alpha b}$  associated with this form. For  $A_{\alpha b}$  the non-positive spectrum is finite and*

$$(5.15) \quad N(0; A_{\alpha b}) = n(\alpha^{-1}; B).$$

This result comes from the variational characterization of  $N(0; A)$ , often referred to as Glazman’s lemma. For us  $\mathcal{H} = L^2(\Omega)$ ,  $D(a) = \tilde{D}(a) = H_0^l(\Omega)$ ,  $a(u) = \int_\Omega |\nabla^l u|^2 dx$ ,  $b(u) = \int_\Omega V|u|^2 dx$ ,  $\alpha = 1$ ,  $B = T_V$ . From the proof of Theorem 2.1, (5.14) is satisfied,  $T_V$  is compact, and the form  $a(u)$  is closed in  $L^2(\Omega)$ . We thus obtain  $N(0, (-\Delta)^l - V(x)) = n(1, T_V)$ . So, by Theorem 5.2,  $N(0, (-\Delta)^l - V(x)) \leq C_3 \|V\|_{\mathcal{B}, \Omega}$ .  $\square$

## 6. Problem (R), the radial problem on an annulus

**6.1. The problem and its variational setup.** Here  $\Omega = A_{R_0}^R := \{x \in \mathbb{R}^{2l} : R_0 < |x| < R\}$  will denote an annulus, with  $R_0 > 0$  and  $R < +\infty$ . We seek radial solutions to the problem

$$(R) \quad \begin{cases} (-\Delta)^l u = 2ue^{u^2} + \varphi(x, u) & \text{in } \Omega, \\ \left(\frac{\partial}{\partial \nu}\right)^j u \Big|_{\partial\Omega} = 0 & \text{for } j = 0, \dots, l-1, \end{cases}$$

where  $\varphi(x, u) = \varphi(|x|, u)$ . The proper space for this problem is

$$H_r := \{u \in H_0^l(\Omega) : u(x) = u(|x|) \text{ almost everywhere in } \Omega\}$$

and the corresponding functional is

$$I_1(u) := \frac{1}{2} \int_\Omega |\nabla^l u|^2 dx - \int_\Omega (e^{u^2} - 1) dx - \int_\Omega \Phi(x, u) dx$$

where  $\Phi(x, u) = \int_0^u \varphi(x, t) dt$  is not even in  $u$ . Since  $\varphi(x, u)$  is radial in its explicit dependence on  $x$ , critical points of  $I_1$ , even when restricted to  $H_r$ , still correspond to generalized solutions to (R). This can be seen from a simple direct calculation using spherical coordinates. The general principle behind this fact is called the Principle of Symmetric Criticality, see [21]. Concerning the size of the perturbation, we assume there exists  $\beta < 1$  and  $C > 0$  such that

$$(6.1) \quad |\Phi(x, t)| + |\varphi(x, t)t| \leq Ct^2 e^{\beta t^2}$$

for all  $x \in \Omega$ , and  $t \in \mathbb{R}$  with  $|t|$  large. Similarly to before, the path of functionals we will be concerned with is

$$I_\theta(u) := \frac{1}{2} \int_{\Omega} |\nabla^l u|^2 dx - \int_{\Omega} (e^{u^2} - 1) dx - \theta \int_{\Omega} \Phi(x, u) dx,$$

where  $\theta \in [0, 1]$ .

**6.2. Bolle's and Tanaka's requirements.** The compactness of Palais–Smale sequences in the space  $H_r$  as required by Bolle's condition (H1) follows as in the general case, with the exception that instead of Adam's inequality we use Lemma 7.1, given below in Section 7. Thus the radial setting allows us to consider a nonlinearity which was of critical growth in the earlier unrestricted setting. For the condition (H2) assume  $|I_\theta(u)| \leq b$ . In fact, we only need to assume that  $I_\theta(u) \leq b$ :

$$\begin{aligned} \|I'_\theta(u)\|_{H^{-1}} \|u\|_{H^1} &\geq -\langle I'_\theta(u), u \rangle = - \int_{\Omega} |\nabla^l u|^2 dx + \int_{\Omega} \{2u^2 e^{u^2} + \theta \varphi(x, u) u\} dx \\ &= \int_{\Omega} |\nabla^l u|^2 dx + \int_{\Omega} \{(2u^2 - 4)e^{u^2} + \theta \varphi(x, u) u - 4\theta \Phi(x, u)\} dx - 4I_\theta(u) - C_1 \\ &\geq \|u\|_{H^1(\Omega)}^2 + c \int_{\Omega} u^2 e^{u^2} dx - C_2 - b. \end{aligned}$$

Then, from (6.1) it follows that

$$\|I'_\theta(u)\|_{H^{-1}} \|u\|_{H^1} \geq c \left| \int_{\Omega} \Phi(x, u) dx \right| - C_3 = c \left| \frac{\partial}{\partial \theta} I_\theta(u) \right| - C_3,$$

where  $b$  was absorbed into  $C_3$  and where  $c$  is some small positive constant in the last two inequalities. This verifies condition (H2). Next for condition (H3), let  $u$  be a critical point of  $I_\theta$ . Then

$$\begin{aligned} I_\theta(u) &= I_\theta(u) - \frac{1}{2} \langle I'_\theta(u), u \rangle \\ &= \int_{\Omega} \left\{ (u^2 - 1)e^{u^2} + \frac{\theta}{2} \varphi(x, u) u - \theta \Phi(x, u) \right\} dx - C_4 \geq c \int_{\Omega} u^2 e^{u^2} dx - C_5, \end{aligned}$$

where again  $c$  is some small positive constant. Applying Jensen's inequality and (6.1), we have, for sufficiently large constants  $C_6$  and  $C_7$ ,

$$C_6[|I_\theta(u)| + 1]^\beta \geq \int_{\Omega} (u^2 e^{u^2})^\beta dx + C_7 \geq \left| \int_{\Omega} \Phi(x, u) dx \right| = \left| \frac{\partial}{\partial \theta} I_\theta(u) \right|.$$

So, condition (H3) holds with  $\bar{f}_1(\theta, t) = \bar{f}_2(\theta, t) = f(t) := C[|t| + 1]^\beta$ . Condition (H4) follows as before. If we assume that only the second possibility of Theorem 2.1 holds for sufficiently large  $n \in \mathbb{N}$ , then for some  $K > 0$  such that

$$c_{n+1} - c_n \leq K(\bar{f}_1(c_{n+1}) + \bar{f}_2(c_n) + 1),$$

for sufficiently large  $n$ . More concisely, by enlarging  $K$  if necessary, this means

$$c_{n+1} - c_n \leq K((c_{n+1})^\beta + (c_n)^\beta + 1)$$



for sufficiently large  $n$ . Finally, using the fact that  $\beta < 1$ , this implies that for some  $A > 0$

$$c_n \leq An^{1/(1-\beta)} \quad \text{for } n > n_0,$$

with  $n_0$  sufficiently large. The argument is the same as that used for (3.30).

**6.3. The lower bound.** As before, the goal is to obtain a lower bound for  $c_n$  that will contradict the alleged upper bound. The requirements in Tanaka's Theorem are all verified as before, with no new phenomena appearing. By Tanaka's Theorem, there exists a sequence  $u_n$  in the Hilbert space  $H_r$  such that

- (i)  $I_0(u_n) \leq c_n$ ,
- (ii)  $I_0'(u_n) = 0$ ,
- (iii)  $n \leq \text{index}_0 I_0''(u_n)$ .

For simplicity we denote  $u_n$  as  $u$ , holding  $n$  fixed for the moment. As before,

$$\text{index}_0 I_0''(u) = \text{number of non-positive eigenvalues of } (-\Delta)^l - 2(u^2 + 2)e^{u^2}.$$

By applying Proposition 1.6 we get

$$\begin{aligned} n^{2l} &\leq [\text{index}_0 I_0''(u)]^{2l} \leq C_8 \int_{A_{R_0}^R} 2(u^2 + 2)e^{u^2} \log \left( \frac{R}{|x|} \right)^{2i} dx \\ &\leq C_{R_0} \int_{A_{R_0}^R} 2(u^2 + 2)e^{u^2} dx. \end{aligned}$$

Since  $u = u_n$  is a critical point of  $I_0$  as before we have that

$$c_n \geq I_0(u_n) \geq c \int_{\Omega} u^2 e^{u^2} dx - C_9.$$

Therefore we obtain  $c_n \geq C_{10} \cdot n^{2l}$  for sufficiently large  $n$ . Therefore, if  $2l > 1/(1 - \beta)$  this contradicts the upper bound, and proves Theorem 1.2.  $\square$

### 7. Piecewise-polynomial approximation and spectral estimates in the annular case

Here we prove Proposition 1.6. As mentioned earlier, we first need some appropriate inequalities in the radial case to take the place of the Orlicz-Sobolev inequality of Proposition 5.3. Lemma 7.1 below is a generalization of an inequality by Ni, see [20]. Note, in Lemmas 7.1 and 7.2 below, the constant  $C_d$  refers to a generic constant which changes in a suitable way from one line to another but just depends on  $d$ .

LEMMA 7.1. *Let  $u \in H_r$ .*

(a) *If  $d = 2l = 2$  then*

$$|u(x)| \leq C_d \|\partial_r u\|_{L^2(\Omega)} \cdot \sqrt{\log \left( \frac{R}{|x|} \right)} \quad \text{for } x \in \Omega = A_{R_0}^R.$$

(b) If  $d = 2l > 2$  then

$$|u(x)| \leq C_d \|\partial_r^l u\|_{L^2(\Omega)} \cdot \log \left( \frac{R}{|x|} \right) \quad \text{for } x \in \Omega = A_{R_0}^R.$$

PROOF. (a) For simplicity we write  $u = u(r)$  as a function of the radial variable. By a density argument we may assume that  $u \in C_0^\infty(\Omega)$ . For  $r \in [R_0, R]$

$$-u(r) = u(R) - u(r) = \int_r^R u(\rho) d\rho,$$

so

$$\begin{aligned} |u(r)| &\leq \int_r^R |u(\rho)| d\rho \\ &\leq \left( \int_r^R |u(\rho)|^2 \rho d\rho \right)^{1/2} \left( \int_r^R \frac{1}{\rho} d\rho \right)^{1/2} \leq C_d \|\partial_r u\|_{L^2(\Omega)} \cdot \log \left( \frac{R}{r} \right)^{1/2}, \end{aligned}$$

which is the required result.

(b) Again we take  $u \in C_0^\infty$ .

$$\begin{aligned} u(r) &= u(r) - u(R) = - \int_r^R u'(\rho_1) d\rho_1 \\ &= \int_r^R u'(R) - u'(\rho_1) d\rho_1 = \int_r^R \int_{\rho_1}^R u''(\rho_2) d\rho_2 d\rho_1 \\ &\dots\dots\dots \\ &= (-1)^l \int_r^R \int_{\rho_1}^R \dots \int_{\rho_{l-1}}^R u^{(l)}(\rho_l) d\rho_l \dots d\rho_1. \end{aligned}$$

So

$$\begin{aligned} |u(r)| &\leq \int_r^R \int_{\rho_1}^R \dots \int_{\rho_{l-1}}^R |u^{(l)}(\rho_l)| d\rho_l \dots d\rho_1 \\ &= \int_r^R \int_{\rho_1}^R \dots \int_{\rho_{l-1}}^R |u^{(l)}(\rho_l)| \rho_l^{(d-1)/2} \rho_l^{(1-d)/2} d\rho_l \dots d\rho_1 \\ &\leq \int_r^R \int_{\rho_1}^R \dots \int_{\rho_{l-2}}^R \left( \int_{\rho_{l-1}}^R |u^{(l)}(\rho_l)|^2 \rho_l^{d-1} d\rho_l \right)^{1/2} \\ &\quad \cdot \left( \int_{\rho_{l-1}}^R \rho_l^{1-d} d\rho_l \right)^{1/2} d\rho_{l-1} \dots d\rho_1 \\ &\leq C_d \|\partial_r^l u\|_{L^2(\Omega)} \cdot \int_r^R \int_{\rho_1}^R \dots \int_{\rho_{l-2}}^R \rho_{l-1}^{1-l} d\rho_{l-1} \dots d\rho_1 \\ &\leq C_d \|\partial_r^l u\|_{L^2(\Omega)} \cdot \int_r^R \frac{1}{\rho_1} d\rho_1 = C_d \|\partial_r^l u\|_{L^2(\Omega)} \cdot \log \left( \frac{R}{r} \right) \end{aligned}$$

which is the required estimate.  $\square$

When proving a radial version of Theorem 5.1 it is necessary to have at one's disposal inequalities of the above type, but without the zero boundary conditions. The key is to find the appropriate  $(l - 1)$ th degree polynomial to subtract from  $u$ , so that the remainder can be controlled by the  $l$ th order derivative of  $u$ . It is not surprising that this is the same polynomial approximation which appears in the radial version of Theorem 5.1.

LEMMA 7.2. *Let  $u \in H^l(A)$ , where  $A = A_{R_0}^R$  is an annulus centered at the origin in  $\mathbb{R}^{2l}$ , and  $u(x) = u(|x|)$  almost everywhere in  $A$ .*

(a) *If  $d = 2l = 2$ , then*

$$|u(x) - \bar{u}_A| \leq C_d \|\partial_r u\|_{L^2(A)} \cdot \left[ 1 + \log \left( \frac{R}{|x|} \right)^{1/2} \right]$$

*for  $x \in A$ , where  $\bar{u}_A$  is the average value of  $u$  in  $A$ .*

(b) *When  $d = 2l > 2$ . First define*

$$\tau_l(u)(r, s) := \sum_{n=0}^{l-1} \frac{u^{(n)}(s)}{n!} (r - s)^n$$

*and*

$$P_{l,A}(u)(r) := \frac{1}{|A|} \int_A \tau_l(u)(r, |x|) dx$$

*which is a polynomial in  $r$  of degree  $\leq l - 1$  and linear in  $u$ . Then*

$$|u(x) - P_{l,A}(u)(|x|)| \leq C_d \|\partial_r^l u\|_{L^2(A)} \cdot \left[ 1 + \log \left( \frac{R}{|x|} \right) \right] \quad \text{for } x \in A = A_{R_0}^R.$$

PROOF. We assume  $u \in C^\infty(A)$  and radially symmetric. Then

$$(7.1) \quad |u(r) - \bar{u}_A| = \left| \frac{1}{|A|} \int_A \{u(r) - u(x)\} dx \right| \leq \frac{1}{|A|} \int_A |u(r) - u(x)| dx.$$

Now

$$u(r) - u(x) = \int_{|x|}^r u'(\rho) d\rho.$$

So

$$|u(r) - u(x)| \leq \int_{|x|,r} |u'(\rho)| \rho^{1/2} \rho^{-1/2} d\rho,$$

where the notation  $\int_{a,b}$  denotes unoriented integration over the interval with endpoints  $a$  and  $b$ . So

$$\begin{aligned} |u(r) - u(x)| &\leq \left( \int_{|x|,r} |u'(\rho)|^2 \rho d\rho \right)^{1/2} \left( \int_{|x|,r} \rho^{-1} d\rho \right)^{1/2} \\ &\leq C_d \|\partial_r u\|_{L^2(A)} \cdot \left| \log \left( \frac{r}{|x|} \right) \right|^{1/2}. \end{aligned}$$

Plugging this into the earlier inequality gives

$$\begin{aligned} |u(r) - \bar{u}_A| &\leq \frac{1}{|A|} \int_A C_d \|\partial_r u\|_{L^2(A)} \cdot \left| \log \left( \frac{r}{|x|} \right) \right|^{1/2} dx \\ &= \frac{C_d}{|A|} \|\partial_r u\|_{L^2(A)} \cdot \int_{R_0}^R \left| \log \left( \frac{r}{\rho} \right) \right|^{1/2} \rho d\rho. \end{aligned}$$

It is easy to check that the value of the integral is bounded above by a constant multiple of

$$R(R - R_0) \left[ 1 + \left( \log \left( \frac{R}{r} \right) \right)^{1/2} \right].$$

To see this we evaluate the integral in two parts:

$$I_1 = \int_{R_0}^r \left[ \log \left( \frac{r}{\rho} \right) \right]^{1/2} \rho d\rho \quad \text{and} \quad I_2 = \int_r^R \left[ \log \left( \frac{\rho}{r} \right) \right]^{1/2} \rho d\rho.$$

In  $I_1$  we let  $t = \log(r/\rho)$  and so

$$\begin{aligned} I_1 &= r^2 \int_0^{\log(r/R_0)} t^{1/2} e^{-2t} dt \\ &\leq C_0 r^2 \int_0^{\log(r/R_0)} e^{-t} dt = C_0 r(r - R_0) \leq C_0 R(R - R_0). \end{aligned}$$

For  $I_2$  we simply notice

$$I_2 = \int_r^R \left[ \log \left( \frac{\rho}{r} \right) \right]^{1/2} \rho d\rho \leq \left[ \log \left( \frac{R}{r} \right) \right]^{1/2} R(R - R_0).$$

Since  $A$  is a  $d = 2$  dimensional annulus we have that  $R(R - R_0)/|A| \leq C_d$ . Thus

$$|u(r) - \bar{u}_A| \leq C_d \|\partial_r u\|_{L^2(A)} \cdot \left[ 1 + \left[ \log \left( \frac{R}{r} \right) \right]^{1/2} \right]$$

which is the required result.

(b) As before, we assume  $u$  is smooth. Then

$$(7.2) \quad \begin{aligned} |u(x) - P_{l,A}(u)(|x|)| &= \left| \frac{1}{|A|} \int_A \{u(r) - \tau_l(u)(r, |x|)\} dx \right| \\ &\leq \frac{1}{|A|} \int_A \{|u(r) - \tau_l(u)(r, |x|)|\} dx. \end{aligned}$$

Set  $v(r) := u(r) - \tau_l(u)(r, |x|)$ . Note that, keeping  $|x|$  fixed, we have

$$v^{(n)}(r)|_{r=|x|} = 0 \quad \text{for } 0 \leq n \leq l - 1,$$

where the differentiation is partial differentiation w.r.t.  $r$ . We seek to estimate  $v(r)$  by repeatedly applying this property.

$$\begin{aligned} v(r) &= v(r) - v(|x|) = \int_{|x|}^r v'(\rho_1) d\rho_1 \\ &= \int_{|x|}^r \{v'(\rho_1) - v'(|x|)\} d\rho_1 = \int_{|x|}^r \int_{|x|}^{\rho_1} v''(\rho_2) d\rho_2 d\rho_1 \\ &= \int_{|x|}^r \int_{|x|}^{\rho_1} \{v''(\rho_2) - v''(|x|)\} d\rho_2 d\rho_1 \\ &\dots\dots\dots \\ &= \int_{|x|}^r \int_{|x|}^{\rho_1} \dots \int_{|x|}^{\rho_{l-1}} v^{(l)}(\rho_l) d\rho_l \dots d\rho_1, \end{aligned}$$

thus

$$|v(r)| \leq \int_{|x|,r} \int_{|x|,\rho_1} \dots \int_{|x|,\rho_{l-1}} |v^{(l)}(\rho_l)| d\rho_l \dots d\rho_1.$$

We apply Hölder’s inequality to the inner most integral

$$\begin{aligned} |v(r)| &\leq \int_{|x|,r} \int_{|x|,\rho_1} \dots \int_{|x|,\rho_{l-2}} \left( \int_{|x|,\rho_{l-1}} |v^{(l)}(\rho_l)|^2 \rho_l^{d-1} d\rho_l \right)^{1/2} \\ &\quad \times \left( \int_{|x|,\rho_{l-1}} \rho_l^{1-d} d\rho_l \right)^{1/2} d\rho_{l-1} \dots d\rho_1 \\ &\leq C_d \|\partial_r^l u\|_{L^2(A)} \cdot \int_{|x|,r} \int_{|x|,\rho_1} \dots \int_{|x|,\rho_{l-2}} |\rho_{l-1}^{2-d} - |x|^{2-d}|^{1/2} d\rho_{l-1} \dots d\rho_1 \end{aligned}$$

after using  $v^{(l)}(\rho_l) = \partial_r^l u(\rho_l)$ . Returning to (7.2) we get

$$(7.3) \quad |u(r) - P_{l,A}(u)(r)| \leq \frac{C_d}{|A|} \|\partial_r^l u\|_{L^2(A)} \times \int_A \int_{|x|,r} \int_{|x|,\rho_1} \dots \int_{|x|,\rho_{l-2}} |\rho_{l-1}^{2-d} - |x|^{2-d}|^{1/2} d\rho_{l-1} \dots d\rho_1 dx.$$

We seek to estimate the above integral. First, by converting to polar coordinates the integral becomes (after factoring out a constant depending only on the dimension)

$$\int_{R_0}^R \int_{\rho_0,r} \int_{\rho_0,\rho_1} \dots \int_{\rho_0,\rho_{l-2}} |\rho_{l-1}^{2-d} - \rho_0^{2-d}|^{1/2} d\rho_{l-1} \dots d\rho_1 \rho_0^{d-1} d\rho_0.$$

We divide the integration by  $d\rho_0$  into two pieces. One where  $R_0 \leq \rho_0 \leq r$  and one where  $r \leq \rho_0 \leq R$ . This allows us to properly orient the endpoints. The

first integral is

$$\begin{aligned}
(7.4) \quad & \int_{R_0}^r \int_{\rho_0}^r \int_{\rho_0}^{\rho_1} \cdots \int_{\rho_0}^{\rho_{l-2}} (\rho_0^{2-d} - \rho_{l-1}^{2-d})^{1/2} d\rho_{l-1} \cdots d\rho_1 \rho_0^{d-1} d\rho_0 \\
& \leq \int_{R_0}^r \int_{\rho_0}^r \int_{\rho_0}^{\rho_1} \cdots \int_{\rho_0}^{\rho_{l-2}} \rho_0^{1-d/2} d\rho_{l-1} \cdots d\rho_1 \rho_0^{d-1} d\rho_0 \\
& = \int_{R_0}^r \int_{\rho_0}^r \int_{\rho_0}^{\rho_1} \cdots \int_{\rho_0}^{\rho_{l-3}} \rho_0^{1-d/2} (\rho_{l-2} - \rho_0) d\rho_{l-2} \cdots d\rho_1 \rho_0^{d-1} d\rho_0 \\
& \leq \int_{R_0}^r \int_{\rho_0}^r \int_{\rho_0}^{\rho_1} \cdots \int_{\rho_0}^{\rho_{l-3}} \rho_0^{1-d/2} \rho_{l-2} d\rho_{l-2} \cdots d\rho_1 \rho_0^{d-1} d\rho_0 \\
& \dots \dots \dots \\
& \leq C_l \int_{R_0}^r \int_{\rho_0}^r \rho_0^{1-d/2} \rho_1^{l-2} d\rho_1 \rho_0^{d-1} d\rho_0 \leq C \int_{R_0}^r \rho_0^{1-d/2} r^{l-1} \rho_0^{d-1} d\rho_0 \\
& = C \int_{R_0}^r \rho_0^{d/2} r^{l-1} d\rho_0 \leq C r^{d/2+l-1} (r - R_0) \\
& \leq C R^{d/2+l-1} (R - R_0) = C R^{d-1} (R - R_0),
\end{aligned}$$

where we have used  $d = 2l$  in the final equality.

The second integral is

$$\begin{aligned}
& \int_r^R \int_r^{\rho_0} \int_{\rho_1}^{\rho_0} \cdots \int_{\rho_{l-2}}^{\rho_0} (\rho_{l-1}^{2-d} - \rho_0^{2-d})^{1/2} d\rho_{l-1} \cdots d\rho_1 \rho_0^{d-1} d\rho_0 \\
& \leq \int_r^R \int_r^{\rho_0} \int_{\rho_1}^{\rho_0} \cdots \int_{\rho_{l-2}}^{\rho_0} \rho_{l-1}^{1-d/2} d\rho_{l-1} \cdots d\rho_1 \rho_0^{d-1} d\rho_0 \\
& \dots \dots \dots \\
& \leq C_d \int_r^R \int_r^{\rho_0} \rho_1^{l-1-d/2} d\rho_1 \rho_0^{d-1} d\rho_0.
\end{aligned}$$

Now, using  $l - 1 - d/2 = -1$ ,

$$(7.5) \quad = C_d \int_r^R \log\left(\frac{\rho_0}{r}\right) \rho_0^{d-1} d\rho_0 \leq C_d R^{d-1} (R - R_0) \log\left(\frac{R}{r}\right).$$

Combining (7.3), (7.4), and (7.5) gives

$$(7.6) \quad |u(r) - P_{l,A}(u)(r)| \leq C_d \frac{R^{d-1}(R - R_0)}{|A|} \|\partial_r^l u\|_{L^2(A)} \left[1 + \log\left(\frac{R}{r}\right)\right].$$

Since  $A$  is an annulus,  $R^{d-1}(R - R_0)/|A| \leq C_d$ . So

$$(7.7) \quad |u(r) - P_{l,A}(u)(r)| \leq C_d \|\partial_r^l u\|_{L^2(A)} \left[1 + \log\left(\frac{R}{r}\right)\right]$$

which is the required estimate.  $\square$

We will apply Lemma 7.2 to our main region  $\Omega = A_{R_0}^R$ , where  $R < \infty$  and  $R_0 > 0$  are, respectively, the fixed outer and inner radii of  $\Omega$ . Let  $\tilde{u}(x)$ ,  $x \in \Omega$ ,

be as in that lemma. We have that, for  $\tilde{r} \in [R_0, R]$ ,

$$|\tilde{u}(\tilde{r}) - P_{l,\Omega}\tilde{u}(\tilde{r})|^2 \leq c_d \left[ 1 + \log \left( \frac{R}{\tilde{r}} \right) \right]^{2i} \int_{R_0}^R |\partial_{\tilde{r}}^l \tilde{u}|^2 \tilde{\rho}^{d-1} d\tilde{\rho}.$$

So

$$(7.8) \quad |\tilde{u}(\tilde{r}) - P_{l,\Omega}\tilde{u}(\tilde{r})|^2 \leq \kappa_0 \int_{R_0}^R |\partial_{\tilde{r}}^l \tilde{u}|^2 \tilde{\rho}^{d-1} d\tilde{\rho}$$

where

$$\kappa_0 = \kappa_0(d, R_0, R) = c_d \left[ 1 + \log \left( \frac{R}{R_0} \right) \right]^{2i}.$$

Now consider a change of variable, replacing the domain  $\Omega = A_{R_0}^R$  with a smaller annulus contained in it,  $A$ . It is centered at the origin, with inner radius  $R_A$ , and outer radius  $R^A$ ,  $R_0 \leq R_A < R^A \leq R$ :

Let  $r = \mu\tilde{r} + \beta$ , where  $\mu := (R^A - R_A)/(R - R_0)$  and  $\beta := (R_0R^A - RR_A)/(R^A - R_A)$ . For a radially symmetric  $\tilde{u} \in H^l(\Omega)$ , let  $u(r) := \tilde{u}(\tilde{r})$ . Define

$$(7.9) \quad P_{l,A}u(r) := P_{l,\Omega}\tilde{u}(\tilde{r}).$$

Inequality (7.8) becomes

$$(7.10) \quad |u(r) - P_{l,A}u(r)|^2 \leq \kappa_0 \mu^{2l-1} \int_{R_A}^{R^A} |\partial_r^l u|^2 \tilde{\rho} d\rho,$$

where  $\rho = \mu\tilde{\rho} + \beta$ . Since  $R_0 > 0$  we have  $R_0/R \leq \tilde{\rho}/\rho \leq R/R_0$ . This and inequality (7.10) give

$$(7.11) \quad |u(r) - P_{l,A}u(r)|^2 \leq \kappa' \mu^{2l-1} \int_A |\partial_r^l u|^2 dx, \quad \kappa' := \kappa_0 (R/R_0)^{d-1}.$$

But clearly

$$\mu^{2l-1} \leq \left( \frac{c_d}{R_0^{d-1}(R - R_0)} \right)^{2l-1} |A|^{2l-1}.$$

So finally, inequality (7.11) gives

$$(7.12) \quad |u(r) - P_{l,A}u(r)|^2 \leq \kappa'' |A|^{2l-1} \int_A |\partial_r^l u|^2 dx,$$

$$\kappa'' := \kappa' \left( \frac{c_d}{R_0^{d-1}(R - R_0)} \right)^{2l-1},$$

for all radially symmetric  $u \in H^l(A)$ . We will use this result as the basis for establishing a radial analogue of Theorem 5.1. Before we proceed with that, we need a lemma on functions of sets.

Let  $\mathcal{J}$  be a non-negative function of half-open annuli  $A \subseteq \Omega$  (always taken to be centered at the origin), which is super-additive. That is, if an annulus  $A$  is

partitioned into finitely many annuli  $\{A_j\}$ , then  $\sum_j \mathcal{J}(A_j) \leq \mathcal{J}(A)$ . The  $\mathcal{J}$  that we are interested in is

$$\mathcal{J}(A) := \int_A V(x) \left[ 1 + \log \left( \frac{R}{|x|} \right) \right]^{2i} dx.$$

For a partition  $\Xi = \{A_j\}$  of  $\Omega$  into (half-open) annuli define

$$G(\mathcal{J}, \Xi) := \max_{A \in \Xi} |A|^{2l-1} \mathcal{J}(A).$$

Then, by Theorem 1.5 in [7], we have

LEMMA 7.3. *For any natural number  $n$  there exists a partition  $\Xi$  of  $\Omega$  into (half-open) annuli such that*

$$\text{card}(\Xi) \leq n \quad \text{and} \quad G(\mathcal{J}, \Xi) \leq C(l, d)n^{-2l} \mathcal{J}(\Omega).$$

Its proof is given in Section 2.2 of that reference. Let  $\Xi$  be the partition guaranteed by Lemma 7.3. For a set  $A$  let  $\chi_A$  denote its characteristic function. Define

$$K_{\Xi, l} := \sum_{A \in \Xi} \chi_A P_{l, A},$$

the operator of piecewise-polynomial approximation. Note that

$$\text{rank}(K_{\Xi, l}) \leq l \cdot \text{card}(\Xi) \leq ln.$$

THEOREM 7.4. *With the above notation we have*

$$(7.13) \quad \int_{\Omega} V(x) |u - K_{l, \Xi} u|^2 dx \leq \kappa n^{-2l} \mathcal{J}(\Omega) \int_{\Omega} |\partial_r^l u|^2 dx.$$

where  $\kappa = \kappa(d, l, R_0, R)$ .

PROOF. First

$$\begin{aligned} \int_{\Omega} V(x) |u - K_{l, \Xi} u|^2 dx &= \sum_{A \in \Xi} \int_A V(x) |u - P_{l, A} u|^2 dx \\ &\leq \sum_{A \in \Xi} \sup_{x \in A} \frac{|u(x) - P_{l, A} u(x)|^2}{[1 + \log(R/|x|)]^{2i}} \cdot \int_A V(x) [1 + \log(R/|x|)]^{2i} dx \\ &\leq \sum_{A \in \Xi} \sup_{x \in A} |u(x) - P_{l, A} u(x)|^2 \cdot \mathcal{J}(A), \end{aligned}$$

where we have used  $[1 + \log(R/|x|)]^{2i} \geq 1$  in the last inequality. So, by equation (7.12),

$$\begin{aligned} \int_{\Omega} V(x) |u - K_{l, \Xi} u|^2 dx &\leq \kappa'' \sum_{A \in \Xi} \int_A |\partial_r^l u|^2 dx \cdot |A|^{2l-1} \cdot \mathcal{J}(A) \\ &\leq G(\mathcal{J}, \Xi) \cdot \kappa'' \cdot \sum_{A \in \Xi} \int_A |\partial_r^l u|^2 dx \leq \kappa n^{-2l} \mathcal{J}(\Omega) \int_{\Omega} |\partial_r^l u|^2 dx, \end{aligned}$$

where we used Lemma 7.3 in the last line.  $\square$



On the space  $H_r = H_r(\Omega)$  endowed with the norm  $(\int_{\Omega} |\nabla^l u|^2 dx)^{1/2}$  we consider the quadratic form

$$b_V(u) := \int_{\Omega} V(x)|u(x)|^2 dx,$$

where  $V(x)$  is radial and integrable when weighted with the weight

$$[1 + \log(R/|x|)]^{2i}.$$

If  $b_V$  is bounded on  $H_r$ , then it generates a bounded, self-adjoint, non-negative operator on  $H_r$ , which we will denote by  $T_V$ . By definition, for  $f \in H_r$ ,

$$u = T_V f \Leftrightarrow u \in H_r; \quad \int_{\Omega} \nabla^l u \cdot \nabla^l w dx = \int_{\Omega} V u w dx, \quad \text{for all } w \in H_r.$$

As before we let

$$n(\lambda; T_V) = N(-\lambda; -T_V) = \# \{\text{eigenvalues of } T_V \text{ that are } \geq \lambda\}.$$

**THEOREM 7.5.** *Let  $V(x)$  and  $\Omega$  be as above. Then the operator  $T_V$  is compact on  $H_r$ , and there exists a constant  $C_4 = C_4(\Omega)$  such that, for any  $\lambda > 0$ ,*

$$n(\lambda; T_V)^{2l} \leq C_4 \lambda^{-1} \int_{\Omega} V(x) [1 + \log(R/|x|)]^{2i} dx.$$

**PROOF.** We will actually prove an upper bound on for the eigenvalues of  $T_V$  and use this to derive the estimate for  $n(\lambda; T_V)$ . Fix a natural number  $n$ . By Theorem 7.4, there exists a partition  $\Xi$  of  $\Omega$  into smaller annuli such that  $\text{card}(\Xi) \leq n$  and for any radially symmetric  $u \in H^l(\Omega)$  the estimate (7.13) holds. Let  $\mathcal{F} := \ker(K_{\Xi, l})$ . We have

$$\text{codim}(\mathcal{F}) = \text{rank}(K_{\Xi, l}) \leq ln.$$

For  $u \in \mathcal{F}$  we compute

$$\begin{aligned} \langle T_V u, u \rangle &= b_V(u) = \int_{\Omega} V(x)|u|^2 dx \\ &= \int_{\Omega} V(x)|u - K_{\Xi, l} u|^2 dx \leq \kappa n^{-2l} \mathcal{J}(\Omega) \int_{\Omega} |\nabla^l u|^2 dx, \end{aligned}$$

where Theorem 7.4 was applied in the last line. Thus by taking  $n \rightarrow \infty$  we see that  $T_V$  is the norm-limit of finite rank operators, hence compact. Also

$$(7.14) \quad \max_{u \in \mathcal{F}} \frac{\langle T_V u, u \rangle_{H_0^l(\Omega)}}{\langle u, u \rangle_{H_0^l(\Omega)}} = \max_{u \in \mathcal{F}} \frac{\int_{\Omega} V(x)|u|^2 dx}{\int_{\Omega} |\nabla^l u|^2 dx} \leq \kappa n^{-2l} \cdot \mathcal{J}(\Omega).$$

So, by Courant's Minimax Theorem for general symmetric compact operators, we have

$$\lambda_{nl+1} \leq \kappa n^{-2l} \cdot \mathcal{J}(\Omega),$$

where  $\lambda_j$  denotes the  $j$ -th of the positive operator  $T_V$  in  $H_r$ . Hence there is a constant  $C = C(\Omega)$  such that

$$(7.15) \quad \lambda_n \leq C(\Omega)n^{-2l} \cdot \mathcal{J}(\Omega), \quad n = 1, 2, \dots$$

To prove the required estimate we proceed as follows:

$$\begin{aligned} n(\lambda; T_V) &= \#\{n : \lambda_n \geq \lambda\} \leq \#\{n : C(\Omega)\mathcal{J}(\Omega)n^{-2l} \geq \lambda\} \\ &= \#\{n : n \leq C(\Omega)^{1/2l} \mathcal{J}(\Omega)^{1/2l} \cdot \lambda^{-1/2l}\} \leq C(\Omega)^{1/2l} \mathcal{J}(\Omega)^{1/2l} \cdot \lambda^{-1/2l}. \end{aligned}$$

Raising both sides to the power  $2l$  gives the result.  $\square$

PROOF OF PROPOSITION 1.6. To finish the proof of Proposition 1.6 we again use the Birman–Swchinger Principle with  $\mathcal{H} = L^2_{\text{rad}}(\Omega)$ ,  $D(a) = \tilde{D}(a) = H_r$ ,  $a(u) = \int_{\Omega} |\nabla^l u|^2 dx$ ,  $b(u) = \int_{\Omega} V|u|^2 dx$ , and  $B = T_V$  to obtain

$$N(0; (-\Delta)^l - V)^{2l} = n(1; T_V)^{2l} \leq C_{11}\mathcal{J}(\Omega). \quad \square$$

### 8. Problem (H), the radial problem with Hardy potential

Finally, we consider problem (H). That is we seek weak radial solutions to

$$(H) \quad \begin{cases} (-\Delta)^l u + \frac{\bar{b}}{|x|^{2l}} u = g(x, u) + \varphi(x) & \text{in } B_R(0), \\ \left(\frac{\partial}{\partial \nu}\right)^j u \Big|_{\partial B_R(0)} = 0 & \text{for } j = 0, \dots, l-1, \end{cases}$$

where  $\bar{b} > 0$ , and  $g$  and  $\varphi$  are radially symmetric in their dependence on  $x$ . Notice that the Hardy exponent is the critical one. That is the corresponding Hardy inequality in  $H_0^l(B_R)$  does not hold in general. Define the space  $\mathcal{C}_r$  as the set of  $u \in C_0^\infty(B_R \setminus \{0\})$  with  $u(x) = u(|x|)$  almost everywhere.

Let  $\mathcal{H}_r :=$  the closure of  $\mathcal{C}_r$  in the norm

$$(8.1) \quad \|u\|_{\mathcal{H}_r}^2 := \int_{B_R} |\nabla^l u|^2 dx + \bar{b} \int_{B_R} \frac{|u|^2}{|x|^{2l}} dx.$$

Note that clearly  $\mathcal{H}_r \hookrightarrow H_0^l(B_R)$  continuously, something to keep in mind. We will prove the existence of solutions by looking for critical points of the functional

$$I_1(u) := \frac{1}{2} \int_{B_R} |\nabla^l u|^2 dx + \frac{\bar{b}}{2} \int_{B_R} \frac{|u|^2}{|x|^{2l}} dx - \int_{B_R} G(x, u) dx - \int_{B_R} \varphi u dx$$

on  $\mathcal{H}_r$ , where  $G(x, u) = \int_0^u g(x, t) dt$ . Because  $g$  and  $\varphi$  are radially symmetric in their explicit dependence on  $x$ , the critical points of this functional on  $\mathcal{H}_r$  are also critical points on  $H_0^l(B_R)$ . Hence they are generalized solutions for (H). Since  $G(x, u)$  has the same growth restrictions in this problem as it did in (P) all conditions in Theorem 2.1 and in Tanaka’s Theorem follow exactly the same reasoning as they did earlier. Since  $\bar{b} > 0$  Palais–Smale sequences are bounded in the norm of  $\mathcal{H}_r$ , instead of merely in the norm of  $H_0^l$ . In any case the argument

proceeds as before and we need to contradict the possibility that the minimax values  $c_n$  of the unperturbed functional satisfy equation (3.30)

$$c_n \leq An[\ln(n)]^{1/\alpha_1} \quad \text{for } n > n_0.$$

Earlier in Section 4 we define the smoother functional  $J(u)$  by equation (4.1), and its minimax levels

$$b_n := \inf_{g \in \Gamma_n} \sup_{u \in g(E_n)} J(u).$$

We do exactly the same thing here, except  $J(u)$  is now defined as

$$J(u) = \frac{1}{2} \int_{B_R} |\nabla^l u|^2 dx + \frac{\bar{b}}{2} \int_{B_R} \frac{|u|^2}{|x|^{2l}} dx - \int_{B_R} H(u) dx.$$

That is, the form

$$\frac{1}{2} \int_{B_R} |\nabla^l u|^2 dx$$

is replaced by the form

$$\frac{1}{2} \int_{B_R} |\nabla^l u|^2 dx + \frac{\bar{b}}{2} \int_{B_R} \frac{|u|^2}{|x|^{2l}} dx.$$

Since  $J(u) \leq I_0(u)$  by construction, we have  $b_n \leq c_n$ . So it will suffice to obtain a good lower bound on  $b_n$ . By Tanaka [32], Theorem B, there exists a sequence  $u_n$  such that

- (i)  $J(u_n) \leq b_n$ ,
- (ii)  $J'(u_n) = 0$ ,
- (iii)  $n \leq \text{index}_0 J''(u_n)$ .

Therefore  $n \leq \text{index}_0 J''(u) = N_-((-\Delta)^l + b|x|^{-2l} - H''(u_n(x)))$ .

The eigenvalue estimate we apply is the following result from [17].

**LEMMA 8.1** (Laptev–Netrusov). *Consider the unbounded linear operator  $H := (-\Delta)^l + b|x|^{-2l} - V(x)$  acting on  $L^2(B_R)$ ,  $B_R$  the ball of radius  $R$  in  $\mathbb{R}^{2l}$ . Suppose that  $V(x) = V(|x|) \geq 0$  and  $V(x) \in L^1(B_R)$ . Then*

$$N_-((-\Delta)^l + b|x|^{-2l} - V(x)) \leq C(l, b) \int_{B_R} V(x) dx.$$

So

$$n \leq C \int_{B_R} H''(u_n(x)) dx \leq C \int_{B_R} (|u_n| + 1)^{2\alpha_2 - 2} e^{(u_n^2 + 1)^b} dx.$$

Since  $u_n$  is a critical point of  $J$  the last term in the above line can be controlled by  $J(u_n)$ . Like in Section 4, this gives

$$Cn[\log(n)]^\gamma \leq J(u_n) \leq b_n \leq c_n$$

for sufficiently large  $n$ , where this time  $\gamma = 2/\alpha_2 - 1$ . Thus if  $1/\alpha_1 < 2/\alpha_2 - 1$  this contradicts the alleged upper bound, and proves that  $I_1$  an unbounded sequence of radial critical points.

For more multiplicity results in a setting of radial symmetry see the paper [11]. In particular, Lemma 4.1 from [11] deals with the case where  $l = 1$  and  $d \geq 3$ , unlike this paper where we dealt with the critical case where  $d = 2l$ .

**Acknowledgements.** The author would like to thank Marcello Lucia, Mythily Ramaswamy, and Kyril Tintarev for some helpful discussions.

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*Manuscript received May 12, 2016*

*accepted May 3, 2017*

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