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ALMOST PERIODIC SOLUTIONS OF EVOLUTION EQUATIONS

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Abstract. We state existence theorems for almost periodic solutions of evolution problems, namely, quasi-autonomous problems and more generally, time dependent evolution equations. We apply these theorems firstly, to a boundary value quasilinear hyperbolic equation of first order, and secondly, to a boundary value quasi-parabolic equation.

1. Introduction

In this paper we draw general conditions of existence for almost periodic solutions of evolution problems in a real Banach space X.

Firstly, we prove (in Section 3) the existence of almost periodic solutions of the following quasi-autonomous evolution problem:

$$\text{QP}(f) \qquad \frac{du}{dt}(t) \in Au(t) + f(t), \quad t \in \mathbb{R},$$

where $f\colon \mathbb{R} \to X$ is an almost periodic function and $A\colon X \to X$ a multivalued nonlinear densely defined operator such that $A+\omega I$ is, for some $\omega > 0$, dissipative with compact resolvent.

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Secondly, we extend (in Section 4) this existence result for almost periodic solutions to abstract evolution equations (EV) governed by suitable families $(A(t))_{t\in\mathbb{R}}$ of nonlinear multivalued operators from X to X

$$\text{(EV)} \qquad \frac{du}{dt}(t) \in A(t)u, \quad t \in \mathbb{R}.$$

Finally, these theoretical results are applied to two examples of boundary value problems (Sections 5 and 6).

A continuous function $f\colon \mathbb{R} \to X$ is said to be almost periodic if for each $\varepsilon > 0$, the set of almost ε periods is relatively dense in $\mathbb{R}$. An almost ε period of f is a real number p satisfying

$$\text{(1.1)} \qquad \sup_{t\in\mathbb{R}} \|f(t+p) - f(t)\| \leq \varepsilon.$$

A subset $D \subset \mathbb{R}$ is said to be relatively dense if there exists an $l > 0$ such that any interval $[a, a+l]$ has a nonempty intersection with D.

An equivalent definition is that a continuous function f is almost periodic if and only if the set of translated functions $T_s f\colon t \mapsto f(s+t)$ is precompact in $C_{\mathrm{b}}(\mathbb{R}, X)$, the set of bounded functions from $\mathbb{R}$ to X endowed with the supremum norm.

Evidently, a periodic function is an almost periodic function. Almost periodic functions arise naturally in vibrating phenomena (as superposition of harmonics with frequencies that are not all multiple of the same fundamental frequency).

The existence theorems given in this paper complement for instance the existence of almost periodic trajectories given in the autonomous case, in [12] or in [1] and [2] (where X is additionally assumed to be a Hilbert space), associated with autonomous equations $\dot{u}(t) = Au(t)$, $t \in \mathbb{R}^+$, generating suitable semigroups of contractions. More generally, in this paper, we shall consider time dependent problems involving evolution operators $S(s,t)$.

It was underlined in [13] that in the quasi-autonomous case $\mathrm{QP}(f)$, the existence of almost periodic solutions is an open problem. We provide an answer to this question when $A+\omega I$ is, for some $\omega > 0$, dissipative with compact resolvent.

Similar problems were studied by many researchers (see for example [4], [19], [17], [14] and [3]). The closest one is [4]. But there the conditions are formulated in terms of Yosida's approximation, so it seems to be difficult to apply the result of [4] to problems, which we consider in the present paper. In the present paper, conditions are imposed on the operator itself.

We generalize also, in particular, the existence result for almost periodic solutions obtained in [20] in the non-autonomous case, to the special case where the state space X is a Hilbert space and the operator $(t,u) \mapsto A(t,u)$ is uniformly continuous in u and $(-\omega)$-dissipative in t, with $\omega > 0$.

The present approach is based on nonlinear methods in semi-groups theory: more precisely, two fundamental ingredients are used, namely, on the one hand, integral inequalities involving the evolution operator $S(s,t)$ associated with suitable time-dependent families $(A(t))_{t\in\mathbb{R}}$, as introduced in [8] and [7], and on the other hand, a general compactness result deduced from [6].

Examples studied in the two last sections underline the relevance of the abstract framework drawn out in Sections 3 and 4. In Section 6, dedicated to the boundary value quasi-parabolic equation, we will need technical arguments from the topological degree method developed in [16]. It is well known that these methods are not applicable to almost periodic problems (see [18]) but we use them only to prove that the corresponding operator is m-dissipative. Section 2 contains useful notations and preliminaries.

2. Notations and preliminaries

Let $J \subset \mathbb{R}$ be an interval. We denote by $C_{\mathrm{b}}(J,X)$ the Banach space of bounded continuous functions from $\mathbb{R}$ to X endowed with the supremum norm $\|\cdot\|_\infty$. The symbol I stands for the identity map on X. Let us start with some definitions.

Definition 2.1 (Evolution operator). Let $\Gamma = \{(s,t)\in\mathbb{R}^2 : s\le t\}$ and denote by $C(X,X)$ the set of continuous functions from X to X. Then a map $S\colon \Gamma\to C(X,X)$ is said to be an *evolution operator* on X if the following two conditions hold:

(a) For any fixed $x\in X$, the function $S(\,\cdot\,,\,\cdot\,)x$ is continuous on Γ.

(b) For all $(t,s),(s,r)\in\Gamma$, we have $S(s,t)\circ S(r,s)=S(r,t)$ and $S(t,t)=I$.

Definition 2.2 (Complete trajectory. Solution of (EV)). The continuous function $u\colon\mathbb{R}\to X$ is said to be a *complete trajectory* for the evolution operator S if, for all $s,t\in\mathbb{R}$, we have

$$s\le t \;\Rightarrow\; u(t)=S(s,t)u(s).$$

If S is the evolution operator associated with $(A(t))_{t\in\mathbb{R}}$ (see Definition 2.3 below), such a complete trajectory u is called a *solution* of (EV).

Let $(A(t))_{t\in\mathbb{R}}$ be a family of multivalued m-dissipative operators on X. Let $u_0\in X$ and $J=[a,b)$ with $-\infty<a\le b\le+\infty$, a subinterval of $\mathbb{R}$. Let us denote by $\mathrm{EV}(J,u^0)$ the evolution problem

$$\begin{cases}\dfrac{du}{dt}(t)\in A(t)u & \text{for } t\in J,\\ u(a)=u^0.\end{cases}\tag{2.1}$$

The concept of solution of $\mathrm{EV}([a,b),u^0)$ should always be considered (and in particular in the quasi-autonomuos case) as a mild solution (see for instance [8],

[6], [9], [10]). Such a solution is a continuous (uniform on compact subintervals of J) limit of time-implicit discrete schemes. In this paper, regularity assumptions on the time dependence will allow to restrict to the case of constant path subdivisions in the discrete schemes. In particular, such a solution is unique. In addition, if $J' = [c, d]$ (with $c \leq d$) is a compact subinterval of $J = [a, b)$, and u is a solution of $\mathrm{EV}(J, u^0)$, then the restriction $u|_{J'}$ is a solution of $\mathrm{EV}(J', u(c))$.

Definition 2.3 (Associated evolution operator). The evolution operator $S(s,t)u^0$ associated with $(A(t))_{t\in\mathbb{R}}$ is defined as the value at $t \geq s$ of the unique solution u of $\mathrm{EV}([s,t], u^0)$.

With assumptions given further, for $s < t$, the evolution operator $S(s,t)$ will can be described as

$$S(s,t)u^0 = \lim_n \prod_{i=1}^{n} J_{(t-s)/n}\bigg(s + i\,\frac{t-s}{n}\bigg)u^0, \quad u^0 \in X,$$

where, we have set $J_\lambda(t) = (I - \lambda A(t))^{-1}$. In the quasi-autonomous case

$$A(t)u = Au + f(t), \quad t \in \mathbb{R},\ u \in X,$$

the evolution equation (EV) will be denoted by $\mathrm{QP}(f)$ and the quasi-autonomous problem

$$\begin{cases} \dfrac{du}{dt}(t) \in Au + f(t) & \text{for } t \in J = [a,b), \\ u(a) = u^0. \end{cases} \tag{2.2}$$

will be denoted by $\mathrm{QP}(f, J, u^0)$.

Let us recall that a multivalued operator A on X is dissipative if $-A$ is accretive, namely, for all $(u, \xi_u(s)) \in A(s)$, $(v, \xi_v(s)) \in A(s)$,

$$-[u - v, -\xi_u(s) + \xi_v(t)] \leq 0, \tag{2.3}$$

where, the bracket (see [9] for instance), is defined, for $u, v \in X$, as

$$[u,v] = \lim_{\lambda\downarrow 0} \frac{\|u + \lambda v\| - \|u\|}{\lambda}.$$

An operator A on X is m-dissipative if A is dissipative and $R(I - \lambda A) = X$ for all $\lambda > 0$. It is well known that $\mathrm{QP}(f, J, u^0)$ has a unique solution for all $u^0 \in X$, when A is m-dissipative.

In the sequel, the following definition (adapted from [6]) is needed.

Definition 2.4 (The set W). Notation $\theta \in W$ will mean that for some l, there is a finite number of continuous functions $g_k\colon \mathbb{R} \to X$, for $k = 1, \ldots, l$, satisfying

$$\theta(s,t) = \sum_{k=1}^{l} \|g_k(t) - g_k(s)\|, \qquad s, t \in \mathbb{R}.$$

This set W is a subset of the set W defined in [6]. Let $\theta \in W$. Then we introduce the following three properties (see [6]).

(i) For $\omega > 0$ given, each operator $A(t) + \omega I$ is a nonlinear densely defined multivalued m-dissipative operator on X.

(ii) For all $s, t \in \mathbb{R}^+$ and all $(u, \xi_u(s)) \in A(s)$, $(v, \xi_v(t)) \in A(t)$,

$$-[u - v, -\xi_u(s) + \xi_v(t)] \le \theta(s, t) - \omega\|u - v\|. \tag{2.4}$$

(iii) For all $\lambda > 0$ and all bounded subsets $K \subseteq X$, the set

$$\bigcup_{t\ge 0} (I - \lambda A(t))^{-1}(K) \tag{2.5}$$

is relatively compact.

Definition 2.5 (Conditions $\mathcal{C}(\theta,\omega)$ and $\mathcal{CK}(\theta,\omega)$). Let $\omega > 0$. We say that $(A(t))_{t\in\mathbb{R}}$ satisfies $\mathcal{C}(\theta,\omega)$ if conditions (i), (ii) hold. We say that $(A(t))_{t\in\mathbb{R}}$ satisfies $\mathcal{CK}(\theta,\omega)$ if conditions (i), (ii) and (iii) hold.

Proposition 2.6 (Condition $\mathcal{CK}$, quasi-autonomous case). *In the quasi-autonomous case $A(t)u = Au + f(t)$, condition $\mathcal{CK}(\theta,\omega)$ holds with*

$$\theta(s, t) := \|f(s) - f(t)\|, \quad s, t \in \mathbb{R},$$

if the following three properties hold:

(a) *$A + \omega I$ is densely defined, m-dissipative with $\omega > 0$;*
(b) *for each $\lambda > 0$, the operator $J_\lambda = (I - \lambda A)^{-1}$ with $\lambda > 0$ is compact*;
(c) *$f \in C_{\rm b}(\mathbb{R}, X)$.*

Proof (see [6]). Relations (i) and (ii) of $\mathcal{CK}(\theta,\omega)$ follow immediately from assumptions. Claim (iii) of $\mathcal{CK}(\theta,\omega)$ is provided by the equivalence

$$(v = (I - \lambda A(t))^{-1} w) \;\Leftrightarrow\; (v = (I - \lambda A)^{-1}(w + \lambda f(t))).$$

and the fact that $\{\lambda f(t) : t \in \mathbb{R}^+\}$ is bounded. □

Let us recall now Bénilan's integral inequalities for mild solutions in the quasi-autonomous case, when $A + \omega I$ is densely defined, m-dissipative and $f \in \mathbb{L}_{\rm loc}(\mathbb{R}, X)$. Let u be a solution of $\mathrm{QP}(f, [s, T], u^0)$ and v be a solution of $\mathrm{QP}(g, [\tau + s, \tau + T], v^0)$ for a given $\tau \in \mathbb{R}$; then we have for $t \in [s, T)$,

$$\|u(t) - v(t+\tau)\| \le e^{-\omega(t-s)}\|u^0 - v^0\| + e^{-\omega(t-s)} \int_0^{t-s} e^{\omega\sigma} \|f(\sigma+s) - T_\tau g(\sigma+s)\|\, d\sigma,$$

where T_τ is the translation operator defined in Introduction.

In particular, with $(w_0, \widehat{w_0}) \in A$, taking $g(t) := -\widehat{w_0}$, we obtain

$$\|u(t) - w_0\| \le e^{-\omega(t-s)}\|u^0 - w^0\| + e^{-\omega(t-s)} \int_0^{t-s} e^{\omega\sigma} \|f(\sigma + s) + \widehat{w_0}\|\, d\sigma. \tag{2.6}$$

Therefore if f is bounded, it follows from the last inequality (according to $\omega > 0$, $T \to +\infty$) that u is bounded on $[s, +\infty[$.

3. The quasi-autonomous case

We restrict our attention to the quasi-autonomous case QP(f). We assume that

- $A+\omega I$ is nonlinear densely defined, m-dissipative for some $\omega > 0$.
- For all $\lambda > 0$, the operator $J_\lambda = (I-\lambda A)^{-1}$ is compact.
- $f\colon \mathbb{R} \to X$ is almost periodic.

Theorem 3.1 (Existence of an almost periodic solution). *Under the above assumptions, the quasi-autonomous equation QP(f) possesses an almost periodic solution.*

The proof of Theorem 3.1 will result from the following auxiliary results.

Proposition 3.2 (Initial sequence). *There is a strictly increasing sequence of positive real numbers $(r_k)_k$, satisfying $\lim_k r_k = +\infty$ and for all $k \in \mathbb{N}$, the real number r_k is a $1/2^k$ almost period of f, namely*

$$\|f - T_{r_k}(f)\|_\infty \leq \frac{1}{2^k}. \tag{3.1}$$

Proof. The proof follows by induction using the almost periodic assumption on f. □

Proposition 3.3 (Suitable shift). *Fix $u^0 \in X$. Let $u(t) = S(0,t)u^0$ for $t \geq 0$ be the solution of* QP$(0,+\infty,u^0)$ *and let $(r_k)_k$ be the sequence built in Proposition* 3.2*. There is a strictly increasing sequence of positive real numbers $(p_k)_k$, satisfying*:

(a) *$(p_k)_k$ is a subsequence of $(r_k)_k$;*
(b) *$(u(p_k-m))_k$ is convergent in X for all $m \in \mathbb{N}$, with the convention that $u(p_k-m) = u^0$ if $p_k - m < 0$.*

Lemma 3.4 (Precompactness of the positive orbit). *Let $u(t) = S(0,t)u^0$ for $t \geq 0$ be the solution of* QP$(0,+\infty,u^0)$*. Then, $u(\mathbb{R}^+)$ is precompact in X.*

Proof of Lemma 3.4. The lemma a consequence of Theorem 4.1 in [6]. Nevertheless, we give here a sketch of the proof.

Let us show that for all $u^0 \in X$, the pointed family $(A+f(t),u^0)_{t\geq 0}$ satisfies $\overline{\mathrm{BV}}(\theta,\omega,\omega,0)$, (notation explained below) with $\theta(s,t) := \|f(t)-f(s)\|$ for $s,t \in [0,+\infty[$. Let $(u_n^0)_n$ be a sequence satisfying $\lim_n u_n^0 = u^0$ and $u_n^0 \in D(A)$. As claimed in the previous section, the family $(A+f(t))_{t\in\mathbb{R}}$ fulfills $\mathcal{CK}(\theta,\omega)$ with

$$\theta(s,t) := \|f(t)-f(s)\|.$$

Now, for all $n \in \mathbb{N}$, let us introduce the continuous function $f_n\colon [0,+\infty[$, which is affine on each interval $[k/(n+1),(k+1)/(n+1)]$ with $k \in \mathbb{N}$, with nodal values

$$f_n\bigg(\frac{k}{n+1}\bigg) := f\bigg(\frac{k}{n+1}\bigg), \quad k \in \mathbb{N}.$$

Since the almost periodic function f is uniformly continuous, each function f_n (for n sufficiently large) is Lipschitz on $[0, +\infty[$. Setting

$$\varepsilon_n := \sup_{|t-s|\le 1/(n+1)} \|f(t) - f(s)\| \quad \text{and} \quad \eta_n(t) := \|f(t) - f_n(t)\|, \tag{3.2}$$

we obtain for $t \ge 0$, with $k/(n+1) \le t < (k+1)/(n+1)$,

$$\begin{aligned}\eta_n(t) &\le \left\| f(t) - f\left(\frac{k}{n+1}\right)\right\| + \left\| f\left(\frac{k}{n+1}\right) - f_n(t)\right\| \\ &\le \varepsilon_n + (n+1)\left\| f\left(\frac{k+1}{n+1}\right) - f\left(\frac{k}{n+1}\right)\right\| \left(t - \frac{k}{n+1}\right) \le 2\varepsilon_n.\end{aligned}$$

Thus

$$\lim_n \sup_{t\ge 0} \eta_n(t) = 0. \tag{3.3}$$

For each $n \in \mathbb{N}$, let us define for $s, t \ge 0$,

$$A_n(t) := A + f_n(t), \qquad \theta_n(s,t) := \|f_n(t) - f_n(s)\|. \tag{3.4}$$

Then the family $(A_n(t), u_n^0)_{t\ge 0}$ satisfies $\mathrm{BV}(\theta_n, \omega, \omega)$, namely the following definition.

Definition 3.5 (Condition $\mathrm{BV}(\theta_n, \omega, \omega)$). The family $(A_n(t), u_n^0)_{t\ge 0}$ satisfies $\mathrm{BV}(\theta_n, \omega, \omega)$ if the following four conditions are fulfilled:

(a) $\mathcal{C}(\theta_n, \omega)$ (see Definition 2.5),

(b) for any $\widehat{u_n^0} \in Au^0$,

$$\limsup_{h\downarrow 0} \frac{1}{h} \|u_n(h) - u_n^0\| \le \|\widehat{u_n^0}\| < +\infty, \tag{3.5}$$

(c)

$$\theta_n(0+, 0) = \limsup_{h\downarrow 0} \frac{1}{h} \int_0^h \theta_n(\tau, 0)\, d\tau < +\infty, \tag{3.6}$$

(d)

$$V_n = \sup_{t\ge 0} \limsup_{h\downarrow 0} e^{-\omega t} \int_0^t e^{\omega\tau}\, \frac{\theta_n(\tau + h, \tau)}{h}\, d\tau < +\infty. \tag{3.7}$$

Indeed, (3.5) is provided by (2.6); (3.6) holds (by continuity and definition of θ_n) with $\theta_n(0+, 0) = 0$; (3.7) holds with $V_n := L_n/\omega$, where L_n is a Lipschitz constant of f_n.

Now, condition $\overline{\mathrm{BV}}(\theta, \omega, \omega, 0)$ (see [6]) holds for $(A + f(t), u^0)_{t\ge 0}$.

Definition 3.6 (Condition $\overline{\mathrm{BV}}(\theta, \omega, \omega, 0)$). We say that $\overline{\mathrm{BV}}(\theta, \omega, \omega, 0)$ holds for $(A(t), u^0)_{t\ge 0}$ if there is a sequence $(A_n(\cdot), x_n^0, \theta_n)_n$ fulfilling the following conditions:

(a) For each n, the pointed family $(A_n(\,\cdot\,), u_n^0)$ satisfies $\mathrm{BV}(\theta_n, \omega, \omega)$ and $\mathcal{CK}(\theta_n, \omega)$, and $\lim_n u_n^0 = u^0$.

(b) There is a sequence of functions $(\eta_n)_n$ such that

$$(3.8)\qquad -[u - v, -\xi_u(s) + \xi_v^n(t)] \le \theta(s,t) + \eta_n(s) + \eta_n(t) - \omega\|u - v\| \quad \text{and} \quad \lim_n N_{\omega,\mathbb{R}}(\eta_n) = 0,$$

for all $(u, \xi_u(s)) \in A(s)$, $(v, \xi_v^n(t)) \in A_n(t)$ and all $s, t \in \mathbb{R}^+$.

Indeed, according to Proposition 2.6, condition $\mathcal{CK}(\theta_n, \omega)$ holds for the family $(A + f_n(t))_{t\ge 0}$ introduced in (3.4). Thanks to (3.3), relation (3.8) is verified by functions η_n introduced in (3.2). Therefore, the sequence $(A + f_n(\,\cdot\,), u_n^0, \theta_n)_n$ is a $\overline{\mathrm{BV}}(\theta, \omega, \omega, 0)$ approximation of $(A + f(t), u^0)_{t\ge 0}$ in the sense given in [6]. Consequently, the claim preceding Definition 3.6 is shown to be true. Thus Theorem 4.1 in [6] can be applied: we can now conclude that $u(\mathbb{R}^+)$ is precompact in X, ending the proof of Lemma 3.4. □

Remark 3.7 (On BV and compactness conditions). When the family $(A(t), u^0)_{t\ge 0}$ of m-dissipative operators densely defined fulfills $\mathrm{BV}(\theta, \omega, \omega)$, $\overline{\mathrm{BV}}(\theta, \omega, \omega, 0)$ holds automatically for $(A(t), u_n^0)_n$ with $u_n^0 \in D(A(0))$ such that

$$\lim_n u_n^0 = u^0 \quad \text{and} \quad \eta_n \equiv 0.$$

In addition, (3.7) holds for instance if $\theta_n(s,t) = \|f_n(t) - f_n(s)\|$ with f_n Lipschitz. Conditions $\overline{\mathrm{BV}}$ and the compactness condition (2.5) on the resolvent are used in this paper only to insure the relative compactness of the positive orbit of solutions of (EV). In [4], no such condition is invoked, but almost periodicity of $t \mapsto J_\lambda(t)x$ is required for all $x \in X$.

Proof of Proposition 3.3 (Diagonal process). By the above Lemma 3.4, $u(\mathbb{R}^+)$ is precompact in X. Consequently, for all $m \in \mathbb{N}$, a strictly increasing sequence $(\sigma_m(j))_j$ of real numbers such that $(u(\sigma_m(j) - q))_k$ is convergent in X for all $q \in \{0, 1, \ldots, m\}$ can be found by induction. With this goal, we will take as $(\sigma_q(j))_j$ a suitable subsequence of $(\sigma_{q-1}(j))_j$, with the initialization $(\sigma_{-1}(k))_k := (r_k)_k$. Now the diagonal sequence $(\sigma_k(k))_k := (p_k)_k$ fulfills the announced conditions (a) and (b) of Proposition 3.3. □

Proposition 3.8 (Existence of a precompact complete trajectory). *Let $u(t) = S(t,0)u^0$ for $t \ge 0$ be a positive trajectory. There is a precompact trajectory which is complete for S.*

Proof. (a) Let $(p_k)_k$ be the sequence of Proposition 3.3. We are going to prove that there is a function $w \in C(\mathbb{R}, X)$ such that the sequence $(T_{p_k}u)_k$ converges uniformly to w on all compact subsets of $\mathbb{R}$. For all $m \in \mathbb{N}$ set $I_m := [-m, +\infty[$ and put $v_k(t) := u(p_k + t)$, for $t \ge -p_k$.

Let $m \in \mathbb{N}$. For $t \in I_m$ and $i, j \in \mathbb{N}$ with $p_i \geq m$ and $p_j \geq m$, we have

$$\begin{aligned}
\|v_i(t) - v_j(t)\| &= \|S(0, t + p_i)u^0 - S(0, t + p_j)u^0\| \\
&= \|S(0, t + m + p_i - m)u^0 - S(0, t + m + p_j - m)u^0\| \\
&\leq \|S(p_i - m, t + m + p_i - m) \circ S(0, p_i - m)u^0 \\
&\quad - S(p_j - m, t + m + p_j - m) \circ S(0, p_j - m)u^0\| \\
&\leq \|S(p_i - m, (t + m) + p_i - m)u(p_i - m) \\
&\quad - S(p_j - m, (t + m) + p_j - m)u(p_j - m)\| \\
&\leq e^{-\omega(t+m)}\|u(p_i - m) - u(p_j - m)\| \\
&\quad + e^{-\omega(t+m)} \int_0^{t+m} e^{\omega s}\|f(p_i - m + s) - f(p_j - m + s)\|\, ds \\
&\leq e^{-\omega(t+m)}\|u(p_i - m) - u(p_j - m)\| \\
&\quad + e^{-\omega(t+m)} \int_0^{t+m} e^{\omega s}\|T_{p_i} f(-m + s) - T_{p_j} f(-m + s)\|\, ds \\
&\leq e^{-\omega(t+m)}\|v_i(-m) - v_j(-m)\| \\
&\quad + e^{-\omega(t+m)} \int_0^{t+m} e^{\omega s}(\|T_{p_i} f(-m + s) - f(-m + s)\| \\
&\quad + \|f(-m + s) - T_{p_j} f(-m + s)\|)\, ds \\
&\leq e^{-\omega(t+m)}\|v_i(-m) - v_j(-m)\| + \left(\frac{1}{2^i} + \frac{1}{2^j}\right)\frac{1 - e^{-\omega(t+m)}}{\omega}.
\end{aligned}$$

We used Bénilan's inequalities and relation (3.1) (in view of $p_k \geq r_k$).

These computations and Proposition 3.3 imply that $(v_k)_k$ is a Cauchy sequence in $C_{\mathrm{b}}(I_m, X)$. Therefore, for all $m \in \mathbb{N}$, the sequence $(v_k)_k$ converges towards some w_m in $C_{\mathrm{b}}(I_m, X)$. Since $w_m(t) = w_p(t)$ whenever $m \leq p$ and $t \in I_m$, we can define $w\colon \mathbb{R} \to X$ as $w(t) := w_m(t)$, for $t \in \mathbb{R}$, where $m := m(t)$ is any positive integer satisfying $t \geq -m$.

(b) Let $\tau, t \in \mathbb{R}$ with $\tau \leq t$. By using the continuity of $x \mapsto S(\tau, t)x$, we obtain

$$\begin{aligned}
\text{(3.9)} \qquad w(t) = \lim_k S(0, t + p_k)u^0 &= \lim_k S(\tau + p_k, t + p_k) \circ S(0, \tau + p_k)u^0 \\
&= \lim_k S(\tau + p_k, t + p_k)(v_k(\tau)).
\end{aligned}$$

But, thanks to Bénilan's inequalities, we have

$$\begin{aligned}
\text{(3.10)} \qquad \|S(\tau + p_k, t + p_k)v_k(\tau) - S(\tau, t)w(\tau)\| &\leq e^{-\omega(t-\tau)}\|v_k(\tau) - w(\tau)\| \\
&\quad + e^{-\omega(t-\tau)} \int_0^{t-\tau} e^{\omega s}\|T_{p_k} f(\tau + s) - f(\tau + s)\|\, ds \\
&\leq e^{-\omega(t-\tau)}\left(\|v_k(\tau) - w(\tau)\| + \frac{1}{2^k}\frac{1 - e^{-\omega(t-\tau)}}{\omega}\right).
\end{aligned}$$

Since by definition we have $w(\tau) = \lim_k v_k(\tau)$, using (3.9) and (3.10), we then obtain

$$S(\tau,t)w(\tau) = w(t)\Big(= \lim_k S(\tau+p_k, t+p_k)(v_k(\tau))\Big).$$

Therefore w is a solution of the quasi-autonomous equation QP(f).

(c) Invoking Lemma 3.4 and the inclusion $w(\mathbb{R}) \subset \overline{u(\mathbb{R}^+)}$, we can claim that the trajectory $w(\mathbb{R})$ is relatively compact (or equivalently precompact). $\square$

End of proof of Theorem 3.1. Let us consider the function w built in Proposition 3.8. It remains to prove that w is an almost periodic function. Let $\varepsilon > 0$ and $p \in \mathbb{R}$. Set

$$M := \sup_{s\in\mathbb{R}} \|w(s)\|, \qquad \delta_p := \|T_p f - f\|_\infty.$$

Fix $\alpha > 0$ (sufficiently large) such that $4Me^{-\omega\alpha} < \varepsilon$ and with $\tau \in \mathbb{R}$, put $t := \tau - \alpha$. We have

$$\begin{aligned}\|T_p(w)(\tau) \ - w(\tau)\| &= \|S(t+p, t+p+\alpha)w(t+p) - S(t,t+\alpha)w(t)\| \\ &\leq e^{-\omega\alpha}\|w(t+p) - w(t)\| + e^{-\omega\alpha}\int_0^\alpha e^{\omega s}\|T_p f(s+t) - f(s+t)\|\,ds \\ &\leq 2Me^{-\omega\alpha} + \delta_p\,\frac{1-e^{-\omega\alpha}}{\omega} \leq \frac{\varepsilon}{2} + \delta_p\,\frac{1}{\omega}.\end{aligned}$$

Thus, each η almost period p of f, such that $\eta := \varepsilon\omega/2$, is an ε almost period of w. Consequently, the set of ε almost periods of w is relatively dense in $\mathbb{R}$. Thus w is almost periodic, ending the proof of Theorem 3.1. $\square$

4. A generalization

We are going to generalize the problem of existence of almost periodic solutions to more general families of operators than the previous ones. In this section we consider the evolution problem (EV), where, the family $(A(t))_{t\in\mathbb{R}}$ satisfies assumption $\mathcal{H}(\theta,\omega)$, constituted by the following two conditions:

- $\mathcal{CK}(\theta,\omega)$, with $\omega > 0$.
- The function θ is defined for some l by

$$\theta(s,t) = \sum_{k=1}^{l} \|g_k(s) - g_k(t)\|, \tag{4.1}$$

with $g_k \colon \mathbb{R} \to X$ almost periodic and Lipschitz for all $k = 1,\ldots,l$.

Then we have the following existence theorem.

Theorem 4.1 (Existence result). *Under assumption $\mathcal{H}(\theta,\omega)$, there exists an almost periodic solution of* (EV).

Proof. The proof is analogous to the proof in the previous section. Indeed, the operator evolution $S(s,t)$ associated with $(A(t))_{t\in\mathbb{R}}$ enjoys the same properties that the evolution operator of the quasi-autonomous case. Indeed, first we have inequalities analogous to the Bénilan's translation inequality, namely, if u is the solution of $\mathrm{EV}([s,T],u^0)$ and v is the solution of $\mathrm{EV}([\tau+s,\tau+T],v^0)$, for a given $\tau\in\mathbb{R}$, then we have for $t\in[s,T)$ (see [6] or [8] or [7])

$$\|u(t)-v(t+\tau)\|\le e^{-\omega(t-s)}\|u^0-v^0\|+e^{-\omega(t-s)}\int_0^{t-s}e^{\omega\sigma}\theta(\sigma+s,\sigma+s+\tau)\,d\sigma.$$

Precompactness of the positive orbit stated in Lemma 3.1 follows from [6] in the same way: indeed, $\mathcal{H}(\theta,\omega)$ is exactly what we have used in this lemma for checking the requirements of Theorem 4.1 in [6]. Thanks to the Lipschitz assumption (see Remark 3.7) on g_k, condition $\overline{\mathrm{BV}}$ is satisfied with $A_n(t)=A(t)$ for all n. Moreover, the new expression (4.1) for θ is not an obstruction for building suitable sequences of positive integers $(r_k)_k$ and $(p_k)_k$ as in the quasi-autonomous case: indeed, under the assumption that each g_k is almost periodic, $f:=(g_1,\dots,g_l)\colon\mathbb{R}\to X^l$ is almost periodic (see [11]) and thus, for each $\varepsilon>0$, the set of common (to all g_k) ε periods is relatively dense in $\mathbb{R}$. □

5. A quasilinear hyperbolic equation

In order to show the relevance of the general framework defined in the previous section, let us now consider the following quasilinear hyperbolic boundary value problem, set up for instance, in [10] (in the case when the time interval is bounded):

$$(\mathrm{BVP1})=\begin{cases}v_t=-\varphi(v)_x-\omega v & \text{for } (t,x)\in\mathbb{R}\times[0,1],\\ v(t,0)=g(t) & \text{for } t\in\mathbb{R}.\end{cases}$$

Under suitable assumptions, we are going to prove the existence of an almost periodic solution of this problem.

Assumption (HBVP1). The function g is here assumed to be almost periodic on $\mathbb{R}$, and φ to be (continuous) strictly increasing with $\varphi(\mathbb{R})=\mathbb{R}$, such that $\varphi\circ g$ is Lipschitz on $\mathbb{R}$.

Take $X=\mathbb{L}^1([0,1])$ endowed with its usual norm denoted by $\|\cdot\|_1$. Set $S':=[0,1]$. Define for all t, the family $(A(t))_{t\in\mathbb{R}}$ as follows:

$$\begin{cases}A(t)v=-\varphi(v)_x-\omega v,\\ D(A(t))=\{v\in X: v(0)=g(t) \text{ and } \varphi(v) \text{ is absolutely continuous on } S'\}.\end{cases}$$

We will prove further that for $v \in D(A(s))$ and $w \in D(A(t))$, we have

$$\begin{aligned}(5.1)\qquad -[v - w, -A(s)v + A(t)w] &= |\varphi(g(s)) - \varphi(g(t))| - |\varphi(v(1)) - \varphi(w(1))| - \omega \int_0^1 |v(x) - w(x)|\,dx \\ &\leq |\varphi(g(s)) - \varphi(g(t))| - \omega \int_0^1 |v(x) - w(x)|\,dx.\end{aligned}$$

As it was claimed in [10], $A(t) + \omega I$ is, for almost all $t \geq 0$, m-dissipative with dense domain.

Let us show that condition $\mathcal{CK}(\theta, \omega)$ holds for $(A(t))_{t\in\mathbb{R}}$, with

$$(5.2)\qquad \theta(s,t) = |\varphi(g(t)) - \varphi(g(s))| = \|\varphi(g(t)) - \varphi(g(s))\|_1.$$

We underline that for fixed s, t, the function $|\varphi(g(t)) - \varphi(g(s))|$ is constant in $x \in [0,1]$, justifying the last equality.

(a) First in order to prove (5.1), we are going to use the computation of the bracket in $X = \mathbb{L}^1([0,1])$, namely

$$[u, v] = \max_{u^* \in J(u)} \int_0^1 u^* v\,dx,$$

where $u, v \in X$ and the multivalued duality map operator J is the sign operator defined as

$$J(u) = \{u^* \in \mathbb{L}^\infty([0,1]) : |u^*| \leq 1,\ u^* u = |u|\}.$$

Set $u := v - w$ and $y := \varphi(v) - \varphi(w)$. We have $J(u) = J(y)$ since φ is strictly increasing. We then obtain

$$\begin{aligned}[u, (\varphi(v) - \varphi(w))_x] &= \max_{u^* \in J(u)} \int_0^1 u^* (\varphi(v) - \varphi(w))_x\,dx \\ &= \max_{u^* \in J(y)} \int_0^1 u^* y_x\,dx = \int_0^1 \frac{d}{dx}|y|\,dx = |y(1)| - |y(0)|.\end{aligned}$$

We have used that $|y|$ is absolutely continuous since y is; in particular, we have

$$\frac{d}{dx}|y| = \frac{dy}{dx} = 0$$

almost everywhere on the set $\{y = 0\}$ (in other words the set $\{v = w\}$). Now it is easy to prove (5.1).

(b) Let us prove the precompactness condition on the resolvent operators. Consider the dissipative operator $B(t) := A(t) + \omega I$. First, let us remark that for $\lambda > 0$,

$$(v = (I - \lambda A(t))^{-1} u) \ \Leftrightarrow\ (v = (I - \lambda B(t))^{-1}(u - \lambda\omega v)).$$

Consequently, since $(I+\lambda A(t))^{-1}$ is contractive, we have to prove that for each bounded subset $K \subset X$, the set

$$\bigcup_{t\in\mathbb{R}} (I-\lambda B(t))^{-1}(K)$$

is relatively compact. With this goal in sight, let $K \subset X$ be a bounded set. The relationship

$$v = (I-\lambda B(t))^{-1}(u), \quad u \in K,$$

is equivalent to the following one:

$$\begin{cases} v + \lambda\varphi(v)' = u & \text{for } u \in K, \\ v(0) = g(t) & \text{for } \varphi(v) \in \mathrm{AC}(S'), \end{cases}$$

where $\mathrm{AC}(S')$ denotes the set of absolutely continuous functions on S'.

Set $w := \varphi(v)$. We obtain

$$\begin{cases} w' + \dfrac{1}{\lambda}\,\varphi^{-1}(w) = \dfrac{u}{\lambda}, \\ w(0) = \varphi(g(t)), \end{cases}$$

Since the operator $w \mapsto \varphi^{-1}(w)/\lambda$ is m-accretive continuous on $\mathbb{R}$, the above equation has a unique (continuous) mild solution w, (which is a strong solution and is) given by

$$w(x) = \varphi(g(t)) - \int_0^x \frac{1}{\lambda}\,\varphi^{-1}(w)(\xi)\,d\xi + \int_0^x \frac{u(\xi)}{\lambda}\,d\xi, \tag{5.3}$$

for $x \in S'$. Denote by $W_0 \subset X$ the set of functions w satisfying (5.3) with $u \in K$. We only need to prove that the set $V := \{v = \varphi^{-1}(w) : w \in W_0\}$ is precompact in X. This will be done in four steps.

(i) V is bounded in X, since dissipativity of $B(t)$ gives $\|v\|_1 \le \|u\|_1$.

(ii) Owing to (5.3), a consequence of (i) is that

$$\|w\|_\infty \le C + \frac{2}{\lambda}\,\|u\|_1 := L, \tag{5.4}$$

where we have set $C := \max(|\varphi(-\|g\|_\infty)|, |\varphi(\|g\|_\infty)|)$. Therefore W_0 is bounded in X.

(iii) Putting

$$z(\xi) := -\frac{1}{\lambda}\,\varphi^{-1}(w)(\xi) + \frac{u(\xi)}{\lambda},$$

we obtain for any $h \in \mathbb{R}$,

$$\|T_h w - w\|_1 \le \int_0^1 \int_0^{|h|} |z(x+\xi)|\,d\xi\,dx,$$

where, we have set $w(\xi) = 0$ for $\xi \notin S'$. As a consequence

$$\lim_{h\to 0} \sup_{w\in W_0} \|T_h w - w\|_1 = 0,$$

which proves that W_0 is relatively compact because it is bounded.

(iv) The map $\Psi\colon W_0 \to X$ defined as $\Psi(w) := \varphi^{-1}(w)$ is continuous on the compact subset $\overline{W_0}$. Let us prove it by contradiction. Suppose that Ψ is not continuous at $w_* \in \overline{W_0}$. Then, there exist $\varepsilon > 0$ and a sequence $(w_n)_n$ with

$$w_n \in W_0, \quad \lim_n \|w_n - w_*\|_1 = 0, \quad \|\varphi^{-1}(w_n) - \varphi^{-1}(w_*)\|_1 \geq \varepsilon. \tag{5.5}$$

These conditions imply that there is a subsequence $(w_{n_k})_k$ such that

$$\lim_n w_{n_k}(x) = w_*(x) \quad \text{a.a. } x \in S'.$$

Invoking (5.4), we finally deduce

$$\lim_n \Psi(w_{n_k})(x) = \Psi(w_*(x)) \quad \text{a.a. } x \in S' \quad \text{and} \quad |\Psi(w_{n_k})(x)| \leq L \quad \text{a.a. } x \in S'.$$

As L is an integrable function on S, the Lebesgue dominated convergence theorem yields

$$\lim_n \|\Psi(w_{n_k}) - \Psi(w_*)\|_1 = 0,$$

which contradicts (5.5).

In conclusion, Ψ is continuous and thus $V = \Psi(W_0) \subset \Psi(\overline{W_0})$ is relatively compact. In this, verification of $\mathcal{CK}(\theta,\omega)$ is fully completed. In addition, because g is almost periodic, $\varphi \circ g$ is almost periodic in $\mathbb{R}$ (see Proposition 4.2.5. in [12]) and, consequently, $\varphi \circ g$ as a function $t \mapsto \varphi \circ g(t)$, with values in X (by assimilating the constant $\varphi \circ g(t)$ with the corresponding constant function in X), is almost periodic. We then conclude that assumption $\mathcal{H}(\theta,\omega)$ is checked and applying Theorem 4.1 we can claim that there is an almost periodic solution of (BVP1). □

6. A quasi-parabolic equation

Consider now the following quasiparabolic equation with boundary value problem:

$$(\text{BVP2}) = \begin{cases} v_t = \varphi(v)_{xx} - \omega v & \text{for } (t,x) \in \mathbb{R} \times [0,1], \\ (\varphi(v))_x(t,0) = g_0(t) & \text{for } t \in \mathbb{R}, \\ (\varphi(v))_x(t,1) = g_1(t) & \text{for } t \in \mathbb{R}. \end{cases}$$

Let us remark that this boundary value problem is different from Example 4.9 in [15, p. 112] since the boundary conditions do not coincide.

We are going to show that in this application, condition $\mathcal{H}(\theta,\omega)$ holds again (with suitable assumptions on the data) and then that there exists an almost periodic solution.

Assumption (HBVP2). We assume that g_0 and g_1 are almost periodic and Lipschitz on $\mathbb{R}$, and $\varphi \in C^2(\mathbb{R})$ is strictly increasing with $\varphi(\mathbb{R}) = \mathbb{R}$. Set $S' = [0, 1]$ and consider here again $X = \mathbb{L}^1(S')$. Define for all t, the family $(A(t))_{t\in\mathbb{R}}$ as follows:

$$\begin{cases} A(t)v = \varphi(v)_{xx} - \omega v, \\ D(A(t)) = \{v \in X : (\varphi(v))_x(0) = g_0(t),\, (\varphi(v))_x(1) = g_1(t) \\ \qquad\qquad \text{and } \varphi(v)_x \text{ is absolutely continuous on } S'\}. \end{cases}$$

Let us introduce the function

$$\theta(s,t) := |g_1(s) - g_1(t)| + |g_0(s) - g_0(t)|, \quad s, t \in \mathbb{R}. \tag{6.1}$$

Taking into account (5.2) and assumption (HBVP2), we remark that θ satisfies (4.1). So, in order to check $\mathcal{H}(\theta, \omega)$, we just need to prove that $\mathcal{CK}(\theta, \omega)$ holds.

Step 1. The bracket condition. Let us begin by establishing the following estimate:

$$-[u - v, -A(s)u + A(t)v] \le \theta(t,s) - \omega\|u - v\|_1.$$

Let $u \in D(A(s))$, $v \in D(A(t))$ and $\lambda > 0$. Then for every monotone increasing Lipschitz continuous function $p\colon \mathbb{R} \to \mathbb{R}$ satisfying $|p| \le 1$ with $p(0) = 0$, we have

$$\begin{aligned} &\int_0^1 (\varphi(u) - \varphi(v))'' p(\varphi(u) - \varphi(v))\,dx \\ &= (\varphi(u) - \varphi(v))' p(\varphi(u) - \varphi(v))\big|_0^1 - \int_0^1 [\varphi(u)' - \varphi(v)'t]^2 p'(\varphi(u) - \varphi(v))\,dx \\ &\le |g_1(s) - g_1(t)||p(\varphi(u(1)) - \varphi(v(1)))| + |g_0(s) - g_0(t)||p(\varphi(u(0)) - \varphi(v(0)))| \\ &\le |g_1(s) - g_1(t)| + |g_0(s) - g_0(t)|. \end{aligned}$$

Since $|p| \le 1$, we obtain

$$\begin{aligned} &\int_0^1 |u - v\, - \lambda(A(s)u - A(t)v)|\,dx \\ &\qquad = \int_0^1 |(1 + \lambda\omega)(u - v) - \lambda(\varphi(u)'' - \varphi(v)'')|\,dx \\ &\qquad \ge \int_0^1 (1 + \lambda\omega)(u - v)p(\varphi(u) - \varphi(v))\,dx \\ &\qquad\quad - \lambda \int_0^1 (\varphi(u)'' - \varphi(v)'')p(\varphi(u) - \varphi(v))\,dx \\ &\qquad \ge (1 + \lambda\omega) \int_0^1 (u - v)p(\varphi(u) - \varphi(v))\,dx - \lambda\theta(s,t). \end{aligned}$$

Replacing p by p_n defined by

$$p_n(s) := \begin{cases} ns & \text{if } |s| \le 1/n, \\ \operatorname{sign} s & \text{if } |s| > 1/n, \end{cases}$$

and letting $n \to \infty$, since $(u-v)p_n(\varphi(u) - \varphi(v)) \to |u-v|$, we obtain

$$\|u - v - \lambda(A(s)u - A(t)v)\|_1 \ge \|u-v\|_1 + \lambda(\omega\|u-v\|_1 - \theta(s,t))$$

or

$$\lambda^{-1}(\|u - v - \lambda(A(s)u - A(t)v)\|_1 - \|u-v\|_1) \ge \omega\|u-v\|_1 - \theta(s,t).$$

Thus

$$-[v-u, -A(s)v + A(t)u] \le \theta(t,s) - \omega\|v-u\|_1.$$

Step 2. The m-dissipativity condition. Let $B(t) := A(t) + \omega I$, as in the example of Section 5, let us now prove that $B(t)$ is m-dissipative. Since $B(t)$ is dissipative, we need just to prove that

$$R(I - \lambda B(t)) = \mathbb{L}^1(S'), \tag{6.2}$$

for all $\lambda > 0$. It is actually sufficient (see Miyadera [15, p. 22]) to show that (6.2) holds at least for a certain $\lambda > 0$.

So set $\lambda = 1$. As in the previous section, we denote by $\mathrm{AC}(S')$ the set of absolutely continuous functions on $S' = [0,1]$. Let an arbitrary $H \in \mathbb{L}^1(S')$ be given. In order to solve $u - B(t)u = H$, we introduce $v = \varphi(u)$ and look for a function v which satisfies $v_x \in \mathrm{AC}(S')$ and

$$\begin{cases} -v'' + \varphi^{-1}(v) = H, \\ v'(0) = g_0(t), \\ v'(1) = g_1(t). \end{cases} \tag{6.3}$$

The function

$$k(x) := \frac{1}{2}\,(x^2 g_1(t) - (1-x)^2 g_0(t)) \tag{6.4}$$

obviously satisfies the zero boundary conditions. Then v is a solution of (6.3) if and only if $z := v - k$ satisfies the following linear boundary value problem:

$$\begin{cases} -z'' + \varphi^{-1}(z) = h, \\ z'(0) = 0, \\ z'(1) = 0 \end{cases} \tag{6.5}$$

with $h := H + g_0 - g_1$.

Let us introduce $A_0 = A_0(t)$ the (linear) operator $A(t)$ with zero boundary conditions (this operator does not depend on t) and $B_0 = A_0 + \omega I$. If z is a solution of (6.5), set $z := \varphi(w)$. The dissipativity of A_0 gives

(6.6) $$\|\varphi^{-1}(z)\|_1 = \|w\|_1 \leq \|w - 0 + 1 \cdot (A_0 w - A_0 0)\|_1 = \|\varphi^{-1}(z) - z''\|_1 = \|h\|_1.$$

With the help of Index Theory let us now establish that (6.5) has a solution. Recall that Index Theory is not applicable to almost periodic problems and we use it only to prove that the operator $A(t)$ is maximal for each t. Let us recall the following known properties (see [16]).

Proposition 6.1 (Abstract linear decomposition). *Let* $L\colon \mathbb{L}^1(S') \to \mathbb{L}^1(S')$ *be a linear operator with domain* $D(L)$. *Let* $P\colon \mathbb{L}^1(S') \to \mathbb{L}^1(S')$ *be a linear projection operator satisfying* $\operatorname{Im} P = \ker L$. *Then:*

(a) *the restriction operator* $L_P\colon D(L) \cap \ker P \to \operatorname{Im} L$, *given as*

$$L_P(z) = L(z), \quad \text{for all } z \in D(L) \cap \ker P,$$

is a linear isomorphism;

(b) *the operator* $K_P\colon \operatorname{Im} L \to D(L) \cap \ker P$, *given as*

$$K_P := L_P^{-1}, \quad \text{for all } z \in \operatorname{Im} L,$$

satisfies $K_P \circ Lz := z - Pz$ *for all* $z \in D(L)$.

Proposition 6.2 (Linear Fredholm operator of zero index). *Let* $L\colon D(L) \to \mathbb{L}^1(S')$ *be a linear Fredholm operator of zero index such that* $\operatorname{Im} L$ *is a closed subspace of* $\mathbb{L}^1(S')$. *Then:*

(a) *there exist linear continuous projection operators* $P, Q\colon \mathbb{L}^1(S') \to \mathbb{L}^1(S')$ *satisfying* $\operatorname{Im} P = \ker L$ *and* $\operatorname{Im} L = \ker Q$;

(b) *the canonical projection* $\Pi\colon \mathbb{L}^1(S') \to \mathbb{L}^1(S')/\operatorname{Im} L$, *given by*

$$\Pi y = y + \operatorname{Im} L,$$

is a continuous linear operator;

(c) *there exists a continuous linear isomorphism* $\Lambda\colon \operatorname{coker} L \to \ker L$;

(d) *the equation* $Lx = y$, *for* $y \in \mathbb{L}^1(S')$, *is equivalent to*

$$(I - P)x = (\Lambda\Pi + K_{P,Q})y,$$

where I *is the identity in* $\mathbb{L}^1(S')$ *and the operator* $K_{P,Q}\colon \mathbb{L}^1(S') \to \mathbb{L}^1(S')$ *is given by the relation*

$$K_{P,Q}(y) = K_P(y - Qy).$$

In our application, we are concerned with the case

(6.7) $$\begin{cases} Lz = z'', \\ D(L) = \{z \in \mathbb{L}^1(S') : z'' \in \mathbb{L}^1(S'),\ z'(0) = z'(1) = 0\}. \end{cases}$$

It is well known that L is a Fredholm operator of zero index. One can find in the literature a proof of this fact in the case of $\mathbb{L}^2(S')$ but not in $\mathbb{L}^1(S')$. Let us give a sketch of proof for the $\mathbb{L}^1$ case.

(6.7) implies $Lz = 0$ if and only if $z \equiv \text{const}$. Hence $\dim \ker L = 1$.

Let us show that $\mathbb{L}^1(S') = \operatorname{Im} L \oplus \ker L$. We can represent an arbitrary $z \in \mathbb{L}^1(S')$ as $z = z_1 + z_2$ where $z_1 \in \ker L$ and $z_2 \in \operatorname{Im} L$ with $z_1 \equiv \int_0^1 v\,dx$ and $z_2 = z - z_1$. Routine computations show that

$$\operatorname{Im} L = \left\{ h \in \mathbb{L}^1(S') : \int_0^1 h(s)\,ds = 0 \right\}.$$

Hence $\operatorname{Im} L \cap \ker L = \{0\}$ and $\dim \operatorname{coker} L = 1$. Thus, in our case

$$Pz = Qz = \int_0^1 z(s)\,ds$$

and

$$\Lambda \overline{z} = \int_0^1 \overline{z}(s)\,ds \quad \textit{with} \quad \overline{z} \in \operatorname{coker} L.$$

Consider the equation $y - Qy = Lz$ ($z \in D(L) \cap \ker P$), namely

$$y - \int_0^1 y(s)\,ds = z'' \quad \text{with } z'(0) = z'(1) = 0,$$

where $z \in D(L) \cap \ker P$. Solving the last one, we obtain the following form for the operator: $K_{P,Q} \colon \mathbb{L}^1(S') \to \mathbb{L}^1(S')$:

$$K_{P,Q}(y)(t) = \int_0^1 \left(\max(t,s) - \frac{t^2 + s^2}{2} - \frac{1}{3} \right) y(s)\,ds,$$

establishing its compactness. For $V \subset \mathbb{L}^1(S')$, set $\operatorname{Coin}(L,F,V) := \{z \in V : Lz = F(z)\}$. The following theorem is proved in [16] .

Theorem 6.3 (Degree theory). *Let U be an open bounded subset of $\mathbb{L}^1(S')$. Let $F \colon \overline{U} \to \mathbb{L}^1(S')$ be such that the maps ΠF and $K_{P,Q}F$ are compact and the following conditions are fulfilled*:

(a) *$Lz \neq \mu F(z)$ for all $\mu \in (0,1]$, $z \in D(L) \cap \partial U$;*
(b) *$0 \neq \Pi F(z)$ for all $z \in \ker L \cap \partial U$;*
(c) *$\deg_{\ker L}(\Lambda \Pi F|_{\overline{U}_{\ker L}}, \overline{U}_{\ker L}) \neq 0$, where the symbol $\deg_{\ker L}$ means the topological degree evaluated in the space $\ker L$ and $\overline{U}_{\ker L} = \overline{U} \cap \ker L$.*

Then we have $\emptyset \neq \operatorname{Coin}(L,F,\overline{U}) \subset U \cap D(L)$.

We apply this theorem with the operator L given in (6.7) and F defined by

$$F(z) = \varphi^{-1}(z) - h.$$

The choice of U will be done further.

Let us already notice that, in view of the definitions, the conclusion of the theorem, namely $\operatorname{Coin}(L,F,\overline{U})$, implies that (6.5) has a (unique) solution.

Let us check in our application condition (a) of Theorem 6.3. Suppose that there exists $z_\mu \in D(L)$ such that we have

$$Lz_\mu = z''_\mu = \mu F(z) = \mu(\varphi^{-1}(z) - h).$$

Then, according to (6.6),

$$\|\varphi^{-1}(z_\mu)\|_1 \le \|h\|_1,$$

the following inequalities hold:

$$|z'_\mu(x)| \le \int_0^1 |z''_\mu(s)|\, ds = \mu \int_0^1 |\varphi^{-1}(z) - h|\, ds \le 2\|h\|_1, \tag{6.8}$$

$$|z_\mu(x)| \le \left| z_\mu(0) + \int_0^x z'_\mu(s)\, ds \right| \le |z_\mu(0)| + 2\|h\|_1. \tag{6.9}$$

The set $\{z_\mu(0) : \mu \in (0,1]\} \subset \mathbb{R}$ is bounded. Indeed, by contradiction, without loss of generality, suppose $z_\mu(0) \to +\infty$. As far as $\max\limits_{x\in[0,1]} |z'_\mu(x)| \le 2\|h\|_1$, we have $\min\limits_{x\in[0,1]} z_\mu \to +\infty$. As φ is strictly monotonous, we obtain

$$\varphi^{-1}\Big(\min_{x\in[0,1]} z_\mu\Big) \to +\infty,$$

in contradiction with

$$\|h\|_1 \ge \|\varphi^{-1}(z_\mu)\|_1 = \int_0^1 \varphi^{-1}(z_\mu(s))\, ds \ge \varphi^{-1}\Big(\min_{x\in[0,1]} z_\mu(x)\Big). \tag{6.10}$$

Let us now choose $R > \sup\limits_{\mu\in(0,1]} \{z_\mu(0)\} + 2\|h\|_1$ and set

$$U = \{z \in \mathbb{L}^1(S') : \|z\|_1 < R\}. \tag{6.11}$$

Note that any function $z \in D(L)$ such that $Lz = \mu F(z)$ satisfies (6.9) and, as a consequence, $\|z\|_1 < R$. Hence $Lz \ne \mu F(z)$, for every $\mu \in (0,1]$ and $z \in D(L) \cap \partial U$.

Now we are going to prove condition (b) of Theorem 6.3, where Π is defined by

$$\Pi\colon \mathbb{L}^1(S') \to \mathbb{L}^1(S')/\operatorname{Im} L, \qquad \Pi y = y + \operatorname{Im} L.$$

The compactness of ΠF is obvious. As $\ker L \cap \operatorname{Im} L = \{0\}$, so

$$\Pi F(z) = \int_0^1 F(z(s))\, ds + \operatorname{Im} L.$$

Pick $R \in \mathbb{R}^1$ such that $|\varphi(R)| > \|h\|_1$ and $|\varphi(-R)| > \|h\|_1$. Note that

$$(\ker L \cap \partial U) = \{f_1, f_2\} \quad \text{and} \quad \int_0^1 |f_1|\, ds = \int_0^1 |f_2|\, ds = R.$$

Then will have

$$0 < \left| \int_0^1 |\varphi(f_i)|\, ds - \|h\|_1 \right| \le \left| \int_0^1 \varphi(f_i) - h(s)\, ds \right|, \quad i = 1, 2,$$

i.e. $\Pi F(z) \neq 0$ when $z \in \ker L \cap \partial U$. We have

$$\Lambda \Pi F(z) = \int_0^1 \varphi(z(s)) - h(s)\, ds.$$

Arguing in a similar way, we can show that there exist $R > 0$ and $z_1, z_2 \in \partial U \cap \ker L$ such that $\Lambda\Pi F(z_1) > 0$ and $\Lambda\Pi F(z_2) < 0$, where U is given in (6.11). So the third condition of Theorem 6.3 is fulfilled.

Since all conditions of Theorem 6.3 are satisfied, the boundary value problem (6.5) has a unique solution, hence (6.3) also has a unique solution. Therefore, $B(t) + \omega I$ is m-dissipative.

Step 3. The compactness condition. Now suppose that $K \subset \mathbb{L}^1(S')$ is an arbitrary bounded set and $\lambda > 0$. Then, for every $h \in K$, there exists $z_h \in \mathbb{L}^1(S')$, a solution of

$$\begin{cases} -\lambda z'' + \varphi^{-1}(z) = h, \\ z'(0) = 0, \\ z'(1) = 0, \end{cases}$$

and the set $V = \{z_h : h \in K\}$ is relatively compact since (see Miyadera, p. 113) we have

$$|z_h'(x)| \leq 2\|K\|, \qquad |z_h(x)| \leq C + 2\|K\|,$$

where $C = \sup\{|z_h(0)| : h \in K\}$ (the existence of C can be established by using (6.10)) and $\|K\| = \sup_{h\in K} \|h\|_1$.

Let us define (see (6.4))

$$G := \bigcup_{t\in\mathbb{R}} V(t), \quad \text{with } V(t) = V + \{k_t\},$$

where, we have set

$$k_t(x) = \frac{1}{2}\,(x^2 g_1(t) - (1-x)^2 g_0(t)).$$

The set G is clearly relatively compact. Then, as we have already shown in the example in Section 5, the set $\varphi^{-1}(G) = (I - \lambda B(t))^{-1}(K)$ is also relatively compact in X.

Step 4. The densely defined condition. Recall that φ is assumed strictly monotone satisfying $\varphi \in C^2(\mathbb{R})$, $\varphi(\mathbb{R}) = \mathbb{R}$ and $\varphi(0) = 0$. The corresponding operator $\widetilde{\varphi}\colon C(S') \to C(S')$ defined by

$$(\widetilde{\varphi} y)(s) = \varphi(y(s)), \quad y \in C(S'),$$

is thus bijective and continuous.

We are going to prove that the set

$$D(A(t)) := \{u \in C(S') : (\varphi(u))'(k) = g_k(t),\ k \in \{0,1\} \text{ and } (\varphi \circ u)' \in \mathrm{AC}(S')\}$$

is dense in $\mathbb{L}^1(S')$. Since $C^2(S')$ is dense in $\mathbb{L}^1(S')$, it is sufficient to show that for every function $y \in C^2(S')$ there exists a sequence $(u_n)_n$ of functions satisfying

$$\{u_n\} \subset D(A(t)), \quad n \in \mathbb{N}, \quad \text{and} \quad \lim_n \|y - u_n\|_1 = 0.$$

Let y be an arbitrary function $y \in C^2(S')$. We construct the approximative sequence $(u_n)_n$ in the following way. Set $v = \varphi \circ y$. Since $v \in C^2(S')$, it follows that $v' \in \mathrm{AC}(S')$. Set $a_n = 1/n$ and $b_n = 1 - 1/n$. Consider the following sequence of functions $(v_n)_n$:

$$v_n(s) = \begin{cases} v(a_n) + \dfrac{1}{2}\left(v'(a_n)\left(\dfrac{s^2}{a_n} - a_n\right) - \dfrac{(a_n - s)^2}{a_n}\, g_0(t)\right) & \text{if } s \in [0, a_n), \\ v(s) & \text{if } s \in [a_n, b_n], \\ v(b_n) + \dfrac{1}{2}\left[v'(b_n)\left(\dfrac{(s-1)^2}{b_n - 1} - (b_n - 1)\right) - \dfrac{(s - b_n)^2}{b_n - 1}\, g_1(t)\right] & \\ & \text{if } s \in (b_n, 1]. \end{cases}$$

Obviously, each v_n is continuous. Note that $v_n'(a_n) = v'(a_n)$ and $v_n'(b_n) = v'(b_n)$, hence $\{v_n\} \subset C^1(S')$. We also have

$$\begin{aligned} v_n'(0) &= \left(v'(a_n)\,\frac{s}{a_n} + \frac{(a_n - s)}{a_n}\, g_0(t)\right)\Bigg|_{s=0} = g_0(t), \\ v_n'(1) &= \left(v'(b_n)\,\frac{s-1}{b_n - 1} - \frac{(s - b_n)}{b_n - 1}\, g_1(t)\right)\Bigg|_{s=1} = g_1(t). \end{aligned}$$

Their derivatives are uniformly bounded in $C(S')$, i.e.

$$\sup_n \|v_n'\|_{C(S')} < +\infty.$$

Indeed, if $s \in [0, a_n]$, we have

$$|v_n'(s)| = |v'(a_n)ns + (1 - ns)g_0(t)| \le \|v'\|_{C(S')} + |g_0(t)|,$$

and, if $s \in [b_n, 1]$, then

$$|v_n'(s)| = |n(1-s)v'(b_n) + n(s - b_n)g_1(t)| \le \|v'\|_{C(S')} + |g_1(t)|.$$

Note that

$$\begin{aligned} v_n(0) &= v(a_n) - \frac{1}{2}\left(\frac{v'(a_n)}{n} + \frac{g_0(t)}{n}\right), \\ v_n(1) &= v(b_n) + \frac{1}{2}\left(\frac{v'(b_n)}{n} + \frac{g_1(t)}{n}\right). \end{aligned}$$

Thus, the set $\{v_n : n \in \mathbb{N}\}$ is bounded in $C(S')$, i.e. there exists $M > 0$ such that

$$\sup_n \|v_n\|_{C(S')} < M. \tag{6.12}$$

Since $v_n(s) \to v(s)$, we have $(\varphi^{-1} \circ v_n)(s) \to y(s)$. From (6.12), we deduce

$$|\varphi^{-1} \circ v_n)(s)| \le \max(|\varphi^{-1}(-M)|, |\varphi^{-1}(M)|).$$

Then, the Lebesgue dominated convergence theorem gives

$$\lim_n \|y - \varphi^{-1} \circ v_n\|_1 = 0.$$

The set $\{u_n : n \in \mathbb{N}\} = \{\varphi^{-1}(v_n) : n \in \mathbb{N}\}$ is a subset of $D(A(t))$ since the following conditions hold:

(1) $u_n = \varphi^{-1}(v_n) \in C(S')$,
(2) $(\varphi \circ u_n)'(0) = v_n'(0) = g_0(t)$ and $(\varphi(u_n))'(1) = v_n'(1) = g_1(t)$,
(3) $(\varphi \circ u_n)' = v_n' \in \mathrm{AC}(S')$.

The density assertion is now proved. □

Remark 6.4. The condition $\varphi \in C^2(\mathbb{R})$ is only required to guarantee $(\varphi \circ y)'$ in $\mathrm{AC}(S')$ since it is not always true, as shown in the following example.

Example 6.5. Let φ be such that

$$\varphi'(x) = \begin{cases} x \sin \dfrac{1}{x} + 1 & \text{if } x \in (0,1], \\ 1 & \text{if } x = 0, \end{cases}$$

then φ is continuous and strictly monotone on S' but we have $(\varphi \circ y)' \notin \mathrm{AC}(S')$ with $y(s) = s$.

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