

NODAL SOLUTIONS FOR A CLASS OF DEGENERATE ONE DIMENSIONAL BVP'S

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ABSTRACT. In [7], a family of degenerate one dimensional boundary value problems was studied and the existence of positive (and negative) solutions and solutions that possess one interior node was shown for a range of values of a parameter, λ . It was conjectured that there is a natural extension of these results giving solutions with any prescribed number of interior nodes. This conjecture will be established here.

1. Introduction

In the recent papers [7], [8] and [9], the existence of nodal solutions for the degenerate boundary value problem

$$(1.1) \quad \begin{cases} -du'' = \lambda u - a(x)f(u)u & \text{for } x \in [0, L], \\ u(0) = u(L) = 0, \end{cases}$$

was studied. The functions a and f satisfy

$$(1.2) \quad 0 \leq a \in C[0, 1], \quad a^{-1}(0) = [\alpha, \beta], \quad 0 < \alpha < \beta < L,$$

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and

$$(1.3) \quad f \in \mathcal{C}^1(\mathbb{R}), \quad f(0) = 0, \quad \xi f'(\xi) > 0 \quad \text{for } \xi \neq 0, \quad \text{and} \quad \lim_{|\xi| \rightarrow \infty} f(\xi) = \infty.$$

By (1.3), $f(\xi) > 0$ for $\xi \neq 0$.

In (1.1), d is a positive constant, which without loss of generality can be set equal to 1. Thus instead of (1.1), the problem

$$(1.4) \quad \begin{cases} -u'' = \lambda u - a(x)f(u)u & \text{for } x \in [0, L], \\ u(0) = u(L) = 0, \end{cases}$$

will be considered. Problem (1.4) is degenerate in the sense that the function a vanishes on a subinterval of $[0, L]$. The analysis of the classical case when $a(x) > 0$ for all $x \in [0, L]$ can be found in [12], where it was established that for every integer $n \geq 1$, (1.4) admits a solution with $n - 1$ (interior) zeroes, or nodes, in $(0, L)$ if and only if $\lambda > (n\pi/L)^2$, the n -th eigenvalue of $-D^2$, $D = d/dx$, in $(0, L)$ under Dirichlet boundary conditions. Later it was shown in [2] that the degenerate problem (1.4) admits a positive solution if and only if

$$\left(\frac{\pi}{L}\right)^2 < \lambda < \left(\frac{\pi}{\beta - \alpha}\right)^2.$$

Note that $(\pi/(\beta - \alpha))^2$ is the first eigenvalue of $-D^2$ in (α, β) under Dirichlet boundary conditions. More recent results, Theorems 4.1 and 4.2 of [7], tell us that (1.4) has a solution with one node in $(0, L)$ if and only if

$$\left(\frac{2\pi}{L}\right)^2 < \lambda < \left(\frac{2\pi}{\beta - \alpha}\right)^2.$$

It was further conjectured in [7] that in the general case when $n \geq 1$, (1.4) possesses a solution with $n - 1$ (interior) nodes in $(0, L)$ if and only if

$$(1.5) \quad \left(\frac{n\pi}{L}\right)^2 < \lambda < \left(\frac{n\pi}{\beta - \alpha}\right)^2.$$

Note that for every $n \geq 1$, $(n\pi/L)^2$ and $(n\pi/(\beta - \alpha))^2$ are the n -th eigenvalues of $-D^2$ in $(0, L)$ and (α, β) , respectively, under Dirichlet boundary conditions. Our main goal here is to establish this conjecture as well as to further study the structure of the set of solutions of (1.4). To do so, first some preliminaries will be carried out in Section 2. Then, to obtain the conjecture, some nonstandard a priori bounds for solutions will be obtained in Section 3 and the existence argument, which employs Leray–Schauder degree theory, will be given in Section 4. In fact a stronger result will be obtained: for each λ satisfying (1.5), (1.4) has at least two solutions with $n - 1$ interior nodes, u^+ and u^- , such that $(u^+)'(0) > 0$ and $(u^-)'(0) < 0$.

To describe our further results and state our main result precisely, some definitions and notation are required. Throughout this paper, the solutions

of (1.4) are regarded as pairs, $(\lambda, u) \in \mathbb{R} \times \mathcal{C}^2[0, L]$. As in [7], for every integer $n \geq 1$, S_n^\pm stands for the set of functions $u \in \mathcal{C}^2[0, L]$ with $\pm u'(0) > 0$ such that u has exactly $n - 1$ simple zeroes in $(0, L)$. If $u \in S_n^\pm$ solves (1.4), then all its zeroes must be simple due to the uniqueness of the solution to the initial value problem for (1.4). In particular if u has a double zero, i.e. $u(\xi) = 0 = u'(\xi)$, then $u \equiv 0$.

Let \mathcal{P} be the λ -projection operator: $\mathcal{P}(\lambda, u) = \lambda$, for $(\lambda, u) \in \mathbb{R} \times \mathcal{C}[0, L]$. Set

$$(1.6) \quad T_n^\pm \equiv \{(\lambda, u) \in \mathbb{R} \times S_n^\pm : (\lambda, u) \text{ is a solution of (1.4)}\}$$

and

$$\Lambda_n := \left(\left(\frac{n\pi}{L} \right)^2, \left(\frac{n\pi}{\beta - \alpha} \right)^2 \right).$$

Now our main result can be stated:

THEOREM 1.1. $\mathcal{P}(T_n^\pm) = \Lambda_n$ for all $n \geq 1$.

The analysis carried out in [7] showed that when $n \geq 2$, the problem of determining the structure of T_n^\pm is subtle. Indeed, by Proposition 4.1 of [2] and Corollaries 3.1 and 3.2 of [7], T_1^\pm consists of a differentiable curve, (λ, u_λ^\pm) , bifurcating from the family of trivial solutions, $\{(\lambda, 0) : \lambda \in \mathbb{R}\}$ at $\lambda = (\pi/L)^2$ with

$$\lim_{\lambda \uparrow (\pi/(\beta - \alpha))^2} u_\lambda^\pm = \pm\infty \quad \text{in } [\alpha, \beta].$$

However, when $n = 2$, there are (numerical) examples, [9], where T_2^\pm consists of at least two components, \mathfrak{C}_2^\pm and \mathfrak{D}_2^\pm , such that

$$(1.7) \quad \mathcal{P}\left(\mathfrak{C}_2^\pm \setminus \left\{ \left(\left(\frac{2\pi}{L} \right)^2, 0 \right) \right\} \right) = \left(\left(\frac{2\pi}{L} \right)^2, \left(\frac{\pi}{\beta - \alpha} \right)^2 \right)$$

and, for some $\sigma > 0$,

$$(1.8) \quad \mathcal{P}(\mathfrak{D}_2^\pm) = \left[\left(\frac{\pi}{\beta - \alpha} \right)^2 - \sigma, \left(\frac{2\pi}{\beta - \alpha} \right)^2 \right).$$

A component is a closed and connected subset of T_n^\pm maximal with respect to inclusion. Naturally, $\mathfrak{C}_2^\pm \cap \mathfrak{D}_2^\pm = \emptyset$ if $\mathfrak{C}_2^\pm \neq \mathfrak{D}_2^\pm$. Thus, although T_1^\pm is always connected, T_n^\pm can have two or more components if $n \geq 2$. Studying the topological structure of T_n^\pm in the general case when $n \geq 2$ is the second problem treated in this paper. Our main result, a partial one, sharpens Theorems 4.1 and 4.2 of [7] by establishing that, in the general case when $n \geq 2$, T_n^\pm might consist of several unbounded components. More precisely, let $j \in \{1, \dots, n\}$ be the unique integer for which

$$(1.9) \quad \left[\frac{(j-1)\pi}{\beta - \alpha} \right]^2 \leq \left(\frac{n\pi}{L} \right)^2 < \left(\frac{j\pi}{\beta - \alpha} \right)^2,$$

or equivalently

$$(1.10) \quad \frac{(j-1)L}{n} \leq \beta - \alpha < \frac{jL}{n}.$$

Then, T_n^\pm might consist (at least) of $n+1-j$ unbounded components. The components are unbounded at some, or several, of the eigenvalues $(i\pi/(\beta-\alpha))^2$. Whether any of these possible situations can actually occur is an extremely challenging open problem that is being investigated numerically by M. Molina-Meyer, [10]. Her numerical experiments for $n=3$ led us to the results presented here in Section 5.

2. Preliminary results

In this section, some preliminary results about nodal solutions and their properties will be obtained. For the remainder of this paper, by a non-trivial solution of (1.4) we mean a classical solution (λ, u) with $u \not\equiv 0$. Let S denote the closure in $\mathbb{R} \times \mathcal{C}^2[0, L]$ of the set of non-trivial solutions of (1.4). By Theorem 2.1 of [7], for every solution $(\lambda, u) \in S$, there is an integer $n \geq 1$ such that u possesses $n-1$ simple zeroes in $(0, L)$ and hence, $u \in S_n^\pm$. Moreover, in such a case, (1.5) holds. Since the special case when $n=1$ is well understood, throughout the rest of this paper, it will be assumed that $n \geq 2$.

The next result, Theorem 2.4 of [7], gives us some partial information about the sets T_n^\pm .

THEOREM 2.1. *Suppose a and f satisfy (1.2)–(1.3). Then, for each integer $n \geq 2$, there is a component \mathfrak{C}_n of S such that if $\mathfrak{C}_n^\pm \equiv \mathfrak{C}_n \cap \overline{T}_n^\pm$, then*

$$\mathfrak{C}_n^+ \cap \mathfrak{C}_n^- = \left\{ \left(\left(\frac{n\pi}{L} \right)^2, 0 \right) \right\}, \quad \mathfrak{C}_n^\pm \setminus \left\{ \left(\left(\frac{n\pi}{L} \right)^2, 0 \right) \right\} \subset T_n^\pm,$$

and \mathfrak{C}_n^\pm is unbounded in S .

The numerical experiments of [9] reveal that, even for the simplest case of $n=2$, $\mathcal{P}(\mathfrak{C}_n^\pm)$ can be a proper subinterval of

$$(2.1) \quad \Lambda_n := \left(\left(\frac{n\pi}{L} \right)^2, \left(\frac{n\pi}{\beta-\alpha} \right)^2 \right), \quad n \geq 1.$$

Thus, in order to establish that

$$(2.2) \quad \mathcal{P}(T_n^\pm) = \Lambda_n \quad \text{for all } n \geq 2,$$

the first goal of this paper, more global information about T_n^\pm is needed. In this direction, the next result provides us with a component of T_n^\pm , denoted by \mathfrak{D}_n^\pm , which may be different from \mathfrak{C}_n^\pm .

THEOREM 2.2. *Suppose a and f satisfy (1.2)–(1.3), $n \geq 2$, and*

$$(2.3) \quad \left(\frac{n\pi}{L}\right)^2 < \left[\frac{(n-1)\pi}{\beta-\alpha}\right]^2, \quad \lambda \in I_{n-1} := \left[\left[\frac{(n-1)\pi}{\beta-\alpha}\right]^2, \left(\frac{n\pi}{\beta-\alpha}\right)^2\right).$$

- (a) *Then there exists a unique solution, (λ, u_λ^\pm) , of (1.4) in T_n^\pm and the $n-1$ interior zeroes of u_λ^\pm lie in (α, β) .*
- (b) *Each of the curves, $\lambda \mapsto u_\lambda^\pm$, $\lambda \in I_{n-1}$, is continuous.*
- (c) *There is a (connected) component of S , \mathfrak{D}_n^\pm , such that $\mathfrak{D}_n^\pm \subset T_n^\pm$ and $(\lambda, u_\lambda^\pm) \in \mathfrak{D}_n^\pm$ for all $\lambda \in I_{n-1}$.*
- (d) *As $\lambda \rightarrow (n\pi/(\beta-\alpha))^2$, $\|u_\lambda^\pm\|_{L^\infty[0,L]} \rightarrow \infty$.*

PROOF. Assertions (a) and (d) follow directly from results in [7] and the sets \mathfrak{D}_n^\pm are the components of T_n^\pm to which the curves belong so (c) is a consequence of (a) and (b). Thus the only novelty here is (b). We will prove the + case, the other proof being the same. These continuity properties are based on the fact that the solutions of (1.4) can be regarded as fixed points for a compact operator and on the existence of a priori bounds for u_λ^\pm in any compact subset of I_{n-1} .

To be more precise, let G denote the Green's function for $-D^2$ under 0 boundary conditions on $[0, L]$. Then, (1.4) is equivalent to the nonlinear integral equation

$$(2.4) \quad u = \int_0^L G(\cdot, y)[\lambda - a(y)f(u(y))]u(y) dy \equiv T(\lambda, u),$$

where $u \in E \equiv \{u \in C^1[0, L] : u(0) = 0 = u(L)\}$. It is well known that $T: \mathbb{R} \times E \rightarrow E$ is a compact operator. Thus to prove (b), let $\lambda_k \in I_{n-1}$, $k \geq 1$, be such that

$$(2.5) \quad \lambda_\infty := \lim_{k \rightarrow \infty} \lambda_k \in I_{n-1}.$$

Then

$$(2.6) \quad u_{\lambda_\infty} = \lim_{k \rightarrow \infty} u_{\lambda_k} \quad \text{in } E$$

provided $\{u_{\lambda_k}\}_{k \geq 1}$ is bounded in $C[0, L]$. Indeed, if $\{u_{\lambda_k}\}_{k \geq 1}$ is bounded in $C[0, L]$, the problem (1.4) shows that the sequence is also bounded in $C^2[0, L]$. Hence, $\{u_{\lambda_k}\}_{k \geq 1}$ is bounded in E and the compactness of T and (2.4) then show that along some subsequence, also labeled by k ,

$$u_\infty := \lim_{k \rightarrow \infty} u_{\lambda_k} \quad \text{in } C^2[0, L]$$

and $(\lambda_\infty, u_\infty)$ is a solution of (1.4). By (2.3), $\lambda_\infty > (n\pi/L)^2$. Thus, $u_\infty \not\equiv 0$, since $((n\pi/L)^2, 0)$ is the unique bifurcation point from $(\lambda, 0)$ for solutions in T_n^\pm . Consequently, $(\lambda_\infty, u_\infty) \in T_n^+$ so by the uniqueness assertion of (a), $u_\infty = u_{\lambda_\infty}$, completing the proof of (2.6), because the same argument can be repeated along any subsequence.

To establish the continuity of $\lambda \mapsto u_\lambda$, it remains to prove that, for every sufficiently small $\varepsilon > 0$, there exists a constant C_ε such that

$$(2.7) \quad \|u_\lambda\|_{L^\infty[0,L]} \leq C_\varepsilon \quad \text{for all } \lambda \in I_{n-1}^\varepsilon \equiv \left[\left[\frac{(n-1)\pi}{\beta-\alpha} \right]^2, \left(\frac{n\pi}{\beta-\alpha} \right)^2 - \varepsilon \right].$$

Arguing indirectly, suppose there is a sequence of solutions, $(\lambda_k, u_k) \in T_n^+$, where $u_k := u_{\lambda_k}$, such that

$$(2.8) \quad \lim_{k \rightarrow \infty} \lambda_k = \mu, \quad \lim_{k \rightarrow \infty} \|u_k\|_{C[0,L]} = \infty$$

and

$$(2.9) \quad \mu < \left(\frac{n\pi}{\beta-\alpha} \right)^2.$$

We will show that this is impossible using variants of arguments from [7]. Let $z_{k,i}$, $1 \leq i \leq n-1$, denote the $n-1$ interior zeroes of u_k , $k \geq 1$, ordered so that

$$(2.10) \quad \alpha < z_{k,1} < \dots < z_{k,n-1} < \beta.$$

By Corollary 5.3 of [7],

$$(2.11) \quad \zeta_i := \lim_{k \rightarrow \infty} z_{k,i} = \alpha + \frac{\beta-\alpha}{n} i, \quad 1 \leq i \leq n-1.$$

Therefore

$$(2.12) \quad \mu < \left(\frac{n\pi}{\beta-\alpha} \right)^2 = \left(\frac{\pi}{\zeta_1 - \alpha} \right)^2.$$

Then, there are constants $\delta > 0$ and $\sigma > 0$ such that

$$\mu + \delta < \left(\frac{\pi}{\zeta_1 + \sigma - \alpha} \right)^2.$$

Thus, due to Theorems 2.2 and 3.1 of [7], the problem

$$\begin{cases} -v'' = (\mu + \delta)v - af(v)v & \text{in } (0, \zeta_1 + \sigma), \\ v(0) = v(\zeta_1 + \sigma) = 0, \end{cases}$$

has a unique positive solution v on $(0, \zeta_1 + \sigma)$. Moreover, for sufficiently large k , say $k \geq k_0$, v satisfies

$$\begin{cases} -v'' = (\mu + \delta)v - af(v)v > \lambda_k v - af(v)v & \text{in } (0, \zeta_1 + \sigma), \\ v(0) = 0, \quad v(z_{k,1}) > 0. \end{cases}$$

Hence v is a positive supersolution of

$$(2.13) \quad \begin{cases} -w'' = \lambda_k w - af(w)w & \text{in } (0, z_{k,1}), \\ w(0) = w(z_{k,1}) = 0 \end{cases}$$

so by a standard comparison argument, see e.g. [5, Theorem 3.5], $v \geq u_k$ in $[0, z_{k,1}]$. Therefore

$$(2.14) \quad \|u_k\|_{C[0, z_{k,1}]} \leq \|v\|_{C[0, \zeta_1 + \sigma]}$$

which with (2.11) shows that functions u_k are uniformly bounded in $[0, \alpha]$. By (1.4), the same is true of u_k'' and hence u_k' . Observing that $-u_k'' = \lambda_k u_k$ in $[\alpha, \beta]$, and multiplying this equation by u_k' gives

$$(2.15) \quad (u_k'(x))^2 + \lambda_k u_k^2(x) = (u_k'(z_{k,i}))^2 = (u_k'(\alpha))^2 + \lambda_k u_k^2(\alpha)$$

for all $k \geq k_0$, $1 \leq i \leq n - 1$, and $x \in [\alpha, \beta]$. Now (2.14)–(2.15) and (1.4) give the existence of k -independent L^∞ bounds for u_k, u_k' and u_k'' in $[0, \beta]$. This together with the basic existence-uniqueness theorem for the initial value problem for ordinary differential equations implies that there are constants $M, \eta > 0$, independent of k , such that

$$(2.16) \quad \|u_k\|_{C[0, \beta + \eta]} \leq M.$$

Thus (2.8) and (2.16) imply $\lim_{k \rightarrow \infty} \|u_k\|_{C[\beta + \eta, L]} = \infty$. Consequently, for sufficiently large k , say $k \geq k_1 \geq k_0$, the solution u_k has a positive maximum or negative minimum at a point $\xi_k \in (\beta + \eta, L)$ with

$$(2.17) \quad \lim_{k \rightarrow \infty} |u_k(\xi_k)| = \infty.$$

The arguments being essentially the same, suppose $u_k(\xi_k) > 0$. Then, since $u_k'(\xi_k) = 0$ and $u_k''(\xi_k) \leq 0$, by (1.4),

$$\left(\min_{[\beta + \eta, L]} a \right) f(u_k(\xi_k)) \leq a(\xi_k) f(u_k(\xi_k)) \leq \lambda_k \leq \mu + \delta$$

so

$$u_k(\xi_k) \leq f^{-1} \left((\mu + \delta) / \min_{[\beta + \eta, L]} a \right),$$

contradicting (2.17). Thus (2.12) and (2.9) are impossible and (b) has been verified. \square

By (d), \mathfrak{D}_n^\pm becomes unbounded as $\lambda \uparrow (n\pi/(\beta - \alpha))^2$. Even in the simplest situation when $n = 2$, there are (numerical) examples where $\mathfrak{C}_n^\pm \cap \mathfrak{D}_n^\pm = \emptyset$, or, alternatively, $\mathfrak{C}_n^\pm = \mathfrak{D}_n^\pm$, [9].

The next result complements Theorem 2.2, items (a) and (b) being contained in Theorem 5.4 of [7], (c) following from the argument of the proof of Theorem 2.2, and (d) from (a)–(c).

THEOREM 2.3. *Suppose a and f satisfy (1.2)–(1.3), $n \geq 2$, and*

$$(2.18) \quad \left[\frac{(n-1)\pi}{\beta - \alpha} \right]^2 \leq \left(\frac{n\pi}{L} \right)^2, \quad \lambda \in J := \left(\left(\frac{n\pi}{L} \right)^2, \left(\frac{n\pi}{\beta - \alpha} \right)^2 \right).$$

(a) *Then there exists a unique solution, (λ, u_λ^\pm) , of (1.4) in T_n^\pm .*

- (b) *The $n - 1$ interior zeroes of u_λ^\pm lie in (α, β) .*
- (c) *Each of the curves $\lambda \mapsto u_\lambda^\pm$, $\lambda \in J$, is continuous so*

$$(2.19) \quad \mathfrak{C}_n^\pm = \{(\lambda, u_\lambda^\pm) : \lambda \in J\} = T_n^\pm.$$

- (d) *In particular, $\lim_{\lambda \downarrow (n\pi/L)^2} u_\lambda^\pm = 0$ uniformly in $[0, L]$ and (2.2) holds.*

REMARK 2.4. The first estimate of (2.18) can be rewritten as

$$(2.20) \quad \beta - \alpha \geq \frac{n - 1}{n} L,$$

which cannot be satisfied for sufficiently large $n \geq 2$.

Theorem 2.3 provides a proof of our main result when (2.20) holds. It remains to prove (2.2) when

$$(2.21) \quad \beta - \alpha < \frac{n - 1}{n} L,$$

which is equivalent to the first estimate of (2.3). So, without lost of generality, (2.21) can be assumed for what follows.

3. A priori bounds for the nodal solutions

In this section, a priori L^∞ bounds will be obtained for the solutions of (1.4), the bounds depending on λ and the number of nodes. As a tool, we will use the next result that provides us with such bounds for any non-trivial solution, (λ, u) , of (1.4) in terms of the positive and negative solutions of (1.4) in the interval where they exist. It is a direct consequence of Theorem 2.2 and Corollary 3.1 of [7].

THEOREM 3.1. *Suppose a and f satisfy conditions (1.2)–(1.3) and $\lambda \in ((\pi/L)^2, (\pi/(\beta - \alpha))^2)$. Let $u_{\lambda,1}^\pm$ denote the unique solution of the problem (1.4) with $\pm u_{\lambda,1}^\pm > 0$. Then, for every solution (λ, u) of (1.4),*

$$(3.1) \quad u_{\lambda,1}^- \leq u \leq u_{\lambda,1}^+.$$

REMARK 3.2. For the special case where

$$\left(\frac{n\pi}{L}\right)^2 < \left(\frac{\pi}{\beta - \alpha}\right)^2,$$

as a consequence of Theorem 3.1, we have

$$(3.2) \quad \left(\left(\frac{n\pi}{L}\right)^2, \left(\frac{\pi}{\beta - \alpha}\right)^2\right) \subset \mathcal{P}(\mathfrak{C}_n^\pm).$$

Since, by Corollary 3.2 of [7], the mappings $\lambda \mapsto u_{\lambda,1}^\pm$ are of class \mathcal{C}^1 , for each $\lambda \in ((\pi/L)^2, (\pi/(\beta - \alpha))^2)$ and every solution (λ, u) of (1.4), the following estimate holds:

$$(3.3) \quad \|u\|_{\mathcal{C}[0,L]} \leq (\|u_{\lambda,1}^+\|_{L^\infty[0,L]} + \|u_{\lambda,1}^-\|_{L^\infty[0,L]}) \equiv M(\lambda).$$

Note that $M(\lambda)$ is continuous.

The next result gives us a priori bounds for any solution (λ, u) with $\lambda \in I_{n-1}$ provided that the number of nodes of u in $(0, L)$ is at least $n - 1$. It makes strong use of the uniqueness portion of Theorem 2.2, and extends the estimate (3.3) to the setting that will be needed in the next section.

THEOREM 3.3. *Under the assumptions of Theorem 2.2, for every $\lambda \in I_{n-1}$ there is a constant $M_{n-1}(\lambda)$ such that for any solution, (λ, u) , of (1.4) having $j \geq n - 1$ zeroes in $(0, L)$,*

$$(3.4) \quad \|u\|_{C[0,L]} < M_{n-1}(\lambda).$$

Moreover, the map $\lambda \mapsto M_{n-1}(\lambda)$ can be chosen to be continuous for $\lambda \in I_{n-1}$.

PROOF. Since $n \geq 2$, $j \geq n - 1 \geq 1$. Let (λ, u) be a solution of (1.4) having j nodes in $(0, L)$ and $\lambda \in I_{n-1}$. If $j = n - 1$, then $u \in \{u_{\lambda,n}^+, u_{\lambda,n}^-\}$, where the functions $u_{\lambda,n}^\pm$ are the unique solutions with $n - 1$ nodes constructed in Theorem 2.2 and the $n - 1$ nodes lie in $[\alpha, \beta]$. Thus for this case the estimate follows from the continuity of the curves $\lambda \mapsto u_{\lambda,n}^\pm$, given by Theorem 2.2. Hence for what follows, it can be assumed that $j > n - 1 \geq 1$, so $j \geq 2$.

Next observe that since $\lambda \in I_{n-1}$, u possesses at least one node z in $[\alpha, \beta]$. Otherwise $u \neq 0$ in $[\alpha, \beta]$ so there are constants A, B such that $0 \leq A < \alpha < \beta < B \leq L$ and the problem

$$(3.5) \quad \begin{cases} -w'' = \lambda w - a(x)f(w)w & \text{for } x \in (A, B), \\ w(A) = w(B) = 0, \end{cases}$$

admits a solution without nodes in (A, B) . Then, by Theorem 2.1 in [7], $\lambda < (\pi/(\beta - \alpha))^2$. But if $\lambda \in I_{n-1}$, then

$$\lambda \geq \left[\frac{(n-1)\pi}{\beta - \alpha} \right]^2 \geq \left(\frac{\pi}{\beta - \alpha} \right)^2,$$

a contradiction. Consequently, $u(z) = 0$ for some $z \in [\alpha, \beta]$.

By Theorem 2.2, the $n - 1$ nodes z_k^\pm , $1 \leq k \leq n - 1$, of $u_{\lambda,n}^\pm$ lie in (α, β) . Order them so that $\alpha < z_1^\pm < \dots < z_{n-1}^\pm < \beta$. Note that, in general, $z_k^+ \neq z_k^-$. Since $u_{\lambda,n}^+$ is the unique positive solution of

$$\begin{cases} -w'' = \lambda w - a(x)f(w)w & \text{for } x \in (0, z_1^+), \\ w(0) = w(z_1^+) = 0, \end{cases}$$

$u_{\lambda,n}^+(x) > 0$ for all $x \in (0, z_1^+)$ and $(u_{\lambda,n}^+)'(0) > 0$, $(u_{\lambda,n}^+)'(z_1^+) < 0$. Similarly, $u_{\lambda,n}^-(x) < 0$ for all $x \in (0, z_1^-)$ and $(u_{\lambda,n}^-)'(0) < 0$, $(u_{\lambda,n}^-)'(z_1^-) > 0$.

Let $z \in [\alpha, \beta]$ denote the smallest node, of u in $[\alpha, \beta]$. It being a simple node $u'(z) \neq 0$. To continue, a case analysis depending on the sign of $u'(z)$ and the relative position of z in $[\alpha, \beta]$ with respect to z_1^\pm is needed.

Case 1. Suppose that

$$(3.6) \quad u'(z) < 0 \quad \text{and} \quad z \leq z_1^+.$$

On any subinterval (r, s) of $(0, z_1^+)$, where $u > 0$ and $u(r) = u(s) = 0$, the restriction $u_{\lambda,n}^+|_{(r,s)}$ is a positive supersolution for the problem

$$(3.7) \quad \begin{cases} -w'' = \lambda w - a(x)f(w)w & \text{for } x \in (r, s), \\ w(r) = w(s) = 0. \end{cases}$$

Since for sufficiently small $\varepsilon > 0$,

$$\underline{u}(x) = \varepsilon \sin \left[\frac{\pi(x-r)}{s-r} \right], \quad x \in [r, s],$$

is a positive subsolution of (3.7) such that $\underline{u} < u_{\lambda,n}^+$, we deduce that

$$0 < u < u_{\lambda,n}^+ \quad \text{in } (r, s).$$

Because of (3.6), there exists $r \in [0, \alpha]$ such that $u(x) > 0$ for all $x \in (r, z)$ and $u(r) = u(z) = 0$. Thus it follows that

$$0 < u < u_{\lambda,n}^+ \quad \text{in } (r, z).$$

By similar reasoning, on each subinterval (\tilde{r}, \tilde{s}) of $(0, r)$, where $u(x) < 0$ for all $x \in (\tilde{r}, \tilde{s})$ and $u(\tilde{r}) = u(\tilde{s}) = 0$, the restriction $u_{\lambda,n}^-|_{(\tilde{r}, \tilde{s})}$ is a negative subsolution of

$$\begin{cases} -w'' = \lambda w - a(x)f(w)w & \text{for } x \in (\tilde{r}, \tilde{s}), \\ w(\tilde{r}) = w(\tilde{s}) = 0, \end{cases}$$

so $u_{\lambda,n}^- < u < 0$ in (\tilde{r}, \tilde{s}) . Therefore,

$$(3.8) \quad u_{\lambda,n}^- \leq u \leq u_{\lambda,n}^+ \quad \text{in } (0, z).$$

Let x_0 be the closest point in $(0, z)$ to z at which $u'(x_0) = 0$. Then

$$(3.9) \quad u'(z) = \int_{x_0}^z u''(x) dx = \int_{x_0}^z [a(x)f(u(x)) - \lambda]u(x) dx,$$

so by (3.8) and (3.9), there exists a constant $C = C(\lambda) > 0$ independent of u such that $|u'(z)| \leq C$. Since $-u'' = \lambda u$ in (α, β) , it follows that

$$|u'(x)|^2 + \lambda u^2(x) = |u'(z)|^2 \leq C \quad \text{for all } x \in [\alpha, \beta].$$

Therefore, there exists a constant $C_1(\lambda) > 0$, independent of u , such that

$$(3.10) \quad \|u\|_{C[0,\beta]} + \|u'\|_{C[0,\beta]} < C_1(\lambda).$$

Moreover, by Theorem 2.2, $C_1(\lambda)$ can be chosen to be continuous in $\lambda \in I_{n-1}$.

Next using (3.9), uniform a priori bounds will be obtained for $\|u\|_{C[\beta,L]}$. As in (2.16), an application of the basic existence-uniqueness theorem for the initial value problem for ordinary differential equations shows there are constants $\delta, C_2(\lambda) > 0$, such that $\|u\|_{C^1[0,\beta+\delta]} < C_2(\lambda)$ and $C_2(\lambda)$ can be chosen to be

continuous in λ . Now, again as in the proof of Theorem 2.2, suppose u has a positive maximum at a point $x_0 \in (\beta + \delta, L)$. Then $u'(x_0) = 0$ and $u''(x_0) \leq 0$. Thus, it follows from (1.4) that

$$\left(\min_{[\beta+\delta, L]} a \right) f(u(x_0)) \leq a(x_0) f(u(x_0)) \leq \lambda$$

and hence,

$$u(x_0) \leq f^{-1} \left(\lambda / \min_{[\beta+\delta, L]} a \right).$$

Combining this estimate with a similar one at a negative minimum of u yields

$$\|u\|_{C[0, L]} \leq \max \left\{ C_2(\lambda), f^{-1} \left(\lambda / \min_{[\beta+\delta, L]} a \right) \right\}.$$

Using this bound with (1.4) yields the desired $C^1[0, L]$ bound for u and concludes the proof of the theorem under (3.6).

Case 2. Suppose that

$$(3.11) \quad u'(z) < 0 \quad \text{and} \quad z_1^+ < z.$$

Then, since u and $u_{\lambda, n}^+$ satisfy $-w'' = \lambda w$ in (α, β) , there are constants, C and D , such that

$$u(x) = C \sin [\sqrt{\lambda}(x-z)] \quad \text{and} \quad u_{\lambda, n}^+(x) = D \sin [\sqrt{\lambda}(x-z_1^+)] \quad \text{for all } x \in [\alpha, \beta],$$

and all zeroes of these functions in $[\alpha, \beta]$ are simple. Note that the distance between two consecutive zeroes of u or of $u_{\lambda, n}^+$ in $[\alpha, \beta]$ is $\pi/\sqrt{\lambda}$. Thus there is a zero of u between each pair of zeroes of $u_{\lambda, n}^+$ in $[\alpha, \beta]$ and by Theorem 2.2, $u_{\lambda, n}^+$ has exactly $n - 1$ zeroes in $[\alpha, \beta]$. We claim that u must also have exactly $n - 1$ zeroes in $[\alpha, \beta]$, with the largest, ζ , satisfying

$$(3.12) \quad z_{n-1}^+ < \zeta \leq \beta.$$

If (3.12) is false, u has $n - 2$ zeroes in $[z, z_{n-1}^+)$ and $\zeta < z_{n-1}^+$. Since $\lambda \in I_{n-1}$,

$$(3.13) \quad (n - 1) \frac{\pi}{\sqrt{\lambda}} \leq \beta - \alpha < n \frac{\pi}{\sqrt{\lambda}}.$$

Writing

$$\beta - \alpha = z - \alpha + (n - 3) \frac{\pi}{\sqrt{\lambda}} + \beta - \zeta,$$

inequality (3.13) implies

$$2 \frac{\pi}{\sqrt{\lambda}} \leq z - \alpha + \beta - \zeta < 3 \frac{\pi}{\sqrt{\lambda}}.$$

Since z is the first zero of u in $[\alpha, \beta]$ and $\alpha < z$, it follows that $z - \alpha < \pi/\sqrt{\lambda}$. Hence $\beta - \zeta \geq \pi/\sqrt{\lambda}$ so u has an additional zero in $(\zeta, \beta]$, contrary to the definition of ζ . Consequently $z_{n-1}^+ < \zeta \leq \beta$ and u possesses at least $n - 1$ zeros in $(\alpha, \beta]$. But $z_1^+ < z$, so $u_{\lambda, n}^+$ would have n zeroes in this interval if u had n zeroes. Thus, u has exactly $n - 1$ (simple) zeros in $(\alpha, \beta]$.

Next observe that $u'(z)(u_{\lambda,n}^+)'(z_1^+) > 0$. Therefore $u'(\zeta)(u_{\lambda,n}^+)'(z_{n-1}^+) > 0$. Hence, the argument given in Case 1 can be adapted to complete Theorem 3.3 under condition (3.11). Indeed, suppose $n - 1$ is odd. Then, $(u_{\lambda,n}^+)'(z_{n-1}^+) < 0$, $u'(\zeta) < 0$, and $u_{\lambda,n}^+(x) < 0$ for all $x \in (z_{n-1}^+, L)$. Moreover, there exists $s \in (\beta, L]$ such that $u(x) < 0$ for all $x \in (\zeta, s)$, $u(s) = 0$ and $u'(s) > 0$. Since $u_{\lambda,n}^+|_{[\zeta,s]}$ is a subsolution of

$$\begin{cases} -w'' = \lambda w - a(x)f(w)w & \text{for } x \in (\zeta, s), \\ w(\zeta) = w(s) = 0, \end{cases}$$

as earlier, $u_{\lambda,n}^+ < u < 0$ in (ζ, s) . In addition, $0 < u < u_{\lambda,n}^-$ in any subinterval (\tilde{s}, \tilde{t}) of (s, L) where $u(\tilde{s}) = u(\tilde{t}) = 0$ and $u(x) > 0$ for all $x \in (\tilde{s}, \tilde{t})$, while $u_{\lambda,n}^+ < u < 0$ in any subinterval (\hat{s}, \hat{t}) of (s, L) , where $u(\hat{s}) = u(\hat{t}) = 0$ and $u(x) < 0$ for all $x \in (\hat{s}, \hat{t})$. Therefore,

$$(3.14) \quad u_{\lambda,n}^+ \leq u \leq u_{\lambda,n}^- \quad \text{in } (\zeta, L).$$

Similarly, when n is even, $(u_{\lambda,n}^+)'(z_{n-1}) > 0$, $u'(\zeta) > 0$ and

$$(3.15) \quad u_{\lambda,n}^- \leq u \leq u_{\lambda,n}^+ \quad \text{in } (\zeta, L).$$

As for Case 1, since (1.4) is linear in $[\alpha, \beta]$, (3.14), or (3.15), provide uniform a priori bounds for u and u' in $[\alpha, L]$, and then as earlier, we obtain the a priori bounds in the entire interval $[0, L]$. This completes the proof of the theorem when $u'(z) < 0$.

Two cases remain: If $u'(z) > 0$ and $z \leq z_1^-$, then, arguing as in Case 1, it is easily seen that $u_{\lambda,n}^- \leq u \leq u_{\lambda,n}^+$ in $(0, z)$ and the proof of the theorem follows the general pattern of Case 1, while if $u'(z) > 0$ and $z_1^- < z$, the argument of Case 2 can be adapted easily to complete the proof for this case. \square

REMARK 3.4. For the function, $M_{n-1}(\lambda)$, we have

$$\limsup_{\lambda \uparrow (n\pi/(\beta-\alpha))^2} M_{n-1}(\lambda) = \infty.$$

Otherwise there exists a constant $C > 0$ such that $M_{n-1}(\lambda) \leq C$ for all $\lambda \in I_{n-1}$. Then, $\|u_{\lambda,n}^+\|_{C[0,L]} \leq C$ for all $\lambda \in I_{n-1}$ and straightforward arguments show there exists $u \in T_n^+$ such that $u_{\lambda,n}^+ \rightarrow u$ in $C^2[0, L]$ as $\lambda \uparrow (n\pi/(\beta - \alpha))^2$. But then $((n\pi/(\beta - \alpha))^2, u)$ is a solution of (1.4) having $n - 1$ zeroes in $(0, L)$, which is impossible.

REMARK 3.5. Suppose $\psi \in C([0, 1], C[0, L])$ with

$$\psi(0) = a, \quad \psi(s) \geq 0, \quad \psi(s)^{-1}(0) = [\alpha_s, \beta_s],$$

and $0 < \alpha_s < \beta_s < L$, for $s \in [0, 1]$. Set $a_s = \psi(s)$ for $s \in [0, 1]$. Then the proof of Theorem 3.3 holds equally well with $M_{n-1} = M_{n-1}(\lambda, s)$ also continuous

in $s \in [0, 1]$ provided that

$$\lambda \in I_{n-1}(s) \equiv \left[\left[\frac{(n-1)\pi}{\beta_s - \alpha_s} \right]^2, \left(\frac{n\pi}{\beta_s - \alpha_s} \right)^2 \right).$$

The final result of this section is a technical assertion that will be required in the next section.

PROPOSITION 3.6. *Suppose a and f satisfy (1.2)–(1.3) and $n \geq 1$. Then there exists a function, $r(\lambda) > 0$, continuous on Λ_n with $r(\lambda) \rightarrow 0$ as $\lambda \rightarrow (n\pi/L)^2$ and such that if $(\lambda, u) \in T_n^\pm$, then $\lambda \in \Lambda_n$ and $\|u\|_{C^1[0,L]} > r(\lambda)$.*

PROOF. That $(\lambda, u) \in T_n^\pm$, implies $\lambda \in \Lambda_n$ was proved in [7]. The second statement follows since $((n\pi/L)^2, 0)$ is the unique bifurcation point for solutions of (1.4) in T_n^\pm . Therefore for each $\lambda \in \Lambda_n$, there is $\rho(\lambda) > 0$ such that if $(\lambda, u) \in T_n^\pm$, then $\|u\|_{C^1[0,L]} > \rho(\lambda)$. That $\rho(\lambda)$ can be modified to get a continuous $r(\lambda)$ with $r(\lambda) \rightarrow 0$ as $\lambda \rightarrow (n\pi/L)^2$ is a straightforward exercise. \square

REMARK 3.7. For any small $\eta > 0$, $r(\lambda)$ can be chosen to be constant in $[(n\pi/L)^2 + \eta, (n\pi/(\beta - \alpha))^2)$.

4. The main existence result

In this section, our main result,

THEOREM 4.1. $\mathcal{P}(T_n^\pm) = \Lambda_n$ for all $n \geq 1$.

will be proved. This result goes back to [2] in case $n = 1$, to [7] in case $n = 2$, and it is a direct consequence of Theorem 2.3 under condition (2.20). The proof of Theorem 4.1 also provides more information about the structure of T_n^\pm , but such a discussion will be postponed to Section 5. The proof will be carried out for T_n^+ ; the proof for T_n^- is the same.

Note that there is a unique $j \in \{1, \dots, n\}$ depending on $\beta - \alpha$ and L such that

$$(4.1) \quad \left[\frac{(j-1)\pi}{\beta - \alpha} \right]^2 \leq \left(\frac{n\pi}{L} \right)^2 < \left(\frac{j\pi}{\beta - \alpha} \right)^2$$

or equivalently

$$(4.2) \quad \frac{(j-1)L}{n} \leq h \equiv \beta - \alpha < \frac{jL}{n}.$$

If $j = n$, then (4.2) becomes (2.20) and Theorem 4.1 is a consequence of Theorem 2.3. Consequently, it suffices to prove Theorem 4.1 when $j \leq n - 1$ which will be assumed for what follows. Set

$$(4.3) \quad I_0 := \left(\left(\frac{n\pi}{L} \right)^2, \left(\frac{j\pi}{h} \right)^2 \right).$$

Observe that Theorem 2.1 gives the existence of an unbounded continuum, \mathfrak{C}_n^+ , of solutions of (1.4) in T_n^+ . By (4.1) and Theorem 3.3 (with n replaced by j), if $(\lambda, u) \in \mathfrak{C}_n^+$, then $\|u\|_{L^\infty[0,L]} \leq M_{j-1}(\lambda)$. Therefore $I_0 \subset \mathcal{P}(\mathfrak{C}_n^+)$. In addition, Theorem 2.2 also shows there is an unbounded continuum \mathfrak{D}_n^+ of solutions of (1.4) in T_n^+ which for $\lambda \in I_{n-1}$ is a curve parameterized by λ . Thus $I_{n-1} \subset \mathcal{P}(\mathfrak{D}_n^+)$. If

$$(4.4) \quad \mathcal{P}(\mathfrak{C}_n^+ \cup \mathfrak{D}_n^+) = \Lambda_n,$$

there is nothing more to prove. However (4.4) will not hold in general and to handle the remaining cases, Leray–Schauder degree theory will be employed. As was shown earlier, (1.4) is equivalent to the integral equation (2.4). Define

$$\Phi(\lambda, u) := u - T(\lambda, u), \quad (\lambda, u) \in \mathbb{R} \times E.$$

Since $T: \mathbb{R} \times E \rightarrow E$ is a compact mapping, the Leray–Schauder degree of $\Phi(\lambda, \cdot)$ with respect to a bounded open set \mathcal{O} and a point $b \in E$, denoted by

$$\deg(\Phi(\lambda, \cdot), \mathcal{O}, b),$$

is well defined whenever $\Phi(\lambda, u) \neq b$ for all $u \in \partial\mathcal{O}$.

To choose a suitable set, \mathcal{O} , let $\lambda \in I_p$ for $p \in \{0, j, \dots, n-1\}$. By Theorem 3.3, there is a function $M_p(\lambda) > 1$, continuous for $\lambda \in I_p$ and such that whenever $(\lambda, u) \in I_p \times S_n^+$ and is a solution of (1.4), $\|u\| \equiv \|u\|_E < M_p(\lambda)$. By Remark 3.4, the function $M_p(\lambda) \rightarrow \infty$ as $\lambda \uparrow (p\pi/h)^2$, the right endpoint of I_p . Let $B_r(w)$ denote an open ball of radius r in E about w . Proposition 3.6 gives a function, $r(\lambda) > 0$, continuous on I_p with $r(\lambda) \rightarrow 0$ as $\lambda \downarrow (n\pi/L)^2$ such that whenever $(\lambda, u) \in I_p \times \overline{B}_{r(\lambda)}(0)$, then $u \notin S_n^+$. Note further that S_n^+ is an open set in E . These observations show that

$$\deg(\Phi(\lambda, \cdot), (B_{M_p(\lambda)}(0) \setminus \overline{B}_{r(\lambda)}(0)) \cap S_n^+, 0)$$

is defined for $\lambda \in I_p$ and, by the homotopy invariance property of the degree,

$$(4.5) \quad \deg(\Phi(\lambda, \cdot), (B_{M_p(\lambda)}(0) \setminus \overline{B}_{r(\lambda)}(0)) \cap S_n^+, 0) \equiv \text{constant} = c_p$$

for all $\lambda \in I_p$. The next result calculates c_p .

PROPOSITION 4.2. $c_p = (-1)^{n-1}$ for every $p \in \{0, j, \dots, n-1\}$ and $\lambda \in I_p$.

PROOF. The case of $p = 0$ corresponds to the interval in question being the one where bifurcation from the n -th eigenvalue of the associated linear problem occurs. Here, (1.3) implies bifurcation is in the direction of increasing λ . Hence a standard degree theoretic computation as in [12] shows that $c_0 = (-1)^{n-1}$. Thus suppose that $p \neq 0$. This case will be reduced to that of $p = 0$ by a homotopy argument. Let

$$h(s) = (1-s)h + s \frac{pL}{n}, \quad s \in [0, 1],$$

so $h(0) = h$ and $h(1) = pL/n$. Let $\psi \in C([0, 1], C[0, L])$ be such that

$$\psi(0) = a(x), \quad \psi(s) \geq 0, \quad \psi(s)^{-1}(0) = [\alpha_s, \beta_s],$$

with $0 < \alpha_s < \beta_s < L$ and $\beta_s - \alpha_s = h(s)$. It is straightforward to construct such a function ψ . Set

$$I_p(s) := \left[\left[\frac{p\pi}{h(s)} \right]^2, \left[\frac{(p+1)\pi}{h(s)} \right]^2 \right)$$

and define, for every $(s, \lambda, u) \in [0, 1] \times I_p(s) \times E$,

$$\Psi(s, \lambda, u) := u - \int_0^L G(\cdot, y)[\lambda - \psi(s)(y)f(u(y))]u(y) dy.$$

Then, as for Φ , by Remark 3.5, there exist functions $M_p(\lambda, s), r(\lambda, s) > 0$ that are defined and continuous for $(s, \lambda) \in [0, 1] \times I_p(s)$, except for $s = 1$ and $\lambda = [p\pi/h(1)]^2 = (n\pi/L)^2$ where $r = 0$. Let $\mu(s)$ be the midpoint of $I_p(s)$:

$$\mu(s) = \frac{1}{2} \left[\frac{\pi}{h(s)} \right]^2 [p^2 + (p+1)^2], \quad s \in [0, 1].$$

Then, for $s \in [0, 1]$ and $\lambda \in I_p(s)$,

$$\deg(\Psi(s, \lambda, \cdot), (B_{M_p(s, \lambda)}(0) \setminus \overline{B}_{r(s, \lambda)}(0)) \cap S_n^+, 0)$$

is defined and, by the homotopy invariance property of degree again, for every $s \in [0, 1]$,

$$\begin{aligned} c_p &= \deg(\Phi(\lambda, \cdot), (B_{M_p(\lambda)}(0) \setminus \overline{B}_{r(\lambda)}(0)) \cap S_n^+, 0) \\ &= \deg(\Phi(\mu(0), \cdot), (B_{M_p(\mu(0))}(0) \setminus \overline{B}_{r(\mu(0))}(0)) \cap S_n^+, 0) \\ &= \deg(\Psi(s, \mu(s), \cdot), (B_{M_p(s, \mu(s))}(0) \setminus \overline{B}_{r(s, \mu(s))}(0)) \cap S_n^+, 0) \\ &= \deg(\Psi(1, \mu(1), \cdot), (B_{M_p(1, \mu(1))}(0) \setminus \overline{B}_{r(1, \mu(1))}(0)) \cap S_n^+, 0). \end{aligned}$$

Since

$$I_p(1) = \left[\left(\frac{n\pi}{L} \right)^2, \left[\frac{(p+1)\pi}{h(1)} \right]^2 \right),$$

and $\mu(1) \in I_p(1)$, we are back in the setting for c_0 with $\Psi(1, \mu(1), \cdot)$ replacing $\Phi(\lambda, \cdot)$. Therefore $c_p = (-1)^{n-1}$ and the proposition is proved. \square

Since $c_p \neq 0$ for $\lambda \in I_p$, as a standard consequence of Proposition 4.2, by the Leray–Schauder continuation theorem [3], we have

COROLLARY 4.3. *For each $p \in \{0, j, \dots, n-1\}$, T_n^+ contains a component of solutions, $\mathfrak{C}_{n,p}^+$, such that $P(\mathfrak{C}_{n,p}^+) \supset I_p$.*

Since we can take $\mathfrak{C}_{n,0}^+ = \mathfrak{C}_n^+$ and $\mathfrak{C}_{n,n-1}^+ = \mathfrak{D}_n$, Theorem 4.1 is an immediate consequence of Corollary 4.3.

REMARK 4.4. (a) $\mathfrak{C}_{n,p}^+$ may be unbounded as λ approaches the right endpoint of I_p .

(b) T_n^+ may consist of a single component, e.g. when $\mathfrak{C}_n^+ = \mathfrak{D}_n^+$. By Theorem 2.3, this occurs in particular when (2.20) is satisfied, i.e. $h \geq (n - 1)L/n$.

5. More on T_n^+ , $n \geq 2$

In this section, as a further consequence of the above ideas, a closer study of T_n^+ will be made when $\mathfrak{C}_n^+ \cap \mathfrak{D}_n^+ = \emptyset$.

THEOREM 5.1. *Suppose that $\mathfrak{C}_n^+ \cap \mathfrak{D}_n^+ = \emptyset$ and j is as in (4.2). Then, for some $p \in \{j, \dots, n - 1\}$, \mathfrak{D}_n^+ is unbounded at $(p\pi/h)^2$, i.e. as $\lambda \rightarrow (p\pi/h)^2$ for $\lambda < (p\pi/h)^2$ and $(\lambda, u) \in \mathfrak{D}_n^+$, $\|u\| \rightarrow \infty$. Moreover \mathfrak{D}_n^+ has at least two solutions for each λ less than and near $(p\pi/h)^2$.*

PROOF. For $\lambda \in I_{n-1}$, \mathfrak{D}_n^+ is a curve: $A_n^+ \equiv \{(\lambda, u_\lambda^+) : \lambda \in I_{n-1}\}$. Due to Theorem 2.2, the only solution of (1.1) in T_n^+ for $\lambda = [(n - 1)\pi/h]^2$ is $([(n - 1)\pi/h]^2, v)$ where $v = u_{[(n-1)\pi/h]^2}^+$. This implies there are constants $\sigma, \varrho > 0$ such that there are no members $(\lambda, u) \in T_n^+$ with $|\lambda - [(n - 1)\pi/h]^2| \leq \sigma$ and $\|u - v\| = \varrho$. Hence, $\deg(\Phi(\lambda, \cdot), B_\varrho(v) \cap S_n^+, 0)$ is defined for $|\lambda - [(n - 1)\pi/h]^2| \leq \sigma$ and due to Proposition 4.2,

$$(5.1) \quad \deg(\Phi(\lambda, \cdot), B_\varrho(v) \cap S_n^+, 0) = (-1)^{n-1} \quad \text{if } \left| \lambda - \left[\frac{(n-1)\pi}{h} \right]^2 \right| \leq \sigma.$$

As in the proof of the global bifurcation theorem, [12, Theorem 1.3], there exists an open neighbourhood, \mathcal{O} , of \mathfrak{D}_n^+ having the property that $\partial\mathcal{O} \cap T_n^+ = \emptyset$. Since $\mathfrak{C}_n^+ \cap \mathfrak{D}_n^+ = \emptyset$, the open set \mathcal{O} can be constructed with the additional property that $\mathcal{O} \cap \mathfrak{C}_n^+ = \emptyset$. Set $\mathcal{O}_\lambda = \{u \in E : (\lambda, u) \in \mathcal{O}\}$. Suppose that

$$(5.2) \quad \deg(\Phi(\lambda, \cdot), \mathcal{O}_\lambda \cap S_n^+, 0) = 0 \quad \text{for } \lambda = \left[\frac{(n-2)\pi}{h} \right]^2.$$

This will be the case, for example, if $\mathcal{O}_\lambda = \emptyset$ for $\lambda = [(n - 2)\pi/h]^2$ and, in particular, if $h \geq (n - 2)L/n$ since then $[(n - 2)\pi/h]^2 \leq (n\pi/L)^2$. By (5.2) and the homotopy invariance of degree,

$$(5.3) \quad \deg(\Phi(\lambda, \cdot), \mathcal{O}_\lambda \cap S_n^+, 0) = 0 \quad \text{for all } \lambda \in I_{n-2}.$$

Combining (5.3) with (5.1) we get, for every $\lambda \in [[(n - 1)\pi/h]^2 - \sigma, [(n - 1)\pi/h]^2)$, that there is a second solution of (1.1) in \mathfrak{D}_n^+ . Moreover the subcontinuum $\mathfrak{D}_n^+ \setminus A_n^+$ of \mathfrak{D}_n^+ must be unbounded as $\lambda \uparrow [(n - 1)\pi/h]^2$, for otherwise $\mathfrak{D}_n^+ \setminus A_n^+$ remains bounded as $\lambda \uparrow [(n - 1)\pi/h]^2$ and hence, $\mathfrak{D}_n^+ \setminus A_n^+$ must contain a point, $([(n - 1)\pi/h]^2, w)$, with $w \neq v$, contrary to the uniqueness of $([(n - 1)\pi/h]^2, v)$ as a solution of (1.1) in T_n^+ when $\lambda = [(n - 1)\pi/h]^2$.

Next suppose that

$$(5.4) \quad \deg(\Phi(\lambda, \cdot), \mathcal{O}_\lambda \cap S_n^+, 0) \neq 0 \quad \text{for } \lambda = \left[\frac{(n-2)\pi}{h} \right]^2.$$

Then \mathfrak{D}_n^+ continues into the region where $\lambda \in I_{n-3}$. If

$$\deg(\Phi(\lambda, \cdot), \mathcal{O}_\lambda \cap S_n^+, 0) = 0 \quad \text{for } \lambda = \left[\frac{(n-3)\pi}{h} \right]^2,$$

the argument above shows that \mathfrak{D}_n^+ has at least two solutions for each λ in a left neighbourhood of $\lambda = [(n-2)\pi/h]^2$ and \mathfrak{D}_n^+ becomes unbounded at $\lambda = [(n-2)\pi/h]^2$. If, on the contrary,

$$\deg(\Phi(\lambda, \cdot), \mathcal{O}_\lambda \cap S_n^+, 0) \neq 0 \quad \text{for } \lambda = \left[\frac{(n-3)\pi}{h} \right]^2,$$

the continuation process goes on for another step. Thus after a finite number of steps, the conclusion of the theorem follows. \square

REMARK 5.2. Suppose that $\mathcal{P}(\mathfrak{C}_n^+ \cup \mathfrak{D}_n^+) \neq \Lambda_n$ and j is as in (4.2). Then for any $p \in \{0, 1, \dots, n-1\}$ such that $\mathcal{P}(\mathfrak{C}_n^+ \cup \mathfrak{D}_n^+) \not\supset I_p$, by Corollary 4.3, there is a component, $\mathfrak{C}_{n,p}^+ \subset T_n^+$ such that $\mathcal{P}(\mathfrak{C}_{n,p}^+) \supset I_p$. Thus, $\mathfrak{C}_{n,p}^+ \cap (\mathfrak{C}_n^+ \cup \mathfrak{D}_n^+) = \emptyset$. As in the proof of Theorem 5.1, there is an open neighbourhood \mathcal{O} of $\mathfrak{C}_{n,p}^+$ such that $\partial\mathcal{O} \cap T_n^+ = \emptyset$ and $\mathcal{O} \cap (\mathfrak{C}_n^+ \cup \mathfrak{D}_n^+) = \emptyset$. It can be further assumed that

$$d_p \equiv \deg(\Phi(\lambda, \cdot), \mathcal{O}_\lambda \cap S_n^+, 0) \neq 0 \quad \text{for } \lambda \in I_p,$$

for otherwise

$$\deg(\Phi(\lambda, \cdot), (B_{M_p(\lambda)}(0) \setminus \overline{B}_{r(\lambda)}(0)) \cap S_n^+, 0) = 0,$$

contrary to Proposition 4.2. Since $d_p \neq 0$, the argument of Theorem 5.1 shows there is $q \leq p-1$ such that $\mathfrak{C}_{n,p}^+$ is unbounded in $I_q \times S_n^+$ and (1.4) has at least two solutions in $\mathfrak{C}_{n,p}^+$ for each $\lambda < (q\pi/h)^2$ and near $(q\pi/h)^2$. Likewise $\mathfrak{C}_{n,p}^+$ must be unbounded as $\lambda \uparrow (t\pi/h)^2$ for some $t \in \{p, \dots, n-1\}$ since if it were bounded, it could be continued up to $\lambda = [(n-1)\pi/h]^2$. But for $\lambda \in I_{n-1}$, all solutions lie in \mathfrak{D}_n^+ .

REMARK 5.3. An interesting open question is to determine the dependence of the structure of the components of T_n^+ on the oscillation properties of a . According to [2] or [7], T_1^+ is a C^1 curve, like T_n^+ if $\beta - \alpha \geq (n-1)L/n$ via Theorem 2.3. But when, e.g. $n = 2$ and $\beta - \alpha < L/2$, numerical studies show that T_2^+ may consist of at least two components $\mathfrak{C}_2^+ \neq \mathfrak{D}_2^+$, plus, possibly, some additional component under appropriate assumptions on a , as was discussed in [9]. In the case when $n = 3$, computations by M. Molina-Meyer [10] show that there are examples with at least three (different) components, \mathfrak{C}_3^+ , $\mathfrak{C}_{3,1}^+$ and \mathfrak{D}_3^+ , while in other situations two of these components, and even the three, may coincide.

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