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NIELSEN FIXED POINT THEORY ON INFRA-SOLVMANIFOLDS OF SOL

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Abstract. Using averaging formulas, we compute the Lefschetz, Nielsen and Reidemeister numbers of maps on infra-solvmanifolds modeled on Sol, and we study the Jiang-type property for those infra-solvmanifolds.

1. Introduction

In Nielsen fixed point theory, given a continuous selfmap f on a closed manifold M, there are three important homotopy invariants $L(f), N(f)$ and $R(f)$ which are the Lefschetz, Nielsen and Reidemeister numbers of f, respectively. The non-vanishing of the Lefschetz number of f implies the existence of a fixed point, while the Nielsen number is a lower bound for the number of fixed points, and the Reidemeister number is an upper bound of the Nielsen number. By a classical result of Wecken [22], the Nielsen number coincides with the minimal number of fixed points in the homotopy class of the map when the dimension of M is at least three. The Nielsen number gives better information concerning the number of fixed points than the Lefschetz number, but the computation of the Nielsen number is in general much more difficult than that of the Lefschetz

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number. The problem of finding useful and practical formulas for Nielsen numbers is still the subject of great deal of research in Nielsen fixed point theory. There have been some developments to find algebraic and practical computation formulas, which are usually called averaging formulas for the Lefschetz, Nielsen and Reidemeister numbers [2], [7], [8], [13]–[17]. We recall in Theorem 2.5 the averaging formulas for the Lefschetz, Nielsen and Reidemeister numbers for a continuous map on an infra-solvmanifold M of type (R) as appeared in the form of [2, Theorem 1.1].

On the other hand, there is the notion of the class of Jiang spaces which includes simply connected spaces, generalized lens spaces, H-spaces and homogeneous spaces of the form G/G_0, where G is a topological group and G_0 a closed connected subgroup [10], [24]. It is well-known that if f is a continuous selfmap on a Jiang space which is a compact polyhedron, then the converse of Lefschetz–Hopf theorem holds, and either (i) $L(f) = 0 \Rightarrow N(f) = 0$ or (ii) $L(f) \neq 0 \Rightarrow N(f) = R(f)$ holds. The notion of Jiang spaces has been generalized to the notion of Jiang-type spaces. By definition, a closed manifold M is said to be a *Jiang-type space* if for all continuous maps $f\colon M \to M$, either (i) or (ii) holds [24]. It is known that nilmanifolds except tori are not Jiang spaces but are of Jiang-type, and every special solvmanifold of type (R) is of Jiang-type [2, Theorem 7.2].

Recently, Gonçalves and Wong [3] computed the Nielsen and Reidemeister numbers of a selfmap on infra-solvmanifolds modeled on Sol which is one of the eight geometries in the Thurston's geometrization theorem. Their computations are based on the fact that any infra-solvmanifold modeled on Sol is either a torus bundle over S^1 or a sapphire space [20].

In this paper, using averaging formulas, we compute the Lefschetz, Nielsen and Reidemeister numbers of continuous maps on infra-solvmanifolds modeled on Sol, correct Gonçalves and Wong's results in [3] and answer to their question in [3, Remarks 2.1 and 3.3]. In Example 3.15, we provide an example showing that their conjecture in [3, Remark 3.3] is not correct. Moreover, we study the Jiang-type property for those infra-solvmanifolds.

The paper is organized as follows. We review in Section 2 necessary terminologies and basic facts on infra-solvmanifolds of type (R) and averaging formulas of Lefschetz, Nielsen and Reidemeister numbers of continuous maps on them. In Section 3, we study the lattices of Sol and the structure of SB-groups modeled on Sol. In the following subsections, on infra-solvmanifolds modeled on Sol, we investigate holonomy groups, Lefschetz, Nielsen and Reidemeister numbers of continuous maps and the Jiang-type property.

2. Averaging formulas on infra-solvmanifolds of type (R)

Let G be a connected Lie group and let $\mathrm{Aut}(G)$ be the group of continuous automorphisms of G. The affine group of G is the semi-direct product $\mathrm{Aff}(G) = G \rtimes \mathrm{Aut}(G)$ with the multiplication $(a, A)(b, B) = (aA(b), AB)$. It has a Lie group structure and acts on G by $(a, A) \cdot x = aA(x)$ for all $x \in G$. Suppose that G has a linear connection defined by left-invariant vector fields. It turns out [12] that $\mathrm{Aff}(G)$ is the group of connection preserving diffeomorphisms of G.

Let S be a connected and simply connected solvable Lie group. A discrete subgroup Γ of S is a *lattice* of S if the orbit space $\Gamma \setminus S$ is compact, and in this case, we say that $\Gamma \setminus S$ is a *special* solvmanifold. Let $\pi \subset \mathrm{Aff}(S)$ be a torsion-free finite extension of a lattice Γ. Then π acts freely on S and the manifold $\pi \setminus S$ is called an *infra-solvmanifold* modeled on S. The finite group $\Phi := \pi/\Gamma$ is the *holonomy* group of π or $\pi \setminus S$. It sits naturally in $\mathrm{Aut}(S)$. Thus every infra-solvmanifold is finitely covered by a special solvmanifold. An infra-solvmanifold $M = \pi \setminus S$ is of type (R) if S is of type (R), that is, $\mathrm{ad}(X)\colon \mathfrak{S} \to \mathfrak{S}$ has only real eigenvalues for all X in the Lie algebra $\mathfrak{S}$ of S.

Recall that a connected solvable Lie group S contains a sequence of closed subgroups $1 = N_1 \subset \ldots \subset N_k = S$ such that N_i is normal in N_{i+1} and $N_{i+1}/N_i \cong \mathbb{R}$ or $N_{i+1}/N_i \cong S^1$. If the groups $N_1, \ldots, N_k$ are normal in S, the group S is called *supersolvable*. The supersolvable Lie groups are the Lie groups of type (R) and vice versa by the following.

Proposition 2.1. *For a connected Lie group S, the following are equivalent*:

(a) *S is supersolvable.*
(b) *All elements of* $\mathrm{Ad}(S)$ *have only positive eigenvalues.*
(c) *S is of type* (R).

Proof. (a) $\Leftrightarrow$ (b) is exactly [23, Lemma 4.1]. Since $\exp \mathrm{ad}(X) = \mathrm{Ad}(\exp X)$, (b) implies that $\mathrm{ad}(X)$ has only real eigenvalues for all $X \in \mathfrak{S}$. Thus (b) $\Rightarrow$ (c) holds. Assume (c) holds. Then S is of type (E), or $\exp\colon \mathfrak{S} \to S$ is surjective. For any $g \in S$, since exp is surjective, $g = \exp X$ for some X. Hence $\mathrm{Ad}(g) = \mathrm{Ad}(\exp X) = \exp \mathrm{ad}(X)$ has only positive eigenvalues, which implies (b). $\square$

An important property of solvable Lie groups of type (R) related to our paper is the Rigidity of Lattices ([9, Theorem 2.2], [23, Corollary 8.3]):

Theorem 2.2 (Rigidity of Lattices). *Let S and S' be connected and simply connected solvable Lie groups of type* (R)*, and let Γ be a lattice of S. Then any homomorphism from Γ to S' extends uniquely to a Lie group homomorphism of S to S'.*

Let $M = \pi \setminus S$ be an infra-solvmanifold of type (R) and let $f\colon M \to M$ be a continuous map. Then f induces a homomorphism φ on the group π of covering transformations of the universal covering projection $S \to M$, which is given by the identity

$$\varphi(\alpha) \circ \widetilde{f} = \widetilde{f} \circ \alpha, \quad \text{for all } \alpha \in \pi,$$

where $\widetilde{f}\colon S \to S$ is a (fixed) lifting of f.

By [17, Lemma 2.1], we choose a lattice $\Lambda(\subset \Gamma := \pi \cap S)$ of S which is a fully invariant subgroup of π so that any homomorphism $\theta\colon \pi \to \pi$ maps Λ into Λ. Thus, φ maps Λ into Λ and f can be lifted to $\overline{f}\colon \Lambda \setminus S \to \Lambda \setminus S$ on a special solvmanifold $\Lambda\backslash S$. Since S is of type (R), by Theorem 2.2, the restriction $\varphi|_\Lambda\colon \Lambda \to \Lambda$ extends to a homomorphism on the Lie group S in a unique way. Now its differential is a Lie algebra homomorphism, denoted by $\overline{f}_*\colon \mathfrak{S} \to \mathfrak{S}$ or simply f_*. On the other hand, by [17, Theorem 2.2], the homomorphism $\varphi\colon \pi \to \pi$ is semi-conjugate by an "affine map". That is, there exist $d \in S$ and a homomorphism $D\colon S \to S$ such that

$$\varphi(\alpha) \circ (d, D) = (d, D) \circ \alpha, \quad \text{for all } \alpha \in \pi.$$

This implies that the affine map $(d, D)\colon S \to S$ induces a map $M \to M$, which is homotopic to f. For $\lambda \in \Lambda$, the above identity is reduced to the identity $\varphi(\lambda) = \tau_d D(\lambda)$, where τ_d is the conjugation by d. That is, $\varphi|_\Lambda = \tau_d D\colon \Lambda \to \Lambda$. In particular, $\overline{f}$ is homotopic to the map induced by the homomorphism $\tau_d D\colon S \to S$, and so $f_* = \mathrm{Ad}(d) D_*$. We call such a homomorphism D or its differential D_* a *linearization* of f. In particular we obtain:

THEOREM 2.3.

(a) *Any continuous map $f\colon \pi \setminus S \to \pi \setminus S$ on an infra-solvmanifold $\pi \setminus S$ of type* (R) *has an affine map $(d, D)\colon S \to S$ as a homotopy lift.*
(b) *Any continuous map $f\colon \Gamma \setminus S \to \Gamma \setminus S$ on a special solvmanifold $\Gamma \setminus S$ of type* (R) *has a Lie group homomorphism $D\colon S \to S$ as a homotopy lift.*

Let $f\colon \Gamma \setminus S \to \Gamma \setminus S$ be a continuous map on a special solvmanifold $\Gamma \setminus S$ of type (R). By Theorem 2.3, f has a Lie group homomorphism $D\colon S \to S$ as homotopy lift. Let $\overline{f}$ denote the map induced by D so that $f \simeq \overline{f}$. Since the type (R) implies the type (NR), it follows from [13, Theorem 1] that $N(\overline{f}) = |L(\overline{f})|$. Noting that the Nielsen and Lefschetz numbers are homotopy invariants, we have the following corollary which says that the Anosov relation holds for any continuous map on a special solvmanifold $\Gamma \setminus S$ of type (R).

COROLLARY 2.4. *Let $f\colon \Gamma \setminus S \to \Gamma \setminus S$ be a continuous map on a special solvmanifold $\Gamma \setminus S$ of type* (R)*. Then $N(f) = |L(f)|$.*

The following are averaging formulas for the Lefschetz, Nielsen and Reidemeister numbers of a continuous map on an infra-solvmanifold M of type (R).

THEOREM 2.5 ([17, Theorem 4.3], [2, Theorem 1.1], [6]). *Let $f\colon M \to M$ be a continuous map on an infra-solvmanifold M of type* (R) *with holonomy group Φ and linearization D_*. Then*

$$\begin{aligned} L(f) &= \frac{1}{\#\,\Phi} \sum_{A\in\Phi} \det(I - A_*D_*), \\ N(f) &= \frac{1}{\#\,\Phi} \sum_{A\in\Phi} |\det(I - A_*D_*)|, \\ R(f) &= \frac{1}{\#\,\Phi} \sum_{A\in\Phi} \sigma(\det(I - A_*D_*)), \end{aligned}$$

where $\sigma\colon \mathbb{R} \to \mathbb{R}\cup\{\infty\}$ is defined by $\sigma(0) = \infty$ and $\sigma(x) = |x|$ for all $x \neq 0$.

3. Nielsen fixed point theory on infra-solvmanifolds of Sol

Recall that $\mathrm{Sol} = \mathbb{R}^2 \rtimes_\varphi \mathbb{R}$ where

$$\varphi(t) = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}.$$

Then Sol is a connected and simply connected unimodular 2-step solvable Lie group of type (R). It has a faithful representation into $\mathrm{Aff}(\mathbb{R}^3)$ as follows:

$$\mathrm{Sol} = \left\{ \begin{bmatrix} e^t & 0 & 0 & x \\ 0 & e^{-t} & 0 & y \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix} : x, y, t \in \mathbb{R} \right\}.$$

Thus its Lie algebra $\mathfrak{sol}$ can be identified with

$$\mathfrak{sol} = \left\{ \begin{bmatrix} s & 0 & 0 & a \\ 0 & -s & 0 & b \\ 0 & 0 & 0 & s \\ 0 & 0 & 0 & 0 \end{bmatrix} : a, b, s \in \mathbb{R} \right\}$$

with a (linear) basis

$$\mathbf{b}_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{b}_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{3.1}$$

The diffeomorphism $\exp\colon \mathfrak{sol} \to \mathrm{Sol}$ is given by

$$\begin{bmatrix} s & 0 & 0 & a \\ 0 & -s & 0 & b \\ 0 & 0 & 0 & s \\ 0 & 0 & 0 & 0 \end{bmatrix} \mapsto \begin{bmatrix} e^s & 0 & 0 & a(e^s - 1)/s \\ 0 & e^{-s} & 0 & b(1 - e^{-s})/s \\ 0 & 0 & 1 & s \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

It is known that a closed 3-manifold has a Sol-geometry if and only if it is an infra-solvmanifold modeled on Sol [4]. Moreover, it is also known that if M is a closed 3-manifold with Sol-geometry, then M is the suspension of a diffeomorphism of the torus $\mathbb{R}^2/\mathbb{Z}^2$ defined by a hyperbolic linear map, or M is a sapphire space (see, e.g., [20, Theorem 2.1]). On the other hand, the fundamental group Π of $M = \Pi\backslash\mathrm{Sol}$ is a Bieberbach group of Sol or an SB-group. It can be embedded into $\mathrm{Aff}(\mathrm{Sol}) = \mathrm{Sol} \rtimes \mathrm{Aut}(\mathrm{Sol})$ so that there is an exact sequence

$$1 \longrightarrow \Gamma \longrightarrow \Pi \longrightarrow \Pi/\Gamma \longrightarrow 1,$$

where $\Gamma = \Pi \cap \mathrm{Sol}$ is a lattice of Sol and $\Phi = \Pi/\Gamma$ is a finite group, called the *holonomy group* of Π or M, which sits naturally inside $\mathrm{Aut}(\mathrm{Sol})$.

The lattices Γ of Sol are determined by 2×2-integer matrices

$$A = \begin{bmatrix} \ell_{11} & \ell_{12} \\ \ell_{21} & \ell_{22} \end{bmatrix}$$

of determinant 1 and trace > 2, see for example [18, Lemma 2.1]. Namely,

$$\begin{aligned} \Gamma = \Gamma_A &= \langle a_1, a_2, \tau : [a_1, a_2] = 1, \tau a_i \tau^{-1} = a_1^{\ell_{1i}} a_2^{\ell_{2i}} \rangle \\ &= \langle a_1, a_2, \tau : [a_1, a_2] = 1, \tau a_i \tau^{-1} = \mathbf{a}^{Ae_i} \rangle \\ &= \langle a_1, a_2, \tau : [a_1, a_2] = 1, \tau a_i \tau^{-1} = A(a_i) \rangle = \mathbb{Z}^2 \rtimes_A \mathbb{Z}. \end{aligned}$$

Here we use the notation $\mathbf{a}^{\mathbf{x}}$ which means $a_1^x a_2^y$ for $\mathbf{a} = (a_1, a_2)$ and $\mathbf{x} = (x, y)$. For a 2×2-matrix M, we also denote $\mathbf{a}^{Me_i}$ by the notation $M(a_i)$. The defining matrix A of the lattice Γ of Sol has two distinct irrational eigenvalues $(\ell_{11} + \ell_{22} \pm \sqrt{(\ell_{11}+\ell_{22})^2 - 4})/2$. With two corresponding eigenvectors, we form a real invertible matrix

$$P = \begin{bmatrix} \dfrac{\ell_{11} - \ell_{22} + \sqrt{(\ell_{11}+\ell_{22})^2 - 4}}{2\ell_{21}}\, c & \dfrac{\ell_{11} - \ell_{22} - \sqrt{(\ell_{11}+\ell_{22})^2 - 4}}{2\ell_{21}}\, d \\ c & d \end{bmatrix}^{-1} =: [\mathbf{x}_1\, \mathbf{x}_2].$$

Let $t_0 = \ln((\ell_{11} + \ell_{22} + \sqrt{(\ell_{11}+\ell_{22})^2 - 4})/2)$. Then

$$PAP^{-1} = \begin{bmatrix} e^{t_0} & 0 \\ 0 & e^{-t_0} \end{bmatrix} =: D.$$

Now we can embed Γ_A into Sol as follows:

$$a_i \mapsto \begin{bmatrix} 1 & 0 & 0 & x_i \\ 0 & 1 & 0 & y_i \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \tau \mapsto \begin{bmatrix} e^{t_0} & 0 & 0 & 0 \\ 0 & e^{-t_0} & 0 & 0 \\ 0 & 0 & 1 & t_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \text{where } \mathbf{x}_i = \begin{bmatrix} x_i \\ y_i \end{bmatrix}.$$

The affine group $\mathrm{Aff}(\mathrm{Sol}) = \mathrm{Sol} \rtimes \mathrm{Aut}(\mathrm{Sol})$ of Sol can be embedded into $\mathrm{Aff}(\mathbb{R}^3) \subset \mathrm{GL}(4,\mathbb{R})$. Recalling that Sol is embedded into $\mathrm{Aff}(\mathbb{R}^3)$ already and using a description of $\mathrm{Aut}(\mathrm{Sol})$ given in [4, Section 2], we can embed $\mathrm{Aut}(\mathrm{Sol})$ into $\mathrm{Aff}(\mathbb{R}^3)$ as follows:

$$\begin{bmatrix}\alpha & 0 & \mu\\ 0 & \beta & \nu\\ 0 & 0 & 1\end{bmatrix} \mapsto \begin{bmatrix}\alpha & 0 & 0 & -\mu\\ 0 & \beta & 0 & \nu\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}, \qquad \begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & -1\end{bmatrix} \mapsto \begin{bmatrix}0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}.$$

It is known from [4, Corollary 8.3] that every SB-group Π is isomorphic to one of the following groups, adopting the notation in [5].

(1) $\Pi_1 = \Gamma_A = \langle a_1, a_2, \tau : [a_1, a_2] = 1, \tau a_i \tau^{-1} = A(a_i)\rangle$,

(2) $\Pi_2^{\pm} = \langle a_1, a_2, \sigma : [a_1, a_2] = 1, \sigma a_i \sigma^{-1} = N_{\pm}(a_i)\rangle$, where $N_{\pm}$ are square roots of A:

$$N_{\pm} = -\begin{bmatrix}\dfrac{\ell_{11} \pm 1}{\sqrt{\ell_{11}+\ell_{22}\pm 2}} & \dfrac{\ell_{12}}{\sqrt{\ell_{11}+\ell_{22}\pm 2}}\\ \dfrac{\ell_{21}}{\sqrt{\ell_{11}+\ell_{22}\pm 2}} & \dfrac{\ell_{22} \pm 1}{\sqrt{\ell_{11}+\ell_{22}\pm 2}}\end{bmatrix},$$

(3) $$\Pi_3 = \left\langle a_1, a_2, \tau, \rho : \begin{array}{l}[a_1, a_2] = 1,\ \tau a_i \tau^{-1} = A(a_i),\\ \rho a_i \rho^{-1} = M(a_i),\\ \rho^2 = \mathbf{a}^{\mathbf{e}_2},\ \rho\tau\rho^{-1} = \mathbf{a}^{\mathbf{k}'}\tau^{-1}\end{array}\right\rangle, \text{ where}$$

$$A = \begin{bmatrix}\ell_{11} & \ell_{12}\\ \ell_{21} & \ell_{11}\end{bmatrix} \quad (\ell_{11} \text{ odd; } \ell_{12}, \ell_{21} \text{ even}), \quad M = \begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix},$$

$$\mathbf{k}' - \mathbf{e}_2 \neq \mathbf{0} \quad \text{in} \quad \frac{\ker(A - M)}{\mathrm{im}(A^{-1} + M)} \cong \mathbb{Z}_2,$$

(4) $$\Pi_6 = \left\langle a_1, a_2, \sigma, \rho : \begin{array}{l}[a_1, a_2] = 1,\ \sigma a_i \sigma^{-1} = N(a_i),\\ \rho a_i \rho^{-1} = M(a_i),\\ \rho^2 = \mathbf{a}^{\mathbf{e}_2},\ \rho\sigma\rho^{-1} = \mathbf{a}^{\mathbf{k}'}\sigma^{-1}\end{array}\right\rangle, \text{ where}$$

$$A = \begin{bmatrix}\ell_{11} & \ell_{12}\\ \ell_{21} & \ell_{11}\end{bmatrix} = \begin{bmatrix}2s^2-1 & 2sp\\ 2sq & 2s^2-1\end{bmatrix}, \quad N = -\begin{bmatrix}s & p\\ q & s\end{bmatrix}, \quad M = \begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix},$$

$$(s > 1,\ s^2 - pq = 1,\ s \text{ odd}, p, q \text{ even})$$

$$\mathbf{k}' - \mathbf{e}_2 \neq \mathbf{0} \quad \text{in} \quad \frac{\ker(N - M)}{\mathrm{im}(N^{-1} + M)} \cong \mathbb{Z}_2.$$

From the identity $N_{\pm}^2 = A$, we see that $\langle a_1, a_2, \sigma^2\rangle = \Gamma_A$ is a subgroup of both $\Pi_2^{\pm}$ and Π_6. Recall from [4, Section 9] that $\Pi_1\backslash\mathrm{Sol}$, $\Pi_2^+\backslash\mathrm{Sol}$ and $\Pi_2^-\backslash\mathrm{Sol}$ are torus bundles over S^1 so that their fundamental groups are of the form $\mathbb{Z}^2 \rtimes_\phi \mathbb{Z}$, where $\phi = A$ (so $\det\phi = 1, \mathrm{tr}\,\phi > 2$), $\phi = N_+$ (so $\det\phi = 1$, $\mathrm{tr}\,\phi < -2$) and

$\phi = N_-$ (so $\det\phi = -1$, $\operatorname{tr}\phi < 0$), respectively. The case where $\det\phi = -1$ and $\operatorname{tr}\phi > 0$ is isomorphic to the case where $\det\phi = -1$ and $\operatorname{tr}\phi < 0$.

It is easy to see that every SB-group Π has Γ_A as a characteristic subgroup. In fact, if φ is an automorphism of Π, then it can be conjugated by an affine diffeomorphism (d, D), i.e., $\varphi(\alpha) = (d, D) \circ \alpha \circ (d, D)^{-1}$ for all $\alpha \in \Pi$. Hence if $\alpha = (\alpha, I) \in \Gamma_A$, then

$$\varphi(\alpha) = (d, D)(\alpha, I)(D^{-1}d^{-1}, D^{-1}) = (dD(\alpha)d^{-1}, I) \in \Gamma_A.$$

In the following, we will show that every SB-group Π has Γ_A as a fully invariant subgroup.

LEMMA 3.1. *The* SB-*groups* Π *have* Γ_A *as a fully invariant subgroup.*

PROOF. We note that $\Pi_2^\pm \supset \Gamma_{N_\pm^2} = \Gamma_A$, $\Pi_3 \supset \Gamma_A$ and $\Pi_6 \supset \Pi_2^+$ as index 2 subgroups. First we show that $\Pi_2^\pm$ has Γ_A as a fully invariant subgroup. Let $\varphi\colon \Pi_2^- \to \Pi_2^-$ be a homomorphism. Then

$$\varphi(a_i) = \mathbf{a}^{\mathbf{m}_i}\sigma^{z_i}, \qquad \varphi(\sigma) = \mathbf{a}^{\mathbf{x}}\sigma^{z}$$

for some $\mathbf{m}_i, \mathbf{x} \in \mathbb{Z}^2$ and $z_i, z \in \mathbb{Z}$. Write $N_- = [n_{ij}] = [\mathbf{n}_1\, \mathbf{n}_2]$. Since $\sigma a_i \sigma^{-1} = N_-(a_i) = \mathbf{a}^{\mathbf{n}_i}$, we have

$$\varphi(\sigma a_i \sigma^{-1}) = \varphi(\mathbf{a}^{\mathbf{n}_i}) \;\Rightarrow\; \mathbf{a}^*\sigma^{z_i} = \mathbf{a}^*\sigma^{n_{1i}z_1 + n_{2i}z_2} \;\Rightarrow\; n_{1i}z_1 + n_{2i}z_2 = z_i.$$

Because $I - N_-^t$ is invertible, $z_1 = z_2 = 0$. Hence

$$\varphi(a_i) = \mathbf{a}^{\mathbf{m}_i} \in \Gamma_A = \langle a_1, a_2, \sigma^2\rangle, \qquad \varphi(\sigma^2) = (\mathbf{a}^{\mathbf{x}}\sigma^z)^2 = \mathbf{a}^*(\sigma^2)^z \in \Gamma_A.$$

Therefore Γ_A is a fully invariant subgroup of Π_2^-. In a similar way, we can show that Γ_A is a fully invariant subgroup of Π_2^+.

It is shown in [4, Section 9] that Π_3 and Π_6 are isomorphic to the fundamental group of the sapphires, i.e., the spaces which are the union of two twisted I-bundles over the Klein bottle and in [3, Lemma 3.3] that such fundamental group has Π_1 or Π_2^+ as a fully invariant subgroup. Hence we conclude that the SB-groups Π have Γ_A as a fully invariant subgroup. □

3.1. Holonomy groups. We will explain how we can embed the abstract Bieberbach groups $\Pi_2^\pm, \Pi_3$ and Π_6 of Sol into $\mathrm{Aff}(\mathrm{Sol}) = \mathrm{Sol} \rtimes \mathrm{Aut}(\mathrm{Sol}) \subset \mathrm{Aff}(\mathbb{R}^3)$. As a result, we find the corresponding holonomy groups Φ as a subgroup of $\mathrm{Aut}(\mathrm{Sol}) \cong \mathrm{Aut}(\mathfrak{sol})$. We will express Φ as a subgroup of $\mathrm{Aut}(\mathfrak{sol})$ with respect to the basis $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$, see (3.1). An embedding procedure especially for Π_3 and Π_6 is described below in details.

(1) For Π_2^+ we embed Π_2^+ into $\mathrm{Aff}(\mathbb{R}^3)$ as

$$a_i \mapsto \begin{bmatrix} 1 & 0 & 0 & x_i \\ 0 & 1 & 0 & y_i \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \sigma \mapsto \begin{bmatrix} e^{t_0/2} & 0 & 0 & 0 \\ 0 & e^{-t_0/2} & 0 & 0 \\ 0 & 0 & 1 & t_0/2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Thus the holonomy group of Π_2^+ is

$$\Phi_2^+ = \left\langle \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\rangle$$

and, in particular, Π_2^+ is orientable, i.e., the quotient space $\Pi_2^+ \backslash \mathrm{Sol}$ is orientable.

(2) For Π_2^- we embed Π_2^- into $\mathrm{Aff}(\mathbb{R}^3)$ as

$$a_i \mapsto \begin{bmatrix} 1 & 0 & 0 & x_i \\ 0 & 1 & 0 & y_i \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \sigma \mapsto \begin{bmatrix} e^{t_0/2} & 0 & 0 & 0 \\ 0 & e^{-t_0/2} & 0 & 0 \\ 0 & 0 & 1 & t_0/2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Hence the holonomy group of Π_2^- is

$$\Phi_2^- = \left\langle \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\rangle$$

and, noting that $\det \Phi_2^- = -1$, we see that Π_2^- is non-orientable.

(3) For Π_3, since $\ell_{11}^2 - \ell_{12}\ell_{21} = 1$ and $\ell_{11} > 1$, it follows that a diagonalizing matrix P of A is of the form

$$P = \begin{bmatrix} \dfrac{\sqrt{\ell_{11}^2 - 1}}{\ell_{21}} c & -\dfrac{\sqrt{\ell_{11}^2 - 1}}{\ell_{21}} d \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} \dfrac{\sqrt{\ell_{11}^2 - 1}}{2c\ell_{12}} & \dfrac{1}{2c} \\ -\dfrac{\sqrt{\ell_{11}^2 - 1}}{2d\ell_{12}} & \dfrac{1}{2d} \end{bmatrix}.$$

Then

$$PAP^{-1} = \begin{bmatrix} \ell_{11} + \sqrt{\ell_{12}\ell_{21}} & 0 \\ 0 & \ell_{11} - \sqrt{\ell_{12}\ell_{21}} \end{bmatrix}.$$

Thus we take an embedding $\Gamma_A = \langle a_1, a_2, \tau \rangle$ into $\mathrm{GL}(4, \mathbb{R})$ as:

$$a_1 \mapsto \begin{bmatrix} 1 & 0 & 0 & \dfrac{\sqrt{\ell_{11}^2 - 1}}{2c\ell_{12}} \\ 0 & 1 & 0 & -\dfrac{\sqrt{\ell_{11}^2 - 1}}{2d\ell_{12}} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad a_2 \mapsto \begin{bmatrix} 1 & 0 & 0 & \dfrac{1}{2c} \\ 0 & 1 & 0 & \dfrac{1}{2d} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\tau \mapsto \begin{bmatrix} \ell_{11}+\sqrt{\ell_{12}\ell_{21}} & 0 & 0 & 0 \\ 0 & \ell_{11}-\sqrt{\ell_{12}\ell_{21}} & 0 & 0 \\ 0 & 0 & 1 & \ln(\ell_{11}+\sqrt{\ell_{12}\ell_{21}}) \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Assume that ρ is assigned to an element $(g,S)\in \mathrm{Aff}(\mathrm{Sol})$ so that as an element of $\mathrm{Aff}(\mathbb{R}^3)$ we have

$$\rho \mapsto (g,S) = \begin{bmatrix} e^t & 0 & 0 & x \\ 0 & e^{-t} & 0 & y \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \mathrm{Aff}(\mathbb{R}^3).$$

From the relation $\rho^2 = a_2$, we have $c = 1/(2(x+e^t y))$ and $d = ce^t$. Then we can see that the relation $\rho a_i \rho^{-1} = M(a_i)$ holds.

Using [4, Lemmas 7.1 and 7.2], we can find an integer vector which generates the cyclic group $\ker(A-M)/\mathrm{im}(A^{-1}+M)$ of order 2 as follows: Noting that

$$A - M = \begin{bmatrix} \ell_{11}+1 & \ell_{12} \\ \ell_{21} & \ell_{11}-1 \end{bmatrix}$$

with all nonzero entries, let $h = \gcd(\ell_{11}+1, \ell_{21})$ and choose $u, v \in \mathbb{Z}$ such that $(\ell_{11}+1)u/h + \ell_{21}v/h = 1$. Then the integer vector $(-(\ell_{12}u + (\ell_{11}-1)v)/2, h/2)$ is a generator, so is $\mathbf{k}' - \mathbf{e}_2$. Thus $\mathbf{k}' = (-(\ell_{12}u+(\ell_{11}-1)v)/2, h/2) + (0,1)$. The relation $\alpha\tau\alpha^{-1}\tau = \mathbf{a}^{\mathbf{k}'}$ induces that

$$\begin{aligned} &\begin{bmatrix} (1-\ell_{11}+\sqrt{\ell_{12}\ell_{21}})\,x \\ (1-\ell_{11}-\sqrt{\ell_{12}\ell_{21}})\,y \end{bmatrix} = P\mathbf{k}' \\ \Rightarrow &\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2(1-\ell_{11})}\,(I - PAP^{-1})P\mathbf{k}' \\ &\qquad = \frac{1}{2(1-\ell_{11})}\,(I-PAP^{-1})P \begin{bmatrix} -(\ell_{12}u + (\ell_{11}-1)v)/2 \\ h/2+1 \end{bmatrix}. \end{aligned}$$

This will result in an embedding Π_3 into $\mathrm{Aff}(\mathbb{R}^3)$. Hence the holonomy group of Π_3 is

$$\Phi_3 = \left\langle \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right\rangle$$

and, in particular, Π_3 is orientable.

(4) For Π_6, since $\ell_{11} = \ell_{22} = 2s^2-1$, $\ell_{12} = 2ps$ and $\ell_{21} = 2qs$, it follows that a diagonalizing matrix P of A is of the form

$$P = \begin{bmatrix} \dfrac{\sqrt{s^2-1}}{q}\,c & -\dfrac{\sqrt{s^2-1}}{q}\,d \\ c & d \end{bmatrix}^{-1}.$$

Observe that the relation $\rho\sigma\rho^{-1} = \mathbf{a}^{\mathbf{k}'}\sigma^{-1}$ induces that

$$\rho\sigma^2\rho^{-1} = \mathbf{a}^{\mathbf{k}'}\sigma^{-1}\mathbf{a}^{\mathbf{k}'}\sigma^{-1} = \mathbf{a}^{\mathbf{k}'}\mathbf{a}^{N^{-1}(\mathbf{k}')}\sigma^{-2}.$$

It is showed in [4, Section 7.6] that if

$$(\mathbf{k}, \mathbf{k}' - \mathbf{k}) \in \frac{\ker(I - M)}{\mathrm{im}(I + M)} \oplus \frac{\ker(N - M)}{\mathrm{im}(N^{-1} + M)},$$

then

$$(\mathbf{k}, (I + N^{-1})\mathbf{k}' - \mathbf{k}) \in \frac{\ker(I - M)}{\mathrm{im}(I + M)} \oplus \frac{\ker(A - M)}{\mathrm{im}(A^{-1} + M)}.$$

Here we fix $\mathbf{k} = (0, 1)$ and $\mathbf{k}' - \mathbf{k}$ a generator of $\ker(N - M)/\mathrm{im}(N^{-1} + M)$, so $(I+N^{-1})\mathbf{k}'-\mathbf{k}$ is a generator of $\ker(A - M)/\mathrm{im}(A^{-1} + M)$. Now, we notice that Π_6 has the subgroup $\langle a_1, a_2, \sigma\rangle \cong \Pi_2^+$ and the subgroup $\langle a_1, a_2, \tau := \sigma^2, \rho\rangle \cong \Pi_3$. As we did in the case of Π_2^+ before, we embed a_i and σ into elements of $\mathrm{Aff}(\mathbb{R}^3)$. Next we embed τ and ρ as above in the case of Π_3 using the chosen $\mathbf{k}$ and $(I + N^{-1})\mathbf{k}'$. These will result in a required embedding Π_6 into $\mathrm{Aff}(\mathbb{R}^3)$. Hence the holonomy group of Π_6 is

$$\Phi_6 = \left\langle \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right\rangle$$

and, in particular, Π_6 is orientable.

3.2. Lefschetz and Nielsen numbers. We will compute the Lefschetz, Nielsen and Reidemeister numbers of continuous maps on infra-solvmanifolds modeled on Sol using the averaging formula in Theorem 2.5. In what follows, all maps considered are assumed to be continuous maps.

Let $f\colon M \to M$ be a selfmap on a closed 3-manifold $M = \Pi\backslash \mathrm{Sol}$ with Sol-geometry. Then f induces a homomorphism $\varphi\colon \Pi \to \Pi$ and, since Γ_A is a fully invariant subgroup of Π, φ restricts to a homomorphism $\varphi'\colon \Gamma_A \to \Gamma_A$, which extends uniquely to a Lie group homomorphism $F\colon \mathrm{Sol} \to \mathrm{Sol}$. Its differential is a linearization F_* of f. On the other hand, by [18, Theorem 2.4], the homomorphism $\varphi'\colon \Gamma_A \to \Gamma_A$ is determined by $\varphi(a_i) = \mathbf{a}^{\mathbf{m}_i}$ and $\varphi(\tau) = \mathbf{a}^{\mathbf{n}}\tau^r$ for some $\mathbf{m}_i, \mathbf{n} \in \mathbb{Z}^2$ and $r \in \mathbb{Z}$. We say that φ, φ' or F_* is of type (I) if $r = 1$; of type (II) if $r = -1$; of type (III) if $r \neq \pm1$. When φ is of type (III), we have $\varphi(a_i) = 1$. We will denote by $[\varphi]$ the matrix $[\mathbf{m}_1\, \mathbf{m}_2]$. Then we may assume that F_* is expressed as

$$F_* = \begin{bmatrix} \mathbf{m}_i & \mathbf{n} \\ 0 & r \end{bmatrix} = \begin{bmatrix} [\varphi] & \mathbf{n} \\ 0 & r \end{bmatrix}$$

with respect to the basis $\{a_1, a_2, \tau\}$ of Γ_A. Consequently, by Theorem 2.5,

$$\begin{aligned} L(f) &= \frac{1}{|\Phi|} \sum_{A\in\Phi} \det(I - A_*F_*), \\ N(f) &= \frac{1}{|\Phi|} \sum_{A\in\Phi} |\det(I - A_*F_*)|, \\ R(f) &= \frac{1}{|\Phi|} \sum_{A\in\Phi} \sigma(\det(I - A_*F_*)). \end{aligned}$$

In order to use these formulas, we need to express A_* and F_* with respect to a fixed basis. For this purpose, we will use the basis $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$, see (3.1). For each Π, we also need to describe all possible $[\varphi]$ for homomorphisms $\varphi\colon \Pi \to \Pi$. This in fact gives a homotopy classification of maps on $\Pi \backslash \mathrm{Sol}$.

3.2.1. Case $\Pi = \Pi_1$ *or* $\Pi_2^{\pm}$. Let $f\colon \Pi_2^{\pm} \backslash \mathrm{Sol} \to \Pi_2^{\pm} \backslash \mathrm{Sol}$ be any map and let $\varphi\colon \Pi_2^{\pm} \to \Pi_2^{\pm}$ be a homomorphism induced by f. Since $\Gamma_A = \langle a_1, a_2, \sigma^2\rangle$ is a fully invariant subgroup of $\Pi_2^{\pm}$, it follows that $\varphi(\Gamma_A) \subset \Gamma_A$. Note that the subgroup $\langle a_1, a_2\rangle$ is fully invariant in Γ_A. Thus

$$\varphi(a_1) = \mathbf{a}^{\mathbf{m}_1}, \qquad \varphi(a_2) = \mathbf{a}^{\mathbf{m}_2}, \qquad \varphi(\sigma) = \mathbf{a}^{\mathbf{n}}\sigma^r$$

for some integer vectors $\mathbf{m}_i, \mathbf{n}$ and integer r. Put $N = N_+$ or N_-. Note also that $\varphi(\sigma^2) = \mathbf{a}^*(\sigma^2)^r$. Thus we have

$$\varphi(\sigma a_i \sigma^{-1}) = \varphi(\sigma)\varphi(a_i)\varphi(\sigma)^{-1} = (\mathbf{a}^{\mathbf{n}}\sigma^r)(\mathbf{a}^{\mathbf{m}_i})(\mathbf{a}^{\mathbf{n}}\sigma^r)^{-1} = \sigma^r(\mathbf{a}^{\mathbf{m}_i})\sigma^{-r} = \mathbf{a}^{N^r\mathbf{m}_i}.$$

Since $\sigma a_i \sigma^{-1} = N(a_i)$, we have $\mathbf{a}^{N^r\mathbf{m}_i} = \varphi(N(a_i))$, which is equivalent to $N^r[\varphi] = [\varphi]N$. Now recall from [4, Remark 5.5] that

$$PN_+P^{-1} = -\sqrt{D}, \qquad PN_-P^{-1} = -\sqrt{D}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Let $Q = P[\varphi]P^{-1}$. Then it follows that

$$\sqrt{D}^r Q = (-1)^{r+1} Q\sqrt{D} \quad \text{or} \quad \operatorname{diag}\{(-1)^r, 1\}\sqrt{D}^r Q = Q \operatorname{diag}\{-1, 1\}\sqrt{D}.$$

This yields three possibilities (cf. [4, Remark 3.7]):

(I) $\quad r = 1,\ Q = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$ and $[\varphi] = \begin{bmatrix} u & v \\ \dfrac{\ell_{21}}{\ell_{12}} v & u - \dfrac{\ell_{11} - \ell_{22}}{\ell_{12}} v \end{bmatrix}$;

(II) $\quad r = -1,\ Q = \begin{bmatrix} 0 & \gamma \\ \delta & 0 \end{bmatrix}$ and $[\varphi] = \begin{bmatrix} -u & \dfrac{\ell_{11} - \ell_{22}}{\ell_{21}} u - \dfrac{\ell_{12}}{\ell_{21}} v \\ v & u \end{bmatrix}$

(note that when $N = N_-$, the equality

$$\operatorname{diag}\{-1, 1\}\sqrt{D}^{\,-1} Q = Q \operatorname{diag}\{-1, 1\}\sqrt{D}$$

forces $Q = 0$, i.e. $\gamma = \delta = 0$);

(III) $\quad r \neq \pm 1$ and $Q = 0$;

where

$$\begin{aligned}
\alpha &= u - \frac{\ell_{11} - \ell_{22} + \sqrt{(\ell_{11}+\ell_{22})^2-4}}{2\ell_{12}}\, v,\\
\beta &= u - \frac{\ell_{11} - \ell_{22} - \sqrt{(\ell_{11}+\ell_{22})^2-4}}{2\ell_{12}}\, v,\\
\gamma &= \frac{d}{c}\left(u + \frac{\ell_{11} - \ell_{22} - \sqrt{(\ell_{11}+\ell_{22})^2-4}}{2\ell_{21}}\, v\right),\\
\delta &= \frac{c}{d}\left(u + \frac{\ell_{11} - \ell_{22} + \sqrt{(\ell_{11}+\ell_{22})^2-4}}{2\ell_{21}}\, v\right).
\end{aligned}$$

By the choice of P satisfying $Q = P[\varphi]P^{-1}$, a linearization of f, with respect to the standard ordered (linear) basis $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ of $\mathfrak{sol}$, is one of the following forms:

$$\text{(I)} \qquad \begin{bmatrix} \alpha & 0 & * \\ 0 & \beta & * \\ 0 & 0 & 1 \end{bmatrix};$$

$$\text{(II)} \qquad \begin{bmatrix} 0 & \gamma & * \\ \delta & 0 & * \\ 0 & 0 & -1 \end{bmatrix} \ (\text{if } N = N_- \text{ then } \gamma = \delta = 0);$$

$$\text{(III)} \qquad \begin{bmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & r \end{bmatrix} \ (r \neq \pm 1),$$

where $\alpha+\beta, \alpha\beta, \gamma\delta, r \in \mathbb{Z}$. Moreover, $\alpha = 0$ if and only if $\beta = 0$, and $\gamma = 0$ if and only if $\delta = 0$ (see also [18, Corollary 3.4]). Observe also that for the nontrivial element $S_\pm$ of the holonomy group $\Phi_2^\pm$,

$$S_*^\pm = \begin{bmatrix} -1 & 0 & 0 \\ 0 & \mp 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Hence the Lefschetz and Nielsen numbers of $f\colon \Pi_2^\pm \backslash \mathrm{Sol} \to \Pi_2^\pm \backslash \mathrm{Sol}$ are

$$\begin{aligned}
L(f) &= \frac{1}{2}\big(\det(I - F_*) + \det(I - S_*^\pm F_*)\big)\\
&= \begin{cases} 0 & \text{when } F_* \text{ is of type (I)},\\ 2(1-\det F_*) & \text{when } F_* \text{ is of type (II)},\\ 1-r & \text{when } F_* \text{ is of type (III)}, \end{cases}
\end{aligned}$$

$$N(f) = \begin{cases} 0 & \text{when } F_* \text{ is of type (I)},\\ 2|1-\det F_*| & \text{when } F_* \text{ is of type (II)},\\ |1-r| & \text{when } F_* \text{ is of type (III)}. \end{cases}$$

Let $f\colon \Pi_1\backslash\mathrm{Sol}\to\Pi_1\backslash\mathrm{Sol}$ be any map. By the same reason as above (by simply replacing N by A, we have $PAP^{-1}=D$), a linearization of f, with respect to the standard ordered (linear) basis $\{\mathbf{b}_1,\mathbf{b}_2,\mathbf{b}_3\}$ of $\mathfrak{sol}$, is one of the above forms (I)–(III). Hence the Lefschetz and Nielsen numbers of $f\colon \Pi_1\setminus\mathrm{Sol}\to\Pi_1\setminus\mathrm{Sol}$ are

$$L(f)=\det(I-F_*)=\begin{cases}0 & \text{when } F_* \text{ is of type (I)},\\ 2(1-\det F_*) & \text{when } F_* \text{ is of type (II)},\\ 1-r & \text{when } F_* \text{ is of type (III)},\end{cases}$$

$$N(f)=|\det(I-F_*)|=\begin{cases}0 & \text{when } F_* \text{ is of type (I)},\\ 2|1-\det F_*| & \text{when } F_* \text{ is of type (II)},\\ |1-r| & \text{when } F_* \text{ is of type (III)}.\end{cases}$$

In conclusion, we have

THEOREM 3.2. *Let f be a selfmap on $\Pi_1\setminus\mathrm{Sol}$ or $\Pi_2^{\pm}\setminus\mathrm{Sol}$. Then*

$$L(f)=\begin{cases}0 & \textit{when } F_* \textit{ is of type } \text{(I)},\\ 2(1-\det F_*) & \textit{when } F_* \textit{ is of type } \text{(II)},\\ 1-r & \textit{when } F_* \textit{ is of type } \text{(III)},\end{cases}$$

$$N(f)=\begin{cases}0 & \textit{when } F_* \textit{ is of type } \text{(I)},\\ 2|1-\det F_*| & \textit{when } F_* \textit{ is of type } \text{(II)},\\ |1-r| & \textit{when } F_* \textit{ is of type } \text{(III)}.\end{cases}$$

Noting that $\Pi_1\setminus\mathrm{Sol}$ and $\Pi_2^{+}\setminus\mathrm{Sol}$ are orientable and $\Pi_2^{-}\setminus\mathrm{Sol}$ is nonorientable, we have

COROLLARY 3.3. *Let f be a homeomorphism of $\Pi_1\setminus\mathrm{Sol}$ or $\Pi_2^{+}\setminus\mathrm{Sol}$. Then $L(f)=N(f)=0$ or 4. Furthermore, f is orientation preserving if and only if $L(f)=0$ if and only if $N(f)=0$.*

PROOF. Since f is a homeomorphism, F_* is of type (I) or (II) with $\det F_*=\pm1$. Theorem 3.2 immediately proves the first assertion. Note that f is orientation preserving if and only if $\deg f=1$. Letting $\Pi=\Pi_1$ or Π_2^{+}, we have $f_3^*\colon H^3(\Pi;\mathbb{Q})\to H^3(\Pi;\mathbb{Q})$, which is $\deg f$. We will show that $H_1(\Pi;\mathbb{Q})\cong\mathbb{Q}$. Consider $\Pi=\Pi_1$ first. Then

$$[\Pi,\Pi]=\big\langle[\tau,a_1]=a_1^{\ell_{11}-1}a_2^{\ell_{21}},[\tau,a_2]=a_1^{\ell_{12}}a_2^{\ell_{22}-1}\big\rangle\cong\mathrm{im}(A-I)$$

is a subgroup of the group $\langle a_1,a_2\rangle\cong\mathbb{Z}^2$. The index is $[\mathbb{Z}^2:\mathrm{im}(A-I)]$, which is finite because $A-I$ is invertible as A has no eigenvalue 1. Thus $\langle a_1,a_2\rangle/[\Pi,\Pi]$

is a finite group. Furthermore, we have the following commutative diagram:

$$
\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
1 & \longrightarrow & \langle a_1, a_2\rangle/[\Pi,\Pi] & \longrightarrow & \Pi/[\Pi,\Pi] & \longrightarrow & \mathbb{Z} & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \uparrow{=} & & \\
1 & \longrightarrow & \langle a_1, a_2\rangle & \longrightarrow & \Pi & \longrightarrow & \mathbb{Z} & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & & & \\
 & & [\Pi,\Pi] & \xrightarrow{=} & [\Pi,\Pi] & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 1 & & 1. & & & &
\end{array}
$$

This implies that the finitely generated abelian group $H_1(\Pi;\mathbb{Z}) = \Pi/[\Pi,\Pi]$ has free rank 1. Consequently, we have $H_1(\Pi;\mathbb{Q}) \cong \mathbb{Q}$. When $\Pi = \Pi_2^+$, we can prove the same result simply by repeating the above argument with $\tau = \rho$ and $A = N_+$. Since Π is orientable, it follows that $H^2(\Pi;\mathbb{Q}) \cong H_1(\Pi;\mathbb{Q})$ and $H^1(\Pi;\mathbb{Q}) \cong \hom(H_1(\Pi;\mathbb{Q}),\mathbb{Q})$. This shows that $f_1^* = f_2^*(=\pm 1)$ and $L(f) = 1 - \deg f$. Hence the second assertion follows immediately from Theorem 3.2. □

Corollary 3.4. *Let f be a selfmap on $\Pi_2^- \backslash \mathrm{Sol}$. Then we have*:

(a) *If $f\colon \Pi_2^- \backslash \mathrm{Sol} \to \Pi_2^- \backslash \mathrm{Sol}$ is of type* (II), *then $L(f) = N(f) = 2$.*
(b) *If f is a homeomorphism of $\Pi_2^- \backslash \mathrm{Sol}$, then $L(f) = N(f) = 0$.*

Proof. If f is of type (II), then $\gamma = \delta = 0$ and so $\det F_* = 0$, thus $L(f) = N(f) = 2$ by Theorem 3.2. If f is a homeomorphism, then $\deg F_* = \pm 1$ and so F_* must be of type (I), hence $L(f) = N(f) = 0$ by Theorem 3.2. □

Remark 3.5. Our result is exactly the same as Theorems 2.2, 2.3 and 2.4 and Corollary 2.5 of [3]. See also [11, Proposition 7.5]. We need to remark further the case that $\Pi = \Pi_2^-$ and a linearization of f is of type (II). This case corresponds to the case when $k = -1$ and $\det A = -1$ in Theorem 2.4 of [3]. In our notation, $k = r = -1$ and $A = N_-$. We observed that this case reduces to $Q = 0$ and so $B = 0$ in Theorem 2.4 of [3]. Therefore, $\det F_* = 0$ and so $L(f) = N(f) = 2$.

In [3, Remark 2.1], the authors questioned what the set $n(M) := \{N(f) : f\colon M \to M\}$ would be for any given Sol 3-manifold M that is a torus bundle over S^1. The following is the answer to this question.

Corollary 3.6. *Let M be a closed 3-manifold with Sol-geometry that is a torus bundle over S^1. Then $n(M) = \{0\} \cup \mathbb{N}$.*

Proof. When $M = \Pi_1 \setminus \mathrm{Sol}$, we consider the endomorphism $\varphi\colon \Pi_1 \to \Pi_1$ defined as: $\varphi(a_1) = \varphi(a_2) = 1$ and $\varphi(\tau) = \tau^r$ for some $r \in \mathbb{Z}$. Then the Nielsen number $N(\varphi)$ is $|1-r|$. This shows that $n(\Pi_1 \setminus \mathrm{Sol}) = \{0\} \cup \mathbb{N}$.

When $M = \Pi_2^{\pm} \setminus \mathrm{Sol}$, we consider the endomorphism $\varphi\colon \Pi_2^{\pm} \to \Pi_2^{\pm}$ defined as: $\varphi(a_1) = \varphi(a_2) = 1$ and $\varphi(\sigma) = \sigma^r$ for some $r \in \mathbb{Z}$. Then $\varphi(\tau) = \varphi(\sigma^2) = \sigma^{2r} = \tau^r$. Thus the Nielsen number $N(\varphi)$ is 0 when $r = 1$, is 2 when $r = -1$ and is $|1-r|$ when $r \neq \pm 1$. This shows that $n(\Pi_2^{\pm} \setminus \mathrm{Sol}) = \{0\} \cup \mathbb{N}$. □

3.2.2. Case $\Pi = \Pi_3$. Consider a selfmap f on the closed 3-manifold $\Pi_3 \setminus \mathrm{Sol}$ and let $\varphi\colon \Pi_3 \to \Pi_3$ be the homomorphism induced by f. Since $\Gamma_A = \langle a_1, a_2, \tau\rangle = \Pi_1$ is a fully invariant subgroup of Π_3, we have, as before,

$$\varphi(a_1) = \mathbf{a}^{\mathbf{m}_1}, \qquad \varphi(a_2) = \mathbf{a}^{\mathbf{m}_2}, \qquad \varphi(\tau) = \mathbf{a}^{\mathbf{n}}\tau^r, \qquad \varphi(\rho) = \mathbf{a}^{\mathbf{x}}\tau^z\rho^w$$

for some integer vectors $\mathbf{m}_i, \mathbf{n}, \mathbf{x}$ and integers r, z and $w \in \{0,1\}$. The restriction of φ to Π_1 is a homomorphism $\varphi'\colon \Pi_1 \to \Pi_1$. Hence we can choose a matrix P so that $PAP^{-1} = D$, and $Q = P[\varphi]P^{-1}$ is one of the three types given in the case of Π_1 before. Consequently, a linearization of f, with respect to the standard ordered (linear) basis $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ of $\mathfrak{sol}$, has one of the above forms (I)–(III). Note that a diagonalizing matrix P of A used above is

$$P = \begin{bmatrix} \dfrac{\sqrt{\ell_{11}^2 - 1}}{\ell_{21}}\, c & -\dfrac{\sqrt{\ell_{11}^2 - 1}}{\ell_{21}}\, d \\ c & d \end{bmatrix}^{-1}$$

and so

$$M' := PMP^{-1} = \begin{bmatrix} 0 & d/c \\ c/d & 0 \end{bmatrix}.$$

Next, we will find further necessary conditions on φ. Since φ preserves the relation $\rho a_i \rho^{-1} = M(a_i)$, we have $A^z M^w[\varphi] = [\varphi]M$ and so $D^z {M'}^w Q = QM'$.

Consider first the case when $w = 0$. Then, since $D^z Q = QM'$, we can see easily that $\alpha = \beta = 0$ (when Q is of type (I)) and $\gamma = \delta = 0$ (when Q is of type (II)). Thus $[\varphi] = 0 = Q$. Since $\rho^2 = \mathbf{a}^{\mathbf{e}_2} = a_2$, it follows that $\varphi(\rho)^2 = \varphi(a_2) = 1$ and so $\varphi(\rho) = 1$ because $\varphi(\rho)$ cannot be a torsion element. Furthermore, since $\rho\tau\rho^{-1} = \mathbf{a}^{\mathbf{k}'}\tau^{-1}$, we see that $\varphi(\tau) = \varphi(\tau)^{-1}$ and hence $\varphi(\tau) = 1$ and so in particular $r = 0$. (This proves the first part of [3, Theorem 3.5].) Thus we see that φ is a trivial homomorphism. Hence we conclude that

$$w = 0 \iff Q(=0) \text{ is of type (III) and } r = 0 \iff \varphi \text{ is trivial.}$$

Notice that there is no homomorphism of type (III) with $r \neq 0$, and if F_* is of type (III) then φ is a trivial homomorphism and hence f is homotopic to a constant map.

Suppose now that $w = 1$. Then $Q \neq 0$ by the above observation. Recall that if $Q \neq 0$, then $\alpha\beta \neq 0$ and $\gamma\delta \neq 0$. Thus $r = \pm 1$ and so Q is of type (I) or (II).

The relation $\tau a_i \tau^{-1} = A(a_i)$ yields that $A^r[\varphi] = [\varphi]A$ and $D^r Q = QD$. This implies that $r = 1$ when Q is of type (I) and $r = -1$ if Q is of type (II). Hence we conclude that

$$w = 1 \;\Leftrightarrow\; Q \neq 0 \text{ is of type (I) or (II)} \;\Leftrightarrow\; \varphi \text{ is nontrivial.}$$

The relation $\rho a_i \rho^{-1} = M(a_i)$ together with $M^w = M$ implies $D^z M' Q = QM'$, so we have $\alpha = e^{t_0 z}\beta$ or $\gamma = (d/c)^2 e^{t_0 z}\delta$ according as Q is of type (I) or (II), respectively. Therefore $\det F_* = \alpha\beta = e^{t_0 z}\beta^2 > 0$ (when Q is of type (I)) or $\det F_* = \gamma\delta = (d/c)^2 e^{t_0 z}\delta^2 > 0$ (when Q is of type (II)). Since $\ell_{11} = \ell_{22}$ in Π_3, we see that

$$\alpha = u - \frac{\sqrt{\ell_{11}^2 - 1}}{\ell_{12}}\, v, \qquad \beta = u + \frac{\sqrt{\ell_{11}^2 - 1}}{\ell_{12}}\, v,$$

$$\gamma = \frac{d}{c}\left(u - \frac{\sqrt{\ell_{11}^2 - 1}}{\ell_{21}}\, v\right), \qquad \delta = \frac{c}{d}\left(u + \frac{\sqrt{\ell_{11}^2 - 1}}{\ell_{12}}\, v\right),$$

and so $\det F_* = u^2 - \ell_{21} v^2/\ell_{12}$ (when Q is of type (I)) or $u^2 - \ell_{12} v^2/\ell_{21}$ (when Q is of type (II)). Furthermore, if φ is an isomorphism, then $\det Q = \pm 1$ and $\det F_* = \pm 1$. As $\det F_* > 0$, we have $\det F_* = 1$. Recall that for the nontrivial element T of the holonomy group Φ_3,

$$T_* = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Consequently, we have:

Theorem 3.7. *The Lefschetz and Nielsen numbers of $f\colon \Pi_3 \backslash \mathrm{Sol} \to \Pi_3 \backslash \mathrm{Sol}$ are*

$$L(f) = \begin{cases} 1 - \det F_* & \textit{when } F_* \textit{ is of type } (\mathrm{I}) \textit{ or } (\mathrm{II}), \\ 1 & \textit{when } F_* \textit{ is of type } (\mathrm{III}), \end{cases}$$

$$N(f) = \begin{cases} \det F_* - 1 & \textit{when } F_* \textit{ is of type } (\mathrm{I}) \textit{ or } (\mathrm{II}), \\ 1 & \textit{when } F_* \textit{ is of type } (\mathrm{III}). \end{cases}$$

Corollary 3.8. *Let f be a homeomorphism of $\Pi_3 \backslash \mathrm{Sol}$. Then $L(f) = N(f) = 0$ and f is orientation preserving.*

Proof. Since f is a homeomorphism, we have $\det F_* = 1$ and so $L(f) = N(f) = 0$ (see also [11, Proposition 7.5] and [3, Theorem 3.4]). Observe that

$$[\Pi_3, \Pi_3] = \big\langle [\tau, a_1] = a_1^{\ell_{11}-1} a_2^{\ell_{21}}, [\tau, a_2] = a_1^{\ell_{12}} a_2^{\ell_{22}-1}, [\rho, a_1] = a_1^{-2}, [\rho, \tau] = \mathbf{a}^{\mathbf{k}'}\tau^{-2} \big\rangle.$$

Then $[\Pi_3, \Pi_3]$ is a subgroup of $\langle a_1, a_2, \tau\rangle \cong \Pi_1$ and contains the group

$$\langle a_1^{\ell_{11}-1} a_2^{\ell_{21}}, a_1^{\ell_{12}} a_2^{\ell_{22}-1} \rangle \cong \mathrm{im}(A - I).$$

This implies that $\Pi_1/[\Pi_3, \Pi_3]$ is a finite group. Finally because of the following commutative diagram:

$$\begin{tikzcd}
 & 1 & 1 & & \\
1 \arrow[r] & \Pi_1/[\Pi_3,\Pi_3] \arrow[u] \arrow[r] & \Pi_3/[\Pi_3,\Pi_3] \arrow[u] \arrow[r] & \mathbb{Z}_2 \arrow[r] & 1 \\
1 \arrow[r] & \Pi_1 \arrow[u] \arrow[r] & \Pi_3 \arrow[u] \arrow[r] & \mathbb{Z}_2 \arrow[u, "="'] \arrow[r] & 1 \\
 & {[\Pi_3,\Pi_3]} \arrow[u] \arrow[r, "="] & {[\Pi_3,\Pi_3]} \arrow[u] & & \\
 & 1 \arrow[u] & 1 \arrow[u] & &
\end{tikzcd}$$

we can see that $H_1(\Pi_3; \mathbb{Z}) = \Pi_3/[\Pi_3, \Pi_3]$ is a finite group. Thus $H_1(\Pi_3; \mathbb{Q}) = 0$ and so $H^2(\Pi_3; \mathbb{Q}) = 0 = H^1(\Pi_3; \mathbb{Q})$. Since $L(f) = 1 - \deg f = 0$, it follows that $\deg f = 1$, and hence f is orientation preserving. □

REMARK 3.9. Let f be a selfmap on the closed 3-manifold $M = \Pi_3 \backslash \mathrm{Sol}$ and let $\varphi \colon \Pi_3 \to \Pi_3$ be the homomorphism induced by f. Consider the set $n(M)$. If φ is trivial then $N(f) = 1 \in n(M)$ and we may assume that F_* is of type (I). Then $N(f) = \det F_* - 1 = u^2 - \ell_{21} v^2/\ell_{12} - 1$. Let $\ell = \gcd(\ell_{12}, \ell_{21})$, and write $\ell_{12} = p\ell$ and $\ell_{21} = q\ell$. Since $\ell_{21}v/\ell_{12} = qv/p \in \mathbb{Z}$, we must have $v = pv'$ and so $\det F_*$ is reduced to $u^2 - pq(v')^2$ with $\gcd(p, q) = 1$. Hence the question of finding the set $n(M)$ is reduced to a problem of solving the Diophantine equations of the form $u^2 - pqv^2 = k$ for all positive integers k. These equations are known as Pell-type equations (cf. [1]).

3.2.3. Case $\Pi = \Pi_6$. Consider a closed 3-manifold $\Pi_6 \backslash \mathrm{Sol}$ and a selfmap f, and let $\varphi \colon \Pi_6 \to \Pi_6$ be the homomorphism induced by f. Since $\Gamma_A = \langle a_1, a_2, \sigma^2 \rangle = \Pi_1$ and $\langle a_1, a_2, \sigma \rangle = \Pi_2^+$ are fully invariant subgroups of Π_6, we have, as before,

$$\varphi(a_1) = \mathbf{a}^{\mathbf{m}_1}, \qquad \varphi(a_2) = \mathbf{a}^{\mathbf{m}_2}, \qquad \varphi(\sigma) = \mathbf{a}^{\mathbf{n}}\sigma^r, \qquad \varphi(\rho) = \mathbf{a}^{\mathbf{x}}\sigma^z\rho^w$$

for some integer vectors $\mathbf{m}_i, \mathbf{n}, \mathbf{x}$ and integers r, z and $w \in \{0, 1\}$.

The relations of Π_6 will provide further restrictions on these integers. For example, if $w = 0$ then $z = 0$. Indeed, if $w = 0$ then $\varphi(\rho) = \mathbf{a}^{\mathbf{x}}\sigma^z$. The relation $\rho^2 = \mathbf{a}^{\mathbf{e}_2}$ after taking φ reduces to $(\mathbf{a}^{\mathbf{x}}\sigma^z)^2 = \mathbf{a}^{\mathbf{m}_2}$, hence $\mathbf{a}^*\sigma^{2z} = \mathbf{a}^*$. This yields $z = 0$. However z is not necessarily 0.

Consider the subgroup $\langle a_1, a_2, \sigma^2, \rho\rangle = \Pi_3$ of Π_6. It is easy to see that Π_3 is a normal subgroup of Π_6. Since

$$\varphi(\sigma^2) = (\mathbf{a}^{\mathbf{n}}\sigma^r)^2 = \mathbf{a}^{\mathbf{n}+N^r\mathbf{n}}(\sigma^2)^r,$$

it follows that Π_3 is φ-invariant if and only if $\varphi(\rho) \in \Pi_3$ if and only if z is even. The following is an example showing that Π_3 is not a characteristic subgroup and so not a fully invariant subgroup of Π_6.

EXAMPLE 3.10. Consider Π_6 with defining matrices N, A and $\mathbf{k} = \mathbf{e}_2$, $\mathbf{k}' = \mathbf{e}_1$:

$$N = -\begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}, \qquad A = \begin{bmatrix} 17 & 12 \\ 24 & 17 \end{bmatrix}.$$

We will show that there is an automorphism φ determined by the following identities:

$$\varphi(a_1) = a_1^u a_2^{2v}, \qquad \varphi(a_2) = a_1^u a_2^v, \qquad \varphi(\sigma) = a_1^{n_1} a_2^{n_2}\sigma, \qquad \varphi(\rho) = a_1^x a_2^y \sigma\rho.$$

Because the relations of Π_6 must be preserved by φ, we then have

$$\begin{aligned}
&\sigma a_i \sigma^{-1} = N(a_i) \ \Rightarrow\ N[\varphi] = [\varphi]N,\\
&\rho a_i \rho^{-1} = M(a_i) \ \Rightarrow\ NM[\varphi] = [\varphi]M \Rightarrow u = v,\\
&\rho^2 = a_2 \ \Rightarrow\ \begin{bmatrix} x \\ y \end{bmatrix} + N\begin{bmatrix} -x \\ y \end{bmatrix} + N\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} v \\ u \end{bmatrix} \ \Rightarrow\ 4x - 2y - 3 = u,\\
&\rho\sigma\rho^{-1}\sigma = a_1 \ \Rightarrow\ \begin{bmatrix} x \\ y \end{bmatrix} + N\begin{bmatrix} -n_1 + 1 \\ n_2 \end{bmatrix} + N^{-1}\begin{bmatrix} -x + n_1 \\ -y + n_2 \end{bmatrix} = \begin{bmatrix} u \\ 2v \end{bmatrix}\\
&\qquad\qquad\qquad \Rightarrow\ 4n_1 - 3n_2 - 2x + 2y - 2 = v.
\end{aligned}$$

A choice of $u = v = 1$, $x = 1$, $y = 0$ and $n_1 = 2$, $n_2 = 1$ gives rise to an automorphism φ with $z = 1$. Hence Π_3 is not a characteristic subgroup of Π_6.

Consider the restriction of φ to Γ_A which is a homomorphism $\varphi\colon \Pi_1 \to \Pi_1$. Then we can choose a matrix P so that $PAP^{-1} = D$, and $Q = P[\varphi]P^{-1}$ is one of the three types given in the case of Π_1 before. Consequently, a linearization of f, with respect to the standard ordered (linear) basis $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ of $\mathfrak{sol}$, has one of the above forms (I), (II) and (III). Note that a diagonalizing matrix P of A used above is

$$P = \begin{bmatrix} \dfrac{\sqrt{s^2-1}}{q}c & -\dfrac{\sqrt{s^2-1}}{q}d \\ c & d \end{bmatrix}^{-1}$$

and

$$PAP^{-1} = D, \qquad PNP^{-1} = -\sqrt{D}, \qquad M' := PMP^{-1} = \begin{bmatrix} 0 & d/c \\ c/d & 0 \end{bmatrix}.$$

We can find further necessary conditions on φ. Since φ preserves the relation $\rho a_i\rho^{-1} = M(a_i)$, we have $N^z M^w[\varphi] = [\varphi]M$ and so $(-1)^z\sqrt{D}^z M'^w Q = QM'$. Similarly, the relation $\sigma a_i\sigma^{-1} = N(a_i)$ yields that $N^r[\varphi] = [\varphi]N$ and so $(-1)^r\sqrt{D}^r Q = -Q\sqrt{D}$.

If $w = 0$ (so that $z = 0$ and $Q = QM'$), then we can see easily that $\alpha = \beta = 0$ (when Q is of type (I)) and $\gamma = \delta = 0$ (when Q is of type (II)); hence $[\varphi] = 0 = Q$. In the case when $Q = 0$, $\rho^2 = \mathbf{a}^{\mathbf{k}} = a_2 \Rightarrow \varphi(\rho)^2 = \varphi(a_2) = 1$ and so $\varphi(\rho) = 1$ because $\varphi(\rho)$ cannot be a torsion element. Furthermore, if $\rho\sigma\rho^{-1} = \mathbf{a}^{\mathbf{k}'}\sigma^{-1}$ then $\varphi(\sigma) = \varphi(\sigma)^{-1}$ and hence $\varphi(\sigma) = 1$. Consequently, if $w = 0$ then φ is trivial, in particular $r = 0$. In conclusion,

(i) $w = 0 \Leftrightarrow Q(=0)$ is of type (III) and $r = 0 \Leftrightarrow \varphi$ is trivial.
(ii) $w = 1 \Leftrightarrow Q \neq 0$ is of type (I) or (II) $\Leftrightarrow \varphi$ is nontrivial.

We have observed that $w = 0$ or 1 and in each case $r = 0$ or ± 1, respectively. (This proves the first part of [3, Theorem 3.5], namely, $\deg \overline{f}_1 \in \{0, \pm 1\}$.) Suppose now that $r = \pm 1$. Recalling that if $r = 1$ then Q is of type (I) and if $r = -1$ then Q is of type (II), the identity $(-1)^r\sqrt{D}^r Q = -Q\sqrt{D}$ gives no further restriction on r. However, since $w = 1$, we have that $(-1)^z\sqrt{D}^z M'Q = QM'$. This identity implies that

$$Q = \begin{cases} \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} = \begin{bmatrix} (-1)^z e^{t_0 z/2}\beta & 0 \\ 0 & \beta \end{bmatrix} & \text{when } F_* \text{ is of type (I)}, \\ \begin{bmatrix} 0 & \gamma \\ \delta & 0 \end{bmatrix} = \begin{bmatrix} 0 & (-1)^z (d/c)^2 e^{t_0 z/2}\delta \\ \delta & 0 \end{bmatrix} & \text{when } F_* \text{ is of type (II)}. \end{cases}$$

Hence

$$\det F_* = \begin{cases} \det Q = (-1)^z e^{t_0 z/2}\beta^2 & \text{when } F_* \text{ is of type (I)}, \\ -\det Q = (-1)^z (d/c)^2 e^{t_0 z/2}\delta^2 & \text{when } F_* \text{ is of type (II)}. \end{cases}$$

If f is a homeomorphism, then $\det F_* = \pm 1 = (-1)^z$. Example 3.10 shows that there is a homeomorphism f of $\Pi_6 \setminus \mathrm{Sol}$ with $\det F_* = -1$. Let S and T be generators of the holonomy group Φ_6. Then

$$S_* = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad T_* = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Now we have:

THEOREM 3.11. *The Lefschetz and Nielsen numbers of* $f\colon \Pi_6\backslash\mathrm{Sol} \to \Pi_6\backslash\mathrm{Sol}$ *are*

$$L(f) = \begin{cases} 1-\det F_* & \text{when } F_* \text{ is of type (I) or (II)},\\ 1 & \text{when } F_* \text{ is of type (III)}, \end{cases}$$

$$N(f) = \begin{cases} |1-\det F_*| & \text{when } F_* \text{ is of type (I) or (II)},\\ 1 & \text{when } F_* \text{ is of type (III)}. \end{cases}$$

PROOF. Indeed, if F_* is of type (III) then φ is a trivial homomorphism and hence $F_* = 0$; thus

$$\begin{aligned} L(f) &= \frac{1}{4}\big(\det(I-F_*) + \det(I-T_*F_*) + \det(I-S_*F_*) + \det(I-T_*S_*F_*)\big)\\ &= \frac{1}{4}(1+1+1+1) = 1. \end{aligned}$$

If F_* is of type (I) then

$$\begin{aligned} L(f) &= \frac{1}{4}\big(\det(I-F_*) + \det(I-T_*F_*) + \det(I-S_*F_*) + \det(I-T_*S_*F_*)\big)\\ &= \frac{1}{4}\big(0 + 2(1-\alpha\beta) + 0 + 2(1-\alpha\beta)\big) = 1-\alpha\beta = 1-\det F_*. \end{aligned}$$

A similar computation for $N(f)$ and for F_* of type (II) yields the above identities. □

REMARK 3.12. We should notice that our result is different from the corresponding result, Theorem 3.5 in [3]. In the proof of [3, Theorem 3.5], the authors stated that [3, Corollary 2.5] implies $N(f_1) = N(\tau_* f_1)$ if $\deg \overline{f}_1 = \pm 1$. But this identity is not true in general from [3, Corollary 2.5] since $\deg \overline{\tau_* f_1} = -\deg \overline{f}_1$.

REMARK 3.13. Recall from for example [4, Section 9] that Π_6 is isomorphic to the fundamental group of a torus semi-bundle N_ϕ for some $\phi \in \mathrm{SL}(2,\mathbb{Z})$. When

$$\phi = \begin{bmatrix} a & b\\ c & d \end{bmatrix}$$

with $abc \neq 0$ and $ad - bc = 1$, we can choose an isomorphism so that

$$N = -\begin{bmatrix} s & p\\ q & s \end{bmatrix} = \begin{bmatrix} ad+bc & -2ac\\ -2bd & ad+bc \end{bmatrix}$$

(see for example (3.1) in [3]). Then $ad + bc = -s$, hence $ad = (1-s)/2 < 0$ because $s > 1$.

More explicitly, we consider N_ϕ with $\phi = \begin{bmatrix} -1 & 2\\ -1 & 1 \end{bmatrix}$, then

$$\begin{bmatrix} ad+bc & -2ac\\ -2bd & ad+bc \end{bmatrix} = -\begin{bmatrix} 3 & 2\\ 4 & 3 \end{bmatrix}.$$

Hence the group Π_6 in Example 3.10 is isomorphic to $\pi_1(N_\phi)$. By [21, Theorem 1.7], $\delta(a,d) := ad/\gcd(a,d)^2 = -1$ and odd, so the degrees of all selfmaps of N_ϕ are

$$\{\pm(2\ell+1)^2 : \ell \in \mathbb{Z}\}.$$

Corollary 3.14. *Let f be a homeomorphism of $\Pi_6 \setminus \mathrm{Sol}$. Then $L(f) = N(f) = 0$ or 2. Furthermore, f is orientation preserving if and only if $L(f) = 0$ if and only if $N(f) = 0$.*

Proof. If f is a homeomorphism then we have shown that $\det F_* = \pm 1$. By Theorem 3.11, $L(f) = 1 - \det F_* = 0$ or 2, and $N(f) = |L(f)|$. The first assertion is proved. The second assertion follows from the same argument as in the proof of Corollary 3.8. Indeed, we can show that $H_1(\Pi_6;\mathbb{Q}) = 0$ and so $H^2(\Pi_6;\mathbb{Q}) = 0 = H^1(\Pi_6;\mathbb{Q})$. It now follows that $L(f) = 1 - \deg f$. For a homeomorphism f, we have that f is orientation preserving if and only if $\deg f = 1$ if and only if $L(f) = 0$. By Theorem 3.11, we have $N(f) = |L(f)| = 0$.□

In the following example, we show that there is a selfmap f on $\Pi_6 \setminus \mathrm{Sol}$ with $\deg f < 0$. This example provides a counterexample to [3, Theorem 3.4], see the proof of [3, Theorem 3.4] and [3, Remark 3.2] where it is stated that every homeomorphism of N_ϕ has degree $+1$.

Example 3.15. Consider Π_6 given in Example 3.10. Let φ be the automorphism given in Example 3.10:

$$\varphi(a_1) = a_1a_2^2, \qquad \varphi(a_2) = a_1a_2, \qquad \varphi(\sigma) = a_1^2a_2\sigma, \qquad \varphi(\rho) = a_1\sigma\rho.$$

Let f be a selfmap of $\Pi_6 \setminus \mathrm{Sol}$ inducing φ. Since φ is an automorphism and the Borel conjecture in dimension 3 is true (cf. [19, Remark 4.5]), it follows that f is a homeomorphism. Then

$$F_* = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

and so $L(f) = 1 - \det F_* = 2 = N(f)$. Thus Corollary 3.14 implies that $\deg f = -1$.

Corollary 3.16. *Let φ be an endomorphism of Π_6 given by*

$$\varphi(a_1) = \mathbf{a}^{\mathbf{m}_1}, \qquad \varphi(a_2) = \mathbf{a}^{\mathbf{m}_2}, \qquad \varphi(\sigma) = \mathbf{a}^{\mathbf{n}}\sigma^r, \qquad \varphi(\rho) = \mathbf{a}^{\mathbf{x}}\sigma^z\rho^w$$

for some integer vectors $\mathbf{m}_i, \mathbf{n}, \mathbf{x}$ and integers r, z and $w \in \{0,1\}$. When φ is an automorphism, we have that $N(\varphi) = 0$ if and only if z is even.

Proof. If φ is an automorphism, then we have seen that $\det F_* = (-1)^z = \pm 1$ and so $N(\varphi) = |1 - (-1)^z|$. □

REMARK 3.17. Just like for the set $n(\Pi_3 \setminus \mathrm{Sol})$, the question of finding the set $n(\Pi_6 \setminus \mathrm{Sol})$ is reduced to a problem of solving certain Diophantine equations $u^2 - pqv^2 = k$ for all integers k.

3.3. Jiang-type property and Reidemeister numbers. In this section, we study the Jiang-type property for infra-solvmanifolds modeled on Sol and show that infra-solvmanifolds of type (R) are not necessarily Jiang-type. We also compute the Reidemeister numbers $R(\varphi)$ for all endomorphisms φ of Π.

THEOREM 3.18. *The infra-solvmanifolds* $\Pi_1 \setminus \mathrm{Sol}$ *and* $\Pi_2^{\pm} \setminus \mathrm{Sol}$ *are Jiang-type spaces.*

PROOF. For any selfmap f, suppose $L(f) = 0$. By Theorem 3.2, F_* is of type (I) or F_* is of type (II) with $\det F_* = 1$. In each case, by Theorem 3.2 again, we have $N(f) = 0$.

Next suppose $L(f) \neq 0$. By Theorem 3.2, F_* is of type (II) with $\det F_* \neq 1$ or F_* is of type (III). In each case, by Theorem 3.2 again, we have $N(f) \neq 0$. It remains to show that $N(f) = R(f)$. When $M = \Pi_1 \setminus \mathrm{Sol}$, by Theorem 2.5 we have that $R(f) = \sigma(\det(I - F_*)) = |\det(I - F_*)| = N(f)$ since $N(f) = |\det(I - F_*)| \neq 0$. When $M = \Pi_2^+ \setminus \mathrm{Sol}$, by Theorem 2.5 we have

$$\begin{aligned} R(f) &= \frac{1}{2}\big(\sigma(\det(I - F_*)) + \sigma(\det(I - S_*^+ F_*))\big) \\ &= \frac{1}{2}\big(|\det(I - F_*)| + |\det(I - S_*^+ F_*)|\big) = N(f) \end{aligned}$$

because if F_* is of type (II) with $\det F_* \neq 1$ then $S_*^+ F_*$ is of type (II) with $\det S_*^+ F_* = \det F_* \neq 1$ and hence two terms are (equal and) nonzero; if F_* is of type (III) then $S_*^+ F_* = F_*$ is of type (III) and hence two terms are (equal and) nonzero. Finally we consider the case $M = \Pi_2^- \setminus \mathrm{Sol}$. By Theorem 2.5, we have

$$\begin{aligned} R(f) &= \frac{1}{2}\big(\sigma(\det(I - F_*)) + \sigma(\det(I - S_*^- F_*))\big) \\ &= \frac{1}{2}\big(|\det(I - F_*)| + |\det(I - S_*^- F_*)|\big) = N(f) \end{aligned}$$

because if F_* is of type (II) then $Q = 0$ and so $\det F_* = 0$ and $S_*^- F_* = F_*$ is of type (II) and $\det S_*^- F_* = \det F_* = 0$ and hence each term is equal to 2; if F_* is of type (III) then $S_*^- F_* = F_*$ is of type (III) and hence each term is equal to $|1 - r| \neq 0$. □

When $\varphi\colon \Pi \to \Pi$ is an automorphism of the SB-groups Π, we computed the Reidemeister number $R(\varphi)$ in [5]. In the following, we will compute $R(\varphi)$ for all endomorphisms φ.

PROPOSITION 3.19. *For any endomorphism* $\varphi\colon \Pi_1 \to \Pi_1$*, the following hold*:

(a) *If* φ *is of type* (I) *or of type* (II) *with* $\det F_* = 1$*, then* $R(\varphi) = \infty$.

(b) *If φ is of type* (II) *with* $\det F_* \neq 1$*, then* $R(\varphi) = 2|1 - \det F_*|$.

(c) *If φ is of type* (III)*, then* $R(\varphi) = |1 - r|$.

Proof. By Theorem 2.5, we have $R(\varphi) = \sigma(\det(I - F_*))$. It is immediate that $\det(I - F_*) = 0$ if and only if φ is of type (I) or of type (II) with $\det F_* = 1$. When φ is of type (II), we can see that $\det(I - F_*) = 2(1 - \gamma\delta) = 2(1 - \det F_*)$. It is clear that if φ is of type (III), then $R(\varphi) = |1 - r|$. □

Proposition 3.20. *For any endomorphism* $\varphi\colon \Pi_2^{\pm} \to \Pi_2^{\pm}$*, the following hold*:

(a) *If φ is of type* (I) *or of type* (II) *with* $\det F_* = 1$*, then* $R(\varphi) = \infty$.

(b) *If* $\varphi\colon \Pi_2^{+} \to \Pi_2^{+}$ *is of type* (II)*, then* $R(\varphi) = 2|1 - \det F_*|$.

(c) *If* $\varphi\colon \Pi_2^{-} \to \Pi_2^{-}$ *is of type* (II)*, then* $R(\varphi) = 2$.

(d) *If φ is of type* (III)*, then* $R(\varphi) = |1 - r|$.

Proof. By Theorem 2.5, we have

$$R(\varphi) = \frac{1}{2}\big(\sigma(\det(I - F_*)) + \sigma(\det(I - S_*^{\pm} F_*))\big).$$

Hence $R(\varphi) = \infty$ if and only if $\det(I - F_*) = 0$ or $\det(I - S_*^{\pm} F_*) = 0$.

If F_* is of type (I), then $\det(I - F_*) = 0$ and so $R(\varphi) = \infty$. If F_* is of type (III), then $F_* = S_*^{\pm} F_*$ and $\det(I - F_*) = \det(I - S_*^{\pm} F_*) = 1 - r$, hence $R(\varphi) = |1 - r| < \infty$.

Assume F_* is of type (II). Then we can show that $\det(I - F_*) = \det(I - S_*^{\pm} F_*) = 2(1 - \gamma\delta) = 2(1 - \det F_*)$. Recall that when S_*^{-} is involved, $\gamma = \delta = 0$ and so $\det(I - F_*) = \det(I - S_*^{-} F_*) = 2$, hence $R(\varphi) = 2$ and $\det F_* = 0$. It is clear that if φ is of type (III), then $R(\varphi) = |1 - r|$. □

In [5, Theorem 3.3], we have shown that $R(\varphi) = \infty$ for all automorphisms φ of Π_3 or Π_6. We now show that $R(\varphi) = \infty$ for all *non-trivial* endomorphisms.

Proposition 3.21. *For any endomorphism φ of Π_3 or Π_6, we have*

$$R(\varphi) = \begin{cases} 1 & \text{when } \varphi \text{ is trivial,} \\ \infty & \text{when } \varphi \text{ is non-trivial.} \end{cases}$$

Proof. By Theorem 2.5, we have

$$R(\varphi) = \begin{cases} \dfrac{1}{2}\big(\sigma(\det(I - F_*)) + \sigma(\det(I - T_* F_*))\big) & \text{when } \Pi = \Pi_3, \\ \dfrac{1}{4}\big(\sigma(\det(I - F_*)) + \sigma(\det(I - T_* F_*)) & \\ \qquad + \sigma(\det(I - S_* F_*)) + \sigma(\det(I - S_* T_* F_*))\big) & \text{when } \Pi = \Pi_6. \end{cases}$$

If F_* is of type (III), then, by the results of Sections 3.2.2 and 3.2.3, φ is trivial and vice versa, hence $\det(I - F_*) = \det(I - T_* F_*) = \det(I - S_* F_*) = \det(I - S_* T_* F_*) = 1$ and so $R(\varphi) = 1$. If F_* is of type (I), then $\det(I - F_*) = 0$ and so $R(\varphi) = \infty$. Assume F_* is of type (II), then $T_* F_*$ is of type (I) and so $R(\varphi) = \infty$. □

Theorem 3.22. *The infra-solvmanifolds* $\Pi_3 \backslash \mathrm{Sol}$ *and* $\Pi_6 \backslash \mathrm{Sol}$ *are not Jiang-type spaces.*

Proof. Consider any endomorphism φ of Π_3 or Π_6 of type (I) with $\det F_* \neq 1$. From the results of Sections 3.2.2 and 3.2.3, we have

$$L(f) = 1 - \det F_* \neq 0, \qquad N(f) = |\det F_* - 1|,$$

where f is the selfmap on $\Pi \backslash \mathrm{Sol}$ determined by φ. But by Proposition 3.21, $R(\varphi) = \infty$. Thus the infra-solvmanifolds $\Pi_3 \backslash \mathrm{Sol}$ and $\Pi_6 \backslash \mathrm{Sol}$ are not Jiang-type spaces.

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