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EXISTENCE OF SOLUTIONS TO A SEMILINEAR ELLIPTIC BOUNDARY VALUE PROBLEM WITH AUGMENTED MORSE INDEX BIGGER THAN TWO

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Abstract. Building on the construction of least energy sign-changing solutions to variational semilinear elliptic boundary value problems introduced in [5], we prove the existence of a solution with *augmented Morse index* at least three when a sublevel of the corresponding action functional has nontrivial topology. We provide examples where the set of least energy sign changing solutions is disconnected, hence has nontrivial topology.

1. Introduction

We consider the existence of solutions to the equation

$$
\begin{cases} -\Delta u = f(u) & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}
$$

where Ω is a bounded subset of $\mathbb{R}^n$, its boundary $\partial\Omega$ is Lipschitizian, and f is a differentiable function.

The solvability of (1.1) has motivated fundamental developments in critical point theory in the last fifty years. The *mountain pass lemma* was developed in [2]

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by A. Ambrosetti and P.H. Rabinowitz in order the prove the existence of positive solutions to (1.1). The *saddle point principle* proved by P.H. Rabinowitz in [14] was motivated by the solvability of (1.1) in the presence of resonance. In [16], Z.-Q. Wang studied connections between mountain passes in order to establish the existence of solutions to (1.1) given by critical points with *augmented Morse* index greater that or equal to two, see Definition 1.1. Refinements of the arguments in [16] led to the existence of solutions to (1.1) that change sign exactly once and have Morse index 2, see [5]. This paper builds on the constructions in [5] obtaining solutions with *augmented Morse index* greater than two, see Theorem 1.3.

We assume that there exist $A > 0$ and $p \in [1, (N+2)/(N-2))$ such that

$$|f'(u)| \leq A(|u|^{p-1} + 1) \quad \text{for } u \text{ in } \mathbb{R}. \tag{1.2}$$

Let $\lambda_1 < \lambda_2 \leq \ldots \to +\infty$ denote the eigenvalues of $-\Delta$ with Dirichlet boundary condition in Ω. We also assume the following hypotheses:

- (h_1) $f(0) = 0$, $f'(0) < \lambda_1$.
- (h_2) $\lim_{|u|\to\infty} f(u)/u = \infty$, i.e. f is *superlinear*.
- (h_3) $f'(u) > f(u)/u$ for all $u \neq 0$.
- (h_4) There exist $m \in (0,1)$ and $\rho > 0$ such that $(m/2)uf(u) - F(u) \geq 0$ for $|u| > \rho$, where $F(u) = \int_0^u f(s)\,ds$.

From these hypotheses it follows that there exists a positive constant K such that

$$\alpha t f(\alpha t) \geq K\alpha^{2/m} t f(t) \quad \text{for } \alpha \geq 1 \text{ and } |t| > \rho. \tag{1.3}$$

Let $\mathbb{H}(\Omega) := \mathbb{H}$ denote the Sobolev space of functions vanishing in $\partial\Omega$ and having square integrable first order partial derivatives. The solutions to (1.1) are the critical points of the functional $J\colon \mathbb{H} \to \mathbb{R}$,

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega F(u)\,dx, \tag{1.4}$$

where $F(t) = \int_0^t f(s)\,ds$. The functional J is of class C^2. Its gradient is given by

$$\langle \nabla J(u), v\rangle = \int_\Omega (\nabla u \cdot \nabla v - f(u)v)\,dx, \tag{1.5}$$

for all $u, v \in \mathbb{H}$, and its Hessian is given by

$$\langle D^2 J(u)v, w\rangle = \int_\Omega (\nabla v \cdot \nabla w - f'(u)vw)\,dx, \tag{1.6}$$

for all $u, v, w \in \mathbb{H}$.

Definition 1.1. If u is a critical point of J, we will say that u has *Morse index* k if $D^2J(u)$ has exactly k negative eigenvalues, counting multiplicity; and that u has *augmented Morse index* k if the number of nonpositive eigenvalues of $D^2J(u)$, counting multiplicity, is k. We will denote the Morse index of J at u by $m(J,u)$ and by $m_+(J,u)$ the augmented Morse index of J at u.

Due to the assumptions on f, J satisfies the Palais–Smale condition, i.e. if $\{J(u_k)\}_k$ is a bounded sequence and $\{\nabla J(u_k)\}_k$ converges to 0 then $\{u_k\}_k$ has a converging subsequence, see [5].

Let $h(u) = \langle \nabla J(u), u\rangle$ and

$$\mathcal{N} = \{u \in \mathbb{H} : u \neq 0,\ h(u) = 0\}. \tag{1.7}$$

From (h_3) we have, for all $\mathcal{N}$,

$$\begin{aligned}\langle \nabla h(u), u\rangle &= \int_\Omega t(2|\nabla u|^2 - u^2 f'(u) - uf(u))\,dx \\ &= \int_\Omega (|\nabla u|^2 - u^2 f'(u))\,dx < 0.\end{aligned} \tag{1.8}$$

The set $\mathcal{N}$ is known as the Nehari manifold of (1.1). It is easily seen that every nonzero solution to (1.1) belongs to $\mathcal{N}$.

We make extensive use of the properties of J compared to those of the restriction of J to $\mathcal{N}$, $J_{|\mathcal{N}}$. In particular we make use of the following result.

Lemma 1.2. *For J and $\mathcal{N}$ above, we have*

$$m(J,u) = m(J_{|\mathcal{N}}, u) + 1 \quad \text{and} \quad m_+(J,u) = m_+(J_{|\mathcal{N}}, u) + 1, \tag{1.9}$$

where $J_{|\mathcal{N}}$ denotes the restriction of J to $\mathcal{N}$.

Proof. Let V be a k-dimensional subspace tangent to $\mathcal{N}$ at u on which $D^2J_{|\mathcal{N}}(u)$ is negative definite. Hence, for any $v \in V$, $\langle \nabla h(u), v\rangle = 0$. Therefore

$$\begin{aligned}0 &= \int_\Omega (2\nabla u \cdot \nabla v - f'(u)uv - f(u)v)\,dx \\ &= \int_\Omega (\nabla u \cdot \nabla v - f'(u)uv)\,dx.\end{aligned} \tag{1.10}$$

Thus, for any $\alpha, \beta \in \mathbb{R}^2 \setminus \{(0,0)\}$ and $v \in V$,

$$\begin{aligned}\langle D^2J(u)(\alpha v + \beta u), (\alpha v + \beta u)\rangle &= \alpha^2 \int_\Omega (|\nabla v|^2 - f'(u)v^2)\,dx \\ + \beta^2 \int_\Omega (|\nabla u|^2 - f'(u)u^2)\,dx &+ 2\alpha\beta \int_\Omega (\nabla v \cdot \nabla u - f'(u)uv)\,dx \\ =\alpha^2 \int_\Omega (|\nabla v|^2 - f'(u)v^2)\,dx &+ \beta^2 \int_\Omega (|\nabla u|^2 - f'(u)u^2)\,dx.\end{aligned} \tag{1.11}$$

Therefore, by (1.8), (1.10) and (1.11), D^2J is negative definite in a $(k+1)$-dimensional subspace of $\mathbb{H}$. Thus $m(J,u) \geq k+1$. On the other hand, from the definition of Morse index, $D^2J_{|\mathcal{N}}(u)$ is nonnegative definite in a k-dimensional subspace of the tangent space to $\mathcal{N}$ at u. Since such tangent space is a codimension 1 subspace of $\mathbb{H}$, $m(J,u) \leq k+1$. This proves the first identity in (1.9). The proof of the second identity follows the same pattern and is left for the reader.□

In [5] it was proven that defining

$$\mathcal{E} := \{u \in \mathcal{N} \mid \langle \nabla J(u),\ u_+\rangle = 0\}, \tag{1.12}$$

there exists $w \in \mathcal{E}$ such that

$$c = J(w) = \min\{J(u) : u \in \mathcal{E}\}, \tag{1.13}$$

w changes sign exactly once, and w satisfies (1.1). All functions w satisfying (1.13) are solutions to (1.1) that change sign exactly once. Moreover in [4] it is proven that the Morse index of w is two. Earlier in [7] such a result was obtained under the additional assumption that w an isolated solution. For the sake of simplicity in the text, we will call such solutions CCN-solutions and c the CCN-level, and we will denote

$$\mathbb{W} = \{u \in \mathcal{E} : J(u) = J(w)\}. \tag{1.14}$$

Our main result is:

Theorem 1.3. *Let $\Omega, f, \mathcal{N}, \mathcal{E}$, and w be as above and $a \in \mathbb{R}$. Let $J_a = \{u \in \mathcal{E} : J(u) < a\}$. and $\pi_k(J_a)$ the k-th homotopy group of J_a. If J_a is disconnected or $\pi_k(J_a)$ is nontrivial for some positive integer k, then J has a critical level $c_1 \in [a,\infty)$ and a critical point with augmented Morse index greater than or equal to three.*

The proof of Theorem 1.3 is in the spirit of Theorem 1 of [8] where the result was stated in terms for singular homology. A fundamental ingredient in this proof is that $\mathcal{E}$ is connected and $\pi_k(\mathcal{E})$ is trivial for all positive integers k, see Theorem A.1 in Appendix A.

Remark 1.4. Replacing homotopy groups by singular homology groups in the statement of Theorem 1.3 leads to the same result and the proofs are very similar.

Corollary 1.5. *Let $\Omega, f, \mathcal{N}, \mathcal{E}$, and w be as above. If $\mathbb{W}$, defined as in* (1.14)*, is disconnected then there exist $c_1 > J(w)$ and $u \in \mathcal{E}$ such that $\nabla J(u) = 0$ and $J(u) = c_1$.*

Finally, we show that Theorem 1.3 and Corollary 1.5 are not vacuous, by constructing regions where the level $J(w) \subset \mathcal{E}$ is disconnected. In fact we have the following theorem.

Theorem 1.6. *Let A_1 and A_2 be smooth congruent regions with disjoint closures. Let $\tau\colon [1,2] \to \mathbb{R}^n$ a one-to-one differentiable function such that $\tau(i) \in A_i$,*

$\tau'(i)$ is transversal to the boundary of A_i, $\tau((1,2)) \cap \partial(A_1 \cup A_2) = \emptyset$, $\epsilon > 0$ and $C = \{x \in \mathbb{R}^n : |x - \tau(t)| < \epsilon\}$. If $\epsilon > 0$ is sufficiently small $\Omega = A_1 \cup C \cup A_2$ is symmetric with respect to a hyperplane then $\{u \in \mathcal{E} : J(u) = J(w)\}$ is disconnected.

In Section 2 we prove some preliminary estimates needed later in the paper. In Section 3 we prove a deformation lemma on $\mathcal{E}$. Note that, unlike usual deformation lemmas (see [15]), $\mathcal{E}$ does not have a differentiable structure. We bypass this deficiency by making strong use of the fact that J attains a strict maximum in the radial direction at every point in $\mathcal{N}$. In Section 4 we prove Theorem 1.3. This proof proceeds much like the proof of Theorem 1 in [8]. In Section 5 we prove Theorem 1.6 by establishing that CCN-solutions concentrate away from the handle. Finally in Appendix A we prove that the homotopy groups of $\mathcal{E}$ are trivial.

2. Preliminary results

Using the implicit function theorem, it is easily seen that $\mathcal{N}$ is a differentiable manifold of class C^1. Moreover, it is diffeomorphic to the unit sphere in $\mathbb{H}$. In fact, from (h_1)–(h_3) it follows that for each $u \in \mathbb{H} \setminus \{0\}$ there exists a unique positive real number $P(u)$ such that $P(u)u \in \mathcal{N}$. In other words, $P(u)u$ is the intersection of $\mathcal{N}$ with $\{su : s \in (0,\infty)\}$.

LEMMA 2.1. *If A is a bounded subset of $\mathcal{N}$ then there exist $C_1, C_2 > 0$, and $\delta > 0$ such that if* $\operatorname{dist}(u, A) < \delta$, *$v \in A$, then*

$$\Theta(u) := \int_\Omega (f'(u)u^2 - uf(u))\,dx \geq C_1, \tag{2.1}$$

and

$$J(v) \leq J((1-s)v) + C_2 s^2 \quad \textit{for } |s| < \delta. \tag{2.2}$$

PROOF. We argue by contradiction. Suppose there exists a sequence u_n such that that $\lim\limits_{n\to\infty} \Theta(u_n) = 0$ and $\lim\limits_{n\to\infty} \operatorname{dist}(u_n, A) = 0$. Let $u_n = v_n + w_n$ with $v_n \in A$, and $\lim\limits_{n\to\infty} w_n = 0$. Since $\{u_n\}$ is bounded, we may assume that $\{u_n\}$ converges to $u \in L^{p+1}$. Therefore,

$$\int_\Omega uf(u)\,dx = \lim_{n\to\infty} \int_\Omega u_n f(u_n)\,dx = \lim_{n\to\infty} \|u_n\|^2 \geq C_2,$$

where $C_2 > 0$. Hence $u \neq 0$. Since $t^2 f'(t) - tf(t) > 0$ for $t \neq 0$, $\Theta(u) > 0$. This contradicts the assumption $\lim\limits_{n\to\infty} \Theta(u_n) = 0$ and proves (2.1).

In order to prove (2.2) we assume that $\{v_j\}$ is a sequence in A and $\{s_j\}$ is a sequence of real numbers converging to zero such that $J(v_j) \geq J((1-s_j)v_j) + js_j^2$.

By Taylor's formula, there exists a sequence $\{t_j\}$ with $|t_j| \le s_j$ such that

$$-2j = \frac{d^2}{ds^2} J((1-s)v_j)\Big|_{s=t_j} = \langle D^2 J((1-s)v_j)v_j, v_j\rangle, \tag{2.3}$$

which contradicts that $D^2 J$ is bounded on bounded sets. This proves (2.2) and hence the lemma. □

Lemma 2.2. *Let $A \subset \mathcal{N}$, $C > 1$ and $\delta > 0$ be as in Lemma* 2.1. *There exists $\delta_1 \in (0,\delta)$ such that if $\|v-u\| < \delta_1$, for some $u \in A$, then $|P(v)-1| < C_3\|u-v\|$.*

Proof. Without loss of generality we may assume that $\delta < 1/2$. For $w \in \mathbb{H}$ let

$$I(w) := \int_\Omega |\nabla u|^2\,dx - \int_\Omega w f(u)\,dx. \tag{2.4}$$

Note that

$$\frac{dI(su)}{ds} = \int_\Omega (2s|\nabla u|^2 - f'(su)u^2 - f(su)u)\,dx.$$

Arguing as in the proof of (2.1), we see that there exist $\delta_2 \in (0,\delta)$ and $C_2 > 0$ such that $dI(su)/ds \le -C_2$ for all $s \in (1-\delta_2, 1+\delta_2)$, $u \in A$.

Due to (h_3), there exists $k > 0$ such that if $u \in \mathcal{N}$, $\|v-u\| < \delta$ then $|I(\alpha u) - I(\alpha v)| \le k\|u-v\|$ for $\alpha \in (1-\delta, 1+\delta)$. Hence $I(\alpha v) \le I(\alpha u) + k\|u-v\|$. Also, from (2.1), $I(1+(\delta_2/2)u) \le -C_2\delta_2/2 + k\|u-v\| < 0$ if $\|u-v\| < \delta_1 := C_2\delta_2/(2k)$. Hence $|P(v)-1| < k\|u-v\|/C$ if $\|u-v\| < \delta_1$. Hence $P(v) < 1 + k\|u-v\|/C$. Similarly, $P(v) > 1 - k\|u-v\|/C$, which proves the lemma. □

Lemma 2.3. *If $\{u_j\}_j$ is a sequence in $\mathcal{E}$ such that $\lim_{j\to+\infty} J(u_j) = J(w)$ then $\{\nabla J(u_j)\}_j$ converges to zero. Thus, by the* (PS) *condition, $\{u_j\}_j$ has a subsequence that converges to a* CCN-*solution.*

Proof. Assuming to the contrary, there exist $\alpha > 0$ and a subsequence $\{u_{j_k}\}_k$ such that $\|\nabla J(u_{j_k})\| \ge \alpha$ for all k and $\mathrm{dist}(u_{j_k}, \mathbb{W}) \ge \alpha$, where $\mathbb{W}$ is as in (1.14). Since $\lim_{\|u\|\to+\infty,\, u\in\mathcal{E}} J(u) = +\infty$, $\{u_{j_k}\}_k$ is bounded. Hence there exists $\beta \in (0,\alpha)$ such that $\|\nabla J(u)\| \ge \alpha/2$ for $\|u - u_{j_k}\| < \beta$.

Let $\eta_k := \eta$ denote the solution to

$$\eta'(t) = -\frac{\nabla J(\eta(t))}{\|\nabla J(\eta(t))\|^2}, \quad \eta(0) = u_{j_k}, \quad t \in [0, \alpha\beta/2]. \tag{2.5}$$

Let $t_0(k) := t_0 = 2(J(u_{j_k}) - J(w))$. Hence

$$J(\eta(t_0)) = J(w) - \frac{1}{2}t_0, \qquad \|\eta(t_0)_\pm - (u_{j_k})_\pm\| \le Ct_0, \tag{2.6}$$

where $C > 0$ is a constant independent of k. Let $\lambda_\pm$ be such that $\lambda_\pm\eta(t_0)_\pm \in \mathcal{N}$. By Lemma 2.2 and (2.6), $|P((u_{j_k})_\pm) - P(\eta(t_0))| \le C|J(u_{j_k}) - J(w)|$. Since

$\varphi(s) = J(s(\lambda_\pm \eta(t_0)_\pm))$ attains its maximum at $s = 1$, by (2.6), we have

$$\begin{aligned} J(\lambda_+\eta(t_0)_+ - \lambda_-\eta(t_0)_-) &= J(\lambda_+\eta(t_0)_+) + J(\lambda_-\eta(t_0)_-) \\ &\le J(\eta(t_0)_+) + J(\eta(t_0)_-) + Ct_0^2 \le J(\eta(t_0)) + Ct_0^2 \\ &= J(w) - \frac{1}{2}\, t_0 + Ct_0^2 < J(w) \end{aligned}$$

for k large. This is a contradiction since $\lambda_+\eta(t_0)_+ + \lambda_-\eta(t_0)_- \in \mathcal{E}$ and $J(w) = \min\{J(u) : u \in \mathcal{E}\}$. This contradiction proves the lemma. □

3. A deformation lemma

In this section we prove a *deformation lemma* for J on $\mathcal{E}$. Since $\mathcal{E}$ is not a differentiable manifold several technical issues must be overcome as opposed to the case where the domain is a differentiable manifold (see [15]). In fact we have the following.

Lemma 3.1. *If $b \in \mathbb{R}$ is not a critical value of J then there exists $\epsilon > 0$ such that if $K \subset \{u \in \mathcal{E} : J(u) < b+\epsilon\}$ is compact then there is a continuous function $\sigma\colon [0,2\epsilon] \times K \to \mathcal{E}$ such that $\sigma(0,x) = x$, $J(\sigma(2\epsilon,x)) < b$ for any $x \in K$, and $\sigma(t,x) = x$ for all $t \in [0,2\epsilon]$ if $J(x) < b - 2\epsilon$.*

Proof. Since J satisfies the (PS) condition there exists $\epsilon > 0$ such that $[b-2\epsilon, b+2\epsilon]$ contains no critical values of J. Let $\chi \in C_0^\infty(\mathbb{R})$ be such that $0 \le \chi \le 1$, $\chi \equiv 1$ on $[b-\epsilon, b+\epsilon]$ and 0 on $(-\infty, b-2\epsilon] \cup [b+2\epsilon, \infty)$. Consider the flow defined for $v_0 \in \mathbb{H}$ by

$$\text{(3.1)} \qquad \begin{cases} \dot v = -\chi(J(v))\, \dfrac{\nabla J(v)}{|\nabla J(v)|^2}, \\ v(0) = x. \end{cases}$$

As the vector field $-\chi(J(u))\nabla J(u)/|\nabla J(u)|^2$ is C^1, the flow is continuous. Thus we may define $\mathcal{F}_t\colon \mathbb{H} \to \mathbb{H}$ by $\mathcal{F}_t(x) = v(t)$ where v solves (3.1).

Next define $\lambda_\pm(t,x) := \lambda_\pm(t) = P[\mathcal{F}_t(x)]_\pm$. Thus we have

$$\sigma(t,x) = \lambda_+(t)[\mathcal{F}_t(x)]_+ + \lambda_-(t)[\mathcal{F}_t(x)]_- \quad \text{and} \quad \sigma(0,x) = x.$$

If $J(x) < b - 2\epsilon$ we have $\chi(J(x)) = 0$ implying that $\mathcal{F}_t(x) = x$ giving that $\sigma(t,x) = x$ for all $t \in [0,2\epsilon]$.

Assuming that $J(\mathcal{F}_t(x)) \ge b - \epsilon$ for all $s \in [0,2\epsilon]$, we have

$$\text{(3.2)} \quad \begin{aligned} J(\mathcal{F}_{2\epsilon}(x)) = J(x) - \int_0^{2\epsilon} \left\langle \nabla J(\mathcal{F}_s(x)), \chi(J(\mathcal{F}_s(x)))\, \frac{\nabla(J(\mathcal{F}_s(x)))}{\|\nabla J(\mathcal{F}_s(x))\|^2} \right\rangle ds \\ < b + \epsilon - 2\epsilon, \end{aligned}$$

which is a contradiction. Thus for each x there exists $s \in [0,2\epsilon]$ such that $J(\mathcal{F}_s(x)) < b - \epsilon$. Since $J(\mathcal{F}_s(x))$ defines a decreasing function of s, we have

$J(\mathcal{F}_{2\epsilon}(x)) < b - \epsilon$ for all $x \in K$. Applying Lemma 2.2 and (3.2), and using that $\lambda_\pm$ is a critical value for the function $\varphi(s) = J(s[\mathcal{F}_{2\epsilon}(x)]_\pm)$, we see that

$$J(\sigma(2\epsilon, x)) = J(\lambda_+(2\epsilon)[\mathcal{F}_{2\epsilon}(x)]_+) + J(\lambda_-(2\epsilon)[\mathcal{F}_{2\epsilon}(x)]_-)$$
$$\leq J([\mathcal{F}_{2\epsilon}(x)]_+) + J([\mathcal{F}_{2\epsilon}(x)]_-) + C\epsilon^2 = J(\mathcal{F}_{2\epsilon}(x)) + C\epsilon^2 < b - \epsilon + C\epsilon^2 < b$$

for ϵ sufficiently small. □

4. Proof of Theorem 1.3

Below is a proof of Theorem 1.3. It follows much of the usual methods seen in Theorem 1 of [8].

Proof. Let $\psi\colon S^k \to J_a$ be a nonzero element of $\pi_k(J_a)$. Let B^{k+1} be the closed ball of radius 1 in $\mathbb{R}^{k+1}$ centered at the origin. Define the set

$$\mathcal{B} = \{\varphi : B^{k+1} \to \mathcal{E},\ \varphi \text{ is continuous},\ \varphi(x) = \psi(x) \text{ for } \|x\| = 1\}.$$

By Theorem A.1, $\mathcal{B}$ is not empty. Since ψ defines a nonzero element of $\pi_k(J_a)$,

$$\max_{\|x\|\leq 1} J(\varphi(x)) > a \quad \text{for each } \varphi \in \mathcal{B}. \tag{4.1}$$

Let

$$c_1 = \inf_{\varphi\in\mathcal{B}} \Big(\max_{\|x\|\leq 1} J(\varphi(x))\Big). \tag{4.2}$$

By (4.1), $c_1 \geq a$. Assume that c_1 is not a critical value for the sake of contradiction. Let $\epsilon > 0$ be as in Lemma 3.1 and such that $c_1 - 2\epsilon > \max_{\|x\|=1} J(\psi(x))$. Let $\varphi \in \mathcal{B}$ be such that $\max_{\|x\|\leq 1} J(\varphi(x)) < c_1 + \epsilon$, and $\varphi_1(x) := \sigma(2\epsilon, \varphi(x))$ with σ as in Lemma 3.1. Since σ is continuous and $\sigma(t, v) = v$ for $J(v) < c_1 - 2\epsilon$, $\varphi_1 \in \mathcal{B}$. Hence $\max_{\|x\|\leq 1} J(\varphi_1(x)) < c_1$, which contradicts the definition of c_1 and proves that c_1 is a critical value. Let w_1 be such that $J(w_1) = c_1$ and $\nabla J(\omega_1) = 0$.

For any $u \in \mathcal{E}$, $D^2J(u)$ is negative definite in the two dimensional subspace spanned by $\{u_+, u_-\}$. Assuming that all the critical points of J in $\mathcal{E}$ have augmented Morse index less than three implies that they are nondegenerate Morse index two critical points. Hence J has finitely many critical points in $\mathcal{E}$ and their Morse index restricted to the Nehari manifold is equal to one (see Lemma 1.2). Let $\epsilon \in (0, 1/2)$ be small enough so that

$$\mathcal{E}_1 = \{\alpha u_+ - \beta u_- \in \mathcal{N} : u = u_+ - u_- \in \mathcal{E},\ |\alpha - 1| < \epsilon,\ |\beta - 1| < \epsilon\}$$

is an open submanifold of $\mathcal{N}$. Let c_2 be such that $J(u) < c_2$ for u critical points of J and c as in (1.13). By standard Morse theory (see [9]), we have the exact sequence

$$\cdots \to H_2(J_{c_2} \cap \mathcal{E}_1) \to H_1(J_{c_2} \cap \mathcal{E}_1, J_c \cap \mathcal{E}_1) \to H_1(J_c \cap \mathcal{E}_1) \to H_1(J_{c_2} \cap \mathcal{E}_1) \to \cdots$$

Since J has no critical point in $J_c \cap \mathcal{E}_1$, $H_1(J_c \cap \mathcal{E}_1) = \{0\}$. This and the fact that $H_1(J_{c_2} \cap \mathcal{E}_1, J_c \cap \mathcal{E}_1)$ has at least two generators imply that $H_2(J_{c_2} \cap \mathcal{E}_1)$ is nontrivial. Hence J has an augmented Morse index two critical point in $J_{c_2} \cap \mathcal{E}_1$, which by Lemma 1.2 is an augmented three Morse index critical point of J. This proves the theorem. □

5. Proof of Theorem 1.6

REMARK 5.1. For the sake of simplicity in the proof we assume $N = 2$. The general case, $N \geq 3$, follows by bootstrapping arguments based on successive multiplications by functions of the type $|w|^{r_j} w$, with $r_1 = (N+2)/(N-2) - p$, $r_{j+1} > r_j$, and $\lim\limits_{j \to +\infty} = +\infty$.

PROOF. Let $\epsilon > 0$ be the width of the channel C (see Figure 1). In order to keep track of the width of the channel we will denote $\Omega = \Omega_\epsilon$, $C = C_\epsilon$, $\mathcal{N} = \mathcal{N}_\epsilon$, $\mathcal{E} = \mathcal{E}_\epsilon$, and $\mathbb{W} = \mathbb{W}_\epsilon$, see (1.14). Without loss of generality we may assume that Ω_ϵ is invariant under the transformation $\Phi(x, y) = (-x, y)$.

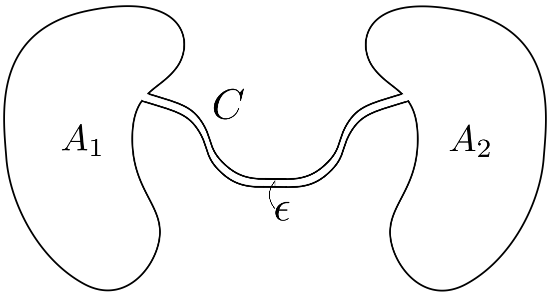

FIGURE 1

Let u_1 be a positive solution to (1.1) in A_1, u_2 a negative solution to (1.1) in A_2. Defining $\widehat{u}_\epsilon(x) = u_1(x)$ in A_1, $\widehat{u}_\epsilon(x) = 0$ for $x \in C$, and $\widehat{u}_\epsilon(x) = u_2(x)$ for $x \in A_2$, we see that $\widehat{u}_\epsilon \in \mathcal{E}_\epsilon$ for any $\epsilon > 0$ sufficiently small. Using that u_i satisfies (1.1) in A_i and that $\widehat{u}_\epsilon$ is identically zero in C, we have

$$\langle \nabla J(\widehat{u}_\epsilon), \widehat{u}_\epsilon \rangle = \int_{A_1 \cup A_2} (|\nabla \widehat{u}_\epsilon|^2 - \widehat{u}_\epsilon f(\widehat{u}_\epsilon))\, dx + \int_C (|\nabla \widehat{u}_\epsilon|^2 - \widehat{u}_\epsilon f(\widehat{u}_\epsilon))\, dx = 0.$$

Hence $\widehat{u}_\epsilon \in \mathcal{E}_\epsilon$, which yields $J(w_\epsilon) \leq J(\widehat{u}_\epsilon)$ for $\epsilon > 0$ sufficiently small. Thus, there exists a positive constant K_1 such that $\|w_\epsilon\| \leq K_1$ for any $w_\epsilon \in \mathcal{E}_\epsilon$. This, the definition of weak solutions, and hypothesis (h_4) give

$$\|w_\epsilon\|^2 = 2 \int_\Omega F(w_\epsilon(x))\, dx + 2K_1 \leq m \int_\Omega w_\epsilon(x) f(w_\epsilon(x))\, dx + K_2 \leq m\|w_\epsilon\|^2 + K_2,$$

for some $K_2 > 0$ independent of (w_ϵ, ϵ). Since $m < 1$ we have $\|w_\epsilon\| \leq K_3$, with $K_3 > 0$ independent of (w_ϵ, ϵ). This and the Sobolev embedding theorem (see [11]) imply $\|f \circ w_\epsilon\|_2 \leq K_4$, again with K_4 independent of (w_ϵ, ϵ).

Let $T = \max\{|x-y| : x,y \in \overline{\Omega}\}$. If $T \le 1/2$, the Green function $G(x,y)$ on Ω is bounded above by $\ln(|x-y|)$. Thus, for any $x \in \Omega$ we have

$$\begin{aligned}(5.1)\qquad |w(x)| = \int_\Omega G(x,y) f(w(y))\,dy &\le \int_\Omega |\ln(|x-y|)||f(w(y))|\,dy \\ &\le \bigg(\int_\Omega \ln^2(|x-y|)\,dy\bigg)^{1/2} \|f\circ w\|_2 := K_5.\end{aligned}$$

On the other hand, if $T > 1/2$, we define $W = \{(1/2T)(x_1,x_2) : (x_1,x_2) \in \Omega\}$ and $w_{1,\epsilon}(x_1,x_2) = w_\epsilon(2Tx_1, 2Tx_2)$. Since $-\Delta w_{1,\epsilon} = (1/4T^2) f(w_{1,\epsilon}(x_1,x_2))$ and $\max\{|x-y| : x,y \in W\} \le 1/2$, the arguments in (5.1) hold for $w_{1,\epsilon}$, hence they hold for w. Thus (5.1) is valid regardless of T.

By a priori estimates for elliptic equations on regions satisfying the uniform exterior cone condition (see [11, Theorem 8.29]), there exist $\alpha \in (0,1)$ and $K_6 > 0$ such that $\|w\|_{C^\alpha(\Omega)} \le K_6$. Hence for each $x \in C$, $|w(x)| \le K_6 \epsilon^\alpha$. Thus

$$(5.2)\qquad \lim_{\epsilon\to 0+} \|w\|_{L^\infty(C)} = 0.$$

Let $\{\epsilon_j\}_j$ is a decreasing sequence of positive numbers converging to zero and $\{w_j\}_j$ a corresponding sequence of CCN-solutions converging in $\mathbb{H}(A_1 \cup A_2)$. From (5.1), (5.2), and regularity for elliptic boundary value problems we may assume that $\{w_j\}_j$ converges to $w \in \mathbb{H}(A_1 \cup A_2)$. Since $(w_j)_+ \in \mathcal{N}_\epsilon$ and $\lim_{j\to\infty} \|w_j\|_{C^\alpha(C_\epsilon)} = 0$ then $w_+ \ne 0$. Similarly $w_- \ne 0$. Hence w changes sign in $A_1 \cup A_2$. If w changes sign in A_1 then taking z_1 as a positive function that minimizes J on A_1 and z_2 as a negative function that minimizes J on A_2, we have $J(z_1+z_2) < J(w)$. Hence for j sufficiently large $J(z_1+z_2) < J(w_j)$, which contradicts that w_j is a CCN-solution in Ω_{ϵ_j}.

Therefore, for $\epsilon > 0$ sufficiently small we may assume that for any CCN-solution,

$$(5.3)\qquad \int_{A_i} u(x,y)\,dx\,dy \ne 0, \quad \text{for } i = 1,2.$$

Let v be a CCN-solution and $\widehat{v}(x,y) = v(-x,y)$. Hence $\widehat{u}$ is also a CCN-solution and

$$(5.4)\qquad \bigg(\int_{A_1} v(x,y)\,dx\,dy\bigg)\bigg(\int_{A_1} \widehat{v}(x,y)\,dx\,dy\bigg) < 0.$$

Suppose for each $t \in [0,1]$ there exists a CCN-solution u_t that depends continuously on t and such that $u_0 = v$ and $u_1 = \widehat{v}$. This and (5.4) imply that, for some $t_0 \in (0,1)$, $\int_{A_1} u_{t_0}(x,y)\,dx\,dy = 0$. Since this contradicts (5.3), the theorem is proved. $\square$

Appendix A. The topology of $\mathcal{E}$

The set $\mathcal{E}$ is connected. In fact, let $P\colon \mathbb{H}\setminus\{0\}\to(0,\infty)$ be the continuous function such that $P(u)u\in\mathcal{N}$. If $u=u_+-u_-$ and $v=v_+-v_-$ are in $\mathcal{E}$, then $P(u_++t(v-u))(u_++t(v-u))+P(u_++t(v-u))(u_++t(v-u))$ defines a continuous path $\mathcal{E}$ from u and v. In addition to being connected, the set $\mathcal{E}$ has the following property.

THEOREM A.1. *For any positive integer k, the homotopy group $\pi_k(\mathcal{E})$ is trivial.*

PROOF. Let S^k denote the unit sphere in $\mathbb{R}^{k+1}$, and $\{\omega_1,\omega_2,\ldots\}$ denote a complete orthonormal set in $\mathbb{H}$ corresponding to the eigenvales $\lambda_1<\lambda_2\leq\ldots\to+\infty$. Since ω_1 does not change sign, we may assume $\omega_1(z)>0$ for all $z\in\Omega$. Let $\phi\colon S^k\to\mathcal{E}$ be a continuous function. By the compactness of $\phi(S^k)$, given $\epsilon>0$, there exists a positive integer $j>2$ such that $|P_1(\phi(x))-\psi(x)|<\epsilon$ with P the orthogonal projection onto the subspace spanned by $\{\omega_1,\ldots,\omega_j\}$.

We let $\Phi(s,x)=\phi(x)+s(P_1(\phi(x))-\phi(x))$. By taking ϵ sufficiently small, we see that Φ changes sign for all $(s,x)\in[0,1]\times S^k$. Letting

$$\phi(1,x)=\sum_{i=1}^{j}a_i(x)\omega_i,$$

we define

$$\Phi(s,x)=\sum_{i=1}^{j}(2-s)a_1(x)\omega_1+\sum_{i=2}^{j}a_i(x)\omega_i.$$

Let us see that Φ is a sign changing function, for all $(s,x)\in[1,2]\times\Omega$. Without loss of generality we assume that $a_1(x)>0$. Since $\Phi(s,x)(z)\leq\Phi(1,x)(z)$ for all $z\in\Omega$, $\Phi(s,x)_-\neq 0$. Also since $\Psi(2,x)$ is L^2-orthogonal to ω_1, $\Phi(s,x)_+\neq 0$. This and $\Phi(s,x)(z)\geq\Phi(2,x)(z)$ for all $z\in\Omega$ imply $\Phi(s,x)_+\neq 0$, which proves the claim.

Finally, for $s\in[2,3]$ we define $\Phi(s,x)=(3-s)\Phi(2,x)+(s-2)\omega_{j+1}$. Since $\Phi(s,x)$ is orthogonal to ω_1, Φ is a sign-changing function also for all $(s,x)\in[2,3]\times S^k$. As $\Phi(3,x)=\omega_{j+1}$ for all $x\in S^k$, we have proven that ϕ is homotopic to a constant function in $V=\{y\in\mathbb{H}-\{0\}: y \text{ changes sign}\}$. Since V can be transformed continuously into $\mathcal{E}$ by $Q(u_+-u_-)=P(u_+)u_++P(-u_-)u_-$, $Q\circ\Phi$ defines a homotopy in $\mathcal{E}$ between ϕ and a constant function. Hence $\pi_k(\mathcal{E})=\{0\}$. □

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