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MULTI-BUMP SOLUTIONS FOR SINGULARLY PERTURBED SCHRÖDINGER EQUATIONS IN $\mathbb{R}^2$ WITH GENERAL NONLINEARITIES

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Abstract. We are concerned with the following equation:

$$-\varepsilon^2\Delta u + V(x)u = f(u), \quad u(x) > 0 \quad \text{in } \mathbb{R}^2.$$

By a variational approach, we construct a solution u_ε which concentrates, as $\varepsilon \to 0$, around arbitrarily given isolated local minima of the confining potential V: here the nonlinearity f has a quite general Moser's critical growth, as in particular we do not require the *monotonicity* of $f(s)/s$ nor the *Ambrosetti–Rabinowitz* condition.

1. Introduction

We are concerned with the existence of positive solutions to the ε-perturbed Schrödinger equation

$$-\varepsilon^2\Delta u + V(x)u = f(u), \quad u > 0, \quad x \in \mathbb{R}^2, \tag{1.1}$$

where $\varepsilon > 0$ and $V \in C(\mathbb{R}^2, \mathbb{R})$. In the past decades, a lot of literature has been devoted to bound states of (1.1) in $\mathbb{R}^N$. From the physical point of view, these solutions represent semi-classical states for small $\varepsilon > 0$, living on the interface between classical and quantum mechanics: for the physics aspects and related

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topics we refer to [4], [7], [14]–[16], [26], [28], [39]–[41], and references therein. In the pioneering work [33], Floer and Weinstein considered problem (1.1) in dimension one and $f(s) = s^3$ and constructed a single-peak solution around any given non-degenerate critical point of V. Motivated by [33], Oh [42] obtained a similar result in the higher dimensional case. A key ingredient of [33] and [42] is a reduction method and a non-degeneracy condition for ground states to the limiting problem with constant potential. To overcome non-degeneracy conditions, Rabinowitz [43] exploited the variational approach which has become an important tool in studying semiclassical states of (1.1). In more recent years, there have been further developments to cover more general nonlinearities, see [48], [24]–[27]. In [24], Del Pino and Felmer used a penalization technique to construct a single-peak solution around a local minimum point of V, with some restrictions on the nonlinearity such as the monotonicity of $f(t)/t$ which is required to be nondecreasing in $(0,\infty)$ as well as the Ambrosetti–Rabinowitz condition. More recently, Byeon and Jeanjean [8] introduced a new penalization approach to show that the Berestycki–Lions conditions, see [5], are almost optimal to get spike solutions around the local minima of V. Closely related results can be found in [12], [13], [22], [49]. In [20], with the Berestycki–Lions conditions, Cingolani, Jeanjean and Tanaka considered the multiplicity of solutions to (1.1) concentrating around the local minima of V in $\mathbb{R}^N$ for $N \geq 3$. Moreover, the authors established the number of solutions related to the topology of the set of minima of V. An interesting class of solutions to (1.1) are semi-classical states which have a spike shape concentrated around some point in $\mathbb{R}^2$, as $\varepsilon \to 0$. In this paper, we focus on localized bound states of (1.1), namely solutions which develop multi bumps around the local minima of V. In the sequel, we assume that V satisfies the following assumptions:

- (V1) $\inf_{x\in\mathbb{R}^2} V(x) = V_0 > 0$;
- (V2) there exist k bounded disjoint open sets O^i, $i = 1,\dots,k$, such that

$$0 < m_i = \inf_{x\in O^i} V(x) < \min_{x\in\partial O^i} V(x), \quad i = 1,\dots,k.$$

In 2008, Byeon, Jeanjean and Tanaka [11] constructed a single-spike solution of (1.1) exploiting the Berestycki–Lions conditions. Precisely, the authors assumed $k = 1$ and $f \in C(\mathbb{R}^+,\mathbb{R}^+)$ satisfies:

- (f_1) $\lim_{t\to 0} f(t)/t = 0$;
- (f_2) for any $\alpha > 0$, there exists $C_\alpha > 0$ such that $|f(t)| \leq C_\alpha \exp(\alpha t^2)$ for $t \geq 0$;
- (f_3) there exists $T > 0$ such that $T^2 m < 2F(T)$, where $m = m_1$ and $F(s) := \int_0^s f(t)\,dt$.

THEOREM A (Theorem 1 in [11]). *Suppose that* (V1)–(V2) *with* $k = 1$, $m_1 = m$, $O^i = O$ *and* (f_1)–(f_3) *hold. Then for sufficiently small* $\varepsilon > 0$, (1.1) *admits a positive solution* u_ε *such that*

(a) *there exists a maximum point* x_ε *of* u_ε *such that* $\lim_{\varepsilon\to 0} \operatorname{dist}(x_\varepsilon, \mathcal{M}) = 0$, *where* $\mathcal{M} := \{x \in O : V(x) = m\}$ *and* (*up to a subsequence*) $U_\varepsilon(x) \equiv u_\varepsilon(\varepsilon x + x_\varepsilon)$ *converges uniformly to a least energy solution of*

$$-\Delta U + mU = f(U), \quad U > 0, \quad U \in H^1(\mathbb{R}^2); \tag{1.2}$$

(b) $u_\varepsilon(x) \le C \exp(-c|x - x_\varepsilon|/\varepsilon)$ *for some* $c, C > 0$.

Hypotheses (f_1)–(f_3) are the so-called Berestycki–Lions conditions (see [5], [6], [8]), which are used to guarantee the existence of ground states to (1.2). In [9], Byeon and Jeanjean considered problem (1.1) in $\mathbb{R}^N$ for $N \ge 3$ and for any $k \in \mathbb{N}^+$, obtained k-bumps solutions provided (V1)–(V2) and the Berestycki–Lions conditions hold. In the same spirit of [9], [8], it is natural to wonder whether the results of Theorem A may hold for any $k \in \mathbb{N}^+$: the first purpose of this paper is to give an affirmative answer to this open problem.

Let $k \in \mathbb{N}^+$ and for any $i \in \{1, \dots, k\}$, $\mathcal{M}^i := \{x \in O^i : V(x) = m_i\}$. Without loss of generality and for the sake of simplicity we may assume $V_0 = 1$. The first result of this paper is the following

THEOREM 1.1. *Suppose that* (V1)–(V2) *and* (f_1)–(f_3) *hold. Then for sufficiently small* $\varepsilon > 0$, (1.1) *admits a positive solution* u_ε, *which has the following properties*:

(a) *there exist* k *local maxima* $x_\varepsilon^i \in O^i$, $i = 1, \dots, k$, *of* u_ε *such that*

$$\lim_{\varepsilon\to 0} \max_{1\le i\le k} \operatorname{dist}(x_\varepsilon^i, \mathcal{M}^i) = 0,$$

and $U_\varepsilon(x) \equiv u_\varepsilon(\varepsilon x + x_\varepsilon^i)$ *converges* (*up to a subsequence*) *uniformly to a least energy solution of*

$$-\Delta U + m_i U = f(U), \quad U > 0, \quad U \in H^1(\mathbb{R}^2); \tag{1.3}$$

(b) $u_\varepsilon(x) \le C \exp\Big(-(c/\varepsilon) \min_{1\le i\le k} |x - x_\varepsilon^i|\Big)$ *for some* $c, C > 0$.

Condition (f_2) casts problem (1.1) in the subcritical setting with respect to the Moser critical growth, see [17], [29], [1], [44] and more recently [18], [35]. The understanding of the limit problem (1.3) is important since it plays a crucial role in the study of semiclassical states of (1.1). In [3], Alves et al. considered the ground state of (1.3) in the Moser critical case, namely when in addition to (f_1) one has the following growth condition:

$$(\mathrm{f}_4) \quad \lim_{s\to+\infty} f(s)\exp(-\alpha s^2) = \begin{cases} 0 & \text{for all } \alpha > 4\pi, \\ +\infty & \text{for all } \alpha < 4\pi. \end{cases}$$

By a constraint minimization variational approach, it was proved in [3] that (1.3) admits a ground state solution provided (f_1), (f_4) and the following hold:

(f_5) there exist $\lambda > 0$ and $p > 2$ such that $f(t) \geq \lambda t^{p-1}$ for $t \geq 0$,

provided λ is sufficiently large. More recently, by means of a truncation argument, the second and third named authors extended Theorem A to the Moser critical case [50]. In [45], Ruf and Sani obtained the result of [3] by replacing condition (f_5) with the following more natural assumption:

(f_5)' $\lim\limits_{|t|\to+\infty} tf(t)/\exp(4\pi t^2) \geq \beta_0$, where $\beta_0 > 0$ is sufficiently large.

It is natural to wonder whether Theorem 1.1 holds in the case when the nonlinearity is in the Moser critical growth range: our second goal is to give a positive answer to this question.

The second result of this paper reads as follows

Theorem 1.2. *Suppose that* (V1)–(V2)*,* (f_1) *and* (f_4)–(f_5)' *hold with*

$$\beta_0 > \frac{e}{2\pi} \max_{1\leq i\leq k} m_i. \tag{1.4}$$

Then for $\varepsilon > 0$ sufficiently small, (1.1) *admits a positive solution v_ε, which satisfies*:

(a) *there exist k local maximum points $x_\varepsilon^i \in O^i$ of v_ε such that*

$$\lim_{\varepsilon\to 0} \max_{1\leq i\leq k} \operatorname{dist}(x_\varepsilon^i, \mathcal{M}^i) = 0,$$

and $w_\varepsilon(x) \equiv v_\varepsilon(\varepsilon x + x_\varepsilon^i)$ converges (*up to a subsequence*) *uniformly to a least energy solution of*

$$-\Delta u + m_i u = f(u), \quad u > 0, \quad u \in H^1(\mathbb{R}^2); \tag{1.5}$$

(b) $v_\varepsilon(x) \leq C \exp\Big(-(c/\varepsilon) \min\limits_{1\leq i\leq k} |x - x_\varepsilon^i|\Big)$ *for some $c, C > 0$.*

2. Proof of Theorem 1.1

In this section, in the spirit of Byeon and Jeanjean [9] (see also [8]), we next prove Theorem 1.1. Since we are interested in the positive solutions of (1.1), from now on we may assume $f(t) = 0$ for $t \leq 0$. By denoting $u_\varepsilon(x) = u(\varepsilon x)$ and $V_\varepsilon(x) = V(\varepsilon x)$, (1.1) is equivalent to

$$-\Delta u_\varepsilon + V_\varepsilon(x) u_\varepsilon = f(u_\varepsilon), \quad u_\varepsilon > 0, \quad u_\varepsilon \in H^1(\mathbb{R}^2). \tag{2.1}$$

Let H_ε be the completion of $C_0^\infty(\mathbb{R}^2)$ with respect to the norm

$$\|u\|_\varepsilon = \bigg(\int_{\mathbb{R}^2} (|\nabla u|^2 + V_\varepsilon u^2)\,dx\bigg)^{1/2}.$$

For any set $S \subset \mathbb{R}^2$ and $\varepsilon, \delta > 0$, we define

$$S_\varepsilon = \{x \in \mathbb{R}^2 : \varepsilon x \in S\} \quad \text{and} \quad S^\delta = \{x \in \mathbb{R}^2 : \operatorname{dist}(x, S) \leq \delta\}.$$

Next we penalize the nonlinearity f of Del Pino and Felmer [24]. Let

$$\mathcal{M}=\bigcup_{i=1}^{k}\mathcal{M}^i \quad \text{and} \quad O=\bigcup_{i=1}^{k}O^i.$$

By (f_1) there exists $a>0$ such that $f(t)\le 1/2t$ for $t\in(0,a)$. For $x\in\mathbb{R}^2$, $t\in\mathbb{R}$, let

$$g(x,t)=\chi_O(x)f(t)+(1-\chi_O(x))\widetilde{f}(t),$$

where $\chi_O(x)=1$ if $x\in O$, $\chi_O(x)=0$ if $x\notin O$ and define

$$\widetilde{f}(t)=\begin{cases} f(t) & \text{if } t\le a,\\ \min\{f(t),1/2t\} & \text{if } t>a.\end{cases}$$

In the following, we consider the modified problem

$$-\Delta u_\varepsilon+V_\varepsilon(x)u_\varepsilon=g(\varepsilon x,u_\varepsilon),\quad u_\varepsilon>0,\quad u_\varepsilon\in H^1(\mathbb{R}^2), \tag{2.2}$$

where $g(\varepsilon x,t)=\chi_{O_\varepsilon}(x)f(t)+(1-\chi_{O_\varepsilon}(x))\widetilde{f}(t)$ and we show that (2.2) has a positive solution u_ε satisfying $u_\varepsilon(x)\le a$ for $x\in\mathbb{R}^N\setminus O_\varepsilon$.

For $u\in H_\varepsilon$, let

$$P_\varepsilon(u)=\frac{1}{2}\int_{\mathbb{R}^2}(|\nabla u|^2+V_\varepsilon u^2)\,dx-\int_{\mathbb{R}^2}G(\varepsilon x,u)\,dx,$$

where $G(x,t)=\int_0^t g(x,s)\,ds$. The following penalization functions were introduced in [15]. Fix $\mu>0$ and set

$$\chi_\varepsilon(x)=\begin{cases}0 & \text{if } x\in O_\varepsilon,\\ \varepsilon^{-\mu} & \text{if } x\in\mathbb{R}^N\setminus O_\varepsilon,\end{cases}\qquad \chi_\varepsilon^i(x)=\begin{cases}0 & \text{if } x\in O_\varepsilon^i,\\ \varepsilon^{-\mu} & \text{if } x\in\mathbb{R}^N\setminus O_\varepsilon^i,\end{cases} \tag{2.3}$$

and

$$Q_\varepsilon(u)=\bigg(\int_{\mathbb{R}^2}\chi_\varepsilon u^2\,dx-1\bigg)_+^2,\qquad Q_\varepsilon^i(u)=\bigg(\int_{\mathbb{R}^2}\chi_\varepsilon^i u^2\,dx-1\bigg)_+^2.$$

Let $\Gamma_\varepsilon,\Gamma_\varepsilon^i\colon H_\varepsilon\to\mathbb{R}$, $i=1,\ldots,k$, be given by

$$\Gamma_\varepsilon(u)=P_\varepsilon(u)+Q_\varepsilon(u),\quad \Gamma_\varepsilon^i(u)=P_\varepsilon(u)+Q_\varepsilon^i(u),$$

which enjoy $\Gamma_\varepsilon,\Gamma_\varepsilon^i\in C^1(H_\varepsilon)$.

Let us recall some results about the ground state solutions of (1.3). In [6], Berestycki, Gallouët and Kavian, under the assumptions on f as in Theorem 1.1, proved that for any $m_i>0$, (1.3) admits a positive ground state solution U_i such that

$$L_{m_i}(U_i)=E_{m_i},\qquad \int_{\mathbb{R}^2}\bigg(F(U_i)-\frac{m_i}{2}U_i^2\bigg)\,dx=0, \tag{2.4}$$

where

$$L_{m_i}(u)=\frac{1}{2}\int_{\mathbb{R}^2}(|\nabla u|^2+m_i\,u^2)\,dx-\int_{\mathbb{R}^2}F(u)\,dx,\quad u\in H^1(\mathbb{R}^2).$$

Moreover, the least energy E_{m_i} turns out to be a mountain pass level, see [36].

Let S_{m_i} be the set of positive ground state solutions U_i of (1.3) normalized as follows:

$$U_i(0) = \max_{x\in\mathbb{R}^2} U_i(x).$$

Next we construct a set of approximate solutions to (2.2). Set

$$\delta = \frac{1}{10}\min\Big\{\operatorname{dist}(\mathcal{M}, O^c), \min_{i\neq j}\operatorname{dist}(O^i, O^j)\Big\}.$$

Let us fix $\beta\in(0,\delta)$ and a cut-off $\varphi\in C_0^\infty(\mathbb{R}^2)$ such that $0\le\varphi\le 1$, $\varphi(x)=1$ for $|x|\le\beta$ and $\varphi(x)=0$ for $|x|\ge 2\beta$. Let $\varphi_\varepsilon(y)=\varphi(\varepsilon y)$, $y\in\mathbb{R}^2$, and for some $x_i\in(\mathcal{M}^i)^\beta$, $1\le i\le k$, and $U_i\in S_{m_i}$, we define

$$U_\varepsilon^{x_1,x_2,\cdots,x_k}(y) = \sum_{i=1}^k \varphi_\varepsilon\Big(y-\frac{x_i}{\varepsilon}\Big)U_i\Big(y-\frac{x_i}{\varepsilon}\Big).$$

From [9], one finds a solution in some neighborhood of the set

$$X_\varepsilon = \{U_\varepsilon^{x_1,\dots,x_k} : x_i\in(\mathcal{M}^i)^\beta,\ U_i\in S_{m_i},\ i=1,\dots,k\},$$

for sufficiently small $\varepsilon>0$ (see Proposition 2.6). From [11] one can construct a family of mountain pass levels E_{m_i}, $1\le i\le k$, as follows.

Proposition 2.1. *For each $1\le i\le k$, there exists $T_i>0$ such that, for any $\delta>0$, there exists a path $\gamma_i^\delta\in C([0,T_i],H^1(\mathbb{R}^2))$ with the following properties*:

- (a) $\gamma_i^\delta(0)=0$, $L_{m_i}(\gamma_i^\delta(T_i))<-1$ *and* $\max_{t\in[0,T_i]} L_{m_i}(\gamma_i^\delta(t))=E_{m_i}$;
- (b) *there exists* $T^i\in(0,T_i)$ *such that* $\gamma_i^\delta(T^i)\in S_{m_i}$, $L_{m_i}(\gamma_i^\delta(T^i))=E_{m_i}$ *and* $L_{m_i}(\gamma_i^\delta(t))<E_{m_i}$ *for* $\|\gamma_i^\delta(t)-\gamma_i^\delta(T^i)\|\ge\delta$;
- (c) *there exist* $C,c>0$ *such that for any* $t\in[0,T_i]$ *one has*

$$|D_x^\alpha(\gamma_i^\delta(t))(x)|\le C\exp(-c|x|),\quad x\in\mathbb{R}^2,\quad |\alpha|=0,1.$$

Without loss of generality, in what follows, we may assume $T_i=1$ for all $i=1,\dots,k$. For any $1\le i\le k$ and some fixed $x_i\in(\mathcal{M}^i)^\beta$, let $\gamma_{\varepsilon,i}^\delta(t)(\,\cdot\,)=(\varphi_\varepsilon\gamma_i^\delta(t))(\,\cdot\,-x_i/\varepsilon)$ for $t>0$, then $\Gamma_\varepsilon(\gamma_{\varepsilon,i}^\delta(t))=P_\varepsilon(\gamma_{\varepsilon,i}^\delta(t))$ for $t\in[0,1]$. Now, define a min-max value C_ε^i as follows

$$C_\varepsilon^i = \inf_{\varphi\in\Phi_\varepsilon^i}\max_{s\in[0,1]}\Gamma_\varepsilon^i(\varphi(s)),$$

where $\Phi_\varepsilon^i=\{\varphi\in C([0,1],H_\varepsilon):\varphi(0)=0,\ \varphi(1)=\gamma_{\varepsilon,i}^\delta(1)\}$. As a consequence of [11], we have

$$\lim_{\varepsilon\to 0} C_\varepsilon^i = E_{m_i}\quad\text{for any } 1\le i\le k.$$

Finally, set

$$\gamma_\varepsilon^\delta(s)=\sum_{i=1}^k\gamma_{\varepsilon,i}^\delta(s_i),\quad s=(s_1,\dots,s_k)\in T,$$

where $T = [0,1]^k$ and define

$$D_\varepsilon^\delta := \max_{s\in T} \Gamma_\varepsilon(\gamma_\varepsilon^\delta(s)). \tag{2.5}$$

PROPOSITION 2.2. *The following hold*:

(a) $\lim_{\varepsilon\to 0} D_\varepsilon^\delta = \sum_{i=1}^k E_{m_i} =: E$;

(b) $\limsup_{\varepsilon\to 0} \max_{s\in\partial T} \Gamma_\varepsilon(\gamma_\varepsilon^\delta(s)) \le \widetilde{E}$, *where* $\widetilde{E} = \max_{1\le j\le k} \Big(\sum_{i\neq j} E_{m_i}\Big)$;

(c) *there exists* $M_0 > 0$ (*independent of* δ) *such that for any* $\delta > 0$, *there exist* $\alpha_\delta > 0$ *and* $\varepsilon_\delta \in (0,1)$ *such that for* $\varepsilon \in (0,\varepsilon_\delta)$:

$$\Gamma_\varepsilon(\gamma_\varepsilon^\delta(s)) \ge D_\varepsilon^\delta - \alpha_\delta \textit{ implies that } \gamma_\varepsilon^\delta(s) \in X_\varepsilon^{M_0\delta}.$$

PROOF. The proof buys the line of [9]. Since $\operatorname{supp}(\gamma_{\varepsilon,i}^\delta) \subset (\mathcal{M}_i^{3\beta})_\varepsilon$ for any $1 \le i \le k$,

$$\Gamma_\varepsilon(\gamma_\varepsilon^\delta(s)) = \sum_{i=1}^k \Gamma_\varepsilon(\gamma_{\varepsilon,i}^\delta(s_i)) = \sum_{i=1}^k P_\varepsilon(\gamma_{\varepsilon,i}^\delta(s_i)), \quad s\in T. \tag{2.6}$$

Moreover, by Proposition 2.1, as $\varepsilon \to 0$, we get

$$\begin{aligned} P_\varepsilon(\gamma_{\varepsilon,i}^\delta(s_i)) &= \frac{1}{2}\int_{\mathbb{R}^2} \big(|\nabla\gamma_{\varepsilon,i}^\delta(s_i)|^2 + V_\varepsilon|\gamma_{\varepsilon,i}^\delta(s_i)|^2\big)\,dx - \int_{O_\varepsilon} F(\gamma_{\varepsilon,i}^\delta(s_i))\,dx \\ &= L_{m_i}(\gamma_{\varepsilon,i}^\delta(s_i)) + \frac{1}{2}\int_{\mathbb{R}^2}(V_\varepsilon - m_i)|\gamma_{\varepsilon,i}^\delta(s_i)|^2\,dx + \int_{\mathbb{R}^2\setminus O_\varepsilon} F(\gamma_{\varepsilon,i}^\delta(s_i))\,dx \\ &= L_{m_i}(\gamma_i^\delta(s_i)) + O(\varepsilon), \end{aligned} \tag{2.7}$$

which implies that $\max_{s_i\in[0,1]} P_\varepsilon(\gamma_{\varepsilon,i}^\delta(s_i)) = E_{m_i} + O(\varepsilon)$. Thus, (a) follows.

For $s \in \partial T$, there exists $1 \le j \le k$ with $s_j = 0$ or $s_j = 1$. Then

$$\max_{s\in\partial T} \Gamma_\varepsilon(\gamma_\varepsilon^\delta(s)) \le \max_{s\in T} \sum_{i\neq j} \Gamma_\varepsilon(\gamma_{\varepsilon,i}^\delta(s_i)).$$

Similarly as above, we have

$$\limsup_{\varepsilon\to 0} \max_{s\in\partial T} \Gamma_\varepsilon(\gamma_\varepsilon^\delta(s)) \le \sum_{i\neq j} E_{m_i} \le \widetilde{E},$$

and also (b) follows.

By Proposition 2.1, there exists $\alpha_\delta > 0$ such that for all $1 \le i \le k$:

$$L_{m_i}(\gamma_i^\delta(s_i)) \ge E_{m_i} - 2\alpha_\delta \quad \text{implies} \quad \|\gamma_i^\delta(s_i) - \gamma_i^\delta(T^i)\| \le \delta. \tag{2.8}$$

From (2.6)–(2.7) we have

$$\sup_{s\in T}\left|\Gamma_\varepsilon(\gamma_\varepsilon^\delta)(s) - \sum_{i=1}^k L_{m_i}(\gamma_i^\delta)(s_i)\right| = O(\varepsilon),$$

and hence there exists $\varepsilon_\delta \in (0,1)$ such that for all $\varepsilon \in (0,\varepsilon_\delta)$, we have $D_\varepsilon^\delta \geq E - \alpha_\delta/2$ and

$$\sup_{s\in T}\left|\Gamma_\varepsilon(\gamma_\varepsilon^\delta)(s) - \sum_{i=1}^k L_{m_i}(\gamma_i^\delta)(s_i)\right| \leq \frac{\alpha_\delta}{2}.$$

It follows that for $\varepsilon \in (0,\varepsilon_\delta)$, $\Gamma_\varepsilon(\gamma_\varepsilon^\delta)(s) \geq D_\varepsilon^\delta - \alpha_\delta$ implies

$$\sum_{i=1}^k L_{m_i}(\gamma_i^\delta)(s_i) \geq D_\varepsilon^\delta - \frac{3\alpha_\delta}{2} \geq E - 2\alpha_\delta.$$

Recalling that for any $1 \leq i \leq k$, $\max_{s_i\in[0,1]} L_{m_i}(\gamma_i^\delta)(s_i) = E_{m_i}$, we get $L_{m_i}(\gamma_i^\delta)(s_i) \geq E_{m_i} - 2\alpha_\delta$ for all $1 \leq i \leq k$, which implies by (2.8):

$$\|\gamma_i^\delta(s_i) - \gamma_i^\delta(T^i)\| \leq \delta, \quad \text{for all } i = 1,\ldots,k.$$

We claim there exists $M_1 > 0$ (independent of ε, δ) such that for all $\varepsilon \in (0,1)$ and $u \in H_\varepsilon$,

$$\|(\varphi_\varepsilon u)(\,\cdot\, - x_i/\varepsilon)\|_\varepsilon \leq M_1\|u\|, \quad i = 1,\ldots,k. \tag{2.9}$$

Indeed, for small $\varepsilon > 0$, we have

$$\begin{aligned}\|(\varphi_\varepsilon u)(\,\cdot\, - x_i/\varepsilon)\|_\varepsilon^2 &= \int_{B(0,2\beta/\varepsilon)} \big(|\nabla(\varphi_\varepsilon u)|^2 + V(\varepsilon x + x_i)\varphi_\varepsilon^2 u^2\big)\,dx \\ &\leq \int_{B(0,2\beta/\varepsilon)} \big(2|\nabla\varphi_\varepsilon|^2u^2 + 2|\nabla u|^2 + V(\varepsilon x + x_i)u^2\big)\,dx \\ &\leq \int_{B(0,2\beta/\varepsilon)} \bigg[2|\nabla u|^2 + \bigg(\sup_{x\in B(x_i,2\beta)} V(x) + 1\bigg)u^2\bigg]\,dx \\ &\leq \bigg(\sup_{x\in B(x_i,2\beta)} V(x) + 2\bigg)\|u\|^2.\end{aligned}$$

Hence, it is enough to choose

$$M_1 := \bigg(\max_{1\leq i\leq k}\sup_{x\in B(x_i,2\beta)} V(x) + 2\bigg)^{1/2}.$$

Thus

$$\big\|\gamma_{\varepsilon,i}^\delta(s_i)(\,\cdot\,) - (\varphi_\varepsilon\gamma_i^\delta(T^i))(\,\cdot\, - x_i/\varepsilon)\big\|_\varepsilon \leq M_1\delta.$$

Let $s_0 = (T^1,\ldots,T^k) \in T$, then $\gamma_\varepsilon^\delta(s_0) \in X_\varepsilon$. Moreover, $\|\gamma_\varepsilon^\delta(s) - \gamma_\varepsilon^\delta(s_0)\|_\varepsilon \leq M_0\delta$, where $M_0 = kM_1$. □

In the following, we construct a special PS-sequence of Γ_ε, which is localized in some neighbourhood X_ε^d of X_ε. Define

$$\Gamma_\varepsilon^\alpha := \{u \in H_\varepsilon : \Gamma_\varepsilon(u) \leq \alpha\}, \quad \alpha \in \mathbb{R},$$

and, for $d > 0$,

$$X_\varepsilon^d := \Big\{u \in H_\varepsilon : \inf_{v\in X_\varepsilon}\|u - v\|_\varepsilon \leq d\Big\}.$$

Proposition 2.3. *Let* $\{\varepsilon_j\}_{j=1}^{\infty}$ *be such that* $\lim_{j\to\infty} \varepsilon_j = 0$ *and* $\{u_{\varepsilon_j}\} \subset X^d_{\varepsilon_j}$ *be such that*

$$\lim_{j\to\infty} \Gamma_{\varepsilon_j}(u_{\varepsilon_j}) \le E \quad \text{and} \quad \lim_{j\to\infty} \Gamma'_{\varepsilon_j}(u_{\varepsilon_j}) = 0.$$

Then, for sufficiently small $d > 0$, *there exists, up to a subsequence,* $\{y^i_j\}_{j=1}^{\infty} \subset \mathbb{R}^2$, $x^i \in \mathcal{M}^i, U_i \in S_{m_i}$, $1 \le i \le k$, *such that*

$$\lim_{j\to\infty} |\varepsilon_j y^i_j - x^i| = 0 \quad \text{and} \quad \lim_{j\to\infty} \Big\| u_{\varepsilon_j} - \sum_{i=1}^{k} \varphi_{\varepsilon_j}(\,\cdot\, - y^i_j) U_i(\,\cdot\, - y^i_j) \Big\|_{\varepsilon_j} = 0.$$

Proof. The proof is similar to [9, Proposition 4] and [11, Proposition 5] but for the convenience of the reader we sketch it. Let us write for simplicity ε in place of ε_i. By the very definition of X^d_ε and the compactness of S_{m_i}, there exist $Z_i \in S_{m_i}$, $x^i_\varepsilon \in \mathcal{M}^\beta_i$ such that $x^i_\varepsilon \to x^i \in \mathcal{M}^\beta_i$ and such that for small $\varepsilon > 0$ one has

$$\Big\| u_\varepsilon - \sum_{i=1}^{k} \varphi_\varepsilon\Big(\,\cdot\, - \frac{x^i_\varepsilon}{\varepsilon}\Big) Z_i\Big(\,\cdot\, - \frac{x^i_\varepsilon}{\varepsilon}\Big) \Big\|_\varepsilon \le 2d. \tag{2.10}$$

Step 1. We claim that choosing $d > 0$ small enough one has

$$\liminf_{\varepsilon\to 0} \sup_{y\in A_\varepsilon} \int_{B(y,1)} |u_\varepsilon|^2 = 0,$$

where $A_\varepsilon = \bigcup_{i=1}^{k} (B(x^i_\varepsilon/\varepsilon, 3\beta/\varepsilon) \setminus B(x^i_\varepsilon/\varepsilon, \beta/2\varepsilon))$, which immediately implies from [11, Lemma 1] that

$$F(u_\varepsilon) \to 0 \ \text{ in } \ L^1(B_\varepsilon), \tag{2.11}$$

where $B_\varepsilon = \bigcup_{i=1}^{k} (B(x^i_\varepsilon/\varepsilon, 2\beta/\varepsilon) \setminus B(x^i_\varepsilon/\varepsilon, \beta/\varepsilon))$. Assume by contradiction that there exists $r > 0$ such that

$$\liminf_{\varepsilon\to 0} \sup_{y\in A_\varepsilon} \int_{B(y,1)} |u_\varepsilon|^2 = 2r > 0,$$

then there exists $y_\varepsilon \in A_\varepsilon$, such that for $\varepsilon > 0$ small enough $\int_{B(y_\varepsilon,1)} |u_\varepsilon|^2 \ge r$. Let $v_\varepsilon(y) = u_\varepsilon(y + y_\varepsilon)$, and then

$$\int_{B(0,1)} |v_\varepsilon|^2 \ge r. \tag{2.12}$$

Assume $v_\varepsilon \to v$ weakly in $H^1(\mathbb{R}^2)$, then $v \not\equiv 0$ and it satisfies

$$-\Delta v + V(x_0) v = f(v) \quad \text{in } \mathbb{R}^2,$$

where $x_0 \in \bigcup_{i=1}^{k} (\mathcal{M}^i)^{4\beta}$ with $\varepsilon y_\varepsilon \to x_0$, as $\varepsilon \to 0$. For sufficiently large $R > 0$,

$$\liminf_{\varepsilon\to 0} \int_{B(y_\varepsilon,R)} |\nabla u_\varepsilon|^2 \ge \frac{1}{2} \int_{\mathbb{R}^2} |\nabla v|^2 = L_{V(x_0)}(v).$$

Clearly, $L_{V(x_0)}(v) \geq E_{V(x_0)} \geq \min\limits_{1\leq i\leq k} E_{m_i}$, from which

$$\liminf_{\varepsilon\to 0} \int_{B(y_\varepsilon,R)} |\nabla u_\varepsilon|^2 \geq \min_{1\leq i\leq k} E_{m_i} > 0,$$

which contradicts (2.10) provided d is small enough. Therefore, Step 1 is proved.

Step 2. Let

$$u_\varepsilon^1(y) = \sum_{i=1}^{k} \varphi_\varepsilon\bigg(y - \frac{x_\varepsilon^i}{\varepsilon}\bigg) u_\varepsilon(y), \qquad u_\varepsilon^2 = u_\varepsilon - u_\varepsilon^1.$$

We claim that $\Gamma_\varepsilon(u_\varepsilon) \geq \Gamma_\varepsilon(u_\varepsilon^1) + \Gamma_\varepsilon(u_\varepsilon^2) + o(1)$, as $\varepsilon \to 0$, provided $d > 0$ is small enough and $\Gamma_\varepsilon(u_\varepsilon^2) \geq 0$ for small $\varepsilon > 0$. On one hand, a direct computation shows that

$$\Gamma_\varepsilon(u_\varepsilon) \geq \Gamma_\varepsilon(u_\varepsilon^1) + \Gamma_\varepsilon(u_\varepsilon^2) - \int_{\mathbb{R}^2} G(\varepsilon y, u_\varepsilon) - G(\varepsilon y, u_\varepsilon^1) - G(\varepsilon y, u_\varepsilon^2) + o(1).$$

Then

$$\begin{aligned}\limsup_{\varepsilon\to 0} \bigg| \int_{\mathbb{R}^2} G(\varepsilon y, u_\varepsilon) - G(\varepsilon y, u_\varepsilon^1) - G(\varepsilon y, u_\varepsilon^2)\bigg| \\ = \limsup_{\varepsilon\to 0} \bigg| \int_{B_\varepsilon} F(u_\varepsilon) - F(u_\varepsilon^1) - F(u_\varepsilon^2)\bigg| = 0,\end{aligned}$$

where we have used (2.11). As a consequence,

$$\Gamma_\varepsilon(u_\varepsilon) \geq \Gamma_\varepsilon(u_\varepsilon^1) + \Gamma_\varepsilon(u_\varepsilon^2) + o(1), \quad \text{as } \varepsilon \to 0.$$

On the other hand, since $G(y, u_\varepsilon) \leq F(u_\varepsilon)$ for any $y \in \mathbb{R}^2$, we have

$$\Gamma_\varepsilon(u_\varepsilon^2) \geq \frac{1}{2}\, \|u_\varepsilon^2\|_\varepsilon^2 - \int_{\mathbb{R}^2} F(u_\varepsilon^2). \tag{2.13}$$

From $u_\varepsilon \in X_\varepsilon^d$, we get $\|u_\varepsilon^2\|_\varepsilon \leq 2d$, provided $\varepsilon > 0$ is small enough. Then, as in [11], by choosing d small enough, we get

$$\int_{\mathbb{R}^2} F(u_\varepsilon^2) \leq \frac{1}{4}\, \|u_\varepsilon^2\|_\varepsilon^2.$$

Thus, it follows from (2.13) that choosing $d > 0$ sufficiently small,

$$\int_{\mathbb{R}^2} F(u_\varepsilon^2) \geq \frac{1}{4}\, \|u_\varepsilon^2\|_\varepsilon^2 \geq 0.$$

Step 3. For any fixed $1 \leq i \leq k$, let

$$u_\varepsilon^{1,i}(y) = \varphi_\varepsilon\bigg(y - \frac{x_\varepsilon^i}{\varepsilon}\bigg) u_\varepsilon(y),$$

then $u_\varepsilon^1 = \sum\limits_{i=1}^{k} u_\varepsilon^{1,i}$. Moreover, $\Gamma_\varepsilon(u_\varepsilon^1) = \sum\limits_{i=1}^{k} \Gamma_\varepsilon(u_\varepsilon^{1,i})$. Set

$$w_\varepsilon^i(y) := u_\varepsilon^{1,i}\bigg(y + \frac{x_\varepsilon^i}{\varepsilon}\bigg) = \varphi_\varepsilon(y) u_\varepsilon\bigg(y + \frac{x_\varepsilon^i}{\varepsilon}\bigg),$$

up to a subsequence, $w_\varepsilon^i \rightharpoonup w^i$ weakly in $H^1(\mathbb{R}^2)$, $w_\varepsilon^i \to w^i$ almost everywhere in $\mathbb{R}^2$. Then it was proved in [11] that for $d > 0$ small enough, for any $1 \le i \le k$, the following hold:

$$\liminf_{\varepsilon\to 0} \sup_{z\in\mathbb{R}^2} \int_{B(z,1)} |w_\varepsilon^i - w^i|^2 = 0$$

and

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^2} F(w_\varepsilon^i)\,dx = \int_{\mathbb{R}^2} F(w^i)\,dx,$$

which in turn gives

$$\begin{aligned}\liminf_{\varepsilon\to 0} \Gamma_\varepsilon(u_\varepsilon^{1,i}) &\ge \liminf_{\varepsilon\to 0} \left(\frac{1}{2}\int_{\mathbb{R}^2} |\nabla w_\varepsilon^i|^2 + V_\varepsilon\left(y + \frac{x_\varepsilon^i}{\varepsilon}\right)|w_\varepsilon^i|^2 - \int_{\mathbb{R}^2} F(w_\varepsilon^i)\right) \\ &\ge \frac{1}{2}\int_{\mathbb{R}^2} |\nabla w^i|^2 + V(x^i)|w^i|^2 - \int_{\mathbb{R}^2} F(w^i).\end{aligned}$$

We know as well that $w^i \not\equiv 0$ (otherwise, if $w^i \equiv 0$, by (2.10) we would get for any $p > 2$ that $\|Z_i\|_p = O(d)$, however, since $Z_i \in S_{m_i}$, by choosing d small enough we get a contradiction). Moreover, it is easy to verify that w^i satisfies

$$-\Delta w + V(x^i)w = f(w), \quad w \in H^1(\mathbb{R}^2).$$

So, $\liminf_{\varepsilon\to 0} \Gamma_\varepsilon(u_\varepsilon^{1,i}) \ge L_{V(x^i)}(w^i) \ge E_{V(x^i)}$. Recalling from [36] that $E_a > E_b$ if $a > b$ which together with Step 1 yields

$$\liminf_{\varepsilon\to 0} \Gamma_\varepsilon(u_\varepsilon) \ge \liminf_{\varepsilon\to 0} \sum_{i=1}^k \Gamma_\varepsilon(u_\varepsilon^{1,i}) \ge \sum_{i=1}^k E_{m_i} = E.$$

Noting that $\limsup_{\varepsilon\to 0} \Gamma_\varepsilon(u_\varepsilon) \le E$ and $\lim_{\varepsilon\to 0} \Gamma_\varepsilon(u_\varepsilon^2) = 0$, $x^i \in \mathcal{M}^i$ and $L_{m_i}(w^i) = E_{m_i}$. Therefore, as in [9, Proposition 4], there exists y_ε^i such that

$$u_\varepsilon^{1,i} \to \varphi_\varepsilon(\,\cdot\, - y_\varepsilon^i)U_i(\,\cdot\, - y_\varepsilon^i) \quad \text{strongly in } H_\varepsilon$$

and consequently

$$u_\varepsilon^1 = \sum_{i=1}^k u_\varepsilon^{1,i} \to \sum_{i=1}^k \varphi_\varepsilon(\,\cdot\, - y_\varepsilon^i)U_i(\,\cdot\, - y_\varepsilon^i),$$

strongly in H_ε. By (2.13), it is easy to see $u_\varepsilon^2 \to 0$ strongly in H_ε and thus the conclusion follows. □

By Proposition 2.3, there exists $d_0 > 0$ small with the following properties: for any $d_1 \in (0, d_0/3)$, there exist $\rho_1 > 0$, $\omega_1 > 0$ and $\varepsilon_1 > 0$, such that for $\varepsilon \in (0, \varepsilon_1)$, $0 \notin X_\varepsilon^{d_0}$, $\inf_{u\in X_\varepsilon^{d_0}} \Gamma_\varepsilon(u) \ge E/2$ and

$$|\Gamma_\varepsilon'(u)| \ge \omega_1 \quad \text{for } u \in \Gamma_\varepsilon^{E+\rho_1} \cap (X_\varepsilon^{d_0} \setminus X_\varepsilon^{d_1}). \tag{2.14}$$

Let $\delta_1 = d_1/M_0$, where M_0 is given in Proposition 2.2. By (2.14) and a deformation argument, Γ_ε admits a Palais–Smale sequence in $\Gamma_\varepsilon^{D_\varepsilon^{\delta_1}} \cap X_\varepsilon^{d_1}$, where $D_\varepsilon^{\delta_1}$ is given in (2.5). Precisely, as in [9], [11] we prove the following

Proposition 2.4. *For sufficiently small* $\varepsilon \in (0,\varepsilon_1)$*, there exists a sequence* $\{u_{n,\varepsilon}\}_{n=1}^{\infty} \subset \Gamma_{\varepsilon}^{D_{\varepsilon}^{\delta_1}} \cap X_{\varepsilon}^{d_1}$ *such that* $|\Gamma_{\varepsilon}'(u_{n,\varepsilon})| \to 0$*, as* $n \to \infty$*.*

Proof. Assume by contradiction that there exists $a(\varepsilon) > 0$ such that $|\Gamma_{\varepsilon}'(u)| \geq a(\varepsilon)$, $u \in \Gamma_{\varepsilon}^{D_{\varepsilon}^{\delta_1}} \cap X_{\varepsilon}^{d_1}$ for small $\varepsilon > 0$. By Proposition 2.2, there exists $\alpha_{\delta_1} \in (0, E - \widetilde{E})$ such that for $\varepsilon > 0$ small enough, one has

$$\Gamma_{\varepsilon}(\gamma_{\varepsilon}^{\delta_1}(s)) \geq D_{\varepsilon}^{\delta_1} - \alpha_{\delta_1} \ \Rightarrow\ \gamma_{\varepsilon}^{\delta_1}(s) \in X_{\varepsilon}^{M_0\delta_1} \subset X_{\varepsilon}^{\delta_1}. \tag{2.15}$$

Then as in Byeon and Jeanjean [8], using a deformation argument, there exist $\widetilde{\mu}_{\delta_1} \in (0, \alpha_{\delta_1})$ and $\gamma^{\delta_1} \in C(T, H_{\varepsilon})$ such that

$$\begin{aligned} \gamma^{\delta_1}(s) &= \gamma_{\varepsilon}^{\delta_1}(s) \quad \text{if } \gamma_{\varepsilon}^{\delta_1}(s) \in \Gamma_{\varepsilon}^{D_{\varepsilon}^{\delta_1} - \alpha_{\delta_1}}, \\ \gamma^{\delta_1}(s) &\in X_{\varepsilon}^{2d_0/3} \quad \text{if } \gamma_{\varepsilon}^{\delta_1}(s) \notin \Gamma_{\varepsilon}^{D_{\varepsilon}^{\delta_1} - \alpha_{\delta_1}}, \end{aligned}$$

and $\Gamma_{\varepsilon}(\gamma^{\delta_1}(s)) < D_{\varepsilon}^{\delta_1} - \widetilde{\mu}_{\delta_1}$ for $s \in T$. Take a cut-off function $\psi \in C_0^{\infty}(O^{2\delta_1}, [0,1])$ with $\psi(x) = 1$ if $x \in O^{\delta_1}$. For $s \in T$, let $\gamma_1^{\delta_1}(s) = \psi_{\varepsilon}\gamma^{\delta_1}(s)$, $\gamma_2^{\delta_1}(s) = \gamma^{\delta_1}(s) - \gamma_1^{\delta_1}(s)$, where $\psi_{\varepsilon}(x) = \psi(\varepsilon x)$. Then one has

$$\begin{aligned} \Gamma_{\varepsilon}(\gamma^{\delta_1})(s) \geq{}& \Gamma_{\varepsilon}(\gamma_1^{\delta_1})(s) + \Gamma_{\varepsilon}(\gamma_2^{\delta_1})(s) + O(\varepsilon) \\ &+ \int_{O_{\varepsilon}^{2\delta_1} \setminus O_{\varepsilon}^{\delta_1}} \widetilde{F}(\gamma_1^{\delta_1}(s)) + \widetilde{F}(\gamma_2^{\delta_1}(s)) - \widetilde{F}(\gamma^{\delta_1}(s)), \end{aligned}$$

where $\widetilde{F}(t) = \int_0^t \widetilde{f}(\tau)\,d\tau$. From the construction of γ^{δ_1}, we have

$$\int_{\mathbb{R}^2 \setminus O_{\varepsilon}} |\gamma^{\delta_1}(s)|^2 \leq C\varepsilon^{\mu}, \quad s \in T,$$

for some $C > 0$ (independent of ε). So that by the very definition of $\widetilde{f}$,

$$\lim_{\varepsilon \to 0} \int_{O_{\varepsilon}^{2\delta_1} \setminus O_{\varepsilon}^{\delta_1}} \widetilde{F}(\gamma_1^{\delta_1}(s)) + \widetilde{F}(\gamma_2^{\delta_1}(s)) - \widetilde{F}(\gamma^{\delta_1}(s)) = 0,$$

and

$$\Gamma_{\varepsilon}(\gamma_2^{\delta_1})(s) \geq -\int_{\mathbb{R}^2 \setminus O_{\varepsilon}} \widetilde{F}(\gamma_2^{\delta_1}(s)) \to 0, \quad \text{as } \varepsilon \to 0.$$

Then

$$\Gamma_{\varepsilon}(\gamma^{\delta_1})(s) \geq \Gamma_{\varepsilon}(\gamma_1^{\delta_1})(s) + O(\varepsilon), \quad s \in T.$$

For any $1 \leq i \leq k$ and $s \in T$, let

$$\gamma_{1,i}^{\delta_1}(s)(x) = \begin{cases} \gamma_1^{\delta_1}(s)(x) & \text{if } x \in (O^i)_{\varepsilon}^{2\delta_1}, \\ \\ 0 & \text{if } x \notin (O^i)_{\varepsilon}^{2\delta_1}, \end{cases}$$

in order to get

$$\gamma_1^{\delta_1}(s)(x) = \sum_{i=1}^{k} \gamma_{1,i}^{\delta_1}(s)(x) \quad \text{and} \quad \Gamma_{\varepsilon}(\gamma_1^{\delta_1})(s) \geq \sum_{i=1}^{k} \Gamma_{\varepsilon}(\gamma_{1,i}^{\delta_1}(s)),$$

and hence

$$\Gamma_\varepsilon(\gamma^{\delta_1})(s) \geq \sum_{i=1}^{k} \Gamma_\varepsilon(\gamma_{1,i}^{\delta_1}(s)) + O(\varepsilon), \quad s \in T. \tag{2.16}$$

Due to the fact that $\alpha_{\delta_1} \in (0, E - \widetilde{E})$, we have $\widetilde{E} < D_\varepsilon^{\delta_1} - \alpha_{\delta_1}$ for sufficiently small ε. By Proposition 2.2, $\max\limits_{s\in\partial T} \Gamma_\varepsilon(\gamma_\varepsilon^{\delta_1}(s)) < D_\varepsilon^{\delta_1} - \alpha_{\delta_1}$ for ε small enough, which implies $\gamma^{\delta_1}(s) = \gamma_\varepsilon^{\delta_1}(s)$ for $s \in \partial T$. Thus, for any $1 \leq i \leq k$,

$$\gamma_{1,i}^{\delta_1} \in \Phi_{\varepsilon,1}^i = \big\{\varphi \in C(T, H_\varepsilon) : \varphi(\widetilde{0}_i) = 0, \varphi(\widetilde{T}_i) = \gamma_{\varepsilon,i}^{\delta}(1)\big\},$$

where $\widetilde{0}_i = (s_1, \ldots, s_{i-1}, 0, s_{i+1}, \ldots, s_k) \in T$, $\widetilde{T}_i = (s_1, \ldots, s_{i-1}, 1, s_{i+1}, \ldots, s_k) \in T$. By [21, Proposition 3.4], we have that there exists $\widetilde{s} \in T$ such that, for any $1 \leq i \leq k$, $\Gamma_\varepsilon(\gamma_{1,i}^{\delta_1}(\widetilde{s})) \geq C_\varepsilon$, where $C_\varepsilon = \inf\limits_{\varphi\in\Phi_\varepsilon} \max\limits_{s\in[0,1]} \Gamma_\varepsilon(\varphi(s))$ and $\Phi_\varepsilon = \{\varphi \in C([0,1], H_\varepsilon) : \varphi(0) = 0,\ \Gamma_\varepsilon(\varphi(1)) < 0\}$. It is easy to see that $C_\varepsilon \geq C_\varepsilon^i$ for any $1 \leq i \leq k$. Then by (2.16)

$$\liminf_{\varepsilon\to 0} \max_{s\in T} \Gamma_\varepsilon(\gamma^{\delta_1}(s)) \geq E,$$

which is a contradiction and this completes the proof. □

In the following, we prove that the PS-sequence $\{u_{n,\varepsilon}\}_n$ obtained in Proposition 2.4 has a nontrivial weak limit u_ε, which is actually a solution to the original problem (2.1). For this purpose, we recall the following inequality due to Cao [17] and do Ó [29] (see also [18] for further results):

LEMMA 2.5. *If $\alpha > 0$ and $u \in H^1(\mathbb{R}^2)$, then*

$$\int_{\mathbb{R}^2} (\exp(\alpha u^2) - 1)\, dx < \infty.$$

Moreover, if $\alpha \in (0, 4\pi)$, then for any positive constant M, there exists $C = C(\alpha, M)$ such that

$$\int_{\mathbb{R}^2} (\exp(\alpha u^2) - 1)\, dx \leq C,$$

for any $u \in H^1(\mathbb{R}^2)$ with $\|\nabla u\|_2 \leq 1$ and $\|u\|_2 \leq M$.

PROPOSITION 2.6. *For sufficiently small $\varepsilon \in (0, \varepsilon_1)$, Γ_ε has a nontrivial critical point $u_\varepsilon \in X_\varepsilon^{d_1} \cap \Gamma_\varepsilon^{D_\varepsilon^{\delta_1}}$.*

PROOF. By Proposition 2.4, Γ_ε admits a PS-sequence $\{u_{n,\varepsilon}\}_{n=1}^\infty \subset X_\varepsilon^{d_1} \cap \Gamma_\varepsilon^{D_\varepsilon^{\delta_1}}$. By the very definition of $X_\varepsilon^{d_1}$, we know that $\{u_{n,\varepsilon}\}_{n=1}^\infty$ is bounded in H_ε. Without loss of generality, we may assume $u_{n,\varepsilon} \rightharpoonup u_\varepsilon$ weakly in H_ε, as $n \to \infty$. By the definition of $g(x,t)$, as a consequence of [15, Proposition 3] one has

$$\lim_{R\to\infty} \sup_{n\geq 1} \int_{|x|\geq R} (|\nabla u_{n,\varepsilon}|^2 + V_\varepsilon |u_{n,\varepsilon}|^2)\, dx = 0. \tag{2.17}$$

This implies that $u_{n,\varepsilon} \to u_\varepsilon$ strongly in $L^2(\mathbb{R}^2)$. Thus $Q_\varepsilon(u_{n,\varepsilon}) \to Q_\varepsilon(u_\varepsilon)$, as $n \to \infty$. Recalling that $u_{n,\varepsilon} \in X_\varepsilon^{d_1}$, by (f$_1$)–(f$_2$) there exist $\alpha > 0$ small and some constant $C > 0$ such that

$$|f(t)| \leq |t| + C(\exp(\alpha t^2) - 1), \quad t \in \mathbb{R},$$

and $\sup_{n\geq 1} \|\nabla u_{n,\varepsilon}\|_2^2 \leq \alpha^{-1}/2$. By Lemma 2.5, $\sup_{n\geq 1} \|f(u_{n,\varepsilon})\|_2 < \infty$, as a consequence

$$\sup_{n\geq 1} \int_{\mathbb{R}^2} |g(\varepsilon y, u_{n,\varepsilon})|^2 \, dy < \infty.$$

Then, for any $\varphi \in C_0^\infty(\mathbb{R}^2)$ we get

$$\int_{\mathbb{R}^2} g(\varepsilon y, u_{n,\varepsilon})(u_{n,\varepsilon} - u_\varepsilon)\varphi \, dy \to 0, \quad n \to \infty. \tag{2.18}$$

Next we prove that actually $u_{n,\varepsilon} \to u_\varepsilon$ strongly in H_ε, as $n \to \infty$, which in turn yields $\Gamma'_\varepsilon(u_\varepsilon) = 0$ in H_ε and $u_\varepsilon \in X_\varepsilon^{d_1} \cap \Gamma_\varepsilon^{D_\varepsilon^{\delta_1}}$.

First notice that by (2.17), for any $\sigma > 0$, there exists $R > 0$ such that for all n,

$$\int_{|x|\geq R} \big(|\nabla u_{n,\varepsilon}|^2 + |\nabla u_\varepsilon|^2 + V_\varepsilon(y)|u_{n,\varepsilon}|^2 + V_\varepsilon(y)|u_\varepsilon|^2\big) \, dy < \sigma. \tag{2.19}$$

Let $\psi \in C_0^\infty(\mathbb{R}^2, [0,1])$ with $\psi(x) = 1$ if $|x| \leq R$ and $\psi(x) = 0$ if $|x| \geq 2R$. Take $(u_{n,\varepsilon} - u_\varepsilon)\psi$ as a test function to get $\langle \Gamma'_\varepsilon(u_{n,\varepsilon}), (u_{n,\varepsilon} - u_\varepsilon)\psi\rangle \to 0$, as $n \to \infty$. Then by (2.17) and (2.18) we obtain

$$\limsup_{n\to\infty} \left| \int_{|x|\leq R} \big(|\nabla u_{n,\varepsilon}|^2 - |\nabla u_\varepsilon|^2 + V_\varepsilon(y)|u_{n,\varepsilon}|^2 - V_\varepsilon(y)|u_\varepsilon|^2\big) \, dy \right| < \sigma,$$

which implies by (2.19) also the following:

$$\limsup_{n\to\infty} \left| \int_{\mathbb{R}^2} \big(|\nabla u_{n,\varepsilon}|^2 - |\nabla u_\varepsilon|^2 + V_\varepsilon(y)|u_{n,\varepsilon}|^2 - V_\varepsilon(y)|u_\varepsilon|^2\big) \, dy \right| \leq 3\sigma,$$

namely $\|u_{n,\varepsilon}\|_\varepsilon \to \|u_\varepsilon\|_\varepsilon$ as $n \to \infty$. □

2.1. Proof for Theorem 1.1 completed. By Proposition 2.6, Γ_ε has a nontrivial critical point $u_\varepsilon \in X_\varepsilon^{d_1} \cap \Gamma_\varepsilon^{D_\varepsilon^{\delta_1}}$ for small $\varepsilon \in (0, \varepsilon_1)$ and $u_\varepsilon \geq 0$ since $f(t) = 0$ for $t \leq 0$.

Step 1. We claim that there exists $C > 0$ (independent of ε) such that

$$\|u_\varepsilon\|_\infty < C, \tag{2.20}$$

which implies by the Harnark inequality (see [34]) that $u_\varepsilon > 0$ in $\mathbb{R}^2$ and $\inf_{\varepsilon\in(0,\varepsilon_1)} \|u_\varepsilon\|_\infty > 0$.

Next we use the Nash–Moser iteration technique (see [47] and also [38]) to prove (2.20). For any $L > 0$ and $\beta \geq 1$, set

$$u_{\varepsilon,L} = \min\{u_\varepsilon, L\} \quad \text{and} \quad v_\varepsilon = u_\varepsilon u_{\varepsilon,L}^{2(\beta-1)}.$$

Let us fix $t > 2$, then by Lemma 2.5, we can choose $\alpha > 0$ sufficiently small such that

$$\sup_{\varepsilon\in(0,\varepsilon_1)} \|\Psi(u_\varepsilon)\|_{L^{t/(t-2)}(\mathbb{R}^2)} < \infty. \tag{2.21}$$

By $\Gamma'_\varepsilon(u_\varepsilon) = 0$ and (f$_1$)–(f$_2$), there exists $C = C(\alpha) > 0$ such that u_ε satisfies

$$-\Delta u_\varepsilon \le C\Psi(u_\varepsilon)u_\varepsilon, \quad u_\varepsilon \ge 0,\ x \in \mathbb{R}^2. \tag{2.22}$$

Then taking v_ε as a test function in (2.22) we obtain

$$\int_{\mathbb{R}^2} |\nabla u_\varepsilon|^2 u_{\varepsilon,L}^{2(\beta-1)}\,dx + 2(\beta-1)\int_{\mathbb{R}^2} |\nabla u_{\varepsilon,L}|^2 u_{\varepsilon,L}^{2(\beta-1)}\,dx \le C\int_{\mathbb{R}^2} \Psi(u_\varepsilon)u_\varepsilon^2 u_{\varepsilon,L}^{2(\beta-1)}\,dx.$$

Let $w_{\varepsilon,L} = u_\varepsilon u_{\varepsilon,L}^{\beta-1}$ and thus $\nabla w_{\varepsilon,L} = \nabla u_\varepsilon u_{\varepsilon,L}^{\beta-1} + (\beta-1)\nabla u_{\varepsilon,L}\nabla u_{\varepsilon,L}^{\beta-1}$. Then

$$\int_{\mathbb{R}^2} |\nabla w_{\varepsilon,L}|^2\,dx \le C\beta^2 \int_{\mathbb{R}^2} \Psi(u_\varepsilon)w_{\varepsilon,L}^2\,dx,$$

where $C > 0$ is independent of ε, L, β. By (2.21), $\|\nabla w_{\varepsilon,L}\|_2 \le C\beta\|w_{\varepsilon,L}\|_t$. For some fixed $s > t$, by the Gagliardo–Nirenberg inequality (see [34]),

$$\|w_{\varepsilon,L}\|_s \le C(\|\nabla w_{\varepsilon,L}\|_2 + \|w_{\varepsilon,L}\|_t) \le C\beta\|w_{\varepsilon,L}\|_t,$$

where C only depends on s, t, N. Letting $L \to \infty$ we have

$$\|u_\varepsilon\|_{L^{s\beta}(\mathbb{R}^2)} \le C^{1/\beta}\beta^{1/\beta}\|u_\varepsilon\|_{L^{t\beta}(\mathbb{R}^2)}.$$

Let $\kappa = s/t > 1$, $\beta = \kappa^n$, so that

$$\|u_\varepsilon\|_{L^{t\kappa^{n+1}}(\mathbb{R}^2)} \le C^{\kappa^{-n}}\kappa^{n\kappa^{-n}}\|u_\varepsilon\|_{L^{t\kappa^n}(\mathbb{R}^2)}.$$

Finally, we obtain

$$\|u_\varepsilon\|_{L^{t\kappa^{n+1}}(\mathbb{R}^2)} \le C^{\sum_{i=0}^{n}\kappa^{-i}}\kappa^{\sum_{i=1}^{n} i\kappa^{-i}}\|u_\varepsilon\|_{L^t(\mathbb{R}^2)},$$

from which recalling that $\sup_{\varepsilon\in(0,\varepsilon_1)}\|u_\varepsilon\|_t < \infty$, (2.20) follows.

Step 2. We now establish the decay of u_ε at infinity. By Proposition 2.3, there exist $\{y_\varepsilon^i\}_{i=1}^k \subset \mathbb{R}^2$, $x^i \in \mathcal{M}^i$, $U_i \in S_{m_i}$ such that for any $1 \le i \le k$,

$$\lim_{\varepsilon\to0} |\varepsilon y_\varepsilon^i - x^i| = 0 \quad \text{and} \quad \lim_{\varepsilon\to0} \Big\|u_\varepsilon - \sum_{i=1}^k U_i(\,\cdot\, - y_\varepsilon^i)\Big\|_\varepsilon = 0,$$

and hence also $\lim\limits_{\varepsilon\to0}\|w_\varepsilon^i - U_i\|_2 = 0$, where $w_\varepsilon^i(y) = u_\varepsilon(y + y_\varepsilon^i)$. Then, for any $\sigma > 0$ there exists $R > 0$ (independent of ε, i) such that

$$\sup_{\varepsilon\in(0,\varepsilon_0)} \int_{\mathbb{R}^2\setminus B(0,R)} (w_\varepsilon^i)^2 \le \sigma.$$

From the uniform boundedness of u_ε, there exists $C > 0$ (independent of i, ε) such that w_ε^i satisfies $-\Delta w_\varepsilon^i \le Cw_\varepsilon^i$ in $\mathbb{R}^2$. Hence from [34, Theorem 8.17], there exists

$C > 0$ (independent of i, ε) such that $w_\varepsilon(y) \le C\sigma^{1/2}$, $\varepsilon \in (0,\varepsilon_1), |y| \ge R + 2$. Then, as in [31], by a comparison principle, for each $1 \le i \le k$, there exist $C, c > 0$ (independent of ε, i) and $y^i_\varepsilon \in \mathbb{R}^2$, such that

$$w^i_\varepsilon(y) \le C\exp(-c|y|) \quad \text{for } y \in \mathbb{R}^2,\ \varepsilon \in (0,\varepsilon_1).$$

Therefore,

$$u_\varepsilon(y) \le C\exp\bigg(-c\min_{1\le i\le k}|y - y^i_\varepsilon|\bigg) \quad \text{for } y \in \mathbb{R}^2,\ \varepsilon \in (0,\varepsilon_1). \tag{2.23}$$

Step 3. It remains to show that u_ε is a solution of (2.1). By the decay estimate (2.23), $Q_\varepsilon(u_\varepsilon) = 0$ and $u_\varepsilon(x) \to 0$, as $\varepsilon \to 0$ uniformly for $x \in \mathbb{R}^2 \setminus O_\varepsilon$. Thus, u_ε is a solution of the original problem (2.1). By elliptic regularity estimates, $w^i_\varepsilon \in C^{1,\alpha}(\mathbb{R}^2)$ for some $\alpha \in (0,1)$ and each $1 \le i \le k$. Then there exists $z^i_\varepsilon \in \mathbb{R}^2$ such that $\|w^i_\varepsilon\|_\infty = w^i_\varepsilon(z^i_\varepsilon) = u_\varepsilon(z^i_\varepsilon + y^i_\varepsilon)$. By Steps 1 and 2, $\{z^i_\varepsilon\}_{i=1}^k \subset \mathbb{R}^2$ is uniformly bounded with respect to ε. Let $x^i_\varepsilon = \varepsilon y^i_\varepsilon + \varepsilon z^i_\varepsilon$, then setting $v_\varepsilon(x) = u_\varepsilon(x/\varepsilon)$, we know $\max\limits_{x\in\mathbb{R}^2} v_\varepsilon(x) = v_\varepsilon(x^i_\varepsilon)$. Since $\varepsilon y^i_\varepsilon \to x^i \in \mathcal{M}^i$ as $\varepsilon \to 0$, we get $\lim\limits_{\varepsilon\to 0}\operatorname{dist}(x^i_\varepsilon, \mathcal{M}^i) = 0$. Finally assuming $z^i_\varepsilon \to z^i$, as $\varepsilon \to 0$, by Proposition 2.3, for each $1 \le i \le k$, $v_\varepsilon(\varepsilon\,\cdot + x^i_\varepsilon) \to U_i(\,\cdot + z^i)$ strongly in $H_\varepsilon(\mathbb{R}^2)$, as $\varepsilon \to 0$.

3. Proof of Theorem 1.2

We consider first the limiting problem (1.3) in the critical case. Assuming (f_1), (f_4), (f_5)' and

(f_6) $0 < 2F(t) \le tf(t)$ for $t \in \mathbb{R} \setminus \{0\}$,

Ruf and Sani [45, Theorem 5] proved that (1.3) admits a positive ground state solution U and that the least energy E_{m_i} is given by a mountain pass level. Here we remark that hypothesis (f_6) can be removed and β_0 in (f_5)' should be large enough. It was shown in [45] that (1.3) possesses a ground state solution by means of the following constraint minimization problem:

$$A_i := \inf\{T(u) : G_i(u) = 0,\ u \in H^1(\mathbb{R}^2) \setminus \{0\}\}, \tag{3.1}$$

where

$$T(u) = \frac{1}{2}\int_{\mathbb{R}^2}|\nabla u|^2\,dx \quad \text{and} \quad G_i(u) = \int_{\mathbb{R}^2}\bigg(F(u) - \frac{m_i}{2}u^2\bigg)\,dx.$$

If problem (3.1) admits a minimizer u_i, then there exists $\theta_i > 0$ such that $u_i(\,\cdot/\sqrt{\theta_i})$ is indeed the ground state solution of (1.3). Following [3], [45], to prove the existence of the minimizer to (1.3), it is enough to prove that $A_i < 1/2$. For this goal, for $1 \le i \le k$, let

$$c_i := \inf_{u\in H^1(\mathbb{R}^2)\setminus\{0\}}\ \max_{t\ge 0} L_{m_i}(tu),$$

then it can be seen in [3] that $A_i \le c_i$. It follows from Lemma 3.2 (see below) that $A_i < 1/2$ for each $1 \le i \le k$, then from [3], [45] one has the following

Lemma 3.1. *Assume* (f$_1$), (f$_4$) *and* (f$_5$)' *with*

$$\beta_0 > \frac{e}{2\pi} \max_{1\le i\le k} m_i, \tag{3.2}$$

then for each $1 \le i \le k$, (1.3) *admits a positive ground state solution. Moreover, the least energy* E_{m_i} *is obtained by a mountain pass value.*

Lemma 3.2. *There exists* $w \in H^1(\mathbb{R}^2) \setminus \{0\}$ *such that* $\max_{t\ge0} L_{m_i}(tw) < 1/2$ *where*

$$L_{m_i}(u) = \frac{1}{2}\int_{\mathbb{R}^2} (|\nabla u|^2 + m_i\,|u|^2)\,dx - \int_{\mathbb{R}^2} F(u)\,dx.$$

Proof. Let us first remark a few facts: by (3.2) we can choose $r > 0$ such that

$$\beta_0 > \max_{1\le i\le k} \frac{e^{r^2 m_i/2}}{\pi r^2}, \tag{3.3}$$

and considering the Moser sequence of functions

$$\widetilde{w}_n(x) := \frac{1}{\sqrt{2\pi}} \begin{cases} \sqrt{\log n} & \text{if } |x| \le \dfrac{r}{n}, \\ \dfrac{\log r/|x|}{\sqrt{\log n}} & \text{if } \dfrac{r}{n} \le |x| \le r, \\ 0 & \text{if } |x| \ge r, \end{cases}$$

it is readily seen that $\|\nabla \widetilde{w}_n\|_2 = 1$ and $\|\widetilde{w}_n\|_2^2 = r^2/(4\log n) + o(r^2/\log n)$. For any $1 \le i \le k$, let

$$|||\widetilde{w}_n|||_i^2 := \|\nabla \widetilde{w}_n\|_2^2 + m_i \|\widetilde{w}_n\|_2^2 = 1 + \frac{d_n(r)}{\log n} m_i,$$

where $d_n(r) := r^2/4 + o_n(1)$ and $o_n(1) \to 0$, as $n \to +\infty$. Set $w_n^i := \widetilde{w}_n / |||\widetilde{w}_n|||_i$, then for n large enough,

$$(w_n^i)^2(x) \ge \frac{1}{2\pi}(\log n - d_n(r) m_i), \quad |x| \le \frac{r}{n}. \tag{3.4}$$

Following the argument of Adimurthi [2] (see also [23], [45], [30]), one can establish the following

Claim. There exists $n \in \mathbb{N}$ such that

$$\max_{t\ge0} L_{m_i}(tw_n^i) < \frac{1}{2}, \quad 1 \le i \le k.$$

Indeed, assume by contradiction that for some i,

$$\max_{t\ge0} L_{m_i}(tw_n^i) \ge \frac{1}{2}, \quad n \in \mathbb{N}.$$

As a consequence of $(f_5)'$, for any $\varepsilon > 0$ there exists $R_\varepsilon > 0$ such that

$$sf(s) \ge (\beta_0 - \varepsilon) e^{4\pi s^2}, \quad \text{for all } s \ge R_\varepsilon, \tag{3.5}$$

which implies that there exist $C_1, C_2 > 0$ such that

$$F(s) \geq C_1 s^4 - C_2, \quad s \geq 0, \tag{3.6}$$

which yields $L_{m_i}(tw_n^i) \to -\infty$, as $t \to \infty$. Thus there exists $t_n > 0$ such that

$$L_{m_i}(t_n w_n^i) = \max_{t \geq 0} L_{m_i}(tw_n^i) \geq \frac{1}{2}, \tag{3.7}$$

which in turn gives

$$\frac{1}{2} \leq \frac{t_n^2}{2} - \int_{\mathbb{R}^2} F(t_n w_n^i) \leq \frac{t_n^2}{2},$$

thus $t_n \geq 1$.

Next we show that actually $\lim\limits_{n\to\infty} t_n = 1$. Observe that

$$t_n^2 = \int_{\mathbb{R}^2} f(t_n w_n^i) t_n w_n^i \, dx, \tag{3.8}$$

and

$$t_n w_n^i = \frac{t_n}{|||\widetilde{w}_n|||_i} \frac{\sqrt{\log n}}{\sqrt{2\pi}} \to +\infty, \quad \text{as } n \to \infty, \quad x \in B_{r/n},$$

for n large enough, and using (3.4)–(3.6) we have

$$\begin{aligned} t_n^2 &\geq (\beta_0 - \varepsilon) \int_{B_{r/n}} e^{4\pi (t_n w_n^i)^2} \, dx - \pi C_2 r^2 \\ &\geq \pi r^2 (\beta_0 - \varepsilon) \, e^{2t_n^2[\log n - d_n(r) m_i] - 2\log n} - \pi C_2 r^2, \end{aligned}$$

which implies that $\{t_n\}$ is bounded and also $\limsup\limits_{n\to\infty} t_n \leq 1$. Thus, the claim is proved.

Noting that $w_n^i \to 0$ almost everywhere in $\mathbb{R}^2$, by the Lebesgue dominated convergence theorem, as $n \to \infty$ one has

$$\int_{\{t_n w_n^i < R_\varepsilon\}} f(t_n w_n^i) t_n w_n^i \, dx \to 0 \quad \text{and} \quad \int_{\{t_n w_n^i < R_\varepsilon\}} e^{4\pi (t_n w_n^i)^2} \, dx \to \pi r^2.$$

Then it follows from (3.8) and (3.5) that

$$\begin{aligned} t_n^2 &= \int_{B_r} f(t_n w_n^i) t_n w_n^i \, dx \\ &\geq (\beta_0 - \varepsilon) \int_{B_r} e^{4\pi (t_n w_n^i)^2} \, dx + \int_{\{t_n w_n^i < R_\varepsilon\}} f(t_n w_n^i) t_n w_n^i \, dx \\ &\quad - (\beta_0 - \varepsilon) \int_{\{t_n w_n^i < R_\varepsilon\}} e^{4\pi (t_n w_n^i)^2} \, dx \\ &= (\beta_0 - \varepsilon) \bigg(\int_{B_r} e^{4\pi (t_n w_n^i)^2} \, dx - \pi r^2 \bigg). \end{aligned}$$

Let us estimate the term $\int_{B_r} e^{4\pi (t_n w_n^i)^2} \, dx$. On one hand, it follows from (3.4) that

$$\int_{B_{r/n}} e^{4\pi (t_n w_n^i)^2} \, dx \geq \pi r^2 e^{2t_n^2[\log n - d_n(r) m_i] - 2\log n}.$$

Noting also that $t_n \geq 1$, we have

$$\liminf_{n\to\infty}\int_{B_{r/n}} e^{4\pi(t_n w_n^i)^2}\,dx \geq \pi r^2 e^{-m_i r^2/2}.$$

On the other hand, using the change of variable $s = re^{-|||\widetilde{w}_n|||_i\sqrt{\log n}\,t}$,

$$\int_{B_r\setminus B_{r/n}} e^{4\pi(w_n^i)^2}\,dx = 2\pi r^2|||\widetilde{w}_n|||_i\sqrt{\log n}\int_0^{\sqrt{\log n}/|||\widetilde{w}_n|||_i} e^{2(t^2-|||\widetilde{w}_n|||_i\sqrt{\log n}\,t)}\,dt$$

$$\geq 2\pi r^2|||\widetilde{w}_n|||_i\sqrt{\log n}\int_0^{\sqrt{\log n}/|||\widetilde{w}_n|||_i} e^{-2|||\widetilde{w}_n|||_i\sqrt{\log n}\,t}\,dt = \pi r^2(1-e^{-2\log n}),$$

Then,

$$\liminf_{n\to\infty}\int_{B_r} e^{4\pi(t_n w_n^i)^2}\,dx \geq \pi r^2(e^{-m_i r^2/2}+1),$$

which implies $1 = \lim_{n\to+\infty} t_n^2 \geq (\beta_0-\varepsilon)\pi r^2 e^{-m_i r^2/2}$. Since ε is arbitrary, we have $\beta_0 \leq e^{r^2 m_i/2}\pi r^2$, which contradicts (3.3) and the proof is complete. ☐

Let S_{m_i} be the set of positive ground state solutions U of (1.3) with $U(0) = \max_{x\in\mathbb{R}^2} U(x)$.

Proposition 3.3. *Assume* (f_1) *and* (f_4) *hold, then one has*

(a) S_{m_i} *is compact in* $H^1(\mathbb{R}^2)$;

(b) *there exists* $\kappa_i > 0$ *such that*

$$0 < \inf\{\|U\|_\infty : U \in S_{m_i}\} \leq \sup\{\|U\|_\infty : U \in S_{m_i}\} < \kappa_i;$$

(c) *there exist* $C, c > 0$, *independent of* $U \in S_{m_i}$, *such that*

$$|D^\alpha U(x)| \leq C\exp(-c|x|), \quad x \in \mathbb{R}^2, \text{ for } |\alpha| = 0, 1.$$

We will use the following lemma from [3].

Lemma 3.4. *Assume that* f *satisfies the same assumptions in Theorem* 1.2 *and let* $\{v_n\}$ *be a sequence in* $H^1_{rad}(\mathbb{R}^2)$ *such that*

$$\sup_n \|\nabla v_n\|^2_{L^2(\mathbb{R}^2)} = \rho < 1 \quad \text{and} \quad \sup_n \|v_n\|^2_{L^2(\mathbb{R}^2)} < \infty.$$

Then, if $v_n \to v$ *weakly in* $H^1_{\mathrm{rad}}(\mathbb{R}^2)$ *as* $n\to\infty$, *we have*

$$\lim_{n\to\infty}\int_{\mathbb{R}^2} F(v_n) = \int_{\mathbb{R}^2} F(v).$$

Proof of Proposition 3.3. Let us set $m = m_i$ and proceed by steps. The proof is similar to [50, Proposition 2.1] but for the convenience of the reader we give the details.

Step 1. We first show that any $U \in S_m$ is such that $U \in L^\infty(\mathbb{R}^2)$. Indeed, for any $r > 0$, U is a weak solution of the following problem:

$$-\Delta u + mu = f(u) \quad \text{in } B_r, \quad u - U \in H^1_0(B_r), \tag{3.9}$$

where $B_r(0) := \{x \in \mathbb{R}^2 : |x| < r\}$. By the Trudinger–Moser inequality of Lemma 2.5, one has $f(U) \in L^2(B_r)$. It follows from the standard Elliptic Theory that $U \in H^2_{\mathrm{loc}}(B_r)$. Moreover, for each open $\Omega \subset\subset B_r$ with $\partial\Omega \in C^1$ one has

$$\|U\|_{H^2(\Omega)} \leq C(\|f(U)\|_{L^2(B_r)} + \|U\|_{L^2(B_r)}), \tag{3.10}$$

where C depends only on Ω, r. Furthermore, by the Sobolev embedding theorem, actually $U \in C^{0,\gamma}(\overline{\Omega})$ for some $\gamma \in (0,1)$ and there exists c (independent of U) such that

$$\|U\|_{C^{0,\gamma}(\overline{\Omega})} \leq c\|U\|_{H^2(\Omega)}. \tag{3.11}$$

Now, we prove that U will vanish at infinity. It suffices to prove that for any $\delta > 0$, there exists $R > 0$ such that $U(x) \leq \delta$, for all $|x| \geq R$. If not, there exists $\{x_j\} \subset \mathbb{R}^2$ with $|x_j| \to \infty$, as $j \to \infty$ and $\liminf\limits_{j\to\infty} U(x_j) > 0$. Let $v_j(x) = U(x + x_j)$, then $\|v_j\| \equiv \|U\|$ and

$$-\Delta v_j + m v_j = f(v_j), \quad v_j \in H^1(\mathbb{R}^2). \tag{3.12}$$

Assume that $v_j \to v$ weakly in $H^1(\mathbb{R}^2)$, we claim that $v \not\equiv 0$. In fact, noting that v_j is a weak solution of (3.9), it follows from (3.10) and (3.11) that, up to a subsequence, $v_j \to v$ uniformly in $\overline{\Omega}$. Hence,

$$v(0) = \liminf_{j\to\infty} v_j(0) = \liminf_{j\to\infty} U(x_j) > 0,$$

which implies that $v \not\equiv 0$.

On the other hand, for any fixed $R > 0$ and j large enough, we have

$$\begin{aligned}\int_{\mathbb{R}^2} U^2 &\geq \int_{B_R(0)} U^2 + \int_{B_R(x_j)} U^2 \\ &= \int_{B_R(0)} U^2 + \int_{B_R(0)} v_j^2 = \int_{B_R(0)} U^2 + \int_{B_R(0)} v^2 + o_j(1),\end{aligned}$$

where $o_j(1) \to 0$, as $j \to \infty$. Since R is arbitrary, we get $v \equiv 0$ which is a contradiction. Thus $U(x) \to 0$, as $|x| \to \infty$. Moreover, since $U \in C(B_r)$ for any $r > 0$, we have $U \in L^\infty(\mathbb{R}^2)$.

Step 2. Here we borrow some results of [10] to prove that any $U \in S_m$ is radially symmetric, which in turn implies that $U \in C^2(\mathbb{R}^2)$. Let

$$T(u) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2\,dx, \qquad G(u) = \int_{\mathbb{R}^N} \left(F(u) - \frac{m}{2}\,u^2\right) dx,$$

and consider the constraint minimization problem

$$T_0 := \inf\big\{T(u) : G(u) = 0,\ u \in H^1(\mathbb{R}^2) \setminus \{0\}\big\}. \tag{3.13}$$

It follows from Lemma 3.2 that $T_0 < 1/2$. Moreover, from [3] T_0 is achieved. On the other hand, for any minimizer u of (3.13) there exists $\theta > 0$ such that

$$\int_{\mathbb{R}^2} \nabla u \nabla \varphi = \theta \int_{\mathbb{R}^2} \Big(f(u) - \frac{m}{2}\, u \Big) \varphi, \quad \text{for all } \varphi \in H^1(\mathbb{R}^2),$$

i.e. u is a weak solution of the following problem:

$$-\Delta u + \theta m u = \theta f(u), \quad u \in H^1(\mathbb{R}^2), \tag{3.14}$$

see [6]. Similarly as above, we know that $u \in C(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ and $u(x) \to 0$, as $|x| \to \infty$. It follows from the C^α-regularity theory (see [37, Theorem 10.1.2]) that $u \in C^{1,\alpha}(\mathbb{R}^2)$ for some $\alpha \in (0,1)$. Moreover, for any solution u of (3.14), $u \in C^{1,\alpha}(\mathbb{R}^2)$ and $u(x) \to 0$, as $|x| \to \infty$. By a classical comparison argument, u decays exponentially at infinity, which implies that u satisfies $G(u) = 0$. By (f_1), $F(s) - m/2s^2 < 0$ for small $|s| > 0$. Therefore, it follows from Proposition 4 in [10] that U is radially symmetric.

Step 3. Let us prove the compactness of S_m. First, we prove that S_m stays bounded in $H^1(\mathbb{R}^2)$. By (2.4), $\{\|\nabla U\|^2_{L^2(\mathbb{R}^2)} : U \in S_m\}$ is bounded. In the following, we claim that $\{\|U\|^2_{L^2} : U \in S_m\}$ is also bounded. Otherwise, there exists $\{U_j\} \subset S_m$ such that $\lambda_j = \|U_j\|_{L^2} \to \infty$, as $j \to \infty$. Let $\widetilde{U}_j(x) = U_j(\lambda_j x)$, then $\widetilde{U}_j$ satisfies $\|\widetilde{U}_j\|_{L^2} = 1$, $\|\nabla \widetilde{U}_j\|^2_{L^2} = 2E_m$ and

$$-\frac{1}{\lambda_j^2}\, \Delta \widetilde{U}_j + m \widetilde{U}_j = f(\widetilde{U}_j) \quad \text{in } \mathbb{R}^2. \tag{3.15}$$

Assume $\widetilde{U}_j \to U_0 \in H^1_{\mathrm{rad}}(\mathbb{R}^2)$ weakly in $H^1(\mathbb{R}^2)$, then it follows from (3.15) that $mU_0(x) = f(U_0(x))$, $x \in \mathbb{R}^2$. By (f_1), as we can see in [11], $U_0 \equiv 0$. Thus, $\widetilde{U}_j \to 0$ weakly in $H^1_{\mathrm{rad}}(\mathbb{R}^2)$, as $j \to \infty$. Noting that $E_m < 1/2$, as a consequence of Lemma 3.4 one has $\int_{\mathbb{R}^2} \widetilde{U}_j f(\widetilde{U}_j) \to 0$, as $j \to \infty$. By (3.15), $\|\widetilde{U}_j\|_2 \to 0$, as $j \to \infty$ which is a contradiction. Therefore, the claim is proved and S_m is bounded in $H^1(\mathbb{R}^2)$.

Next, to prove the compactness of S_m, it is enough to prove that if $\{u_n\} \subset S_m$ and $u_n \to u$ weakly in $H^1(\mathbb{R}^2)$, then $u \in S_m$ and up to a subsequence, $u_n \to u$ strongly in $H^1(\mathbb{R}^2)$. Obviously, each $u_n \in H^1_{\mathrm{rad}}(\mathbb{R}^2)$ and satisfies (2.4). By Lemma 3.4, it is easy to see that $\int_{\mathbb{R}^2} F(u_n) \to \int_{\mathbb{R}^2} F(u)$ and $u \not\equiv 0$, which implies that $L_m(u) \le E_m$. Noting that u is a nontrival solution of (1.3), we get that $u \in S_m$ and

$$\|\nabla u_n\|_2^2 + m\|u_n\|_2^2 \to \|\nabla u\|_2^2 + m\|u\|_2^2, \quad \text{as } n \to \infty.$$

Thus, $u_n \to u$ strongly in $H^1(\mathbb{R}^2)$ and S_m is compact in $H^1(\mathbb{R}^2)$.

Step 4. The fact $\inf\{\|u\|_\infty : u \in S_m\} > 0$ follows directly from $\lim\limits_{t\to 0} f(t)/t = 0$. Noting that S_m is compact in $H^1(\mathbb{R}^2)$, to prove $\sup\{\|u\|_\infty : u \in S_m\} < \infty$, it is enough to prove that for any $\{u_n\} \subset S_m$ with $u_n \to u \in S_m$ strongly in $H^1(\mathbb{R}^2)$, one has $\sup_n \|u_n\|_\infty < \infty$.

By (f$_1$)–(f$_2$), there exist $C > 0$ and $\beta > 4\pi$ such that $0 < f(t) \le mt/2$, $t \in (0,1)$ and $0 < f(t) \le C(\exp(\beta t^2) - 1)$ for $t \ge 1$. Let us now prove that

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} |\exp(2\beta u_n^2) - \exp(2\beta u^2)|^2 = 0. \tag{3.16}$$

In fact, due to $u \in L^\infty(\mathbb{R}^2)$ and $u_n \to u$ strongly in $H^1(\mathbb{R}^2)$, there exists $c > 0$ such that

$$\begin{aligned}
&\int_{\mathbb{R}^2} |\exp(2\beta u_n^2) - \exp(2\beta u^2)|^2 \le c \int_{\mathbb{R}^2} \exp(8\beta|u_n - u|^2)|u_n^2 - u^2|^2 \\
&= c \int_{\mathbb{R}^2} \big[\exp(8\beta|u_n - u|^2) - 1\big] |u_n^2 - u^2|^2 + o_n(1) \\
&\le c\bigg(\int_{\mathbb{R}^2} \big[\exp(16\beta|u_n - u|^2) - 1\big]\bigg)^{1/2} \bigg(\int_{\mathbb{R}^2} |u_n^2 - u^2|^4\bigg)^{1/2} + o_n(1),
\end{aligned}$$

where $o_n(1) \to 0$ as $n \to \infty$. Since $\|u_n - u\| \to 0$, as $n \to \infty$, it follows from the Trudinger–Moser inequality that there exists C such that

$$\int_{\mathbb{R}^2} \big[\exp(16\beta|u_n - u|^2) - 1\big] \le C$$

for n large enough. Thus, (3.16) holds.

Finally, as u_n is a weak solution to (3.9) for $r = 2$, we claim that

$$\sup_n \|f(u_n)\|_2 < \infty. \tag{3.17}$$

Let $A_n := \{x \in \mathbb{R}^2 : u_n(x) \le 1\}$ and $B_n := \{x \in \mathbb{R}^2 : u_n(x) > 1\}$, then

$$\begin{aligned}
\int_{\mathbb{R}^2} |f(u_n)|^2 &= \int_{A_n} |f(u_n)|^2 + \int_{B_n} |f(u_n)|^2 \\
&\le \int_{\mathbb{R}^2} \frac{m^2}{4} |u_n|^2 + C \int_{\mathbb{R}^2} (\exp(2\beta u_n^2) - 1)^2.
\end{aligned}$$

Then, by the Trudinger–Moser inequality (see [31]) and (3.16), it is easy to know that the claim (3.17) is true. Similarly to Step 1, it follows from the *interior H^2-regularity* (see [32]) that

$$\|u_n\|_{H^2(B_1)} \le C\big(\|f(u_n)\|_{L^2(B_2)} + \|u_n\|_{L^2(B_2)}\big), \tag{3.18}$$

where C is independent of n. Meanwhile, by the Sobolev embedding theorem,

$$\|u_n\|_{C^{0,\gamma}(\overline{B_1})} \le c\|u_n\|_{H^2(B_1)}, \tag{3.19}$$

for some $\gamma \in (0,1)$, where c is independent of n. Hence, it follows from (3.17)–(3.19) that $\sup_n \|u_n\|_{C^{0,\gamma}(\overline{B_1})} < \infty$, which implies that, up to a subsequence,

$u_n \to u$ uniformly in $\overline{B_1}$. Thus, due to $u \in L^\infty(\mathbb{R}^2)$, we get $\sup\limits_n \|u_n\|_{L^\infty(\overline{B_1})} < \infty$. Therefore, by the radial lemma, $\sup\limits_n \|u_n\|_{L^\infty(\mathbb{R}^2)} < \infty$.

Step 5. We finally prove the decay estimate of S_m at infinity. By the Strauss radial lemma [46], we know that $u_n(x) \to 0$, as $|x| \to \infty$ uniformly in n. By a classical comparison principle, it follows from $\sup\limits_n \|u_n\|_{L^\infty(\mathbb{R}^2)} < \infty$ that there exist $c, C > 0$ such that

$$U(x) + |\nabla U(x)| \le C \exp(-c|x|), \quad x \in \mathbb{R}^2,$$

for any $U \in S_m$ and the proof is complete. $\square$

PROOF OF THEOREM 1.2 COMPLETED. For any $l > \max\limits_{t\in[0,\kappa]} f(t)$, where $\kappa = \max\limits_{1\le i\le k} \kappa_i$, we modify the nonlinearity f as follows:

$$f_l(t) = \min\{f(t), l\}, \quad t \in \mathbb{R},$$

and consider the following truncated approximating equation:

$$-\Delta u + V_\varepsilon(x)u = f_l(u), \qquad u \in H_\varepsilon. \tag{3.20}$$

Next we construct a multi-peak solution u_ε of (3.20) concentrating around $O_1, \dots, O_k$. Clearly, u_ε is a solution of the original problem provided $\|u_\varepsilon\|_\infty \le \kappa$.

For each $1 \le i \le k$, consider the following limiting problem:

$$-\Delta u + m_i u = f_l(u), \quad u \in H^1(\mathbb{R}^2). \tag{3.21}$$

Denote by $E^l_{m_i}$ the least energy of (3.21) and by $S^l_{m_i}$ the set of positive ground state solutions U of (3.21) with $U(0) = \max\limits_{x\in\mathbb{R}^2} U(x)$. With the assumptions in Theorem 1.2, it is easy to verify that f_l satisfies (f$_1$)–(f$_3$). Moreover, $S^l_{m_i} \ne \emptyset$. By Proposition 3.3, we have

LEMMA 3.5. *For $l > \max\limits_{t\in[0,\kappa]} f(t)$, we have $E^l_{m_i} = E_{m_i}$ and $S^l_{m_i} = S_{m_i}$, for $i = 1, \dots, k$.*

PROOF. Assume for simplicity $k = 1$ and $m = m_i$. It follows from $f_l(s) \le f(s)$ for any $s > 0$ that $E^l_m \ge E_m$. Due to $S_m \subset S^l_m$ for $l > \max\limits_{t\in[0,\kappa]} f(t)$ we get $E^l_m \le E_m$ and hence $E^l_m = E_m$.

Next, to prove $S_m = S^l_m$ for $l > \max\limits_{t\in[0,\kappa]} f(t)$, it is sufficient to show that $S^l_m \subset S_m$ for $l > \max\limits_{t\in[0,\kappa]} f(t)$. Let

$$G_l(u) = \int_{\mathbb{R}^2} \left(F_l(u) - \frac{m}{2}|u|^2 \right) dx,$$

then it is readily seen that

$$E^l_m = \inf\big\{T(u) : G_l(u) = 0,\, u \in H^1(\mathbb{R}^2) \setminus \{0\}\big\}. \tag{3.22}$$

For any $u_l \in S_m^l$, u_l is a minimizer of (3.22). By the definition of f_l and the fact $E_m^l = E_m$, u_l satisfies

$$T(u_l) = E_m \quad \textit{and} \quad G(u_l) \geq 0, \quad \text{where } G(u) = \int_{\mathbb{R}^2} \left(F(u) - \frac{m}{2} |u|^2 \right) dx.$$

At the same time we have

$$E_m = \inf\big\{T(u) : G(u) = 0,\, u \in H^1(\mathbb{R}^2) \setminus \{0\}\big\}. \tag{3.23}$$

Now, we claim $G(u_l) = 0$. Indeed, if not namely $G(u_l) > 0$, there exists $\theta \in (0,1)$ such that $G(\theta u_l) = 0$. However, $T(\theta u_l) = \theta^2 E_m < E_m$, which is a contradiction. Thus, $G(u_l) = 0$, which implies that u_l is a minimizer of (3.23). Therefore, u_l is a ground state solution of (1.3), that is $u_l \in S_m$. $\square$

By Lemma 3.5, let us fix $l > \max\limits_{t \in [0,\kappa]} f(t)$ with $S_{m_i}^l = S_{m_i}$, $i = 1, \ldots, k$. Consider the approximating problem

$$-\varepsilon^2 \Delta v + V(x) v = f_l(v), \quad v > 0,\ x \in \mathbb{R}^2. \tag{3.24}$$

By Theorem 1.1, for sufficiently small $\varepsilon > 0$, there exists a positive solution v_ε of (3.24), such that there exist $U_i \in S_{m_i}$, $1 \leq i \leq k$, and k local maximum points $x_\varepsilon^i \in O^i$ of v_ε, such that

$$\lim_{\varepsilon \to 0} \max_{1 \leq i \leq k} \operatorname{dist}(x_\varepsilon^i, \mathcal{M}^i) = 0,$$

and $v_\varepsilon(\varepsilon\,\cdot\, + x_\varepsilon^i) \to U_i(\,\cdot\, + z_i)$, as $\varepsilon \to 0$ in $H^1(\mathbb{R}^2)$ for some $z_i \in \mathbb{R}^2$. Let $w_\varepsilon^i(\,\cdot\,) := v_\varepsilon(\varepsilon\,\cdot\, + x_\varepsilon^i)$, then w_ε^i satisfies

$$-\Delta w_\varepsilon^i + V_\varepsilon \left(x + \frac{x_\varepsilon^i}{\varepsilon} \right) w_\varepsilon^i = f_l(w_\varepsilon^i), \quad w_\varepsilon^i \in H_\varepsilon.$$

Since $0 \leq f_l(t) \leq k, t \in \mathbb{R}$, by elliptic estimates we obtain $w_\varepsilon^i(\,\cdot\,) \to U_i(\,\cdot\, + z_i)$ uniformly in $B_1(0)$. Hence $\|v_\varepsilon\|_\infty \leq \kappa$ holds as well provided $\varepsilon > 0$ is small enough. $\square$

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