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EXISTENCE OF POSITIVE GROUND STATE SOLUTIONS FOR KIRCHHOFF TYPE EQUATION WITH GENERAL CRITICAL GROWTH

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Abstract. We study the existence of positive ground state solutions for the nonlinear Kirchhoff type equation

$$\begin{cases} -\left(a+b\displaystyle\int_{\mathbb{R}^3}|\nabla u|^2\right)\Delta u+V(x)u=f(u) & \text{in } \mathbb{R}^3,\\ u\in H^1(\mathbb{R}^3), \quad u>0 & \text{in } \mathbb{R}^3,\end{cases}$$

where $a,b>0$ are constants, $f\in C(\mathbb{R},\mathbb{R})$ has general critical growth. We generalize a Berestycki–Lions theorem about the critical case of Schrödinger equation to Kirchhoff type equation via variational methods. Moreover, some subcritical works on Kirchhoff type equation are extended to the current critical case.

1. Introduction

We are concerned with the following Kirchhoff-type equation:

$$\text{(K)}\qquad \begin{cases} -\left(a+b\displaystyle\int_{\mathbb{R}^3}|\nabla u|^2\right)\Delta u+V(x)u=f(u) & \text{in } \mathbb{R}^3,\\ u\in H^1(\mathbb{R}^3), \quad u>0 & \text{in } \mathbb{R}^3,\end{cases}$$

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where $a,b>0$ are constants, $V\in C(\mathbb{R}^3,\mathbb{R})$ and $f\in C(\mathbb{R},\mathbb{R})$ satisfy some conditions to be made precise later.

We recall that u is said to be the ground state (or the least energy) solution of (K) if and only if u solves (K) and minimizes the functional associated with (K) among all possible nontrivial solutions. Almost sufficient and necessary conditions for the existence of ground state solutions to the following nonlinear elliptic equation:

$$\begin{cases} -\Delta u = h(u) & \text{in } \mathbb{R}^N, \\ u\in H^1(\mathbb{R}^N), \quad u>0 & \text{in } \mathbb{R}^N, \end{cases} \tag{1.1}$$

are given by Berestycki and Lions in [6] when $N\geq 3$ and Berestycki et al. in [7] when $N=2$. In particular, in [6], the following existence result is obtained.

Theorem 1.1. *Suppose $N\geq 3$ and h satisfies the following conditions:*

- (H_1) $h\in C(\mathbb{R},\mathbb{R})$ *is odd*;
- (H_2) $-\infty<\liminf\limits_{s\to 0^+} h(s)/s\leq \limsup\limits_{s\to 0^+} h(s)/s=-m<0$;
- (H_3) $-\infty\leq \limsup\limits_{s\to 0^+} h(s)/s^l\leq 0$, *where* $l=(N+2)/(N-2)$;
- (H_4) *there exists* $\zeta>0$ *such that* $H(\zeta):=\int_0^\zeta h(s)\,ds>0$.

Then (1.1) *possesses a positive radial ground state solution.*

This problem was studied in [6] in the space $H^1_r(\mathbb{R}^N)$ of radical symmetric functions, in which case the nonlinear term h is independent of $x\in\mathbb{R}^N$. More importantly, the imbedding of $H^1_r(\mathbb{R}^3)\hookrightarrow L^r(\mathbb{R}^N)$ is compact for $r\in(2,2^*)$. Note also that (H_3) implies that the nonlinear term has subcritical growth.

In [34], Zhang and Zou studied problem (1.1) for

$$h=h(x,u)=-V(x)u+f(u),$$

i.e. h has critical growth and depends on x. Under conditions

- (f_1) $f\in C^1(\mathbb{R}^N,\mathbb{R})$;
- (f_2) $f(t)=o(t)$ as $t\to 0^+$;
- (f_3) $\lim\limits_{t\to+\infty} f(s)/s^{(N+2)/(N-2)}=K>0$;
- (f_4) there exist $D>0$ and $q\in(2,2^*)$ such that $f(s)\geq Kt^{(N+2)/(N-2)}+Dt^{q-1}$ for all $t\geq 0$, where $2^*=2N/(N-2)$;
- (f_5) $|f'(t)|\leq C(1+|t|^{4/(N-2)})$ for $t\geq 0$ and some $C>0$.

they proved that (1.1) has a ground state solution if the potential V satisfies certaiin reasonable hypotheses. These results of Zhang and Zou can be regarded as a generalization of the Berestycki–Lions theorem to critical and non-radial case. Conditions (f_3) and (f_4) characterize equation (1.1) to be of critical growth. Azzollini in [4] studied a class of Kirchhoff equations and extended the Berestycki–Lions theorem to problem (K) by using minimizing arguments on a suitable natural constraint (the Pohozăev's manifold $\mathcal{P}$) in $H^1_r(\mathbb{R}^3)$. Recently, Liu and Guo [22] made an attempt to complement the study initiated in [4] by considering a class of nonlinearities with general critical growth. To the best of our knowledge, so far no result similar to [34] for critical and non-radial case for Kirchhoff equations is established. So our interest in the present paper is to extend the Berestycki–Lions theorem to critical and non-radial case to Kirchhoff equations.

We shall pose the following conditions on the the potential V:

(V1) $V \in C(\mathbb{R}^3, \mathbb{R})$ is weakly differentiable and satisfies $(\nabla V(x), x) \in L^\infty(\mathbb{R}^3) \cup L^{3/2}(\mathbb{R}^3)$ and there exists $\mu > 2$ such that

$$\frac{\mu - 2}{\mu} V(x) - (\nabla V(x), x) \geq 0.$$

(V2) For almost every $x \in \mathbb{R}^3$, $V(x) \leq \lim_{|y|\to+\infty} V(y) = V_\infty < +\infty$ and the inequality is strict on a subset of positive Lebesgue measure.

(V3) $\inf \sigma(-\Delta + V(x)) > 0$, where $\sigma(-\Delta + V(x))$ denotes the spectrum of the self-adjoint operator $-\Delta + V(x)\colon H^1(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$, i.e.

$$\inf \sigma(-\Delta + V(x)) = \inf_{u \in H^1(\mathbb{R}^3)\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2)\, dx}{\displaystyle\int_{\mathbb{R}^3} |u|^2\, dx} > 0,$$

where $(\,\cdot\,,\,\cdot\,)$ is the usual inner product in $\mathbb{R}^3$.

Moreover, we assume that the nonlinear term $f \in C(\mathbb{R}, \mathbb{R})$ satisfies the following hypotheses:

(F1) $\lim_{t\to+\infty} f(t)/t^5 = K > 0$.

(F2) There exist $D > 0$ and $q \in (2, 6)$ such that $f(t) \geq Kt^5 + Dt^{q-1}$ for all $t \geq 0$.

(F3) Denoting $g(t) := f(t) - Kt^5$, there holds $g(t)t - \mu G(t) \geq 0$ for all $t \in \mathbb{R}$, where $G(t) = \int_0^t g(s)\, ds$.

Now we state our main results.

THEOREM 1.2. *Under assumptions* (f_2)*,* (F1)–(F3) *and* (V1)–(V3)*, assume that either* $q \in (4, 6)$ *or* $q \in (2, 4]$ *and* D *is sufficiently large. Then problem* (K) *possesses at least one positive ground state solution in* $H^1(\mathbb{R}^3)$*.*

REMARK 1.3. There are many functions satisfying (V1)–(V3). For instance, $V(x) = V_\infty - (\mu - 2)/\mu(|x| + 3)$, where $V_\infty > 1$ is a positive constant.

REMARK 1.4. In comparison with [22], [35], in the proof of Theorem 1.2 we use a different method, which allows to drop radial restrictions. A local compactness lemma established for critical case plays a crucial role in our arguments. We believe that it can be used to deal with other similar problems. Moreover, it is interesting to learn whether the ground state solution from Theorem 1.2 and the radial ground state solution obtained in [22] are the same. Yet, we do not know the answer to this question.

As is known, problem (K) is a variant of Dirichlet problem of Kirchhoff type. Indeed, if $\mathbb{R}^3$ is replaced by a smooth bounded domain Ω, then (K) reduces to the following Dirichlet problem:

$$\begin{cases} -\left(a+b\displaystyle\int_\Omega |\nabla u|^2\right)\Delta u = f(x,u) & \text{in } \Omega,\\ u=0 & \text{in } \partial\Omega, \end{cases} \tag{1.2}$$

which is related to the stationary analogue of the following equation:

$$\begin{cases} u_{tt}-\left(a+b\displaystyle\int_\Omega |\nabla u|^2\,dx\right)\Delta u = f(x,u) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

It was proposed by Kirchhoff in [15] studying existence of the classical D'Alembert wave equations for free vibration of elastic strings, particularly, taking into account the subsequent change in the string length caused by oscillations. Such a class of problems is viewed as being nonlocal because of the presence of the term $(\int_\Omega |\nabla u|^2\,dx)\Delta u$, which implies that the equation in (1.2) is no longer a pointwise identity. Indeed, such a phenomenon provokes some mathematical difficulties, which make the study of such a class of problems particularly interesting. Let us mention that equation (1.3) received much attention only after Lions [18] introduced an abstract framework to this problem. We refer interested readers to the papers [10], [24], [33], [1]–[3], [9], [27], [25], [5] and references therein.

Recently, there has been increasing interest in studying the following Kirchhoff problem:

$$\begin{cases} -\left(a+b\displaystyle\int_{\mathbb{R}^3} |\nabla u|^2\right)\Delta u + V(x)u = f(x,u) & \text{in } \mathbb{R}^3,\\ u\in H^1(\mathbb{R}^3) & \text{in } \mathbb{R}^3, \end{cases} \tag{1.4}$$

especially the existence of positive solutions, multiple solutions, ground states and semiclassical states, see for example, [32], [17], [14], [30], [26], [20], [21], [16], [29], [12], [11] and references therein. In these papers mainly the case where the nonlinear term behaves as $f(u)\sim |u|^{p-1}u$ with $p\in(3,5)$ and satisfies the following Ambrosetti–Rabinowitz type condition:

(AR) there exists $\mu\ge 4$ such that $0<\mu F(s)\le f(s)s$ for all $s\in\mathbb{R}$,

where $F(u)=\int_0^u f(s)\,ds$ has been investigated. While there are very few results on the existence of solutions for the case $p\in(1,3)$ and without any compactness assumptions. Recently, under certain assumptions on V, Li and Ye [16] proved that problem (1.4) with $f(u)=|u|^{p-1}u$, $p\in(2,5)$, has a positive ground state solution by using a monotonicity trick and a global compactness lemma. Next, Liu and Guo [23] extended this result to the case $p \in (1,5)$. However, in [16], [23] only the subcritical case was considered. It seems natural to expect that there exists a corresponding solution in the critical case. In the present paper, we give an affirmative answer to this question (see Theorem 1.2). It is known that the critical exponent growth makes the problem very tough due to the lack of compactness. Therefore, the method of [16, 23] cannot be used directly and some additional tricks are needed. Indeed, we will employ a monotonicity trick together with establishing a local compactness lemma of critical case to obtain our result.

REMARK 1.5. Note that in the case $f(u) = |u|^{p-1}u + |u|^{5-1}u$ our conditions (F1)–(F4) cover the full subcritical range of $p \in (1,5)$. Therefore, Theorem 1.2 can also be regarded as a generalization of the results in [16] and [23] to the critical case.

Throughout this paper, $C > 0$ denotes various positive generic constants. The remainder of this paper is organized as follows. In Section 2, some preliminary results are presented. In Section 3, we provide proofs of the main results.

2. Preliminaries

We denote by $E := H^1(\mathbb{R}^3)$ the usual Sobolev space equipped with the standard norm

$$\|u\|_H := \left(\int_{\mathbb{R}^3} (|\nabla u|^2 + u^2)\,dx\right)^{1/2}$$

and denote the norm of $D^{1,2}(\mathbb{R}^3)$ by

$$\|u\|_{D^{1,2}} := \left(\int_{\mathbb{R}^3} |\nabla u|^2\,dx\right)^{1/2}.$$

In E, we also define the inner product and norm

$$\langle u,u\rangle := \int_{\mathbb{R}^3} (a|\nabla u|^2 + V(x)u^2)\,dx, \quad \|u\| := \langle u,u\rangle^{1/2},$$

which is equivalent to $\|\cdot\|_H$ under hypotheses (V1)–(V3). $\|\cdot\|_q$ stands for the usual L^q-norm, $q \in [1,+\infty]$. The letter S denotes the best Sobolev constant, i.e.

$$S := \inf_{u \in D^{1,2}(\mathbb{R}^3)\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^3} |\nabla u|^2\,dx}{\left(\displaystyle\int_{\mathbb{R}^3} u^6\,dx\right)^{1/3}}.$$

For any $\rho > 0$ and $z \in \mathbb{R}^3$, $B_\rho(z) := \{x \in \mathbb{R}^3 : |x - z| \le \rho\}$.

For brevity, hereafter we omit the symbol dx in the integrals over $\mathbb{R}^3$ when no confusion can arise, moreover, we also assume that $K = 1$.

Since our interest is to establish the existence of positive solutions for problem (K), we assume $f(t) = 0$ for $t \leq 0$. So $g(t) = f(t) - (t^+)^5$, where $t^+ = \max\{0, t\}$.

Define the energy functional $I\colon E \to \mathbb{R}$ as

$$I(u) := \frac{1}{2}\int_{\mathbb{R}^3} (a|\nabla u|^2 + V(x)u^2) + \frac{b}{4}\bigg(\int_{\mathbb{R}^3} |\nabla u|^2\bigg)^2 - \int_{\mathbb{R}^3} G(u) - \frac{1}{6}\int_{\mathbb{R}^3} (u^+)^6, \tag{2.1}$$

which is a well-defined C^1 functional. Moreover, for any $u, \varphi \in E$, we have

$$I'(u)\varphi = \langle u, \varphi\rangle + b\int_{\mathbb{R}^3} |\nabla u|^2 \int_{\mathbb{R}^3} \nabla u \nabla \varphi - \int_{\mathbb{R}^3} g(u)\varphi - \int_{\mathbb{R}^3} (u^+)^5\varphi. \tag{2.2}$$

Clearly, weak solutions to (K) correspond to critical points of the functional I.

In order to prove our theorem, the following abstract result established in [13] will be needed.

THEOREM 2.1. *Let $(E, \|\cdot\|)$ be a real Banach space with its dual space E^{-1} and $J \in \mathbb{R}^+$ an interval. Consider the family of C^1 functionals on E:*

$$I_\lambda = A(u) - \lambda B(u), \quad \textit{for all } \lambda \in J,$$

with B nonnegative and either $A(u) \to +\infty$ or $B(u) \to +\infty$ as $\|u\| \to \infty$, satisfying $I_\lambda(0) = 0$. For any $\lambda \in J$ we set

$$\Gamma_\lambda = \{\gamma \in C([0,1], E) \mid \gamma(0) = 0,\ \gamma(1) < 0\}.$$

If for every $\lambda \in J$ the set Γ_λ is nonempty and

$$c_\lambda = \inf_{\gamma \in \Gamma_\lambda} \max_{s \in [0,1]} I_\lambda(\gamma(s)) > 0, \tag{2.3}$$

then for almost every $\lambda \in J$ there is a bounded Palais–Smale sequence $\{u_n\}$, i.e. $\{u_n\}$ is bounded and satisfies $I_\lambda(u_n) \to c_\lambda$ and $I'_\lambda(u_n) \to 0$ in E^{-1}. Moreover, the map $\lambda \mapsto c_\lambda$ is continuous from the left.

In our arguments, set

$$A(u) := \frac{1}{2}\,\|u\|^2 + \frac{b}{4}\bigg(\int_{\mathbb{R}^3} |\nabla u|^2\bigg)^2, \qquad B(u) := \int_{\mathbb{R}^3} \bigg(G(u) + \frac{1}{6}\,(u^+)^6\bigg)$$

and $J = [1/2, 1]$. We also will make use of the following Pohozăev type identity, whose proof is standard and can be found in [6].

LEMMA 2.2. *Let u be a critical point of I_λ in E for $\lambda \in J$, then*

$$\frac{a}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + \frac{b}{2}\bigg(\int_{\mathbb{R}^3} |\nabla u|^2\bigg)^2 + \frac{3}{2}\int_{\mathbb{R}^3} V(x)u^2 + \frac{1}{2}\int_{\mathbb{R}^3} (\nabla V(x), x)u^2 - 3\lambda\int_{\mathbb{R}^3} F(u) = 0,$$

where $F(u) = \int_0^u f(s)\,ds$.

Now we give a lemma which will be used later.

LEMMA 2.3. *If all assumptions of Theorem* 1.2 *are satisfied, then the conclusions of Theorem* 2.1 *hold. Moreover,*

$$c_\lambda < c_\lambda^* := \frac{ab}{4\lambda} S^3 + \frac{[b^2S^4 + 4\lambda aS]^{3/2}}{24\lambda^2} + \frac{b^3S^6}{24\lambda^2}.$$

PROOF. It is easy to see that $B(u) \geq 0$ for all $u \in E$ and $A(u) \to +\infty$ as $\|u\| \to \infty$. On the other hand, conditions (f_2) and (F1)–(F2) imply that for all $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that $F(u) \leq \varepsilon|u|^2 + C_\varepsilon|u|^6$. Therefore, there exists $r > 0$ such that for all $\|u\| = r$, $I_\lambda(u) \geq \alpha > 0$, where r, α are independent of λ. From (F2),

$$I_\lambda(u) \leq \frac{1}{2}\|u\|^2 + \frac{b}{4}\bigg(\int_{\mathbb{R}^3} |\nabla u|^2\bigg)^2 - \frac{D}{2q}\int_{\mathbb{R}^3} (u^+)^q - \frac{1}{12}\int_{\mathbb{R}^3} (u^+)^6.$$

Set $v \in E \setminus \{0\}$ such that $v > 0$. Then $\lim\limits_{t\to+\infty} I_\lambda(tv) = -\infty$. Thus, there exists $t_0 > 0$ such that $\|t_0v\| > r$ and $I_\lambda(t_0v) < 0$ for all $\lambda \in [1/2, 1]$. Set $\gamma(0) = 0$, $\gamma(1) = t_0v$, then $\Gamma_\lambda \neq \emptyset$ and $c_\lambda > 0$. Therefore, the conclusions of Theorem 2.1 hold. The argument that $c_\lambda < c_\lambda^*$ is similar to that in [22], so we omit it. □

REMARK 2.4. In the present paper, we assume that for $\lambda \in [1/2, 1]$, if $\{u_n\} \subset E$ is a sequence satisfying

$$\|u_n\| < \infty, \quad I_\lambda(u_n) \to c_\lambda, \quad I'_\lambda(u_n) \to 0,$$

then we can assume that $u_n \geq 0$ in E. Indeed, we have $I'_\lambda(u_n)u_n^- = o(1)$, where $u_n^- = \min\{u_n, 0\}$. Thus, $\|u_n^-\| = o(1)$ from which we derive that $\|u_n^+\| < \infty$, $I_\lambda(u_n^+) \to c_\lambda$ and $I'_\lambda(u_n^+) \to 0$.

3. Proofs of main results

Now we define the limit functional corresponding to I_λ as

$$I_\lambda^\infty := \frac{a}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + \frac{1}{2}\int_{\mathbb{R}^3} V_\infty u^2 + \frac{b}{4}\bigg(\int_{\mathbb{R}^3} |\nabla u|^2\bigg)^2 - \lambda\int_{\mathbb{R}^3} F(u).$$

In view of Lemma 2.2, we have the Pohozǎev type identity corresponding to I_λ^∞

$$P_\lambda(u) := \frac{a}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + \frac{b}{2}\bigg(\int_{\mathbb{R}^3} |\nabla u|^2\bigg)^2 + \frac{3}{2}\int_{\mathbb{R}^3} V_\infty u^2 - 3\lambda\int_{\mathbb{R}^3} F(u) = 0, \tag{3.1}$$

where u is a critical point of I_λ^∞.

REMARK 3.1. It is easy to see that for any nontrivial critical point u of I_λ^∞,

$$I_\lambda^\infty(u) = \frac{a}{3}\int_{\mathbb{R}^3} |\nabla u|^2 + \frac{b}{12}\bigg(\int_{\mathbb{R}^3} |\nabla u|^2\bigg)^2 > 0.$$

In [22], we have proved that there exists a critical point $v \in E \setminus \{0\}$ of I_λ^∞ such that $I_\lambda^\infty(v) = m_\lambda$ for $\lambda \in [1/2, 1]$, where

$$m_\lambda = \inf\{I_\lambda^\infty(u) : P_\lambda(u) = 0,\ u \in E \setminus \{0\}\}.$$

The following lemma can also be found in [22], it embodies the relationship between c_λ and m_λ.

Lemma 3.2. *There exists a continuous path $\gamma_1 \colon [0,1] \to E$ such that $I_\lambda^\infty(\gamma_1(0)) = 0$, $I_\lambda^\infty(\gamma_1(1)) < 0$ and $v \in \gamma_1([0,1])$ with $I_\lambda^\infty(v) = \max_{t \in [0,1]} I_\lambda^\infty(\gamma_1(t))$. Moreover, $c_\lambda < m_\lambda$.*

Now we state the following local compactness lemma for the Palais–Smale sequence at the energy level c_λ.

Lemma 3.3. *Assume all conditions of Theorem 1.2 hold. For $\lambda \in [1/2, 1]$, let $\{u_n\} \subset E$ be a sequence such that $\|u_n\| < \infty$, $I_\lambda(u_n) \to c_\lambda$ and $I'_\lambda(u_n) \to 0$. Then there exist $u_0 \in E$ and $A \in \mathbb{R}$ such that $\overline{I}'_\lambda(u_0) = 0$, where*

$$\overline{I}_\lambda(u) = \frac{a + bA^2}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{1}{2} \int_{\mathbb{R}^3} V(x) u^2 - \lambda \int_{\mathbb{R}^3} F(u), \tag{3.2}$$

a number $k \in \mathbb{N} \cup \{0\}$, nontrivial solutions $w^1, \ldots, w^k$ of the following problem:

$$-(a + bA^2)\Delta u + V_\infty u = \lambda f(u),$$

and k sequences of points $\{y_n^j\} \subset \mathbb{R}^3$, $1 \le j \le k$, such that

(a) $|y_n^j| \to +\infty$, $|y_n^j - y_n^i| \to +\infty$ *if* $i \ne j$, $1 \le i, j \le k$, $n \to +\infty$,

(b) $w^j \ne 0$ *and* $(\overline{I}_\lambda^\infty)'(w^j) = 0$ *for* $1 \le j \le k$,

(c) $\Big\| u_n - u_0 - \sum_{j=1}^{j=k} w^j(\,\cdot\, - y_n^j) \Big\| \to 0$ *as* $n \to \infty$,

(d) $c_\lambda + (bA^4/4) = \overline{I}_\lambda(u_0) + \sum_{j=1}^{j=k} \overline{I}_\lambda^\infty(w^j)$, *where*

$$\overline{I}_\lambda^\infty(u) = \frac{a + bA^2}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{1}{2} \int_{\mathbb{R}^3} V_\infty |u|^2 - \lambda \int_{\mathbb{R}^3} F(u),$$

(e) $A^2 = \|\nabla u_0\|_2^2 + \sum_{j=1}^{j=k} \|w^j\|_2^2$.

Moreover, we agree that in the case $k = 0$ the above holds without w^j and $\{y_n^j\}$.

Proof. $\|u_n\| < \infty$ means that there exist $u_0 \in E$ and $A \in \mathbb{R}$ such that $u_n \rightharpoonup u_0$ in E and $\int_{\mathbb{R}^3} |\nabla u_n|^2 \to A^2$ after extracting a subsequence. Then it follows from $I'_\lambda(u_n) \to 0$ that $\overline{I}'_\lambda(u_0) = 0$. Since

$$\begin{aligned}\overline{I}_\lambda(u_n) &= \frac{a + bA^2}{2} \int_{\mathbb{R}^3} |\nabla u_n|^2 + \frac{1}{2} \int_{\mathbb{R}^3} V(x)|u_n|^2 - \lambda \int_{\mathbb{R}^3} F(u_n) \\ &= I_\lambda(u_n) + \frac{bA^4}{4} + o(1),\end{aligned}$$

and it is easy to prove that $\overline{I}_\lambda(u_n) \to c_\lambda + bA^4/4$ and $\overline{I}'_\lambda(u_n) \to 0$ in E^{-1}. In what follows, we will consider the functional $\overline{I}_\lambda$.

Set $v_n^1 = u_n - u_0$. From the Brezis–Lieb lemma we have

$$\begin{aligned} \|\nabla u_n\|_2^2 &= \|\nabla v_n^1\|_2^2 + \|\nabla u_0\|_2^2, \\ \|u_n\|_s^s &= \|v_n^1\|_s^s + \|u_0\|_s^s, \quad s \in [2,6]. \end{aligned} \tag{3.3}$$

Note that (f_2) and (F1)–(F2) imply that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|g(u)| \le \varepsilon(|u| + |u|^5) + C_\varepsilon |u|^{q-1}. \tag{3.4}$$

Therefore, similarly to the arguments of Lemma 3.2 in [22], it follows from the Strauss compactness lemma [28] and $u_n \rightharpoonup u_0$ in E that

$$\begin{aligned} \int_{\mathbb{R}^3} G(u_n) &= \int_{\mathbb{R}^3} G(u_0) + \int_{\mathbb{R}^3} G(v_n^1) + o(1), \\ \int_{\mathbb{R}^3} g(u_n)u_n &= \int_{\mathbb{R}^3} g(u_0)u_0 + \int_{\mathbb{R}^3} g(v_n^1)v_n^1 + o(1). \end{aligned} \tag{3.5}$$

The remaining proof will be divided into four steps.

Step 1. If

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_1(y)} |v_n^1|^2 = 0,$$

then $u_n \to u_0$ in E and the conclusions hold with $k = 0$.

In view of the Lions lemma in [19], we have

$$v_n^1 \to 0 \quad \text{in } L^s(\mathbb{R}^3), \text{ for all } s \in (2,6). \tag{3.6}$$

It follows from the definition of $\overline{I}_\lambda$ that

$$\begin{aligned} \overline{I}'_\lambda(u_n)u_n =& (a + bA^2) \int_{\mathbb{R}^3} (|\nabla v_n^1|^2 + |\nabla u_0|^2) + \int_{\mathbb{R}^3} V(x)((v_n^1)^2 + |u_0|^2) \\ &- \lambda \int_{\mathbb{R}^3} (g(u_0)u_0 + g(v_n^1)v_n^1) - \lambda \int_{\mathbb{R}^3} (|v_n^1|^6 + |u_0|^6) + o(1) \\ =& \overline{I}'_\lambda(v_n^1)v_n^1 + \overline{I}'_\lambda(u_0)u_0 + o(1). \end{aligned} \tag{3.7}$$

From $\overline{I}'_\lambda(u_0) = 0$ and (3.6), (3.4) we deduce that

$$o(1) = (a + bA^2) \int_{\mathbb{R}^3} |\nabla v_n^1|^2 + \int_{\mathbb{R}^3} V(x)|v_n^1|^2 - \lambda \int_{\mathbb{R}^3} |v_n^1|^6. \tag{3.8}$$

Moreover, it follows from (3.3)–(3.8) that

$$
\begin{aligned}
(3.9)\qquad c_\lambda + \frac{bA^4}{4} - \overline{I}_\lambda(u_0) &= \overline{I}_\lambda(u_n) - \overline{I}_\lambda(u_0) + o(1) \\
&= \frac{(a+bA^2)}{2}\int_{\mathbb{R}^3}|\nabla v_n^1|^2 + \frac{1}{2}\int_{\mathbb{R}^3}V(x)|v_n^1|^2 \\
&\quad - \lambda\int_{\mathbb{R}^3}G(v_n^1) - \lambda\int_{\mathbb{R}^3}|v_n^1|^6 + o(1) \\
&= \frac{(a+bA^2)}{2}\int_{\mathbb{R}^3}|\nabla v_n^1|^2 + \frac{1}{2}\int_{\mathbb{R}^3}V(x)|v_n^1|^2 - \lambda\int_{\mathbb{R}^3}|v_n^1|^6 + o(1).
\end{aligned}
$$

We show that

$$
(3.10)\qquad \overline{I}_\lambda(u_0) \geq \frac{1}{4}\,bA^2\int_{\mathbb{R}^3}|\nabla u_0|^2.
$$

Indeed, since $\overline{I}'_\lambda(u_0) = 0$, by Lemma 2.2, we have the following identity:

$$
\begin{aligned}
(3.11)\qquad \frac{a+bA^2}{2}\int_{\mathbb{R}^3}|\nabla u_0|^2 &+ \frac{3}{2}\int_{\mathbb{R}^3}V(x)|u_0|^2 \\
&+ \frac{1}{2}\int_{\mathbb{R}^3}(\nabla V(x),x)|u_0|^2 - 3\lambda\int_{\mathbb{R}^3}F(u_0) = 0
\end{aligned}
$$

Therefore, this, together with (V2) and (F3), implies that

$$
\begin{aligned}
\bigg(\frac{3}{2} - \frac{1}{\mu}\bigg)(a+bA^2)\int_{\mathbb{R}^3}|\nabla u_0|^2 &+ \int_{\mathbb{R}^3}\bigg(\frac{\mu-2}{2\mu}\,V(x) - \frac{1}{2}\,(\nabla V(x),x)\bigg)u_0^2 \\
&+ \lambda\int_{\mathbb{R}^3}\bigg(\frac{1}{\mu}\,f(u_0)u_0 - F(u_0)\bigg) = 4\overline{I}_\lambda(u_0).
\end{aligned}
$$

Then it follows that

$$
\overline{I}_\lambda(u_0) \geq \frac{1}{4}\bigg(\frac{3}{2} - \frac{1}{\gamma}\bigg)(a+bA^2)\int_{\mathbb{R}^3}|\nabla u_0|^2 \geq \frac{1}{4}\,bA^2\int_{\mathbb{R}^3}|\nabla u_0|^2.
$$

So (3.10) holds. On the other hand, from (3.9) we have

$$
\begin{aligned}
(3.12)\qquad c_\lambda + \frac{bA^4}{4} - \frac{bA^2}{4}\int_{\mathbb{R}^3}|\nabla u_0|^2 &> \frac{(a+bA^2)}{2}\int_{\mathbb{R}^3}|\nabla v_n^1|^2 \\
&+ \frac{1}{2}\int_{\mathbb{R}^3}V(x)|v_n^1|^2 - \lambda\int_{\mathbb{R}^3}|v_n^1|^6 + o(1).
\end{aligned}
$$

It follows from (3.8) that we may assume that there exist $l_i \geq 0$ $(i = 1,2,3)$ such that

$$
\|v_n^1\|^2 \to l_1, \quad bA^2\int_{\mathbb{R}^3}|\nabla v_n^1|^2 \to l_2, \quad \lambda\int_{\mathbb{R}^3}|v_n^1|^6 \to l_3, \quad \text{as } n \to \infty,
$$

then $l_1 + l_2 = l_3$. If $l_1 = 0$, then Step 1 is complete. Else if $l_1 > 0$, then $l_2, l_3 > 0$. Therefore, in view of the above facts, (3.12) can be reduced to $c_\lambda \geq l_1/3 + l_2/12$.

Note that by the Sobolev inequality, we have

$$a^3 \int_{\mathbb{R}^3} |v_n^1|^6 \leq a^3 \bigg(S^{-1} \int_{\mathbb{R}^3} |\nabla v_n^1|^2 \bigg)^3 \leq S^{-3} \|v_n^1\|^6$$

and

$$b\bigg(\int_{\mathbb{R}^3} |v_n^1|^6 \bigg)^{2/3} \leq b\bigg(S^{-1} \int_{\mathbb{R}^3} |\nabla v_n^1|^2 \bigg)^2 = bS^{-2}\bigg(\int_{\mathbb{R}^3} |\nabla v_n^1|^2 \bigg)^2,$$

which imply that

$$l_1 \geq a\lambda^{-1/3} S (l_1 + l_2)^{1/3} \quad \text{and} \quad l_2 \geq b\lambda^{-2/3} S^2 (l_1 + l_2)^{2/3}.$$

It follows from the arguments of Lemma 3.4 of [22] that $c_\lambda \geq c_\lambda^*$, which contradicts Lemma 2.3. Therefore,

$$o(1) = \frac{a + bA^2}{2} \int_{\mathbb{R}^3} |\nabla v_n^1|^2 + \int_{\mathbb{R}^3} V(x) |v_n^1|^2$$

which implies that $u_n \to u_0$ in E and Step 1 follows.

STEP 2. If there exists a sequence $\{z_n\} \subset \mathbb{R}^3$ such that $\int_{B_1(z_n)} |v_n^1|^2 \to d > 0$, then after extracting a subsequence, if necessary, the following hold

(1) $|z_n| \to \infty$.

(2) There exists $w^1 \in E$ such that $u_n(\,\cdot\, + z_n) \rightharpoonup w^1 \neq 0$ in E and

$$(\overline{I}_\lambda^\infty)'(w^1) = 0.$$

The proof is standard, so we omit it.

STEP 3. If there exist a positive integer $m \geq 1$, $w^1, \ldots, w^m \in E$ and m sequences of points $\{y_n^j\} \subset \mathbb{R}^3$, $1 \leq j \leq m$, such that

(i) $|y_n^j| \to +\infty$, $|y_n^j - y_n^i| \to +\infty$ if $i \neq j$, $1 \leq i, j \leq m$, $n \to +\infty$,

(i') $u_n(\,\cdot\, + y_n^j) \rightharpoonup w^j$ in E for $1 \leq j \leq m$,

(ii) $w^j \neq 0$ and $(\overline{I}_\lambda^\infty)'(w^j) = 0$ for $1 \leq j \leq m$,

then one of the following cases must hold:

(1) If $\displaystyle\sup_{z \in \mathbb{R}^3} \int_{B_1(z)} \bigg| u_n - u_0 - \sum_{j=1}^{j=m} \omega^j (x - y_n^j) \bigg|^2 \to 0$, then

$$\bigg\| u_n - u_0 - \sum_{j=1}^{j=m} \omega^j (x - y_n^j) \bigg\| \to 0.$$

(2) If there exists $\{z_n\} \subset \mathbb{R}^3$ such that

$$\int_{B_1(z_n)} \bigg| u_n - u_0 - \sum_{j=1}^{j=m} \omega^j (x - y_n^j) \bigg|^2 \to d > 0,$$

then, after extracting a subsequence, if necessary, the following hold:

(I) $|z_n| \to +\infty$, $|z_n - y_n^j| \to +\infty$ if $1 \leq j \leq m$,

(II) $u_n(\,\cdot\, + z_n) \rightharpoonup w^{m+1} \neq 0$ in E,

(III) $(\overline{I}_\lambda^\infty)'(w^{m+1}) = 0$.

We firstly consider case (1). Set $\eta_n = u_n - u_0 - \sum_{j=1}^{j=m} \omega^j(x - y_n^j)$. It is clear that the sequence $\{\eta_n\}$ is bounded in E. The Lions lemma [19] implies that

$$\eta_n \to 0 \quad \text{in } L^s(\mathbb{R}^3), \quad \text{for all } s \in (2,6). \tag{3.13}$$

By using (i) and (i') of Step 3, we have

$$\eta_n(\,\cdot\, + y_n^j) \rightharpoonup 0 \quad \text{in } E; \quad \eta_n(\,\cdot\, + y_n^j) \to 0 \quad \text{in } L^s_{\rm loc}(\mathbb{R}^3),\ 0 \le j \le m, \tag{3.14}$$

where $y_n^0 = 0$. A direct calculation shows that

$$\begin{aligned} \|\eta_n\|^2 + bA^2 \int_{\mathbb{R}^3} |\nabla \eta_n|^2 &= \overline{I}'_\lambda(u_n)\eta_n - \sum_{j=1}^{j=m} (\overline{I}_\lambda^\infty)'(w^j)\eta_n(\,\cdot\, + y_n^j) \\ &+ \lambda \int_{\mathbb{R}^3} (g(u_n) - g(u_0))\eta_n - \lambda \sum_{j=1}^{j=m} \int_{\mathbb{R}^3} g(w^j)\eta_n(\,\cdot\, + y_n^j) \\ &+ \lambda \int_{\mathbb{R}^3} (|u_n|^5 - |u_0|^5)\eta_n - \lambda \sum_{j=1}^{j=m} \int_{\mathbb{R}^3} |w^j|^5 \eta_n(\,\cdot\, + y_n^j). \end{aligned} \tag{3.15}$$

It follows from the standard elliptic estimates that $u_0, w^j \in L^\infty(\mathbb{R}^3)$ for $j = 1, \ldots, m$. This, together with (3.13), (3.14) and (3.4), implies that

$$\int_{\mathbb{R}^3} (g(u_n) - g(u_0))\eta_n \to 0, \qquad \sum_{j=1}^{j=m} \int_{\mathbb{R}^3} g(w^j)\eta_n(\,\cdot\, + y_n^j) \to 0$$

as $n \to \infty$. Therefore, it follows from Lemma 8.9 of [31] that

$$\begin{aligned} \|\eta_n\|^2 &+ bA^2 \int_{\mathbb{R}^3} |\nabla \eta_n|^2 \\ &= \lambda \int_{\mathbb{R}^3} (|u_n|^5 - |u_0|^5)\eta_n - \lambda \sum_{j=1}^{j=m} \int_{\mathbb{R}^3} |w^j|^5 \eta_n(\,\cdot\, + y_n^j) + o(1) \\ &= \lambda \int_{\mathbb{R}^3} |u_n - u_0 - w^1(\,\cdot\, - y_n^1)|^5 \eta_n - \lambda \sum_{j=2}^{j=m} \int_{\mathbb{R}^3} |w^j|^5 \eta_n(\,\cdot\, + y_n^j) + o(1). \end{aligned}$$

Continuing this process, we have

$$\|\eta_n\|^2 + bA^2 \int_{\mathbb{R}^3} |\nabla \eta_n|^2 = \lambda \int_{\mathbb{R}^3} |\eta_n|^6 + o(1). \tag{3.16}$$

Note that $v_n^1(\,\cdot\,+y_n^1) \rightharpoonup w^1$ in E. By (3.4), (3.5) and the Brezis–Lieb lemma [8], we obtain that

$$\begin{aligned}
&c_\lambda + \frac{bA^4}{4} - \overline{I}_\lambda(u_0) = \overline{I}_\lambda(u_n) - \overline{I}_\lambda(u_0) + o(1)\\
&= \frac{a+bA^2}{2}\int_{\mathbb{R}^3}|\nabla v_n^1(\,\cdot\,+y_n^1)|^2 + \frac{1}{2}\int_{\mathbb{R}^3}V_\infty|v_n^1(\,\cdot\,+y_n^1)|^2\\
&\quad -\lambda\int_{\mathbb{R}^3}F(v_n^1(\,\cdot\,+y_n^1)) + o(1)\\
&= \frac{a+bA^2}{2}\int_{\mathbb{R}^3}|\nabla(v_n^1(\,\cdot\,+y_n^1)-w^1)|^2 + \frac{1}{2}\int_{\mathbb{R}^3}V_\infty|v_n^1(\,\cdot\,+y_n^1)-w^1|^2\\
&\quad -\lambda\int_{\mathbb{R}^3}\bigg[G(v_n^1(\,\cdot\,+y_n^1)-w^1) + \frac{1}{6}|v_n^1(\,\cdot\,+y_n^1)-w^1|^6\bigg] + \overline{I}_\lambda^\infty(w^1) + o(1)\\
&= \frac{a+bA^2}{2}\int_{\mathbb{R}^3}|\nabla v_n^1 - w^1(\,\cdot\,-y_n^1)|^2 + \frac{1}{2}\int_{\mathbb{R}^3}V_\infty|v_n^1 - w^1(\,\cdot\,-y_n^1)|^2\\
&\quad -\lambda\int_{\mathbb{R}^3}G(v_n^1 - w^1(\,\cdot\,-y_n^1)) - \frac{\lambda}{6}\int_{\mathbb{R}^3}|v_n^1 - w^1(\,\cdot\,-y_n^1)|^6 + \overline{I}_\lambda^\infty(w^1) + o(1).
\end{aligned}$$

With the help of (i) and (i') of Step 3, continuing this process, we obtain that

$$\begin{aligned}
(3.17)\quad c_\lambda + \frac{bA^4}{4} - \overline{I}_\lambda(u_0) - \sum_{j=1}^{j=m}\overline{I}_\lambda^\infty(w^j)&\\
= \frac{a+bA^2}{2}\int_{\mathbb{R}^3}|\nabla\eta_n|^2 &+ \frac{1}{2}\int_{\mathbb{R}^3}V_\infty|\eta_n|^2 - \frac{\lambda}{6}\int_{\mathbb{R}^3}|\eta_n|^6.
\end{aligned}$$

On the other hand, it is not hard to check that

$$A^2 = \|\nabla u_0\|_2^2 + \sum_{j=1}^{j=m}\|\nabla w^j\|_2^2 + \|\nabla\eta_n\|^2 + o(1).$$

It follows from Lemma 2.2 that

$$\overline{I}_\lambda^\infty(w^j) \geq \frac{a+bA^2}{3}\int_{\mathbb{R}^3}|\nabla w^j|^2 \geq \frac{bA^2}{4}\int_{\mathbb{R}^3}|\nabla w^j|^2, \quad j=1,\ldots,m,$$

which, together with (3.17), implies that

$$(3.18)\qquad c_\lambda \geq \bigg(\frac{a}{2}+\frac{bA^2}{4}\bigg)\int_{\mathbb{R}^3}|\nabla\eta_n|^2 + \frac{1}{2}\int_{\mathbb{R}^3}V_\infty|\eta_n|^2 - \frac{\lambda}{6}\int_{\mathbb{R}^3}|\eta_n|^6.$$

It follows from (3.16) that we may assume that there exist $l_i \geq 0$ $(i=4,5,6)$ such that

$$\|\eta_n\|^2 \to l_4, \quad bA^2\int_{\mathbb{R}^3}|\nabla\eta_n|^2 \to l_5, \quad \lambda\int_{\mathbb{R}^3}|\eta_n|^6 \to l_6, \quad \text{as } n\to\infty,$$

then $l_4 + l_5 = l_6$. If $l_4 = 0$, then case (1) is complete. Else, if $l_4 > 0$, then $l_5, l_6 > 0$. Note that from (3.13) and conditions (V1)–(V2) we conclude that

$$\int_{\mathbb{R}^3}V(x)|\eta_n|^2 = \int_{\mathbb{R}^3}V_\infty|\eta_n|^2 + o(1),$$

which, together with the above facts, implies that

$$\int_{\mathbb{R}^3} (a|\nabla\eta_n|^2 + V_\infty|\eta_n|^2) = l_4 + o(1).$$

Therefore, combining (3.16) and (3.18), we have

$$c_\lambda \geq \frac{l_4}{3} + \frac{l_5}{12}. \tag{3.19}$$

Similarly to the arguments of Step 1, we have $c_\lambda \geq c_\lambda^*$ which contradicts Lemma 2.3. Thus, $\|\eta_n\| = o(1)$. Therefore, we have

$$c_\lambda + \frac{bA^2}{4} - \overline{I}_\lambda(u_0) - \sum_{j=1}^{j=m} \overline{I}_\lambda^\infty(w^j) = 0.$$

On the other hand, it is not hard to check that

$$A^2 = \|\nabla u_0\|_2^2 + \sum_{j=1}^{j=m} \|\nabla w^j\|_2^2.$$

In case (2) the argument is standard and we omit it.

STEP 4. Conclusion. By Step 1, we know that $(\overline{I}_\lambda)'(u_0) = 0$ and Lemma 3.3 holds with $k = 0$. If not, the assumption of Step 2 holds. If case (1) of Step 3 holds with $m = 1$, arguing as in Step 3, we obtain the conclusions of Lemma 3.3. Otherwise, case (2) of Step 3 holds. Set $y_n^2 = z_n$ and continue the procedure of Step 3. Remark that

$$\lim_{n\to\infty} \bigg(\|u_n\|^2 - \|u_0\|^2 - \sum_{j=1}^{j=m} \int_{\mathbb{R}^3} (a|\nabla w^j|^2 + V_\infty|w^j|^2) \bigg) = \lim_{n\to\infty} \bigg\| u_n - u_0 - \sum_{j=1}^{j=m} w^j(\,\cdot\, + y_n^j) \bigg\|^2.$$

It is easy to get that there exists $\beta > 0$ independent of $\lambda \in [1/2, 1]$ such that any nontrivial critical point u of $\overline{I}_\lambda^\infty$ satisfies $\|u\| \geq \beta$. Thus case (1) in Step 3 must occur after a finite number of iterations. □

LEMMA 3.4. *Assume that all conditions of Theorem 1.2 hold. For $\lambda \in [1/2, 1]$, let $\{u_n\}$ be a bounded Palais–Smale sequence of I_λ at level c_λ, then there exists a nontrivial $u_0 \in E$ such that $u_n \to u_0$ in E.*

PROOF. It follows from Lemma 3.3 that there exist $u_0 \in E$ and $A \in \mathbb{R}$ such that

$$u_n \rightharpoonup u_0 \quad \text{in } E \qquad \int_{\mathbb{R}^3} |\nabla u_n|^2 \to A^2, \quad \text{as } n \to \infty$$

after extracting a subsequence, if necessary, and $\overline{I}'_\lambda(u_0) = 0$. For each nontrivial critical point w^j $(j = 1, \dots, k)$ of $\overline{I}^\infty_\lambda$, w^j satisfies the Pohozǎev identity

$$\overline{P}_\lambda(w^j) := \frac{a + bA^2}{2} \int_{\mathbb{R}^3} |\nabla w^j|^2 + \frac{3}{2} \int_{\mathbb{R}^3} V_\infty |w^j|^2 - 3\lambda \int_{\mathbb{R}^3} F(w^j) = 0. \tag{3.20}$$

Therefore,

$$\frac{a}{2} \int_{\mathbb{R}^3} |\nabla w^j|^2 + \frac{b}{2} \bigg(\int_{\mathbb{R}^3} |\nabla w^j|^2 \bigg)^2 + \frac{3}{2} \int_{\mathbb{R}^3} V_\infty |w^j|^2 - 3\lambda \int_{\mathbb{R}^3} F(w^j) \leq 0.$$

In view of the above inequality, we can easily see that there exist $t_j \in (0, 1]$ such that

$$\begin{aligned} \frac{a}{2} \int_{\mathbb{R}^3} \bigg| \nabla w^j \bigg(\frac{\cdot}{t_j} \bigg) \bigg|^2 &+ \frac{b}{2} \bigg(\int_{\mathbb{R}^3} \bigg| \nabla w^j \bigg(\frac{\cdot}{t_j} \bigg) \bigg|^2 \bigg)^2 \\ &+ \frac{3}{2} \int_{\mathbb{R}^3} V_\infty \bigg| w^j \bigg(\frac{\cdot}{t_j} \bigg) \bigg|^2 - 3\lambda \int_{\mathbb{R}^3} F\bigg(w^j \bigg(\frac{\cdot}{t_j} \bigg) \bigg) = 0. \end{aligned}$$

Therefore, it follows form the definition of m_λ that $I^\infty_\lambda(w^j(\,\cdot\,/t_j)) \geq m_\lambda$. So, by Remark 3.1 and (3.20), we have

$$\begin{aligned} \overline{I}^\infty_\lambda(w^j) &= \overline{I}^\infty_\lambda(w^j) - \frac{1}{3} \overline{P}_\lambda(w^j) = \frac{a + bA^2}{3} \int_{\mathbb{R}^3} |\nabla w^j|^2 \\ &\geq \frac{a}{3} \int_{\mathbb{R}^3} \bigg| \nabla w^j \bigg(\frac{x}{t_j} \bigg) \bigg|^2 + \frac{b}{12} \bigg(\int_{\mathbb{R}^3} \bigg| \nabla w^j \bigg(\frac{x}{t_j} \bigg) \bigg|^2 \bigg)^2 + \frac{bA^2}{4} \int_{\mathbb{R}^3} |\nabla w^j|^2 \\ &= I^\infty_\lambda \bigg(w^j \bigg(\frac{\cdot}{t_j} \bigg) \bigg) + \frac{bA^2}{4} \int_{\mathbb{R}^3} |\nabla w^j|^2 \geq m_\lambda + \frac{bA^2}{4} \int_{\mathbb{R}^3} |\nabla w^j|^2. \end{aligned} \tag{3.21}$$

Then, from Lemma 3.3, we deduce that

$$\begin{aligned} c_\lambda + \frac{bA^4}{4} &= \overline{I}_\lambda(u_0) + \sum_{j=1}^{j=k} \overline{I}^\infty_\lambda(w^j) \\ &\geq k m_\lambda + \frac{1}{4} bA^2 \int_{\mathbb{R}^3} |\nabla u_0|^2 + \frac{1}{4} bA^2 \sum_{j=1}^{j=k} \int_{\mathbb{R}^3} |\nabla w^j|^2 \geq k m_\lambda + \frac{bA^4}{4}. \end{aligned}$$

If $k \geq 1$, then the above inequality contradicts with Lemma 3.2. Therefore, by Lemma 3.3 with $k = 0$, we have $u_n \to u_0$ in E. □

PROOF OF THEOREM 1.2. In view of Lemmas 2.3 and 3.4, for almost all $\lambda \in [1/2, 1]$, there is a nontrivial critical point $u_\lambda \in H^1(\mathbb{R}^3)$ for I_λ with $I_\lambda(u_\lambda) = c_\lambda$. Choosing a sequence $\{\lambda_n\} \subset [1/2, 1]$ satisfying $\lambda_n \to 1$, we find a sequence of nontrivial critical points $\{u_{\lambda_n}\}$ (still denoted by $\{u_n\}$) of I_{λ_n} with $I_{\lambda_n}(u_n) = c_{\lambda_n}$. Now we show that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. Indeed, since $\lambda \to c_\lambda$ is continuous from the left-hand side, $\{c_{\lambda_n}\}$ is a bounded sequence. Similarly to (3.10), we know that

$$c_{\lambda_n} = I_\lambda(u_n) \geq \frac{1}{4}\, a \int_{\mathbb{R}^3} |\nabla u_n|^2.$$

Therefore, there exists $C > 0$ such that $\int_{\mathbb{R}^3} |\nabla u_n|^2 \le C$. On the other hand, from (f_2) and (F1) and the Sobolev embedding inequality, we can easily deduce that $\|u_n\| < C$ for some positive constant C. Therefore, by Theorem 2.1, we see that

$$\lim_{n\to\infty} I(u_n) = \lim_{n\to\infty} \bigg(I_{\lambda_n}(u_n) + (\lambda_n - 1) \int_{\mathbb{R}^3} F(u_n) \bigg) = \lim_{n\to\infty} c_{\lambda_n} = c_1$$

and, for any $\varphi \in H^1(\mathbb{R}^3)$, we have

$$\lim_{n\to\infty} I'(u_n)\varphi = \lim_{n\to\infty} \bigg(I'_{\lambda_n}(u_n)\varphi + (\lambda_n - 1) \int_{\mathbb{R}^3} f(u_n)\varphi \bigg) = 0.$$

That is to say, $\{u_n\}$ is a bounded Palais–Smale sequence for I at level c_1. Then, by Lemma 3.4, there is a nontrivial critical point $u_0 \in H^1(\mathbb{R}^3)$ for I with $I(u_0) = c_1$.

Set $\nu = \inf\{I(u) \mid u \in H^1(\mathbb{R}^3) \setminus \{0\},\ I'(u) = 0\}$. It is easy to see that $0 < \nu \le c_1 < +\infty$. By the definition of ν, there exists $\{u_n\} \subset H^1(\mathbb{R}^3)$ such that $I(u_n) \to \nu$ and $I'(u_n) = 0$. Using the earlier arguments, we can deduce that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. Arguing as in Lemma 3.4, there exists a nontrivial $u \in H^1(\mathbb{R}^3)$ such that $I(u) = \nu$ and $I'(u) = 0$. Therefore, we see that u is a positive ground state solution of problem (K). $\square$

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