EIGENVALUES, GLOBAL BIFURCATION
AND POSITIVE SOLUTIONS
FOR A CLASS OF NONLOCAL ELLIPTIC EQUATIONS

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ABSTRACT. In this paper, we shall study global bifurcation phenomenon
for the following Kirchhoff type problem:

$$\begin{cases} -\left(a + b \int_\Omega |\nabla u|^2 \, dx\right)\Delta u = \lambda u + h(x, u, \lambda) & \text{in } \Omega, \\ u = 0 & \text{on } \Omega. \end{cases}$$

Under some natural hypotheses on h, we show that $(a\lambda_1, 0)$ is a bifurca-
tion point of the above problem. As an application of the above result, we
shall determine the interval of λ, in which there exist positive solutions for
the above problem with $h(x, u; \lambda) = \lambda f(x, u) - \lambda u$, where f is asymptoti-
cally linear at zero and asymptotically 3-linear at infinity. To study global
structure of bifurcation branch, we also establish some properties of the
first eigenvalue for a nonlocal eigenvalue problem. Moreover, we provide
a positive answer to an open problem involving the case $a = 0$.

1. Introduction

In this paper, we study global bifurcation phenomenon for the following problem:

(1.1)
$$\begin{cases} -\left(a + b\int_\Omega |\nabla u|^2\, dx\right)\Delta u = \lambda u + h(x, u, \lambda) & \text{in } \Omega, \\ u = 0 & \text{on } \Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$, $a, b > 0$ are real constants, λ is a parameter and $h\colon \Omega \times \mathbb{R}^2 \to \mathbb{R}$ satisfies the Carathéodory condition in the first two variable and

(1.2)
$$\lim_{s \to 0} \frac{h(x, s, \lambda)}{s} = 0$$

uniformly for almost every $x \in \Omega$ and λ on bounded sets. Moreover, we assume that h satisfies the following growth restriction:

(G) there exist $c > 0$ and $p \in (1, 2^*)$ such that

$$|h(x, s, \lambda)| \leq c(1 + |s|^{p-1})$$

for almost every $x \in \Omega$ and λ on bounded sets, where

$$2^* = \begin{cases} \dfrac{2N}{N-2} & \text{if } N > 2, \\ +\infty & \text{if } N \leq 2. \end{cases}$$

Problem (1.1) is related to the stationary problem of a model introduced by Kirchhoff in 1883 to describe transversal oscillations of a stretched string [27]. More precisely, Kirchhoff proposed a model given by the equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = f(x, u),$$

where ρ, ρ_0, h, E, L are constants, f is an external force, which extends the classical D'Alembert's wave equation by considering the effect of changing in length of the string during vibration. Problem (1.1) received much attention only after Lions [31] proposed an abstract framework to the problem. Some important and interesting results can be found, for example, in [2], [4], [15], [14], [26]. Recently, many mathematicians were studying problem (1.1) by variational methods, see e.g. [5], [6], [32], [33], [35], [38], [41] and references therein.

The authors of [28] studied problem (1.1) with $h(x, s, \lambda) = \lambda f(x, s) - \lambda s$ by using the topological degree argument and variational method. Under some assumptions on f, they provided a positive answer to the existence of positive solutions to (1.1) for the cases $a, b > 0$ and $a > 0$, $b = 0$. They pointed out that the *case $a = 0$ and $b > 0$ is an open problem*. The study of Kirchhoff type

equations has also been extended to the case involving the p-Laplacian and $p(x)$-Laplacian. We refer the readers to [3], [7]–[11], [17], [18], [20] for an overview and references on this subject.

A distinguishing feature of problem (1.1) is that the first equation contains a nonlocal coefficient $a + b \int_\Omega |\nabla u|^2 \, dx$, and hence the equation is no longer a pointwise identity. Moreover, the first equation of problem (1.1) with $h \equiv 0$ is not homogeneous. So problem (1.1) is a fully nonlinear problem what raises some essential difficulties to study this kind of problems.

The main aim of this paper is to establish the global bifurcation result for (1.1) and to study its applications. Let λ_1 denote the first eigenvalue of the following problem:

$$(1.3) \qquad \begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \Omega. \end{cases}$$

It is well known that λ_1 is simple, isolated and it is the unique principal eigenvalue of (1.3). The first result of this paper is the following theorem.

THEOREM 1.1. *The pair $(a\lambda_1, 0)$ is a bifurcation point of (1.1). Moreover, there is a component $\mathcal{C}$ of the set of nontrivial solutions of (1.1) in $\mathbb{R} \times H_0^1(\Omega)$ whose closure contains $(a\lambda_1, 0)$ and it is either unbounded or contains a pair $(a\overline{\lambda}, 0)$ for some $\overline{\lambda}$, an eigenvalue of (1.3) with $\overline{\lambda} \neq \lambda_1$.*

As is known, this result is proved in the case $a = 1$ and $b = 0$ (see [36]). It has also been extended to the p-Laplacian problem in the case $a = 1$ and $b = 0$ (see [16]). While, (1.1) is a fully nonlinear problem. So the Rabinowitz global bifurcation theorem cannot be used directly to obtain our result. We shall transfer problem (1.1) into a new form and then use the Rabinowitz global bifurcation theorem to prove Theorem 1.1 in Section 2.

In Section 3 we study the component $\mathcal{C}$, obtained in Theorem 1.1, for the following eigenvalue problem:

$$(1.4) \qquad \begin{cases} -\left(\int_\Omega |\nabla u|^2 \, dx \right) \Delta u = \mu u^3 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Let

$$\mu_1 = \inf \left\{ \left(\int_\Omega |\nabla u|^2 \, dx \right)^2 : u \in H_0^1(\Omega), \int_\Omega u^4 \, dx = 1 \right\}.$$

We recall that a nodal domain of u is a connected component of $\Omega \setminus \{x \in \Omega : u(x) = 0\}$. Our second result is

THEOREM 1.2. *Assume that $N \leq 3$. Then $\mu_1 > 0$ is the first eigenvalue of (1.4) and it has the following properties*:

(a) *Any eigenfunction u corresponding to μ_1 belongs to $C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$, and $\partial u(x)/\partial \gamma < 0$ if u is nonnegative, where γ is the outer unit normal at $x \in \partial\Omega$.*

(b) *μ_1 is the principal eigenvalue of (1.4).*

(c) *μ_1 is simple if one of the following is satisfied:*
   (i) *Ω is an open ball in $\mathbb{R}^N$;*
   (ii) *$\Omega \subseteq \mathbb{R}^2$ is symmetric in x and y, and convex in x and y directions;* *or*
   (iii) *$\Omega \subseteq \mathbb{R}^2$ is convex.*

(d) *If one of conditions in (c) is satisfied, (1.4) has a positive solution if and only if $\mu = \mu_1$.*

(e) *Let v be any eigenfunction associated to an eigenvalue $\mu > \mu_1$ and $\mathcal{N}$ be its any nodal domain. Then we have*

(1.5) $$|\mathcal{N}| \geq c,$$

   where $c > 0$ is some constant depending only on N and μ.

(f) *If one of conditions in (c)is satisfied, then μ_1 is isolated.*

The authors of [35] proved the existence of μ_1. In [28], the authors proved parts (c) and (d) in the case when Ω is a ball. Recently, the authors of [29] showed parts (b)–(d). To the best of our knowledge, properties (a), (e) and (f) are the first results on this kind of problems. So the results of Theorem 1.2 improve and generalize the corresponding results of [28], [29] and [35].

In Section 4, we describe the component $\mathcal{C}$ for problem (1.1) with $h(x,s,\lambda) = \lambda f(x,s) - \lambda s$, i.e. for the following problem:

(1.6) $$\begin{cases} -\left(a + b \int_{\Omega} |\nabla u|^2 \, dx\right)\Delta u = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \Omega, \end{cases}$$

where Ω satisfies one of conditions in Theorem 1.2 (c). We assume that f satisfies the following conditions:

(f1) $f \colon \Omega \times \mathbb{R} \setminus \mathbb{R}^- \to \mathbb{R} \setminus \mathbb{R}^-$ is a Carathéodory function such that $f(x,s)s > 0$ for almost every $x \in \Omega$ and any $s > 0$.

(f2) There exists $f_0 \in (0,+\infty)$ such that
$$\lim_{s \to 0^+} \frac{f(x,s)}{a\lambda_1 s} = f_0$$
uniformly with respect to almost every $x \in \Omega$.

(f3) There exists $f_\infty \in (0,+\infty)$ such that
$$\lim_{s \to +\infty} \frac{f(x,s)}{b\mu_1 s^3} = f_\infty$$
uniformly with respect to almost every $x \in \Omega$.

Our third result is

THEOREM 1.3. *Assume that $N \leq 3$, Ω satisfies one of conditions in Theorem 1.2 (c) and f satisfies* (f1)–(f3). *Then* $(1/f_0, 0)$ *is the unique bifurcation point of the set of positive solutions of* (1.6). *Moreover, there is an unbounded component $\mathcal{C}$ in $\mathbb{R} \times H_0^1(\Omega)$ whose closure links* $(1/f_0, 0)$ *to* $(1/f_\infty, \infty)$.

REMARK 1.4. The reason of restriction $N \leq 3$ in Theorem 1.2 is to ensure $u^3 \in L^2(\Omega)$ for any $u \in H_0^1(\Omega)$ which is crucial for our proof. If $N = 4$, the authors of [29] have showed that $\mu_1 > 0$ but it cannot be achieved in $H_0^1(\Omega)$. If $N \geq 5$, they also showed that $\mu_1 = 0$. In addition, we pose the condition $N \leq 3$ as our proof depends Theorem 1.2 and the fact $4 < 2^*$.

REMARK 1.5. Note that Theorem 1.1 of [28] is our corollary of Theorem 1.3 if Ω satisfies one of conditions in Theorem 1.2 (c). We also note that Theorem 1.2 (ii) (with $a > 0$) of [28] is the corollary of Theorem 1.3 if Ω satisfies one of conditions in Theorem 1.2 (c). Clearly, Theorem 1.2 (i) of [28] does not occur because $(1/f_\infty, \infty)$ is the unique bifurcation point of positive solutions set of (1.6) where a bifurcation from infinity occurs. Moreover, our assumptions on f are more concise and weaker than the corresponding ones of [28, Theorems 1.1, 1.2, 5.1 and 5.2].

REMARK 1.6. From Theorem 1.3, we can see that any positive solution of (1.6) lies in $\mathcal{C}$. That is to say, we find the range of all positive solutions. So one only needs to study the structure of $\mathcal{C}$ to find the positive solution of (1.6).

REMARK 1.7. By standard elliptic regularity theory (see [21], [22]), we know that any weak solution of (1.1) or (1.6) belongs to $C^{1,\alpha}(\overline{\Omega})$ with $\alpha \in (0,1)$ under conditions (1.2) and (G) or (f1)–(f3).

In Section 5, we consider the case $a = 0$ in (1.6) and give a positive answer to the above mentioned open problem. Concretely, we pose the following assumption on f:

(f4) there exists $\widetilde{f_0} \in (0, +\infty)$ such that

$$\lim_{s \to 0^+} \frac{f(x,s)}{\lambda_1 s} = \widetilde{f_0}$$

uniformly with respect to almost every $x \in \Omega$.

Our last result is the following theorem.

THEOREM 1.8. *Assume that $a = 0$, $N \leq 3$, Ω satisfies one of conditions in Theorem 1.2 (c) and f satisfies* (f1), (f3) *and* (f4). *Then* $(0,0)$ *is the unique bifurcation point of the set of positive solutions of* (1.6). *Moreover, there is an unbounded component $\mathcal{C}$ in $\mathbb{R} \times H_0^1(\Omega)$ whose closure links* $(0,0)$ *to* $(1/f_\infty, \infty)$.

REMARK 1.9. From Theorem 1.8, we can easily see that (1.6) has a positive solution with $\lambda = 1$ and $f_\infty < 1$. This gives a positive answer to an open problem proposed by the authors in [28].

REMARK 1.10. Note that f is asymptotically linear at zero in Theorem 1.8 which is different from [28]. In [28], f is assumed to be asymptotically 3-linear at zero in the case $a = 0$. We shall consider this situation in our future work.

The last section concludes the paper and outlines our future work.

We end this section by introducing some notation conventions which will be used later in this paper. Let X be the usual Sobolev space $H_0^1(\Omega)$ with the norm $\|u\| = \left(\int_\Omega |\nabla u|^2 \, dx \right)^{1/2}$ and X^* be its dual space. Denote by $\langle \cdot, \cdot \rangle$ the duality pairing between X and X^*. We write $u_n \rightharpoonup u$ and $u_n \to u$ for the weak and strong convergence of the sequence $\{u_n\}$ in X, respectively. Use $q' = q/(q-1)$ to denote the conjugative number of q with $q > 1$. For a measurable set A of $\mathbb{R}^N$ we denote its measure by $|A|$. Also, we denote by c and c_i, $i \in \mathbb{N}$, generic positive constants (the exact value may be different from line to line).

2. Global bifurcation

Firstly, consider the following auxiliary problem:

$$(2.1) \qquad \begin{cases} -\Delta u = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

As is known, problem (2.1) possesses a unique weak solution for each $f \in X^*$. Let us denote by $G(f)$ the unique weak solution of (2.1). Then $G \colon X^* \to X$ is a linear continuous operator. Since the embedding $X \hookrightarrow L^q(\Omega)$ is compact for each $q \in (1, 2^*)$, the restriction of G to $L^{q'}(\Omega)$ is a completely continuous operator.

Clearly, the pair (λ, u) is a solution of (1.1) if and only if (λ, u) satisfies

$$(2.2) \qquad u = G\left(\frac{1}{a + b\|u\|^2} (\lambda u + H(\lambda, u)) \right),$$

where $H(\lambda, \cdot)$ denotes the usual Nemitskiĭ operator associated with h. From condition (G) and $2 < 2^*$, we can see that $G \colon L^{p'}(\Omega) \cup L^2(\Omega) \to X$ is completely continuous.

PROOF OF THEOREM 1.1. Let

$$Lu = \frac{1}{a} G(u), \qquad \widetilde{H}(\lambda, u) = \frac{1}{a + b\|u\|^2} G(H(\lambda, u)) - \frac{\lambda b \|u\|^2}{a(a + b\|u\|^2)} G(u).$$

Clearly, $L \colon X \to X$ is linear and completely continuous operator, $\widetilde{H} \colon \mathbb{R} \times X \to X$ is compact. Moreover, it is easy to see that $a\lambda_1$ is a simple characteristic value

of L. Then equation (2.2) is equivalent to

$$u = \lambda L u + \widetilde{H}(\lambda, u).$$

Next, we show that $\widetilde{H} = o(\|u\|)$ at $u = 0$ uniformly on bounded λ intervals. It is sufficient to show that

$$\lim_{\|u\| \to 0} \frac{H(x, u)}{\|u\|} = 0 \quad \text{in } L^{p'}(\Omega).$$

Without loss of generality, we may assume that $p > 2$. Otherwise, we can consider $\widetilde{p} = cp$, $c > 1$ such that $\widetilde{p} \in (2, 2^*)$. From $p < 2^*$, we can see that

$$\frac{p'(p-2)}{2^*} < \frac{2^* - p'}{2^*}.$$

So we can choose a real number $r > 1$ such that

$$\frac{p'(p-2)}{2^*} \leq \frac{1}{r} \leq \frac{2^* - p'}{2^*}.$$

It follows that

$$(2.3) \qquad\qquad p'r(p-2) \leq 2^* \quad \text{and} \quad p'r' \leq 2^*.$$

For any $\varepsilon > 0$, in view of (1.2) and (G), we can choose positive numbers $\delta = \delta(\varepsilon)$ and $M = M(\delta)$ such that for almost every $x \in \Omega$, the following relations hold:

$$\left| \frac{h(x, s, \lambda)}{s} \right| \leq \varepsilon \qquad \text{for } 0 < |s| \leq \delta,$$

$$\left| \frac{h(x, s, \lambda)}{s} \right| \leq M|s|^{p-2} \quad \text{for } |s| > \delta.$$

Then

$$\int_\Omega \left| \frac{H(\lambda, u)}{u} \right|^{p'r} dx \leq \varepsilon |\Omega| + M^{p'r} \int_\Omega |u|^{p'r(p-2)} dx.$$

From this inequality, (2.3) and $u \to 0$ in X, we get

$$(2.4) \qquad\qquad \left| \frac{H(\lambda, u)}{u} \right|^{p'} \to 0 \quad \text{in } L^r(\Omega).$$

Let $v = u/\|u\|$. By the boundedness of v in X, (2.3) and continuity of the embedding $X \hookrightarrow L^{2^*}(\Omega)$, we have

$$(2.5) \qquad\qquad \int_\Omega |v|^{p'r'} dx \leq c$$

for some constant $c > 0$. Then from (2.4), (2.5) and the Hölder inequality, we obtain

$$\int_\Omega \left| \frac{H(\lambda, u)}{\|u\|} \right|^{p'} dx = \int_\Omega \left| \frac{H(\lambda, u)}{|u|} \right|^{p'} |v|^{p'} dx$$

$$\leq \left(\int_\Omega \left| \frac{H(\lambda, u)}{u} \right|^{p'r} dx \right)^{1/r} \left(\int_\Omega |v|^{p'r'} dx \right)^{1/r'} \to 0.$$

Now, from global bifurcation theory (see [37, Theorem 1.3]), we get the existence of a global branch of the set of nontrivial solution of (1.1) emanating from $(a\lambda_1, 0)$. □

3. Properties of the first eigenvalue of a nonlocal problem

In order to study properties of the component $\mathcal{C}$, obtained in Theorem 1.1, we shall consider the following eigenvalue problem:

$$(3.1) \qquad \begin{cases} -\left(\displaystyle\int_\Omega |\nabla u|^2 \, dx\right)\Delta u = \mu u^3 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Let

$$I(u) = \|u\|^4, \quad u \in S := \left\{u \in X : \int_\Omega u^4 \, dx = 1\right\}.$$

Denote by $\mathcal{A}$ the class of closed symmetric subsets of S, let

$$\mathcal{F}_m = \{A \in \mathcal{A} : i(A) \geq m - 1\},$$

where m is a positive integer and $i(A)$ denotes the Yang index of A. The authors of [35] have proved that problem (3.1) possesses unbounded sequences of minimax eigenvalues $0 < \mu_1 \leq \mu_2 < \ldots$ such that

$$\mu_m = \inf_{A \in \mathcal{F}_m} \max_{u \in A} I(u).$$

In particular, if $m = 1$, taking $A = \{u, -u : u \in S\}$, we get

$$\mu_1 = \inf\left\{\|u\|^4 : u \in X, \ \int_\Omega u^4 \, dx = 1\right\}.$$

Next, we are going to study the properties of μ_1. These properties are important in the study of global bifurcation phenomena.

PROPOSITION 3.1. *Let u be any eigenfunction corresponding to μ_1. Then $u \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$. Furthermore, $\partial u(x)/\partial\gamma < 0$ if u is nonnegative, where γ is the outer unit normal at $x \in \partial\Omega$.*

PROOF. Note that problem (3.1) is homogeneous. So by scaling we may suppose that $\|u\| = 1$. It follows that

$$(3.2) \qquad \begin{cases} -\Delta u = \mu_1 u^3 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

By the embedding $X \hookrightarrow L^{2^*}$ and $N \leq 3$, we have $\mu_1 u^3(\,\cdot\,) := f(\,\cdot\,) \in L^2(\Omega)$. By [25, Theorem 8.12], we know that $u \in W^{2,2}(\Omega)$. Furthermore, by the general Sobolev embedding theorem [19, p. 270], we get $u \in C^\gamma(\overline{\Omega})$ for some $\gamma \in (0, 1)$. Moreover, by the definition of weak derivative, $\nabla f = \mu_1 3u^2 \nabla u$. It is easy to verify that $\nabla f \in L^2(\Omega)$. So we have $f \in W^{1,2}(\Omega)$. By [25, Theorem 8.13], we

know that $u \in W^{3,2}(\Omega)$. Again using the general Sobolev embedding theorem [19, p. 270], we get $u \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$. Furthermore, if $u \geq 0$, by the strong maximum principle [23, Theorem 1.2], $\partial u(x)/\partial \gamma < 0$ for all $x \in \partial\Omega$.□

PROPOSITION 3.2. *Let u be an eigenfunction associated with μ_1, then either $u > 0$ or $u < 0$ in Ω, i.e. μ_1 is the principal eigenvalue of* (3.1).

PROOF. We notice that if u is an eigenfunction, so is $v := |u|$. Without loss of generality, we shall assume that $\|v\| = 1$. So we have

$$\begin{cases} -\Delta v = \mu_1 v^3 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

By the strong maximum principle [25, Theorem 8.19], we know that $v > 0$ in the whole domain. By the continuity of u, either u or $-u$ is positive in the whole domain. □

PROOF OF THEOREM 1.2. Propositions 3.1 and 3.2 imply properties (a) and (b), respectively. So we only need to prove properties (c)–(f). Although property (c) was proved in [28], [29], we shall still give a rough sketch of the proof here for reader's convenience.

(c) Consider the following auxiliary problem:

$$(3.3) \qquad \begin{cases} -\Delta u = u^3 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

If Ω is an open ball in $\mathbb{R}^N$, Lemma 2.3 of [24] implies that the positive solution of (3.3) is unique. Let u, v be two eigenfunctions associated with λ_1. Then $\sqrt[2]{\lambda_1}u/\|u\|$ and $\sqrt[2]{\lambda_1}v/\|v\|$ are both solutions to (3.3). Hence $u = \|u\|v/\|v\|$, which shows that λ_1 is simple. If $\Omega \subseteq \mathbb{R}^2$ is symmetric in x and y, and convex in x and y directions, Theorem 2.1 of [12] and Theorem 5 of [13] imply that the positive solution of (3.3) is unique. Similar argument shows that λ_1 is simple. If $\Omega \subseteq \mathbb{R}^2$ is convex, Theorem 1 of [30] implies that the least energy positive solution of (3.3) is unique, where the least energy solution is the one which achieves

$$\inf_{u \in X, \, u \neq 0} \frac{\displaystyle\int_\Omega |\nabla u|^2 \, dx}{\left(\displaystyle\int_\Omega |u|^4 \, dx \right)^{1/2}}.$$

Suppose that u and v are positive solutions of (1.4) with $\lambda = \lambda_1$, then both u and v are least energy positive solutions of (3.3). So $u = cv$ for some constant $c > 0$.

(d) Suppose on the contrary that (3.1) with $\mu > \mu_1$ has a positive solution v, and let u be a positive eigenfunction corresponding to μ_1. Similarly to Proposition 3.1, we can show that $v \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$ and $\partial v(x)/\partial \gamma < 0$. By this and Proposition 3.1, there exists a constant $t_0 > 0$ large enough such that $t_0 v \geq u$. Clearly, $\widetilde{v} := t_0 v$ is also an eigenfunction of (3.1). Define f^1 and f^2 on X by

$$f^1(w) = \frac{1}{4}\|w\|^4, \qquad f^2(w) = \frac{1}{4}\int_\Omega w^4\,dx.$$

Let $A = (f^1)'$ and $B = (f^2)'$. Then w is a weak solution of (3.1) if and only if $Aw = \mu Bw$. It is not difficult to show that $Bu \leq B\widetilde{v}$. Then

$$Au = \mu_1 Bu \leq \mu_1 B\widetilde{v} = \mu B(\eta\widetilde{v}) = A(\eta\widetilde{v}) \quad \text{with } \eta = \left(\frac{\mu_1}{\mu}\right)^{1/3} < 1.$$

Taking $\varphi = (u - \eta\widetilde{v})^+$ as a test function in $Au \leq A(\eta\widetilde{v})$, it follows from (a) of Proposition 5.2, which will be proved in Section 5, that $\nabla(u - \eta\widetilde{v})^+ = 0$, this implies $(u - \eta\widetilde{v})^+ = 0$ and so $u \leq \eta\widetilde{v}$ in Ω. Repeating this argument n times, we obtain that $0 \leq u \leq \eta^n\widetilde{v}$. Letting $n \to +\infty$, we get $u \equiv 0$. This is a contradiction. So v must change sign.

(e) By an argument similar to that of Proposition 3.1, we know that $v \in C(\Omega)$, then $v_{|\mathcal{N}} \in H^1_0(\mathcal{N})$. Define

(3.4)
$$w = \begin{cases} v & \text{for } x \in \mathcal{N}, \\ 0 & \text{for } x \in \Omega \setminus \mathcal{N}. \end{cases}$$

It is easy to see that $w \in X$. We first consider the case $N = 3$. Then

$$\left(\int_\mathcal{N} |\nabla w|^2\,dx\right)^2 = \mu \int_\mathcal{N} w^4\,dx.$$

By the Hölder inequality and the Sobolev embeddings we have

$$c^4\left(\int_\mathcal{N} |w|^6\,dx\right)^{2/3} \leq \left(\int_\mathcal{N} |\nabla w|^2\,dx\right)^2 = \mu \int_\mathcal{N} w^4\,dx \leq \mu\left(\int_\mathcal{N} w^6\,dx\right)^{2/3} |\mathcal{N}|^{1/3},$$

where $c > 0$ is the best embedding constant of $H^1_0(\mathcal{N}) \hookrightarrow L^6(\mathcal{N})$. It follows that $|\mathcal{N}| \geq c^{12}/\mu^3$. For $N = 1$ or 2, it is well known that $H^1_0(\mathcal{N})$ continuously embeds in $L^\infty(\mathcal{N})$ with the best embedding constant $c > 0$. From this fact and reasoning as above, we can show that $|\mathcal{N}| \geq c^4/\mu$.

(f) Assume by contradiction that there exists a sequence of eigenvalues $\mu_n \in (\mu_1, \delta)$ for some constant $\delta > \mu_1$ which converges to μ_1. Let u_n be the corresponding eigenfunctions. Property (d) implies that u_n changes sign. Integration by parts gives

$$\left(\int_\Omega |\nabla u_n|^2\,dx\right)^2 = \mu_n \int_\Omega (u_n)^4\,dx.$$

Define
$$v_n := \frac{u_n}{\left(\displaystyle\int_\Omega (u_n)^4 \, dx \right)^{1/4}}.$$

Obviously, v_n is bounded in X so there exists a subsequence, denoted again by v_n, and $v \in X$ such that $v_n \rightharpoonup v$ in X and $v_n \to v$ in $L^4(\Omega)$. Since the functional I is sequentially weakly lower semi-continuous, we have that

$$\left(\int_\Omega |\nabla v|^2 \, dx \right)^2 \leq \liminf_{n \to +\infty} \left(\int_\Omega |\nabla v_n|^2 \, dx \right)^2 = \liminf_{n \to +\infty} \mu_n = \mu_1.$$

On the other hand, $\int_\Omega (v_n)^4 \, dx = 1$ and $v_n \to v$ in $L^4(\Omega)$ imply that $\int_\Omega v^4 \, dx = 1$. It follows that

$$\left(\int_\Omega |\nabla v|^2 \, dx \right)^2 \leq \mu_1 \int_\Omega v^4 \, dx.$$

The above inequality and the variational characterization of μ_1 imply that

$$\left(\int_\Omega |\nabla v|^2 \, dx \right)^2 = \mu_1 \int_\Omega v^4 \, dx.$$

Then Proposition 3.2 implies that v is positive or negative. Without loss of generality, we may assume that $v > 0$ in Ω. Since $v_n \rightharpoonup v$ X, passing if necessary to a subsequence, we can assume that

$$v_n \to v \quad \text{in } L^q(\Omega) \text{ with } q \in (1, 2^*),$$
$$v_n \to v \quad \text{in a.e. } \Omega.$$

So we conclude that $|\Omega_n^-| \to 0$, where Ω_n^- denotes the negative set of u_n. This contradicts estimate (1.5). □

4. Positive solutions

In this section we apply Theorems 1.1 and 1.2 to study the existence of positive solutions for (1.6).

LEMMA 4.1. *Assume* (f1)–(f3) *hold. Then* $(1/f_0, 0)$ *is a bifurcation point of* (1.6) *and the associated bifurcation branch* $\mathcal{C}$ *in* $\mathbb{R} \times X$ *whose closure contains* $(1/f_0, 0)$ *is either unbounded or contains a pair* $(\overline{\lambda}/(f_0 \lambda_1), 0)$ *where* $\overline{\lambda}$ *is an eigenvalue of* (1.3) *and* $\overline{\lambda} \neq \lambda_1$.

PROOF. Let $\vartheta \colon \Omega \times \mathbb{R} \setminus \mathbb{R}^- \to \mathbb{R} \setminus \mathbb{R}^-$ be a Carathéodory function such that

$$f(x, s) = a\lambda_1 f_0 s + \vartheta(x, s)$$

with

(4.1) $$\lim_{s \to 0^+} \frac{\vartheta(x, s)}{a\lambda_1 s} = 0 \quad \text{and} \quad \lim_{s \to +\infty} \frac{\vartheta(x, s)}{s^3} = b\mu_1 f_\infty$$

uniformly with respect to almost every $x \in \Omega$. From (4.1), we can see that $\lambda\vartheta$ satisfies assumptions of (1.2) and (G). Now, Theorem 1.1 can be applied to get the results of this lemma. □

Let $P = \{u \in X : u(x) > 0 \text{ for all } x \in \Omega\}$ be the positive cone in X.

LEMMA 4.2. *We have* $\mathcal{C} \subseteq ((\mathbb{R} \times P) \cup \{(1/f_0, 0)\})$ *and the last alternative of Lemma 4.1 is impossible.*

PROOF. By the strong maximum principle [25, Theorem 8.19], we know that $u > 0$ in the whole domain for any nontrivial solution $(\lambda, u) \in \mathcal{C}$. So we have $\mathcal{C} \subseteq ((\mathbb{R} \times P) \cup (\mathbb{R} \times \{0\}))$. Suppose on the contrary that there exists $(\lambda_m, u_m) \to (\overline{\lambda}/(f_0\lambda_1), 0)$ when $m \to +\infty$ with $(\lambda_m, u_m) \in \mathcal{C}$, $u_m \not\equiv 0$ and $\overline{\lambda} \neq \lambda_1$. So we have $u_m \in P$ for each $m \in \mathbb{N}$. Let $v_m := u_m/\|u_m\|$, then (λ_m, v_m) satisfies

$$v_m = G\left(\frac{\lambda_m}{a + b\|u_m\|^2} \left(a\lambda_1 f_0 v_m + \frac{\vartheta(\lambda, u_m)}{\|u_m\|} \right) \right).$$

By an argument similar to that of Theorem 1.1, we obtain that for some convenient subsequence $v_m \to v_0$ as $m \to +\infty$. Now v_0 verifies the equation $-\Delta v = \overline{\lambda}v$ and $\|v_0\| = 1$. Hence v_0 must change its sign, and this is a contradiction. Furthermore, it follows that $\mathcal{C} \subseteq (P \cup \{(1/f_0, 0)\})$ and $\mathcal{C}$ is unbounded in $\mathbb{R} \times X$.□

REMARK 4.3. From the proof of Lemma 4.2, we can see that $(1/f_0, 0)$ is the unique bifurcation point of the set of positive solutions of (1.6).

Next, we give a Sturm type comparison theorem.

THEOREM 4.4. *Assume that* $N \leq 3$, g *and* f_n *are two weight functions such that* $g \in C(\overline{\Omega})$, $f_n \in L^3(\Omega)$ *and* $f_n \not\equiv g$ *almost everywhere in* Ω *for any* n *large enough. Let* u *be a positive weak solution of*

(4.2)
$$\begin{cases} -\|u\|^2 \Delta u = g(x)u^3 & in \ \Omega, \\ u = 0 & on \ \partial\Omega. \end{cases}$$

Let $v_n \in W^{2,1}(\Omega)$ *be any solution of*

$$-\|v\|^2 \Delta v = f_n(x)v^3 \quad in \ \Omega.$$

If $gu^2 \leq f_n v_n^2$ *for almost every* $x \in \Omega$ *and* n *is large enough, then* v_n *must change sign for* n *large enough.*

PROOF. Without loss of generality, we may assume that $\|u\| = \|v_n\| = 1$ for any $n \in \mathbb{N}$. Suppose the contrary, e.g. that $v_n > 0$ for n large enough. Then by the following Picone's identity:

$$|\nabla u|^2 - \nabla\left(\frac{u^2}{v} \right) \nabla v = |\nabla u|^2 + \frac{u^2}{v^2}|\nabla v|^2 - 2\frac{u}{v}\nabla u \nabla v \geq 0$$

and an easy calculation, we obtain, for n large enough,

$$0 \le \int_\Omega \left(|\nabla u|^2 + \frac{u^2}{v_n^2} |\nabla v_n|^2 - 2 \frac{u}{v_n} \nabla u \nabla v_n \right) dx$$
$$= \int_\Omega \left(g(x) u^2 - f_n(x) v_n^2 \right) u^2 \, dx \le 0.$$

Consequently, we have $u = c v_n$ for n large enough. Furthermore, we have $u = \pm v_n$ for n large enough. But this is impossible since $f_n \not\equiv g$ almost everywhere in Ω for n large enough. This accomplishes the proof. $\qquad\square$

PROOF OF THEOREM 1.3. In view of Lemmas 4.1, 4.2 and Remark 4.3, it is sufficient to show that $\mathcal{C}$ joins $(1/f_0, 0)$ to $(1/f_\infty, \infty)$. Let $(\lambda_n, u_n) \in \mathcal{C}$, where $u_n \not\equiv 0$ satisfies $\lambda_n + \|u_n\| \to +\infty$. Since $(0,0)$ is the only solution of (1.6) for $\lambda = 0$, we have $\mathcal{C} \cap (\{0\} \times X) = \emptyset$. It follows that $\lambda_n > 0$ for all $n \in \mathbb{N}$.

We claim that there exists a constant M such that $\lambda_n \in (0, M]$ for $n \in \mathbb{N}$ large enough. Suppose the contrary, that $\lim\limits_{n \to +\infty} \lambda_n = +\infty$. Since $(\lambda_n, u_n) \in \mathcal{C}$, it follows that

$$\Delta u_n + \frac{\lambda_n}{a + b\|u_n\|^2} \frac{f(x, u_n)}{u_n} u_n = 0 \quad \text{in } \Omega.$$

We consider two cases to deduce a contradiction.

Case 1. There exists a constant $c > 0$ such that $\|u_n\| \le c$ for n large enough. In this case, we have

$$\frac{1}{a} \ge \frac{1}{a + b\|u_n\|^2} \ge \frac{1}{a + bc^2}.$$

From (f1)–(f3), we can see that $f(x, u_n)/u_n \ge \sigma$ for some $\sigma > 0$ and almost every $x \in \Omega$ and all $n \in \mathbb{N}$. By (f2) and (f3), $F(x, u_n)/u_n \in L^{5/3}(\Omega)$, where F denotes the usual Nemitskiĭ operator associated with f. Applying Theorem 2.6 of [1], we have that u_n must change its sign in Ω, what contradicts Lemma 4.2.

Case 2. $\|u_n\| \to +\infty$ as $n \to +\infty$.

Now, consider

$$\|u_n\|^2 \Delta u_n + \lambda_n \frac{\|u_n\|^2}{a + b\|u_n\|^2} \frac{f(x, u_n)}{u_n^3} u_n^3 = 0 \quad \text{in } \Omega.$$

For any fixed $\varepsilon \in (0, 1/b)$, obviously, there exists $N_1 > 0$ such that, for any $n > N_1$,

$$(4.3) \qquad\qquad \frac{\|u_n\|^2}{a + b\|u_n\|^2} \ge \frac{1}{b} - \varepsilon.$$

By (f1)–(f3), there exists a constant $\varrho > 0$ such that $f(x, u_n)/u_n \ge \varrho$ for almost every $x \in \Omega$ and n large enough. Let

$$f_n(x) = \lambda_n \frac{\|u_n\|^2}{a + b\|u_n\|^2} \frac{f(x, u_n(x))}{u_n(x)} \frac{1}{u_n^2(x)}.$$

Then we have $\lim\limits_{n\to+\infty} f_n(x)u_n(x) = +\infty$ for almost every $x \in \Omega$. By (f1)–(f3) and [25, Theorem 8.12], we know that $u_n \in W^{2,2}(\Omega)$. So we have $u_n \in W^{2,1}(\Omega)$.

On the other hand, by Theorem 1.1 of [23], we have that for any nonempty compact subset $K \subseteq \Omega$, there exists a positive constant c_n such that $u_n \geq c_n$ almost everywhere in K. Then it is not difficult to show that $f_n \in L^3(K)$ for any fixed $n \in \mathbb{N}$. Applying Theorem 4.4 on K with $g(x) \equiv \mu_1$, we have that u_n must change its sign in K for n large enough. This is a contradiction.

Therefore, we get that $\|u_n\| \to +\infty$. Let $\varpi \colon \Omega \times \mathbb{R} \setminus \mathbb{R}^- \to \mathbb{R} \setminus \mathbb{R}^-$ be a Carathéodory function such that

$$f(x, s) = b\mu_1 f_\infty s^3 + \varpi(x, s)$$

with

$$(4.4) \qquad \lim_{s\to+\infty} \frac{\varpi(x, s)}{s^3} = 0 \quad \text{and} \quad \lim_{s\to 0^+} \frac{\varpi(x, s)}{s} = a\lambda_1 f_0$$

uniformly with respect to almost every $x \in \Omega$. Then (λ_n, u_n) satisfies

$$u_n = G\left(\frac{\lambda_n}{a + b\|u_n\|^2}(b\mu_1 f_\infty u_n^3 + \varpi(x, u_n))\right).$$

Dividing the above equation by $\|u_n\|$ and letting $\overline{u}_n = u_n/\|u_n\|$, we get

$$\overline{u}_n = G\left(\frac{\lambda_n\|u_n\|^2}{a + b\|u_n\|^2}\left(b\mu_1 f_\infty \overline{u}_n^3 + \frac{\varpi(x, u_n)}{\|u_n\|^3}\right)\right).$$

Next, we show

$$(4.5) \qquad \lim_{n\to+\infty} \frac{\varpi(x, u_n(x))}{\|u_n\|^3} = 0 \quad \text{in } L^{q'}(\Omega)$$

for some $q < 2^*$. Without loss of generality, we may assume that $q \geq 4$. Otherwise, we can consider $\widetilde{q} = cq$, $c > 1$ such that $\widetilde{q} \in [4, 2^*)$.

From (4.4), for any $\varepsilon > 0$, we can choose $\delta = \delta(\varepsilon)$ and $M = M(\delta)$ such that for almost every $x \in \Omega$ and any $n \in \mathbb{N}$, the following relations hold:

$$(4.6) \qquad\qquad |\varpi(x, s)| \leq M \qquad \text{for } |s| \leq \delta,$$

$$(4.7) \qquad\qquad |\varpi(x, s)| \leq \varepsilon|s|^3 \quad \text{for } |s| > \delta.$$

By (4.6) and (4.7), we can easily show that

$$\int_\Omega \left|\frac{\varpi(x, u_n(x))}{\|u_n\|^3}\right| dx \leq \frac{c}{\|u_n\|^{3q'}} + \varepsilon \int_\Omega |\overline{u}_n|^{3q'} dx.$$

It follows from $q \geq 4$ that $3q' < 2^*$. From the above inequality, $u_n \to +\infty$ in X, $\|\overline{u}_n\| = 1$, we get the desired result. It is not difficult to show that $\overline{u}_n^3$ is bounded in $L^{q'}(\Omega)$ and $\lambda_n\|u_n\|^2/(a + b\|u_n\|^2)$ is bounded in $\mathbb{R}$ for n large enough. By the compactness of G, we obtain that $-\|\overline{u}\|^2\Delta\overline{u} = \overline{\mu}\mu_1 f_\infty \overline{u}^3$, where $\overline{u} = \lim\limits_{n\to+\infty} \overline{u}_n$ and $\overline{\mu} = \lim\limits_{n\to+\infty} \lambda_n$, again choosing a subsequence and relabeling it if necessary.

It is clear that $\|\bar{u}\| = 1$ and $\bar{u} \in \overline{\mathcal{C}} \subseteq \mathcal{C}$ since $\mathcal{C}$ is closed in $\mathbb{R} \times X$. Therefore, Lemma 4.2 implies $\bar{u} > 0$. Theorem 1.2 shows that $\bar{\mu} = 1/f_\infty$. Therefore, $\mathcal{C}$ joins $(1/f_0, 0)$ to $(1/f_\infty, \infty)$. □

COROLLARY 4.5. *Assume that $N \leq 3$ and f satisfies* (f1)–(f2). *Then for*

$$\lambda \in (1/f_0, 1/f_\infty) \cup (1/f_\infty, 1/f_0),$$

problem (1.6) *possesses at least one positive solution. In particular, if $f_0 > 1$ (< 1) and $f_\infty < 1$ (> 1) then* (1.6) *possesses at least one positive solution with $\lambda = 1$.*

EXAMPLE 4.6. Let $a = b = 1$, $\lambda = 1$, $f(x, s) = \nu s + \mu s^3$. By Corollary 4.5, there is an unbounded component $\mathcal{C}$ in $\mathbb{R} \times H_0^1(\Omega)$ whose closure links $(\lambda_1/\nu, 0)$ to $(\mu_1/\mu, \infty)$. In particular, (1.6) has at least one positive solution when $\lambda = 1$ and $(\nu, \mu) \in \{(\nu, \mu) : 0 < \nu < \lambda_1, \ \mu > \mu_1\}$ or $(\nu, \mu) \in \{(\nu, \mu) : \nu > \lambda_1, \ 0 < \mu < \mu_1\}$.

This result can be seen as a complement of Theorem 3.4 of [29] where the authors considered (1.6) with $a = b = 1$, $\lambda = 1$, $f(x, s) = \nu s + \mu s^3$. They got the local existence result of positive solution when (ν, μ) belongs to a subset of $\{(\nu, \mu) : 0 < \nu < \lambda_1 + \varepsilon^*, \ \mu > \mu_1\}$.

5. Answer to an open problem

In this section, we consider problem (1.6) with $a = 0$, i.e. the following problem:

$$(5.1) \qquad \begin{cases} -b \displaystyle\int_\Omega |\nabla u|^2 \, dx \Delta u = \lambda f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \Omega. \end{cases}$$

To prove Theorem 1.8, we need the following topological lemma.

LEMMA 5.1. *Let $\mathcal{X}$ be a normal space and let $\{\mathcal{C}_n\}$ be a sequence of unbounded connected subsets of $\mathcal{X}$. Assume that*:

(a) *there exists $z^* \in \liminf_{n \to +\infty} \mathcal{C}_n$ with $\|z^*\| < +\infty$;*

(b) *for every $R > 0$, $\left(\bigcup_{n=1}^{+\infty} \mathcal{C}_n \right) \cap B_R$ is a relatively compact set of $\mathcal{X}$, where $B_R = \{x \in \mathcal{X} : \|x\| \leq R\}$.*

Then $\mathcal{D} = \limsup_{n \to +\infty} \mathcal{C}_n$ is unbounded, closed and connected.

PROOF. Let $X_R = \mathcal{X} \cap B_R$ for any $R > 0$. Then X_R is a metric subspace under the induced topology of $\mathcal{X}$. Let $A_n = \mathcal{C}_n \cap B_R$. Clearly, we have $\bigcup_{n=1}^{+\infty} A_n = \left(\bigcup_{n=1}^{+\infty} \mathcal{C}_n \right) \cap B_R$. So $\bigcup_{n=1}^{+\infty} A_n$ is relatively compact in X_R. Furthermore, $z^* \in \liminf_{n \to +\infty} \mathcal{C}_n$ implies that every neighbourhood $U(z^*)$ of z^* contains points of all

but a finite number of sets of $\{\mathcal{C}_n\}$. So there exists a positive integer N such that for $n > N$, $U(z^*) \cap \mathcal{C}_n \neq \emptyset$. Since $\|z^*\| < +\infty$, we can take $R > 0$ large enough such that $U(z^*) \subseteq B_R$. Thus, $U(z^*) \cap A_n = U(z^*) \cap \mathcal{C}_n \neq \emptyset$ for $n > N$. So we have $z^* \in \liminf\limits_{n \to +\infty} A_n$. By Theorem 9.1 of [39], it follows that $A = \limsup\limits_{n \to +\infty} A_n$ is connected in X_R.

We claim that $B := \left(\limsup\limits_{n \to +\infty} \mathcal{C}_n\right) \cap B_R = A$. For $x \in A$, any neighbourhood V in X_R of x contains points of infinitely many sets of $\{\mathcal{C}_n \cap B_R\}$. So there exist $x_{n_i} \in \mathcal{C}_{n_i} \cap B_R$ such that $x_{n_i} \to x$ as $i \to +\infty$. It follows that $x \in B_R$ and $x \in \limsup\limits_{n \to +\infty} \mathcal{C}_n$, i.e. $x \in B$. Conversely, if $x \in B$, any neighbourhood V in X_R of x contains a point z of $\limsup\limits_{n \to +\infty} \mathcal{C}_n$ and thus V, a neighborhood of z, contains points of infinitely many sets of $\mathcal{C}_n \cap B_R$. It follows that $x \in A$. Hence, B is connected. As R is arbitrary, we get that $\mathcal{D}$ is connected. From [39], we know that $\mathcal{D}$ is closed.

Next, we show that $\mathcal{D}$ is unbounded. Suppose on the contrary that $\mathcal{D}$ is bounded. It is easy to see that $\mathcal{D}$ is a compact set of $\mathcal{X}$ by (b) and the fact of $z^* \in \mathcal{D}$. Let U_δ be a δ-neighbourhood of $\mathcal{D}$. So we have that

(5.2) $\partial U_\delta \cap \mathcal{D} = \emptyset.$

By (a) and the connectedness of $\mathcal{C}_n$, there exists an integer $N_0 > 0$ such that for all $n > N_0$, $\mathcal{C}_n \cap \partial U_\delta \neq \emptyset$. Take $y_n \in \mathcal{C}_n \cap \partial U_\delta$, then $\{y_n : n > N_0\}$ is a relatively compact subset of $\mathcal{X}$, so there exist $y^* \in \partial U_\delta$ and a subsequence $\{y_{n_k}\}$ such that $y_{n_k} \to y^*$. The definition of superior limit shows that $y^* \in \mathcal{D}$. Therefore, $y^* \in \partial U_\delta \cap \mathcal{D}$. However, this contradicts (5.2). Therefore, $\mathcal{D}$ is unbounded. $\square$

In order to apply Lemma 5.1 to prove Theorem 1.8, we need to discuss the following Kirchhoff–Laplace operator $\Phi_k(u) := -b\|u\|^2 \Delta u$.

Denote $\Phi(u) := b\|u\|^4/4$. It is obvious that the functional Φ is continuously Gâteaux differentiable and its Gâteaux derivative at the point $u \in X$ is the functional $\Phi'(u) \in X^*$ given by

$$\langle \Phi'(u), v \rangle = b\|u\|^2 \int_\Omega \nabla u \cdot \nabla v \, dx.$$

Obviously, the Kirchhoff–Laplace operator is the derivative operator of Φ in the weak sense. We have the following properties of the derivative operator of Φ.

PROPOSITION 5.2. *Let $L = \Phi'$. Then*:

(a) $L\colon X \to X^*$ *is a continuous and strictly monotone operator*;
(b) L *is a mapping of type* (S_+), *i.e. if $u_n \rightharpoonup u$ in X and*

$$\varlimsup_{n \to +\infty} \langle L(u_n) - L(u), u_n - u \rangle \leq 0,$$

then $u_n \to u$ in X;

(c) $L(u)\colon X \to X^*$ *is a homeomorphism.*

PROOF. (a) Let $u_n \to u$ in X, i.e. $\lim\limits_{n \to +\infty} \|u_n - u\| = 0$. Then we can easily see that $\lim\limits_{n \to +\infty} \|u_n\| = \|u\|$. For any $v \in X$, by using of the Hölder inequality, we have

$$|\langle L(u_n) - L(u), v \rangle|$$

$$= \left| b\|u_n\|^2 \int_\Omega \nabla u_n \cdot \nabla v \, dx - b\|u\|^2 \int_\Omega \nabla u \cdot \nabla v \, dx \right|$$

$$= \left| b\|u_n\|^2 \int_\Omega (\nabla u_n - \nabla u) \cdot \nabla v \, dx + b(\|u_n\|^2 - \|u\|^2) \int_\Omega \nabla u \cdot \nabla v \, dx \right|$$

$$\leq b\|u_n\|^2 \|u_n - u\| \|v\| + b\|u_n\|^2 - \|u\|^2 \|u\| \|v\| \to 0 \quad \text{as } n \to +\infty.$$

It follows that L is a continuous operator.

For any $u, v \in X$ with $u \neq v$ in X, by the Cauchy inequality, we obtain

(5.3) $\langle L(u) - L(v), u - v \rangle$

$$= \langle L(u), u \rangle - \langle L(u), v \rangle - \langle L(v), u \rangle + \langle L(v), v \rangle$$

$$= b\|u\|^4 - b\|v\|^2 \int_\Omega \nabla u \cdot \nabla v \, dx - b\|u\|^2 \int_\Omega \nabla u \cdot \nabla v \, dx + b\|v\|^4$$

$$= b\left(\|u\|^4 + \|v\|^4 - (\|u\|^2 + \|v\|^2) \int_\Omega \nabla u \cdot \nabla v \, dx \right)$$

$$\geq b\left(\|u\|^4 + \|v\|^4 - \frac{(\|u\|^2 + \|v\|^2)^2}{2} \right) = b(\|u\|^2 - \|v\|^2)^2 \geq 0,$$

i.e. L is monotone. In fact L is strictly monotone. Indeed, if $\langle L(u) - L(v), u - v \rangle = 0$, then from (5.3) we have

$$\|u\| = \|v\|, \quad \nabla u \equiv \nabla v \quad \text{a.e. in } \Omega.$$

It follows that $\|u - v\| = 0$, i.e. $u \equiv v$, what contradicts $u \neq v$ in X. Therefore, $\langle L(u) - L(v), u - v \rangle > 0$. It follows that L is a strictly monotone operator in X.

(b) From (a), if $u_n \rightharpoonup u$ and $\varlimsup\limits_{n \to +\infty} \langle L(u_n) - L(u), u_n - u \rangle \leq 0$, then we have

$$\lim\limits_{n \to +\infty} \langle L(u_n) - L(u), u_n - u \rangle = 0.$$

In view of (5.3), we obtain that ∇u_n converges in measure to ∇u in Ω, so we get a subsequence (which we still denote by u_n) satisfying $\nabla u_n(x) \to \nabla u(x)$ for almost every $x \in \Omega$. Moreover, we also have

(5.4) $$\lim\limits_{n \to +\infty} \int_\Omega |\nabla u_n|^2 \, dx = \int_\Omega |\nabla u|^2 \, dx.$$

From (5.4) it follows that the integrals of the functions family $\{|\nabla u_n|^2\}$ possess absolute equi-continuity on Ω (see [34, Chapter 6, Section 3]). Since

$$|\nabla u_n - \nabla u|^2 \leq 2(|\nabla u_n|^2 + |\nabla u|^2),$$

the integrals of the family $\{|\nabla u_n - \nabla u|^2\}$ are also absolutely equi-continuous on Ω and therefore we have

$$\lim_{n \to +\infty} \int_\Omega |\nabla u_n - \nabla u|^2 \, dx = 0.$$

Therefore, $u_n \to u$, i.e. L is of type (S_+).

(c) It is clear that L is an injection. Since

$$\lim_{\|u\| \to +\infty} \frac{\langle L(u), u \rangle}{\|u\|} = \lim_{\|u\| \to +\infty} b\|u\|^3 = +\infty,$$

L is coercive, thus L is a surjection in view of the Minty–Browder theorem (see [40, Theorem 26A]). Hence L has an inverse mapping $L^{-1}\colon X^* \to X$. Therefore, the continuity of L^{-1} is sufficient to ensure L to be a homeomorphism.

If $f_n,\, f \in X^*$, $f_n \to f$, let $u_n = L^{-1}(f_n)$, $u = L^{-1}(f)$, then $L(u_n) = f_n$, $L(u) = f$. The coercive property of L implies that $\{u_n\}$ is bounded in X. We can assume that $u_{n_k} \rightharpoonup u_0$ in X. By $f_{n_k} \to f$ in X^*, we have

$$\lim_{k \to +\infty} \langle L(u_{n_k}) - L(u_0), u_{n_k} - u_0 \rangle = \lim_{n \to +\infty} \langle f_{n_k}, u_{n_k} - u_0 \rangle = 0.$$

Since L is of type (S_+), $u_{n_k} \to u_0$. Furthermore, the continuity of L implies that $L(u_0) = L(u)$. By injectivity of L, we have $u_0 = u$. So $u_{n_k} \to u$. We claim that $u_n \to u$ in X. Otherwise, there would exist a subsequence $\{u_{m_j}\}$ of $\{u_n\}$ in X and $\varepsilon_0 > 0$ such that for any $j \in \mathbb{N}$, we have $\|u_{m_j} - u\| \geq \varepsilon_0$. But reasoning as above, $\{u_{m_j}\}$ would contain a further subsequence $u_{m_{j_l}} \to u$ in X as $l \to +\infty$, which is a contradiction to $\|u_{m_{j_l}} - u\| \geq \varepsilon_0$. Therefore, L^{-1} is continuous. $\square$

Now, consider the following auxiliary problem:

$$(5.5) \qquad \begin{cases} -b\|u\|^2 \Delta u = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

LEMMA 5.3. *For any $f \in X^*$, problem (5.5) has a unique weak solution.*

PROOF. Define $\mathcal{F}(v) = \int_\Omega fv \, dx$ for any $v \in X$. We can easily see that $\mathcal{F}$ is a continuous linear functional on X. Since L is a homeomorphism, (5.5) has a unique solution. $\square$

Let us denote by $S(f)$ the unique weak solution of (5.5). Then $S\colon X^* \to X$ is a continuous operator. Since the embedding of $X \hookrightarrow L^q(\Omega)$ is compact for each $q \in (1, 2^*)$, $S\colon L^{q'}(\Omega) \to X$ is a completely operator.

For any $n \in \mathbb{N}$, we study the following auxiliary problem:

$$(5.6) \qquad \begin{cases} -\left(\dfrac{1}{n} + b \displaystyle\int_\Omega |\nabla u|^2 \, dx \right) \Delta u = \lambda f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \Omega. \end{cases}$$

Clearly, we can see that $f_0 = \widetilde{f_0}n$. Theorem 1.3 implies that there exists a sequence of unbounded continua $\mathcal{C}_n$ of positive solutions to problem (5.6) emanating from $(1/\widetilde{f_0}n, 0)$ and joining to $(1/f_\infty, +\infty)$.

PROOF OF THEOREM 1.8. Let $\mathcal{X} = \mathbb{R} \times X$ under the product topology. Clearly, $\mathcal{X}$ is a Banach space. Now, we verify the assumptions of Lemma 5.1. Taking $z_n = (1/\widetilde{f_0}n, 0)$ and $z^* = (0,0)$, we have that $z^* \in \liminf_{n \to +\infty} \mathcal{C}_n$. So (a) is satisfied.

By (f3) and (f4), $F(x, u(x)) \in L^{4/3}(\Omega)$. Now, we consider two possibilities to verify (b): (1) $n < +\infty$ and (2) $n = +\infty$. If case (1) occurs, the completely continuous property of G implies (b). If case (2) occurs, the compactness of S follows (b). By Lemma 5.1, $\mathcal{C} := \limsup_{n \to +\infty} \mathcal{C}_n$ is an unbounded component such that $(0,0) \in \mathcal{C}$ and $(1/f_\infty, +\infty) \in \mathcal{C}$. This completes the proof. $\square$

COROLLARY 5.4. *Assume that $N \leq 3$, Ω satisfies one of the conditions in Theorem 1.2 (c) and f satisfies (f1), (f3) and (f4). Then for $\lambda \in (0, 1/f_\infty)$, problem (5.1) possesses at least one positive solution. In particular, if $f_\infty < 1$ then (5.1) possesses at least one positive solution with $\lambda = 1$.*

REMARK 5.5. From Remark 1.7, we know that any weak solution of (5.6) belongs to $C^{1,\alpha}(\overline{\Omega})$ with $\alpha \in (0,1)$ under assumptions of (f1), (f3) and (f4). It follows that $u \in C^{1,\alpha}(\overline{\Omega})$ with $\alpha \in (0,1)$ for any $(\lambda, u) \in \mathcal{C}$ which is obtained in Theorem 1.8.

6. Future work

In future we plan:

(1) To study the case $f_0 \notin (0, +\infty)$ ($\widetilde{f_0} \notin (0, +\infty)$) or $f_\infty \notin (0, +\infty)$. This plan origins from Theorem 1.2 and the following special example:

$$(6.1) \qquad \begin{cases} -\left(a + b \int_\Omega |\nabla u|^2 \, dx\right)\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \Omega. \end{cases}$$

It is easy to see that $f_\infty = 0$. While, the positive solution pairs of (6.1) must be $(\lambda_1(a + b\|\varphi_1\|^2), \varphi_1)$, where φ_1 is the corresponding positive eigenfunction to λ_1. Thus $(\lambda_1, 0)$ is a bifurcation point of the set of positive solutions of (6.1) and $(\lambda_1(a + b\|\varphi_1\|^2), \varphi_1)$ is the corresponding unique unbounded branch.

(2) To study the unilateral global bifurcation phenomenon and the existence of one-sign and sign-changing solutions for (1.1).

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