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DECENTLY REGULAR STEADY SOLUTIONS TO THE COMPRESSIBLE NSAC SYSTEM

Šimon Axmann — Piotr Bogusław Mucha

Abstract. We aim at proving existence of weak solutions to the stationary compressible Navier–Stokes system coupled with the Allen–Cahn equation – NSAC system. The model is studied in a bounded three dimensional domain with slip boundary conditions for the momentum equations and the Neumann condition for the Allen–Cahn model. The main result establishes existence of weak solutions with bounded densities. The construction is possible assuming sufficiently large value of the heat capacity ratio γ ($p \sim \varrho^\gamma$). As a corollary we obtain weak solutions for a less restrictive case losing pointwise boundedness of the density.

1. Introduction

Phase transition phenomena are an important subject of applied mathematics. Complexity of physical nature of these processes makes us to choose different models in order to obtain the best description in a concrete studied case. Here we want to concentrate our attention on phenomena with fuzzy phase interfaces. We consider the process of melting or freezing, at the level of almost constant critical temperature. We want to control densities of phase constituents, and

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here we arrive at the Allen–Cahn equation:

$$\varrho\,\frac{\partial f}{\partial c}(\varrho, c) - \Delta c = \varrho\mu. \tag{1.1}$$

The equation above gives us control of one of the phase constituents in terms of the chemical potential μ. In the chosen model this equation is coupled with the compressible Navier–Stokes system, which represents a well established and frequently studied model describing the flows of viscous compressible single constituted fluids.

The mathematical analysis of coupled systems, the compressible Navier–Stokes type and the phase separation, is in its infancy, see [1], [11], [17], [18], [7], [6], although the mathematical theory of each of them separately is quite developed (see e.g. [9], [14], [30]). In this article, we study existence of steady weak solutions. The thermodynamically consistent derivation of the model under consideration, which is a variant of a model proposed by Blesgen [2], was presented by Heida, Málek and Rajagopal in [15]. It is represented by the following system of partial differential equations for three unknowns, density of the fluid ϱ, velocity field $\mathbf{v}$, and concentration of one selected constituent c:

$$\operatorname{div}(\varrho\mathbf{v}) = 0, \tag{1.2}$$

$$\operatorname{div}(\varrho\mathbf{v}\otimes\mathbf{v}) = \operatorname{div}\mathbb{T} + \varrho\mathbf{F}, \tag{1.3}$$

$$\operatorname{div}(\varrho c\mathbf{v}) = -\mu, \tag{1.4}$$

$$\varrho\mu = \varrho\,\frac{\partial f}{\partial c}(\varrho, c) - \Delta c, \tag{1.5}$$

where the stress tensor $\mathbb{T}$ is given by

$$\mathbb{T}(\nabla\mathbf{v}, \nabla c, \varrho, c) = \mathbb{S}(\nabla\mathbf{v}) - \bigg(\nabla c\otimes\nabla c - \frac{|\nabla c|^2}{2}\,\mathbb{I}\bigg) - p(\varrho, c)\mathbb{I},$$

with the thermodynamical pressure $p(\varrho, c) = \varrho^2\frac{\partial f}{\partial\varrho}$ and the viscous stress $\mathbb{S}$ satisfying the Stokes law for the Newtonian fluid

$$\mathbb{S}(\nabla\mathbf{v}) = \nu\left(\nabla\mathbf{v} + \nabla^T\mathbf{v}\right) + \eta\operatorname{div}\mathbf{v}\mathbb{I},$$

with viscosity coefficients $\nu > 0$, and $\eta \geq -2\nu/3$. The free energy is assumed to have the form with the so-called logarithmic potential

$$f(\varrho, c) = \frac{1}{\gamma-1}\varrho^{\gamma-1} + \big(a_1 c + a_2(1-c)\big)\log\varrho + c\log c + (1-c)\log(1-c) + b(c)$$

with some positive γ, $a_1, a_2 \geq 0$, b a smooth bounded function with $|b'(c)| \leq C$. Moreover, we assume without loss of generality that $a_1 \geq a_2$ and we denote, for the sake of simplicity, $a = a_1 - a_2$, $d = a_1$ and $L(c) = c\log c + (1-c)\log(1-c)$. The logarithmic terms related to the entropy of the system assure that $c \in [0, 1]$

almost everywhere, since

$$\begin{aligned}\frac{\partial f}{\partial c}(\varrho, c) &= \log c - \log(1-c) + (a_1 - a_2)\log\varrho + b'(c) = L'(c) + a\log\varrho + b'(c),\\ p(\varrho, c) &= \varrho^\gamma + \varrho(ac+d).\end{aligned}$$

In the literature one can find more general forms of constitutive relations [10], [17], we want to concentrate on the above particular case to avoid tedious technicalities.

The fluid is contained in a smooth bounded domain Ω, we supply the equations in the domain with boundary conditions

$$\mathbf{v}\cdot\mathbf{n} = 0 \quad \text{at } \partial\Omega, \tag{1.6}$$

$$\mathbf{n}\cdot\mathbb{T}(c,\nabla\mathbf{v})\cdot\boldsymbol{\tau}_n + k\mathbf{v}\cdot\boldsymbol{\tau}_n = 0 \quad \text{at } \partial\Omega, \tag{1.7}$$

$$\nabla c\cdot\mathbf{n} = 0 \quad \text{at } \partial\Omega, \tag{1.8}$$

where the parameter $k > 0$ represents the friction on the boundary ([1]), $\boldsymbol{\tau}_n$, $n = 1, 2$, are two linearly independent tangent vectors to $\partial\Omega$, and $\mathbf{n}$ denotes the normal vector.

The solutions are parametrized by means of the condition

$$\int_\Omega \varrho\, dx = M. \tag{1.9}$$

The fluid is driven by an external force $\mathbf{F} \in L^\infty(\Omega, \mathbb{R}^3)$.

We aim at constructing weak solutions to the NSAC system. Let us introduce the following definition of weak solutions to system (1.2)–(1.5).

DEFINITION 1.1. Let $M > 0$ be a given constant, $\gamma > 3$, we say that the quadruple $\varrho, \mathbf{v}, \mu, c$ is a weak solution to the steady Navier–Stokes–Allen–Cahn system, if $\varrho \in L^\gamma(\Omega)$, $\varrho \geq 0$ almost everywhere in Ω, $\mathbf{v} \in W^{1,2}(\Omega)$, $\mu \in L^2(\Omega)$, $c \in W^{1,2}(\Omega)$, $0 \leq c \leq 1$ almost everywhere on $\{\varrho > 0\}$, with $\varrho L'(c) \in L^{2\gamma/(\gamma+1)}(\Omega)$ and $\mathbf{v}\cdot\mathbf{n} = 0$ satisfied on $\partial\Omega$ in the sense of traces, and if the following holds true:

(a) The continuity equation is satisfied in the distributional sense, id est ([2]):

$$\operatorname{div}(\varrho\mathbf{v}) = 0, \quad \text{in } \mathcal{D}'(\mathbb{R}^3).$$

(b) For every $\boldsymbol{\varphi} \in C^\infty(\overline{\Omega}, \mathbb{R}^3)$, $\boldsymbol{\varphi}\cdot\mathbf{n} = 0$ at $\partial\Omega$

$$\int_\Omega (-\varrho(\mathbf{v}\otimes\mathbf{v}) : \nabla\boldsymbol{\varphi} + \mathbb{T}(\varrho, c, \nabla c, \nabla\mathbf{v}) : \nabla\boldsymbol{\varphi})\, dx + \int_{\partial\Omega} k(\mathbf{v}\cdot\boldsymbol{\tau})(\boldsymbol{\varphi}\cdot\boldsymbol{\tau})\, dS = \int_\Omega \varrho\mathbf{F}\cdot\boldsymbol{\varphi}\, dx.$$

([1]) For the slip boundary condition corresponding to the case $k = 0$, we need to assume that the domain Ω is not axially symmetric.

([2]) Note that we do not need the notion of renormalized continuity equation, since our density is regular enough.

(c) For every $\varphi \in C^\infty(\overline{\Omega}, \mathbb{R})$

$$\int_\Omega \varrho \mathbf{v} \cdot \nabla c \, \varphi \, dx = \int_\Omega -\mu\varphi \, dx$$

and

$$\int_\Omega \varrho\mu\varphi \, dx = \int_\Omega \varrho \frac{\partial f(\varrho, c)}{\partial c} \varphi + \nabla c \cdot \nabla \varphi \, dx \tag{1.10}$$

such that $\int_\Omega \varrho \, dx = M$.

The main result of the paper is the following.

Theorem 1.2. *Let $\gamma > 6$, $M > 0$ and $\mathbf{F} \in L^\infty(\Omega, \mathbb{R}^3)$. Then there exists at least one weak solution to system* (1.2)–(1.5) *such that $c \in [0,1]$ in Ω,*

$$\varrho \in L^\infty(\Omega), \quad \mathbf{v} \in W^{1,p}(\Omega, \mathbb{R}^3) \quad \textit{and} \quad c \in W^{2,p}(\Omega) \quad \textit{for } p < \infty. \tag{1.11}$$

As a corollary we obtain the following existence result, however without our decent regularity of the density.

Theorem 1.3. *If we assume only $\gamma > 3$ and $M > 0$, $\mathbf{F} \in L^\infty(\Omega, \mathbb{R}^3)$, then there exists at least one weak solution such that*

$$\varrho \in L^{3\gamma-6}(\Omega), \quad \mathbf{v} \in W^{1,2}(\Omega), \quad \nabla c \in L^{(6\gamma-12)/\gamma}(\Omega). \tag{1.12}$$

The existence of non-stationary solutions to the above mentioned system with the no-slip boundary condition for the velocity was shown by Feireisl et al. in [11]. Ding et al. studied in [7] the global existence of weak, strong, and even classical solutions in 1D with free energy approximated by a suitable bistable function, assuming no vacuum zones in the initial data. Further, Kotschote [17] considered a more advanced model where the extra stress tensor is multiplied by a density function. He showed, however only the local-in-time existence of strong solutions provided positiveness of density, including the thermal effects as well. The existence of travelling waves for this model was shown by Freistühler in [13]. To the best our knowledge there are no results concerning existence of weak solutions to the stationary system (1.2)–(1.5). Concerning the steady solutions to the compressible Navier–Stokes system, we refer to the pioneering work of Lions [20] and its further extensions [27], [29], [3], [12], [16], [28], [31]. We use a technique developed by Mucha and Pokorný (see [22], [23]), which was also modified for the Navier–Stokes–Fourier system [24]. Methods from [22] allow to obtain solutions with pointwise bounded densities, which seem to be the best possible regularity for weak solutions. Here we find an explanation for the solutions called *decently regular* solutions.

The paper is organized as follows. First we compute a formal a priori estimate, it allows us to determine the expected regularity of sought solutions.

In Section 3 we deal with the approximative system, we construct regular approximative solutions together with required estimates in the dependence on approximative parameters. Finally we analyze the limit, showing the strong convergence of approximative densities. The method is based on the fact that for approximative densities ϱ_ε we are able to find such m that

$$\lim_{\varepsilon\to 0} |\{\varrho_\varepsilon > m\}| = 0. \tag{1.13}$$

The proof of Theorem 1.3 is an easy application of nowadays well-understood techniques; it will be sketched in Section 5.

We shall say a word concerning limits of admissible γ's. Large γ's do not fix too well to physical theories. However we can look at such results in the following way. For the evolutionary compressible Navier–Stokes system [9] mathematical results started from nonphysical growth of the pressure function, but after some time the progress of the theory allowed to reach classical constitutive laws. Thus, there is a hope that it is possible to find a better a priori estimate to NSAC. Another viewpoint on this issue is that we treat it as an admissible modification of the standard models. Such interpretation keeps our result in applied mathematics.

Throughout the paper we try to follow the standard notation.

2. A priori estimates

Before the technical part of the proof, we will present a priori estimates on certain norms of the solution derived by purely heuristic approach. All generic constants, which may depend on the given data as well, will be denoted by C, its values can vary from line to line or even in the same formula.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with $C^{2+\zeta}$ boundary. Assume that all the above mentioned hypotheses are satisfied with $\gamma > 3$ and that $\varrho, \mathbf{v}, \mu, c$ is a sufficiently smooth solution satisfying* (1.2)–(1.5) *with boundary conditions* (1.6)–(1.8). *Then*

$$\begin{aligned}\|\mathbf{v}\|_{W^{1,2}(\Omega)} + \|\varrho\|_{L^{3\gamma-6}(\Omega)} + \|\nabla c\|_{L^{(6\gamma-12)/\gamma}(\Omega)}& \\ + \|\mu\|_{L^2(\Omega)} + \|\varrho L'(c)\|_{L^{(6\gamma-12)/(3\gamma-4)}(\Omega)} &\le C,\end{aligned}$$

where C may depend on the data, but is independent of the solution.

Remark 2.2. Note that $(6\gamma-12)/\gamma > 2$, for $\gamma > 3$, and that from the bound of the last norm on the left-hand side, we immediately deduce that $c \in [0,1]$ almost everywhere on the set $\{\varrho > 0\}$. Moreover, if $\gamma > 4$, then according to the Sobolev imbedding c is a continuous function and we conclude from the maximum principle for harmonic functions that in fact $c \in [0,1]$ almost everywhere in Ω. Indeed, our construction of solutions is done via approximation with large growth $\gamma > 6$, hence we keep the limits of c in the case of Theorem 1.3, too.

Proof. First, multiplying the momentum equation by the velocity field $\mathbf{v}$ yields (with usage of Korn's inequality and boundary condition)

$$-\int_\Omega p \operatorname{div} \mathbf{v}\, dx + C\int_\Omega |\nabla \mathbf{v}|^2\, dx \le \int_\Omega \varrho\mu\nabla c\cdot \mathbf{v} - \varrho\frac{\partial f}{\partial c}\nabla c\cdot \mathbf{v} + \varrho \mathbf{F}\cdot \mathbf{v}\, dx, \tag{2.1}$$

where we have used equation (1.5) as well. Further, we conclude from the continuity equation that

$$\int_\Omega p \operatorname{div} \mathbf{v} + \operatorname{div}(\varrho f \mathbf{v})\, dx = \int_\Omega \varrho\,\frac{\partial f}{\partial c}\nabla c\cdot \mathbf{v}\, dx,$$

and according to the constitutive equation for μ we get

$$\int_\Omega \mu^2\, dx = \int_\Omega -\varrho\mu\nabla c\cdot \mathbf{v}\, dx. \tag{2.2}$$

Thus, summing up (2.1)–(2.2) yields for $p > 6/5$

$$\int_\Omega |\nabla \mathbf{v}|^2 + \mu^2\, dx \le C\int_\Omega \varrho\mathbf{F}\cdot\mathbf{v}\, dx \le C\big(1 + \|\varrho\|^2_{L^{6/5}(\Omega)}\big) \le C\big(1 + \|\varrho\|^{p/(3(p-1))}_{L^p(\Omega)}\big).$$

In order to bound the density by means of the Bogovskii estimates, we need ∇c in $L^q(\Omega)$, $q > 2$. This can be deduced from the constitutive relation for μ. We state this for purpose of future references in the following lemma.

Lemma 2.3. *Suppose that $f(\varrho, c)$ is as above, and $\mu \in L^q(\Omega)$ and $\varrho \in L^p(\Omega)$ with $q \ge 2$, $p > 3$, $q < 3p/(p-3)$ satisfy equation* (1.5) *with boundary condition* (1.8). *Then*

$$\begin{aligned}\Big\|\varrho\frac{\partial f}{\partial c}\Big\|_{L^{pq/(p+q)}(\Omega)} &+ \|\varrho\mu\|_{L^{pq/(p+q)}(\Omega)} + \|\Delta c\|_{L^{pq/(p+q)}(\Omega)} \\ &+ \|\nabla c\|_{L^{3pq/(3p+3q-pq)}(\Omega)} \le C(\|\varrho\|_{L^p(\Omega)}(\|\mu\|_{L^q(\Omega)} + 1) + 1).\end{aligned}$$

Proof of Lemma 2.3. Let us test the corresponding equation by $F(L'(c))$, for the increasing function F with growth $\zeta F(\zeta) \sim |\zeta|^{\beta+1}$ with some $\beta > 0$, so (recall that L is convex)

$$\begin{aligned}\|\varrho^{1/(\beta+1)}L'(c)\|^{\beta+1}_{L^{\beta+1}(\Omega)} &\le \int_\Omega \big(|\nabla c|^2 F'(L'(c))L''(c) + \varrho L'(c)F(L'(c))\big)\, dx \\ &= \int_\Omega \varrho(\mu - a\log\varrho - b'(c))F(L'(c))\, dx \\ &\le C\int_\Omega \big(\varrho^{\beta/(\beta+1)}\,|L'(c)|^\beta\,(|\mu|\,\varrho^{1/(\beta+1)} + \varrho^{1/(\beta+1)}\,|\log\varrho|) + \varrho\,|L'(c)|^\beta\,|c|\big)\, dx \\ &\le C\bigg(\|\varrho^{1/(\beta+1)}L'(c)\|^\beta_{L^{\beta+1}}\bigg(\|\mu\|_{L^q}\|\varrho\|^{1/(\beta+1)}_{L^{q/(q-(\beta+1))}} + \bigg(\int_\Omega \varrho\,|\log\varrho|^{\beta+1}\, dx\bigg)^{1/(\beta+1)}\bigg) \\ &\quad + \|\varrho^{1/(\beta+1)}L'(c)\|^{\beta+1/2}_{L^{\beta+1}}\bigg)\end{aligned}$$

which yields

$$
\begin{aligned}
&\|\varrho^{1/(\beta+1)}L'(c)\|_{L^{\beta+1}(\Omega)}\\
&\qquad\le C\bigg(1+\|\mu\|_{L^q}\|\varrho\|^{1/(\beta+1)}_{L^{q/(q-(\beta+1))}}+\bigg(\int_\Omega \varrho\,|\log\varrho|^{\beta+1}\,dx\bigg)^{1/(\beta+1)}\bigg).
\end{aligned}
$$

Now we fix the exponent β such that $q/(q-(\beta+1))=p\in(3,\infty)$, hence

$$
\begin{aligned}
&\|\varrho^{p/(pq-q)}L'(c)\|_{L^{q(p-1)/p}(\Omega)}\\
&\qquad\le C\bigg(1+\|\mu\|_{L^q(\Omega)}\|\varrho\|^{p/(q(p-1))}_{L^p(\Omega)}+\bigg(\int_\Omega \varrho\,|\log\varrho|^{(p-1)q/p}\,dx\bigg)^{p/(q(p-1))}\bigg).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\|\varrho L'(c)\|_{L^{pq/(p+q)}(\Omega)}=\|\varrho^{p/(pq-q)}\varrho^{(pq-q-p)/(pq-q)}L'(c)\|_{L^{pq/(q+p)}(\Omega)}\\
&\le \|\varrho^{p/(pq-q)}L'(c)\|_{L^{q(p-1)/p}(\Omega)}\|\varrho^{(pq-q-p)/(pq-q)}\|_{L^{pq(q-1)/(pq-p-q)}(\Omega)}\\
&\le C\|\varrho\|^{(pq-p-q)/(q(p-1))}_{L^p(\Omega)}\\
&\quad\cdot\bigg(1+\|\mu\|_{L^q(\Omega)}\|\varrho\|^{p/(q(p-1))}_{L^p(\Omega)}+\bigg(\int_\Omega \varrho\,|\log\varrho|^{(p-1)q/p}\,dx\bigg)^{p/(q(p-1))}\bigg)\\
&\le C\|\varrho\|^{(pq-p-q)/(q(p-1))}_{L^p(\Omega)}\Big(1+\|\mu\|_{L^q(\Omega)}\|\varrho\|^{p/(q(p-1))}_{L^p(\Omega)}+\|\varrho\|^{p/(q(p-1))}_{L^p(\Omega)}\Big)\\
&\le C((\|\mu\|_{L^q(\Omega)}+1)\|\varrho\|_{L^p(\Omega)}+1).
\end{aligned}
$$

For the other terms we have

$$
\begin{aligned}
\|\varrho b'(c)\|_{L^{pq/(p+q)}(\Omega)}+\|\varrho\log\varrho\|_{L^{pq/(p+q)}(\Omega)}&\le C(1+\|\varrho\|_{L^p(\Omega)}),\\
\|\varrho\mu\|_{L^{pq/(p+q)}(\Omega)}&\le \|\varrho\|_{L^p(\Omega)}\|\mu\|_{L^q(\Omega)}.
\end{aligned}
$$

Using the classical elliptic estimates on the equation $-\Delta c=\varrho\mu-\varrho\frac{\partial f}{\partial c}$, together with the embedding $W^{1,pq/(p+q)}(\Omega)\hookrightarrow L^{3pq/(3p+3q-pq)}(\Omega)$, we get

$$
\begin{aligned}
&\|\nabla c\|_{L^{3pq/(3p+3q-pq)}(\Omega)}\\
&\quad\le\bigg\|\varrho\frac{\partial f}{\partial c}+\varrho\mu\bigg\|_{L^{qp/(q+p)}(\Omega)}+\|c\|_{L^s(\widetilde\Omega)}\le C((\|\mu\|_{L^q(\Omega)}+1)\|\varrho\|_{L^p(\Omega)}+1),
\end{aligned}
$$

where for $\widetilde\Omega$ we can take $\{\varrho>\varrho_0\}$ which has positive measure and on which $c\in[0,1]$ almost everywhere according to the logarithmic terms in $L'(c)$. This completes the proof of Lemma 2.3. □

Applying Lemma 2.3 with $q=2$ yields

$$
\|\nabla c\|_{L^{6p/(6+p)}(\Omega)}\le C\Big(1+\|\varrho\|^{(7p-6)/(6p-6)}_{L^p(\Omega)}\Big).
$$

Now, we are ready to perform the Bogovskii estimates, id est to test the momentum equation by (3)

$$
\mathbf{\Phi}=\mathcal{B}[\varrho^\alpha-\{\varrho^\alpha\}_\Omega],
$$

---

(3) We use notation $\{g\}_\Omega=(1/|\Omega|)\int_\Omega g\,dx$.

where $\alpha > 0$ will be specified later, and $\mathcal{B} \sim \operatorname{div}^{-1}$ is the Bogovskii operator. The theory implies that $\|\nabla\mathbf{\Phi}\|_{L^p} \leq C\|\varrho^\alpha\|_{L^p}$ with $\mathbf{\Phi}|_{\partial\Omega} = 0$ (see e.g. [4], [5], [27]). We obtain

$$\int_\Omega p(\varrho,c)\varrho^\alpha\,dx \leq \int_\Omega \big(-(\varrho\mathbf{v}\otimes\mathbf{v}) : \nabla\mathbf{\Phi} + \mathbb{S}(\nabla\mathbf{v}) : \nabla\mathbf{\Phi} - \varrho\mathbf{F}\cdot\mathbf{\Phi}\big)\,dx \\ + C\int_\Omega |\nabla c|^2\,|\nabla\mathbf{\Phi}|\,dx + \int_\Omega p(\varrho,c)\{\varrho^\alpha\}_\Omega\,dx.$$

The terms on the left-hand side of the inequality have sign and give the desired estimate of $\varrho^{\gamma+\alpha}$, if the right-hand side will be estimated, thus we set $p = \gamma+\alpha$. We will present only the most difficult and restrictive terms.

$$\begin{aligned}\int_\Omega |\nabla c|^2\,|\nabla\mathbf{\Phi}| &\leq \|\nabla c\|^2_{L^{6p/(6+p)}(\Omega)}\|\nabla\mathbf{\Phi}\|_{L^{3p/(2p-6)}(\Omega)} \\ &\leq C\Big(1+\|\varrho\|^{(7p-6)/(3p-3)}_{L^p(\Omega)}\|\varrho^\alpha\|_{L^{3p/(2p-6)}(\Omega)}\Big) \\ &\leq C\Big(1+\|\varrho\|^{(7p-6)/(3p-3)}_{L^p(\Omega)}\|\varrho\|^{(3\alpha p-2p+6)/(3(p-1))}_{L^p(\Omega)}\Big) \\ &\leq C\Big(1+\|\varrho\|^{(5p+3\alpha p)/(3p-3)}_{L^p(\Omega)}\Big),\end{aligned} \tag{2.3}$$

provided $3p\alpha/(2p-6) \leq p$, or equivalently

$$0 < \alpha \leq 2(\gamma-3), \tag{2.4}$$

which yields the restriction $\gamma > 3$. The condition $(\gamma+\alpha)(5+3\alpha)/(3(\gamma+\alpha)-3) < \gamma+\alpha$ is satisfied even for all $\gamma > 8/3$, so we can put this term to the left-hand side. Further, provided $\alpha < 2\gamma - 3$,

$$\begin{aligned}\int_\Omega (\varrho\mathbf{v}\otimes\mathbf{v}) : \nabla\mathbf{\Phi} &\leq \|\mathbf{v}\|^2_{L^6(\Omega)}\|\varrho\nabla\mathbf{\Phi}\|_{L^{3/2}(\Omega)} \\ &\leq C\Big(1+\|\varrho\|^{p/(3(p-1))}_{L^p(\Omega)}\Big)\|\varrho\|_{L^{3p/(2p-3\alpha)}(\Omega)}\|\nabla\mathbf{\Phi}\|_{L^{p/\alpha}(\Omega)} \\ &\leq C\Big(1+\|\varrho\|^{p/(3(p-1))+\alpha}_{L^p(\Omega)}\Big)\|\varrho\|^{(p+3\alpha)/(3(p-1))}_{L^p(\Omega)},\end{aligned} \tag{2.5}$$

where the condition $(2(\gamma+\alpha)+3\alpha)/(3(\gamma+\alpha-1))+\alpha < \gamma+\alpha$ is less restrictive since it requires only $5+3\alpha < 3(\gamma+\alpha) \Rightarrow \gamma > 5/3$. The other terms can be estimated similarly so we get, taking maximal possible value of $\alpha = 2(\gamma-3)$, that $\|\varrho\|_{L^{3\gamma-6}(\Omega)} \leq C$. Using this in the already derived estimates for $\mathbf{v}, c$ and μ yields the result of Lemma 2.1. $\square$

Now, we will show that for $\gamma > 6$ we can expect principally better regularity of solutions, this is connected to the fact that in this case we can take according to (2.4), $\alpha > \gamma$, so $p(\varrho,c) \in L^s(\Omega)$, for some $s > 2$.

Lemma 2.4. *For $\gamma > 6$ we have for solutions to* (1.2)–(1.8)

$$\|\mathbf{v}\|_{W^{1,p}(\Omega)} + \|\varrho\|_{L^\infty(\Omega)} + \|\nabla c\|_{L^\infty(\Omega)} + \|\mu\|_{L^\infty(\Omega)} + \|\varrho L'(c)\|_{L^\infty(\Omega)} \leq C_p,$$

for any $1 < p < +\infty$ *and* $c \in [0,1]$ *almost everywhere in* Ω.

Proof. First, since $\gamma > 6$ we observe a certain smoothing effect of (1.4) and (1.5). In what follows, we will repeatedly use Hölder's inequality in the third equation and Lemma 2.3. Indeed, since $\varrho \mathbf{v} \cdot \nabla c = -\mu$ and $\varrho \in L^{3\gamma-6}$, $\mathbf{v} \in L^6$ and $\nabla c \in L^{(6\gamma-12)/\gamma}$, we get $\mu \in L^{(3\gamma-6)/\gamma}$, $(1/(3\gamma-6) + 1/6 + \gamma/(6\gamma-12) > 1/2)$, and, applying Lemma 2.3, $\nabla c \in L^{\gamma-2}$, which can be again plugged into the third equation in order to get $\mu \in L^{(6\gamma-12)/(\gamma+6)}$, and again $\nabla c \in L^{(6\gamma-12)/(12-\gamma)}$, at least for $6 < \gamma < 12$, etc. This procedure can be repeated until $\nabla c \in L^\infty$, since there exists no reasonable solution to the following system of algebraic equations, where P corresponds to the expected integrability of ∇c, Q to μ and $3Q(\gamma-2)/(3\gamma+Q-6)$ corresponds to $\varrho\mu$:

$$\frac{1}{3\gamma-6} + \frac{1}{6} + \frac{1}{P} = \frac{1}{Q}, \qquad P = \frac{3Q(\gamma-2)}{3\gamma-6-Q\gamma-Q}. \tag{2.6}$$

So we get that $P \to +\infty$ and $Q \to (6\gamma-12)/\gamma > 3$, id est for any $\gamma > 6$ after finite number of such steps we have $\nabla c \in L^\infty(\Omega)$, $\mu \in L^{(6\gamma-12)/\gamma}(\Omega)$, and $\Delta c \in L^{(6\gamma-12)/(\gamma+2)}(\Omega)$. Thus, to summarize

$$\begin{aligned}\|\mathbf{v}\|_{W^{1,2}(\Omega)} + \|\varrho\|_{L^{3\gamma-6}(\Omega)} + \|\nabla c\|_{L^\infty(\Omega)} + \|\mu\|_{L^{(6\gamma-12)/\gamma}(\Omega)} \\ + \|\Delta c\|_{L^{(6\gamma-12)/(\gamma+2)}(\Omega)} + \|\varrho L'(c)\|_{L^{(6\gamma-12)/(\gamma+2)}(\Omega)} \le C.\end{aligned}$$

From the last norm we can deduce $c \in [0,1]$ almost everywhere on $\{\varrho > 0\}$. On the other hand, c is continuous in Ω and on the set $\{\varrho = 0\}$ it satisfies the Laplace equation, and therefore the maximum principle. Thus, $c \in [0,1]$ almost everywhere in Ω. (4)

Now, we will use the fact that we work with the slip boundary condition, and thus we can deduce from the momentum equation the following relation for vorticity ($\omega = \operatorname{curl} \mathbf{v}$), see [21]:

$$\begin{aligned} -\nu\Delta\omega &= -\operatorname{curl}(\varrho\mathbf{v}\cdot\nabla\mathbf{v}) - \operatorname{curl}(\Delta c\nabla c) + \operatorname{curl}(\varrho\mathbf{F}) && \text{in } \Omega, \\ \omega\cdot\boldsymbol{\tau}_1 &= -\left(2\chi_2 - \frac{k}{\nu}\right)\mathbf{v}\cdot\boldsymbol{\tau}_2 && \text{on } \partial\Omega, \\ \omega\cdot\boldsymbol{\tau}_2 &= \left(2\chi_1 - \frac{k}{\nu}\right)\mathbf{v}\cdot\boldsymbol{\tau}_2 && \text{on } \partial\Omega, \\ \operatorname{div}\omega &= 0 && \text{on } \partial\Omega, \end{aligned}$$

where χ_n are the curvatures related to the vectors $\boldsymbol{\tau}_n$. Note, that $\mathbf{n}\cdot(\nabla c\otimes\nabla c)\cdot\boldsymbol{\tau}_n = 0$ on $\partial\Omega$, since $\nabla c\cdot\mathbf{n}|_{\partial\Omega} = 0$.

(4) More precisely, since c is continuous, the set $U = \{c \notin [0,1]\}$ is open. Considering any ball $B(r,x_0) \subset U$, we get that $\varrho = 0$ almost everywhere in $B(r,x_0)$, hence c satisfies $\Delta c = 0$ in $B(r,x_0)$ and cannot reach neither maximum nor minimum within this ball.

Now, we would like to show that $\|\nabla\omega\|_{L^p(\Omega)} \le C$, for some $p > 1$. First, we have $\mathbf{v}\cdot\boldsymbol{\tau}|_{\partial\Omega} \in W^{1/2,2}(\partial\Omega)$, so we control ω on the boundary. Further, $\varrho\mathbf{F} \in L^{3\gamma-6}(\Omega)$, $\varrho\mathbf{v}\cdot\nabla\mathbf{v} \in L^p(\Omega)$, for any $p \in (1,(6\gamma-12)/(4\gamma-6))$, and finally

$$\|\Delta c\nabla c\|_{L^{(6\gamma-12)/(\gamma+2)}(\Omega)} \le \|\Delta c\|_{L^{(6\gamma-12)/(\gamma+2)}(\Omega)}\|\nabla c\|_{L^\infty(\Omega)} \le C.$$

According to the fact that $1 < (3\gamma-6)/(2\gamma-3) < (6\gamma-12)/(\gamma+2)$ even for all $\gamma > 8/3$, we get

$$\omega \in W^{1,(3\gamma-6)/(2\gamma-3)}(\Omega).$$

Now, we will proceed in the same manner as in [22], using the Helmholtz decomposition of the velocity field $\mathbf{v} = \operatorname{curl}\mathbf{A} + \nabla\phi$, define $G = -(2\nu+\eta)\Delta\phi + p(\varrho,c)$, observe

$$\nabla G = -\varrho\mathbf{v}\cdot\nabla\mathbf{v} + \nu\Delta\operatorname{curl}\mathbf{A} + (\varrho\mathbf{F} + \Delta c\nabla c)$$

and show that $G \in W^{1,(3\gamma-6)/(2\gamma-3)}(\Omega) \hookrightarrow L^{(3\gamma-6)/(\gamma-1)}(\Omega)$. This further yields

$$\varrho^\gamma \in L^{12/5}(\Omega), \quad \mathbf{v} \in W^{1,12/5}(\Omega) \hookrightarrow L^{12}(\Omega)$$

and after more iterations $\varrho \in L^\infty(\Omega)$, $\mathbf{v} \in W^{1,p}(\Omega)$, $c \in W^{1,p}(\Omega)$, for any $1 < p < \infty$ as well. $\square$

3. Approximation

In this section we define a problem approximating the original one and prove the existence of the corresponding solutions. We introduce $h = M/|\Omega|$, $\varepsilon > 0$, a smooth cut-off function

$$K(\varrho) = \begin{cases} 1 & \text{for } \varrho \le m-1, \\ \in (0,1) & \text{for } \varrho \in (m-1,m), \\ 0 & \text{for } \varrho \ge m, \end{cases}$$

a "regularized logarithm" which is a function $l_\varepsilon \in C^1([0,\infty))$ which is bounded from below by $\log(\sqrt[t]{\varepsilon}) - 1$ ($t > 1$ will be specified later) and

$$l_\varepsilon(s) = \begin{cases} \log(s) & \text{for } s \ge \sqrt[t]{\varepsilon}, \\ \text{convex, non-decreasing} & \text{for } s < \sqrt[t]{\varepsilon}, \end{cases}$$

with ([5])

$$\begin{aligned} &0 \le \sqrt[t]{\varepsilon}(2l_\varepsilon'(s) + sl_\varepsilon''(s)) \le C, \\ &0 \le s(2l_\varepsilon'(s) + sl_\varepsilon''(s)) \le C \quad \text{for a.a. } s \in [0,\infty), \end{aligned} \tag{3.1}$$

where C is independent of ε. Further we denote the approximated free energy by

$$f_\varepsilon(\varrho,c) = \Gamma(\varrho) + (ac+d)l_\varepsilon(\varrho) + L_\varepsilon(c) + b(c),$$

([5]) We can get such a function, e.g. by replacing the logarithm by a suitable affine function for small arguments.

where we define $L_\varepsilon(c) = \int_0^c l_\varepsilon(s) - l_\varepsilon(1-s)\,ds$ for $c \in [0,1]$, and then extend it to the whole $\mathbb{R}$ as a convex function with $\|L'_\varepsilon\|_{L^\infty(\Omega)} \le -C \log \sqrt[t]{\varepsilon}$, $\Gamma(\varrho) = (\varrho^{\gamma-1})/(\gamma-1)$ and approximated pressure

$$p_\varepsilon(\varrho, c) = P_b(\varrho) + (ac+d)\int_0^\varrho K(s)\partial_s\big(s^2 l'_\varepsilon(s)\big)\,ds,$$

where $P_b(\varrho) = \int_0^\varrho \gamma s^{\gamma-1}K(s)\,ds$. Our approximation problem then reads

$$\varepsilon\varrho + \operatorname{div}(K(\varrho)\varrho\mathbf{v}) = \varepsilon\Delta\varrho + \varepsilon K(\varrho)h, \tag{3.2}$$

$$\begin{aligned}\frac{1}{2}\operatorname{div}(K(\varrho)\varrho\mathbf{v}\otimes\mathbf{v}) + \frac{1}{2}K(\varrho)\varrho\mathbf{v}\cdot\nabla\mathbf{v} - \nu\Delta\mathbf{v} - (\nu+\eta)\nabla(\operatorname{div}\mathbf{v}) + \nabla p_\varepsilon(\varrho)&\\ = K(\varrho)\varrho\mathbf{F} + \operatorname{div}\left(\nabla c\otimes\nabla c - \frac{|\nabla c|^2}{2}\mathbb{I}\right) - a\nabla c\int_0^\varrho sK'(s)\,ds&,\end{aligned} \tag{3.3}$$

$$K(\varrho)\varrho\mathbf{v}\cdot\nabla c = -\,\mu, \tag{3.4}$$

$$K(\varrho)\varrho\mu = -\,\Delta c + K(\varrho)\varrho\frac{\partial f_\varepsilon}{\partial c} + \varepsilon\varrho K(\varrho)L'(c). \tag{3.5}$$

Moreover, we supply equations with the additional boundary condition

$$\nabla\varrho\cdot\mathbf{n} = \mathbf{0}. \tag{3.6}$$

Proposition 3.1. *Let $\varepsilon > 0$. Suppose that the assumptions of Theorem* 1.2 *are satisfied, then there exists at least one solution $\varrho_\varepsilon, \mathbf{v}_\varepsilon, \mu_\varepsilon, c_\varepsilon$ to system* (3.2)–(3.5) *with* (1.6)–(1.9), (3.6). *Moreover, we have with $1 < q < +\infty$ the following estimates independent of ε:*

$$\begin{aligned}\|\varrho_\varepsilon\|_{L^\infty(\Omega)} + \|\mathbf{v}_\varepsilon\|_{W^{1,q}(\Omega)} + \varepsilon\|\nabla\varrho\|^2_{L^2(\Omega)}&\\ + \varepsilon\|(K(\varrho)\varrho)^{1/2}L'(c)\|_{L^2(\Omega)} \le C(m)&,\end{aligned} \tag{3.7}$$

$$\begin{aligned}\|p_\varepsilon\|_{L^2(\Omega)} + \|\mu\|_{L^2(\Omega)} + \|\nabla c\|_{L^{(6\gamma)/(3+\gamma)}(\Omega)} + \|\mathbf{v}_\varepsilon\|_{W^{1,2}(\Omega)}&\\ + \|(K(\varrho)\varrho)^{(\gamma+1)/(2\gamma)}L'_\varepsilon(c)\|_{L^{(2\gamma)/(\gamma+1)}(\Omega)} \le C&.\end{aligned} \tag{3.8}$$

Proof. The existence of solutions for the approximative system will be deduced by means of the following variant of the Leray–Schauder fixed point theorem, see e.g. [8].

Theorem 3.2. *Let X be a Banach space and $\mathcal{T}$ a continuous and compact mapping $\mathcal{T}\colon X \mapsto X$, such that the possible fixed points $x = t\mathcal{T}x$, $0 \le t \le 1$, are bounded in X. Then $\mathcal{T}$ possesses a fixed point.*

We define spaces

$$\begin{aligned}\mathbf{M}_q &= \left\{\mathbf{w}\in(W^{2,q}(\Omega),\mathbb{R}^3),\ \mathbf{w}\cdot\mathbf{n} = \mathbf{0}\text{ on }\partial\Omega\right\},\\ M_q &= \left\{m\in W^{2,q}(\Omega), \nabla m\cdot\mathbf{n} = 0\text{ on }\partial\Omega\right\},\\ N_q &= \left\{z\in W^{3,q}(\Omega),\ \nabla z\cdot\mathbf{n} = 0\text{ on }\partial\Omega\right\},\end{aligned}$$

and search for $\varrho \in M_p$, $\mathbf{v} \in \mathbf{M}_p$, and $c \in N_p$, with $1 \le p < \infty$, see [27], [9] for similar considerations for the Navier–Stokes system. Let us first concentrate on the continuity equation.

Lemma 3.3. *The solution operator $\mathcal{S}_1(\mathbf{v}) = \varrho$ of the problem*

$$\begin{aligned} \varepsilon\varrho + \operatorname{div}(K(\varrho)\varrho\mathbf{v}) &= \varepsilon\Delta\varrho + \varepsilon K(\varrho)h && \text{in } \Omega, \\ \nabla\varrho \cdot \mathbf{n} &= \mathbf{0} && \text{on } \partial\Omega \end{aligned}$$

is for $p > 3$ a continuous compact operator from $\mathbf{M}_p$ to $W^{2,p}$. Moreover, $\varrho \ge 0$, $\int_\Omega \varrho\,dx \le M$, and

$$\|\varrho\|_{W^{2,p}} \le C(m,\varepsilon)(1 + \|\mathbf{v}\|_{W^{1,p}(\Omega)}).$$

Proof. The proof can be found in [24], see also [22], [19]. We recall here only the idea how to obtain the estimates. First, considering the subset $\{\varrho < 0\} \subset \Omega$, we get $\varrho \ge 0$ almost everywhere in Ω, then integrating the approximate continuity equation over Ω yields

$$\int_\Omega \varrho\,dx = h\int_\Omega K(\varrho)\,dx,$$

so $K(\varrho) = 1$ almost everywhere in Ω. Further, testing the continuity equation by ϱ yields

$$\begin{aligned} \varepsilon\int_\Omega &(\varrho^2 + |\nabla\varrho|^2)\,dx - \varepsilon\int_\Omega K(\varrho)\varrho h\,dx \\ &\le -\int_\Omega \varrho \operatorname{div}(K(\varrho)\varrho\mathbf{v})\,dx = \int_\Omega \mathbf{v}\cdot\nabla\varrho K(\varrho)\varrho\,dx \\ &= \int_\Omega \mathbf{v}\cdot\nabla\bigg(\int_0^\varrho K(s)s\,ds\bigg)\,dx \le C\|K(\varrho)\varrho^2\|_{L^2(\Omega)}\|\nabla\mathbf{v}\|_{L^2(\Omega)}, \end{aligned}$$

since the last term on the left-hand side can be easily bounded we get

$$\varepsilon\|\nabla\varrho\|^2_{L^2(\Omega)} \le C(1 + \|K(\varrho)\varrho^2\|_{L^2(\Omega)}\|\nabla\mathbf{v}\|_{L^2(\Omega)}). \tag{3.9}$$ □

Similarly, for the last two equations we have

Lemma 3.4. *The solution operator $\mathcal{S}_2(\mathbf{v}) = c$ of the problem*

$$\begin{aligned} -\mu &= K(\varrho)\varrho\nabla c\cdot\mathbf{v}, \\ -\Delta c + K(\varrho)\varrho\varepsilon L'(c) &= K(\varrho)\varrho\mu - K(\varrho)\varrho\frac{\partial f_\varepsilon}{\partial c} && \text{in } \Omega, \text{ where } \varrho = \mathcal{S}_1(\mathbf{v}), \\ \nabla c\cdot\mathbf{n} &= \mathbf{0} && \text{on } \partial\Omega \end{aligned}$$

is for $p > 3$ a well-defined compact operator from $\mathbf{M}_p$ to N_p. Moreover, $c \in [0,1]$, and

$$\|c\|_{W^{2,2q/(2+q)}} + \|\nabla c\|_{L^{6q/(6+q)}(\Omega)} \le C(\|\mu\|_{L^2(\Omega)}\|K(\varrho)\varrho\|_{L^q(\Omega)} + 1).$$

Proof. The proof is quite similar to the previous lemma, for the estimates we proceed analogously as in the proof of Lemma 2.3. For constructing the solution we use again the Schauder fixed point theorem. We consider for fixed $\varrho \in M_p$ the mapping $c \mapsto z$ defined as a solution operator to the problem

$$
\begin{aligned}
-\mu &= K(\varrho)\varrho\nabla c\cdot \mathbf{v},\\
-\Delta z + K(\varrho)\varrho\varepsilon L'(z) &= K(\varrho)\varrho\mu - K(\varrho)\varrho\frac{\partial f_\varepsilon}{\partial c}(\varrho, c), \qquad (3.10)\\
\nabla z\cdot \mathbf{n} &= \mathbf{0} \quad \text{on } \partial\Omega.
\end{aligned}
$$

The second equation is for $\varepsilon > 0$ strictly elliptic, furthermore, its right-hand side belongs to $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$, in particular, we deduce that $K(\varrho)\varrho L'(c) \in L^\infty(\Omega)$ and $\nabla c \in W^{1,\infty}(\Omega)$, hence we find $K(\varrho)\varrho L''(z) \in L^\infty(\Omega)$ since ϱ and z are continuous, put the corresponding term to the right-hand side, observe that in fact

$$
K(\varrho)\varrho\mu - K(\varrho)\varrho\frac{\partial f_\varepsilon}{\partial c} - K(\varrho)\varrho\varepsilon L'(z) \in W^{1,p}(\Omega),
$$

and so $z \in W^{3,p}(\Omega)$, and the corresponding mapping is compact by the same reasons as above. In order to get the desired estimate independent of ε, we test the equation

$$
-\Delta c + K(\varrho)\varrho(L'_\varepsilon(c) + \varepsilon L'(c)) = K(\varrho)\varrho\mu - K(\varrho)\varrho(b'(c) + al_\varepsilon(\varrho))
$$

by $F(L'_\varepsilon(c)+\varepsilon L'(c))$, where F is an increasing function such that $\zeta F(\zeta) \sim |\zeta|^{\beta+1}$, $|F| \sim |\zeta|^\beta$, for some $\beta > 0$. Note that $L'_\varepsilon(c)+\varepsilon L'(c)$ is a non-decreasing function, so we get

$$
\begin{aligned}
&\varepsilon\|(K(\varrho)\varrho)^{1/(\beta+1)}L'(c)\|^{\beta+1}_{L^{\beta+1}(\Omega)} + \|(K(\varrho)\varrho)^{1/(\beta+1)}L'_\varepsilon(c)\|^{\beta+1}_{L^{\beta+1}(\Omega)} \qquad (3.11)\\
&\quad\le \int_\Omega \big(|\nabla c|^2 F'(L'_\varepsilon(c) + \varepsilon L'(c))(L''_\varepsilon(c) + \varepsilon L''(c))\\
&\qquad + K(\varrho)\varrho(L'_\varepsilon(c) + \varepsilon L'(c))F(L'_\varepsilon(c) + \varepsilon L'(c))\big)\,dx\\
&\quad\le \left|\int_\Omega K(\varrho)\varrho\mu F(L'_\varepsilon(c) + \varepsilon L'(c))\,dx\right|\\
&\qquad + \left|\int_\Omega K(\varrho)\varrho(b'(c) + al_\varepsilon(\varrho))F(L'_\varepsilon(c) + \varepsilon L'(c))\,dx\right|,
\end{aligned}
$$

hence

$$
\begin{aligned}
&\varepsilon\|(K(\varrho)\varrho)^{1/(\beta+1)}L'(c)\|_{L^{\beta+1}(\Omega)} + \|(K(\varrho)\varrho)^{1/(\beta+1)}L'_\varepsilon(c)\|_{L^{\beta+1}(\Omega)} \qquad (3.12)\\
&\qquad\le C(\|\mu\|_{L^p(\Omega)}\|K(\varrho)\varrho\|_{L^{p/(p-(\beta+1))}(\Omega)} + 1)
\end{aligned}
$$

and, using the classical elliptic estimates, we obtain as in Lemma 2.3

$$
\|c\|_{W^{2,pq/(p+q)}} + \|\nabla c\|_{L^{3pq/(3p+3q-pq)}(\Omega)} \le C(\|\mu\|_{L^p(\Omega)}\|K(\varrho)\varrho\|_{L^q(\Omega)} + 1),
$$

especially for $p=2$,

$$\|c\|_{W^{2,2q/(2+q)}} + \|\nabla c\|_{L^{6q/(6+q)}(\Omega)} \le C(\|\mu\|_{L^2(\Omega)}\|K(\varrho)\varrho\|_{L^q(\Omega)} + 1).$$

The issue of existence of solutions to (3.10) requires some comments. The function $L'(\,\cdot\,)$ is singular and it keeps the value of z in the interval $[0,1]$. Thus, we approximate (3.10) by its regularization substituting $L'(\cdot)$ by $L'_\delta(\,\cdot\,)$ which is obtained in the same manner as for f_ε in (3.1). The estimates are the same, there is no problem to pass to the limit $\delta\to 0$, since we control the second derivatives of z. Hence, we ensure that $z\in[0,1]$ as well. □

Finally, we define the solution operator $\mathcal{T}\colon \mathbf{M}_p \to \mathbf{M}_p$, $\mathcal{T}(\mathbf{v}) = \mathbf{w}$ of the problem

$$\begin{aligned}(3.13)\qquad -\nu\Delta\mathbf{w} - (\nu+\eta)\nabla(\operatorname{div}\mathbf{w}) &= -\frac{1}{2}\operatorname{div}(K(\varrho)\varrho\mathbf{v}\otimes\mathbf{v}) - \frac{1}{2}K(\varrho)\varrho\mathbf{v}\cdot\nabla\mathbf{v}\\ &- \nabla P_\varepsilon(\varrho) + K(\varrho)\varrho\mathbf{F} + \operatorname{div}\left(\nabla c\otimes\nabla c - \frac{|\nabla c|^2}{2}\,\mathbb{I}\right) - a\nabla c\int_0^\varrho sK'(s)\,ds,\end{aligned}$$

where $\varrho = \mathcal{S}_1(\mathbf{v})$ and $c=\mathcal{S}_2(\mathbf{v})$, equipped with the boundary condition

$$\mathbf{w}\cdot\mathbf{n} = 0,\quad \mathbf{n}\cdot\mathbb{T}(\mathbf{w})\cdot\boldsymbol{\tau}_{\boldsymbol{n}} + k\mathbf{w}\cdot\boldsymbol{\tau} = 0,\quad \text{on } \partial\Omega.$$

Lemma 3.5. *$\mathcal{T}$ is a continuous and compact operator from $\mathbf{M}_p$ to $\mathbf{M}_p$ for $p>3$.*

Proof (Idea of the proof). It is again a strictly elliptic system with the right-hand side that belongs at least to $L^p(\Omega)$ and it contains at most first order derivatives of $\varrho,\mathbf{v}$ and at most second order derivatives of c, see e.g. [27] for similar considerations. □

Finally, we will verify that all possible solutions of $t\mathcal{T}(\mathbf{v}) = \mathbf{v}$, for $t\in[0,1]$, are bounded in $\mathbf{M}_p$ independently of t. Testing the momentum equation by $\mathbf{v}$ yields (with usage of the last equation tested by $\mathbf{v}\cdot\nabla c$)

$$\begin{aligned}(3.14)\qquad t\int_\Omega \mathbf{v}\cdot\nabla p_\varepsilon(\varrho,c)\,dx &+ \int_\Omega \big(\nu\,|\nabla\mathbf{v}|^2 + (\nu+\eta)\,|\operatorname{div}\mathbf{v}|^2\big)\,dx\\ &+ \int_{\partial\Omega} k(|\mathbf{v}_1\cdot\boldsymbol{\tau}_1|^2 + |\mathbf{v}_2\cdot\boldsymbol{\tau}_2|^2)\,dS\\ &\le t\int_\Omega K(\varrho)\varrho\mathbf{F}\cdot\mathbf{v}\,dx + t\int_\Omega\left(K(\varrho)\varrho\mu\nabla c\cdot\mathbf{v} - K(\varrho)\varrho\frac{\partial f_\varepsilon}{\partial c}\nabla c\cdot\mathbf{v}\right)dx\\ &+ t\int_\Omega\left(-\varepsilon\varrho K(\varrho)L'(c)\nabla c\cdot\mathbf{v} - a\nabla c\cdot\mathbf{v}\int_0^\varrho sK'(s)\,ds\right)dx,\end{aligned}$$

from the equation for concentration we get

$$\int_\Omega \mu^2\,dx = -\int_\Omega K(\varrho)\varrho\mu\mathbf{v}\cdot\nabla c\,dx.$$

Next,

$$\begin{aligned}(3.15)\quad &\int_\Omega \mathbf{v}\cdot\nabla p_\varepsilon(\varrho,c)\,dx\\ &= \int_\Omega \mathbf{v}\cdot\nabla\varrho\Big(\gamma\varrho^{\gamma-1}K(\varrho)+(ac+d)K(\varrho)\frac{d}{d\varrho}(\varrho^2 l_\varepsilon'(\varrho))\Big)\,dx\\ &\qquad + \int_\Omega a\nabla c\cdot\mathbf{v}\int_0^\varrho K(s)\frac{d}{ds}(s^2 l_\varepsilon'(s))\,ds\;dx\end{aligned}$$

and

$$\begin{aligned}(3.16)\quad &\int_\Omega K(\varrho)\varrho\mathbf{v}\cdot\nabla c\frac{\partial f_\varepsilon}{\partial c}(\varrho,c)\,dx\\ &= \int_\Omega K(\varrho)\varrho\mathbf{v}\cdot\nabla\big(f_\varepsilon+\varrho\frac{\partial f_\varepsilon}{\partial\varrho}\big)\,dx - \int_\Omega K(\varrho)\varrho^2 a l_\varepsilon'(\varrho)\mathbf{v}\cdot\nabla c\,dx\\ &\quad - \int_\Omega K(\varrho)\varrho\mathbf{v}\cdot\nabla\varrho\\ &\quad \cdot\Big(2\Gamma'(\varrho)+\varrho\Gamma''(\varrho)+(ac+d)l_\varepsilon'(\varrho)+(ac+d)\frac{d}{d\varrho}(\varrho l_\varepsilon'(\varrho))\Big)\\ &= \int_\Omega \operatorname{div}\Big(K(\varrho)\varrho\mathbf{v}\Big(f_\varepsilon+\varrho\frac{\partial f_\varepsilon}{\partial\varrho}\Big)\Big)\,dx\\ &\quad - \int_\Omega \operatorname{div}(K(\varrho)\varrho\mathbf{v})\Big(f_\varepsilon+\varrho\frac{\partial f_\varepsilon}{\partial\varrho}\Big)\,dx\\ &\quad - \int_\Omega a\nabla c\cdot\mathbf{v}\int_0^\varrho \frac{d}{ds}\big(K(s)s^2 l_\varepsilon'(s)\big)\,ds\,dx\\ &\quad + \int_\Omega K(\varrho)\mathbf{v}\cdot\nabla\varrho\frac{d}{d\varrho}(\varrho^2\Gamma'(\varrho)+\varrho^2 l_\varepsilon'(\varrho)(ac+d))\,dx.\end{aligned}$$

The first term on the right-hand side of (3.16) can be eliminated by boundary conditions, for the second one we will use the continuity equation (tested by $\big(f_\varepsilon+\varrho\frac{\partial f_\varepsilon}{\partial\varrho}\big)$). The third one cancels with the last terms of (3.14) and (3.15), and the last term is exactly the same as the main part of (3.15). Thus, summing up the resulting inequalities with appropriate powers of t, we get

$$\begin{aligned}(3.17)\quad &\|\mathbf{v}\|^2_{W^{1,2}(\Omega)}+t\|\mu\|^2_{L^2(\Omega)}+t\int_\Omega\varepsilon\varrho\Big(f_\varepsilon+\varrho\frac{\partial f_\varepsilon}{\partial\varrho}\Big)\,dx\\ &\le t\int_\Omega K(\varrho)\varrho\mathbf{F}\cdot\mathbf{v}\,dx - t\int_{\Omega_K}\varepsilon\varrho K(\varrho)L'(c)\mathbf{v}\cdot\nabla c\,dx\\ &\qquad + t\int_\Omega(\varepsilon\Delta\varrho+\varepsilon K(\varrho)h)\Big(f_\varepsilon+\varrho\,\frac{\partial f_\varepsilon}{\partial\varrho}\Big)\,dx,\end{aligned}$$

where we have introduced the notation $\Omega_K=\{K(\varrho)\varrho>0\}\cap\Omega$.

We estimate the ε-dependent terms

$$\begin{aligned}
-\varepsilon \int_{\Omega_K} K(\varrho)\varrho L'(c)\nabla c\cdot \mathbf{v}\,dx &= -\int_{\Omega_K} \varepsilon K(\varrho)\varrho\mathbf{v}\cdot\nabla\big(L(c)\big)\,dx \\
&= \int_{\partial\Omega_K} \varepsilon\varrho K(\varrho)\mathbf{v}\cdot\mathbf{n}L(c)\,dS + \int_{\Omega_K} \varepsilon\,\mathrm{div}(K(\varrho)\varrho\mathbf{v})L(c)\,dx \\
&\le \sqrt{\varepsilon}\|\nabla\varrho\|_{L^2(\Omega)}\|\varrho K'(\varrho)+K(\varrho)\|_{L^3(\Omega_K)}\|\mathbf{v}\|_{L^6(\Omega)}\sqrt[4]{\varepsilon}\|L(c)\|_{L^\infty(\Omega_K)}\sqrt[4]{\varepsilon} \\
&\quad + \|K(\varrho)\varrho\|_{L^2(\Omega)}\|\nabla\mathbf{v}\|_{L^2(\Omega)}\sqrt[4]{\varepsilon}\|L(c)\|_{L^\infty(\Omega_K)}\varepsilon^{3/4}
\end{aligned}$$

and we can use $\sqrt[4]{\varepsilon}\|L(c)\|_{L^\infty(\Omega_K)} \le C(\sqrt[4]{\varepsilon}\|(K(\varrho)\varrho)^{1/(\beta+1)}L'(c)\|^{1/4}_{L^{\beta+1}(\Omega)}+1)$ with estimate (3.12).

Concerning the term $\varepsilon K(\varrho)\big(f_\varepsilon + \varrho\frac{\partial f_\varepsilon}{\partial\varrho}\big)$, we have three parts. First, $\Gamma_\varepsilon(\varrho)+\varrho\Gamma'_\varepsilon(\varrho)$, which can be bounded by the corresponding term on the left-hand side, see $\varepsilon(\varrho f_\varepsilon)$. Second, $(ac+d)(1+l_\varepsilon(\varrho))$, which has good sign on the set $\{\varrho\le \mathrm{e}^{-1}\}$ and is bounded by $c(1+\varrho)K(\varrho)$ on $\{\varrho > \mathrm{e}^{-1}\}$. The third term $\varepsilon(L_\varepsilon(c)+b(c))$ can be bounded according to our definition of l_ε. For $\Delta\varrho\big(f_\varepsilon+\varrho\frac{\partial f_\varepsilon}{\partial\varrho}\big)$ we have

$$\begin{aligned}
&\int_\Omega \varepsilon\Delta\varrho\bigg(f_\varepsilon+\varrho\frac{\partial f_\varepsilon}{\partial\varrho}\bigg)\,dx \\
&\qquad = -\varepsilon\int_\Omega |\nabla\varrho|^2\left(\gamma\varrho^{\gamma-2}+(ac+d)(\varrho l''_\varepsilon(\varrho)+2l'_\varepsilon(\varrho))\right)dx \\
&\qquad\qquad - \varepsilon\int_\Omega \nabla\varrho\cdot\nabla c(l'_\varepsilon(\varrho)\varrho+b'(c)+L'_\varepsilon(c))\,dx.
\end{aligned}\tag{3.18}$$

The first integral has a good sign. Indeed, $\varrho^{\gamma-2}$ as well as $\varrho l''_\varepsilon(\varrho)+2l'_\varepsilon(\varrho)$ are non-negative (for large arguments we have $\varrho l''_\varepsilon(\varrho)+2l'_\varepsilon(\varrho)=1/\varrho\ge 0$, in the other case the conclusion is obtained from the fact that for small arguments l_ε is increasing and convex, see (3.1)). Note also that we have (3.9) and for $\varepsilon>0$ the approximated version of (1.5) yields $c\in[0,1]$ as soon as we control $\|K(\varrho)\varrho\|_{L^{2\gamma}(\Omega)}\|\mu\|_{L^2(\Omega)}$. For the rest we get, denoting $V_\varepsilon(c)=b'(c)+L'_\varepsilon(c)$ and $U=\|\mathbf{v}\|^2_{W^{1,2}(\Omega)}+t\|\mu\|^2_{L^2(\Omega)}$, that

$$\begin{aligned}
t&\int_\Omega \varepsilon\,|\nabla\varrho|\,|\nabla c|\,|l'_\varepsilon(\varrho)\varrho+V_\varepsilon(c)|\,dx \\
&\le t\sqrt[4]{\varepsilon}\|\sqrt{\varepsilon}\nabla\varrho\|_{L^2(\Omega)}\|\nabla c\|_{L^2(\Omega)}\sqrt[4]{\varepsilon}(\|l'_\varepsilon(\varrho)\varrho+V_\varepsilon(c)\|_{L^\infty(\Omega)}) \\
&\le t\sqrt[4]{\varepsilon}C\|K(\varrho)\varrho^2\|^{1/2}_{L^2(\Omega)}\|\nabla\mathbf{v}\|^{1/2}_{L^2(\Omega)}\|K(\varrho)\varrho\|_{L^\gamma(\Omega)}\|\mu\|_{L^2(\Omega)} \\
&\le t\sqrt[4]{\varepsilon}C\|K(\varrho)\varrho^2\|^{1/2}_{L^2(\Omega)}\|K(\varrho)\varrho\|_{L^\gamma(\Omega)}U^{3/4},
\end{aligned}$$

where we have used the choice of l_ε that guarantees

$$\sqrt[4]{\varepsilon}\big(\|l'_\varepsilon(\varrho)\varrho\|_{L^\infty(\Omega)}+\|V_\varepsilon(c)\|_{L^\infty(\Omega)}\big)\le C$$

independently of ε. Thus,

$$U \le C\big(1 + \|K(\varrho)\varrho\|^2_{L^{6/5}(\Omega)} + \varepsilon\|K(\varrho)\varrho\|_{L^2(\Omega)}U^{3/4} + \sqrt[4]{\varepsilon}\|K(\varrho)\varrho^2\|^{1/2}_{L^2(\Omega)}\|K(\varrho)\varrho\|_{L^3(\Omega)}U^{3/4}\big).$$

However $\|K(\varrho)\varrho\|_{L^6(\Omega)}$ is definitely finite, so finally,

$$U \le C\big(1+\|K(\varrho)\varrho\|^2_{L^{6/5}(\Omega)}\big), \quad \|\nabla c\|_{L^{6q/(6+q)}(\Omega)} \le C\big(\|K(\varrho)\varrho\|_{L^q(\Omega)}\|\mu\|_{L^2(\Omega)}+1\big).$$

The Bogovskii estimates go along exactly the same lines as in the a priori approach as soon as we observe that

$$\begin{aligned}\int_\Omega a\mathbf{v}\cdot\nabla c\int_0^\varrho tK'(t)\,dt\mathbf{\Phi}\,dx &\le C\|\mathbf{v}\|_{L^6(\Omega)}\|\nabla c\|_{L^{6\gamma/(3+\gamma)}(\Omega)}\|K(\varrho)\varrho\|_{L^{6\gamma/(3\gamma-3)}(\Omega)}\|\mathbf{\Phi}\|_{L^6(\Omega)}\\ &\le \|K(\varrho)\varrho\|^{1+\gamma+2\gamma/(3(2\gamma-1))}_{L^{2\gamma}(\Omega)},\end{aligned}$$

so $\|\varrho_\varepsilon\|_{L^{3\gamma-6}(\Omega)} \le C$, independently of m and ε. Furthermore, by the same iteration process applied on the last two equations of (1.2)–(1.5) as in the a priori approach, we can deduce that

$$\|\mu_\varepsilon\|_{L^{6\gamma/(\gamma+3)}(\Omega)} + \|\nabla c_\varepsilon\|_{L^\infty(\Omega)} + \|\Delta c_\varepsilon\|_{L^{6\gamma/(6+\gamma)}(\Omega)} \le C. \tag{3.19}$$

Having these estimates in hands and noting that $\|K(\varrho)\varrho\|_{L^\infty(\Omega)} \le C(m)$, we can apply the elliptic theory on equation (3.13) and get the estimate of fixed points of $\mathcal{T}$ in $\mathbf{M}_p$. This completes the proof of Proposition 3.1. □

4. Artificial diffusion limit

This section of the paper is dedicated to the proof of convergence of the constructed approximative solutions to a weak solution to the original system. As usual, the key part is related to the proof of the strong convergence of densities. Thanks to estimates (3.7), (3.19), we extract from the family $(\varrho_\varepsilon, \mathbf{v}_\varepsilon, \mu_\varepsilon, c_\varepsilon)$ subsequences which converge in the corresponding spaces as $\varepsilon \to 0+$. Namely ([6]),

$$\begin{aligned}
\mathbf{v}_\varepsilon &\rightharpoonup \mathbf{v} && \text{in } W^{1,q}(\Omega),\\
\mathbf{v}_\varepsilon &\to \mathbf{v} && \text{in } L^\infty(\Omega),\\
\varrho_\varepsilon &\rightharpoonup^* \varrho && \text{in } L^\infty(\Omega),\\
p_\varepsilon(\varrho,c) &\rightharpoonup^* \overline{p(\varrho,c)} && \text{in } L^\infty(\Omega),\\
K(\varrho_\varepsilon)\varrho_\varepsilon &\rightharpoonup^* \overline{K(\varrho)\varrho} && \text{in } L^\infty(\Omega),\\
K(\varrho_\varepsilon) &\rightharpoonup^* \overline{K(\varrho)} && \text{in } L^\infty(\Omega),\\
\int_0^{\varrho_\varepsilon} tK'(t)\,dt &\rightharpoonup^* \overline{\int_0^\varrho tK'(t)\,dt} && \text{in } L^\infty(\Omega),
\end{aligned}$$

([6]) We denote a weak limit of nonlinear expressions $\{w_\varepsilon(\varrho_\varepsilon, \mathbf{v}_\varepsilon, \mu_\varepsilon, c_\varepsilon)\}$ by $\overline{w(\varrho,\mathbf{v},\mu,c)}$.

$$
\begin{aligned}
h_\varepsilon(\varrho_\varepsilon) &\rightharpoonup^* \overline{h_\varepsilon(\varrho)} && \text{in } L^\infty(\Omega),\\
c_\varepsilon &\rightharpoonup c && \text{in } W^{2,2}(\Omega),\\
\nabla c_\varepsilon &\to \nabla c && \text{in } L^6(\Omega),\\
\varepsilon K(\varrho_\varepsilon)\varrho_\varepsilon L'(c_\varepsilon) &\rightharpoonup \overline{\varepsilon K(\varrho)\varrho} L'(c) && \text{in } L^2(\Omega),\\
K(\varrho_\varepsilon)\varrho_\varepsilon L'_\varepsilon(c_\varepsilon) &\rightharpoonup \overline{K(\varrho)\varrho} L'(c) && \text{in } L^{12/7}(\Omega),\\
\mu_\varepsilon &\rightharpoonup \mu && \text{in } L^2(\Omega),\\
K(\varrho_\varepsilon)\varrho_\varepsilon\mu_\varepsilon &\rightharpoonup \overline{K(\varrho)\varrho\mu} && \text{in } L^2(\Omega).
\end{aligned}
$$

Thus, we get

$$
\begin{gathered}
\operatorname{div}(\overline{K(\varrho)\varrho}\mathbf{v}) = 0,\\
\frac{1}{2}\operatorname{div}(\overline{K(\varrho)\varrho}\mathbf{v}\otimes\mathbf{v}) + \frac{1}{2}\overline{K(\varrho)\varrho}\mathbf{v}\cdot\nabla\mathbf{v} - \nu\Delta\mathbf{v} - (\nu+\eta)\nabla(\operatorname{div}\mathbf{v}) + \nabla\overline{p_\varepsilon(\varrho,c)}\\
= \overline{K(\varrho)\varrho}\mathbf{F} + \operatorname{div}\big(\nabla c\otimes\nabla c - \frac{|\nabla c|^2}{2}\mathbb{I}\big) - a\nabla c\overline{\int_0^\varrho tK'(t)\,dt},\\
\overline{K(\varrho)\varrho}\mathbf{v}\cdot\nabla c = -\mu,\\
\overline{K(\varrho)\varrho\mu} = -\Delta c + \overline{h_\varepsilon(\varrho)} + \overline{K(\varrho)\varrho}(L'(c)+b'(c)) + \overline{\varepsilon K(\varrho)\varrho}L'(c),\\
\mathbf{v}\cdot\mathbf{n} = 0, \qquad \nabla c\cdot\mathbf{n} = 0,\\
\mathbf{n}\cdot\mathbb{T}(c,\mathbf{v})\cdot\boldsymbol{\tau}_n + k\mathbf{v}\cdot\boldsymbol{\tau}_n = 0, \quad \text{on } \partial\Omega,
\end{gathered}
$$

where $h_\varepsilon(s) = as\cdot l_\varepsilon(s)$. Note that, due to the high regularity we have the pointwise convergence of concentrations. In order to show the pointwise convergence of densities as well we need to investigate the momentum equation, especially its potential part defining the effective viscous flux. Let us decompose the velocity field $\mathbf{v}$, using the Helmholtz decomposition, i.e.

$$
\mathbf{v} = \nabla\varphi + \operatorname{curl}\mathbf{A},
$$

where

$$
\begin{aligned}
\operatorname{curl}\operatorname{curl}\mathbf{A} &= \operatorname{curl}\mathbf{v} = \omega && \text{in } \Omega, & \qquad & \Delta\varphi &&= \operatorname{div}\mathbf{v} && \text{in } \Omega,\\
\operatorname{div}\operatorname{curl}\mathbf{A} &= 0 && \text{in } \Omega, & \text{and} \qquad & \nabla\varphi\cdot\mathbf{n} &&= 0 && \text{on } \partial\Omega,\\
\operatorname{curl}\mathbf{A}\cdot\mathbf{n} &= 0 && \text{on } \partial\Omega, & & \int_\Omega\varphi\,dx &&= 0.
\end{aligned}
$$

For the stream function $\mathbf{A}$ we have good estimates, see [25],

$$
\|\nabla\operatorname{curl}\mathbf{A}\|_{L^q(\Omega)} \le \|\omega\|_{L^q(\Omega)}, \qquad \|\nabla^2\operatorname{curl}\mathbf{A}\|_{L^q(\Omega)} \le \|\omega\|_{W^{1,q}(\Omega)},
$$

and since ω solves

$$
\begin{aligned}
-\nu\Delta\omega = &-\operatorname{curl}\big(\overline{K(\varrho)\varrho}\mathbf{v}\cdot\nabla\mathbf{v}\big) + \operatorname{curl}\big(\overline{K(\varrho)\varrho}\mathbf{F}\big)\\
&+ \operatorname{curl}\left(\Delta c\nabla c - a\nabla c\overline{\int_0^\varrho tK'(t)\,dt}\right),
\end{aligned}
$$

we have $\|\omega\|_{W^{1,q}(\Omega)} \leq C(m)$. Similarly, we also decompose the approximative velocity field as $\mathbf{v}_\varepsilon = \nabla\varphi_\varepsilon + \operatorname{curl}\mathbf{A}_\varepsilon$ and deduce due to the slip boundary conditions the following problem for vorticity:

$$\begin{aligned}-\nu\Delta\omega_\varepsilon = \operatorname{curl}\underbrace{\Bigg(K(\varrho_\varepsilon)\varrho_\varepsilon\mathbf{F} - K(\varrho_\varepsilon)\varrho_\varepsilon\mathbf{v}_\varepsilon\cdot\nabla\mathbf{v}_\varepsilon - \frac{1}{2}\varepsilon hK(\varrho_\varepsilon)\mathbf{v}_\varepsilon + \varepsilon\frac{1}{2}\varrho_\varepsilon\mathbf{v}_\varepsilon + \Delta c_\varepsilon\nabla c_\varepsilon - a\nabla c_\varepsilon\int_0^{\varrho_\varepsilon} tK'(t)\,dt\Bigg)}_{=:\mathbf{H}_1}\\ \underbrace{-\operatorname{curl}\left(\frac{1}{2}\varepsilon\Delta\varrho_\varepsilon\mathbf{v}_\varepsilon\right)}_{=:\mathbf{H}_2} =: \mathbf{H}_1 + \mathbf{H}_2,\end{aligned}$$

$$\begin{aligned}\omega_\varepsilon\cdot\boldsymbol{\tau}_1 &= -\left(2\chi_2 - \frac{k}{\nu}\right)\mathbf{v}_\varepsilon\cdot\boldsymbol{\tau}_2 \quad \text{on } \partial\Omega,\\ \omega_\varepsilon\cdot\boldsymbol{\tau}_2 &= \left(2\chi_1 - \frac{k}{\nu}\right)\mathbf{v}_\varepsilon\cdot\boldsymbol{\tau}_2 \quad \text{on } \partial\Omega,\\ \operatorname{div}\omega_\varepsilon &= 0 \quad \text{on } \partial\Omega.\end{aligned}$$

The structure of the right-hand side of the first equation enables us to consider $\omega_\varepsilon = \omega_\varepsilon^0 + \omega_\varepsilon^1 + \omega_\varepsilon^2$ as a sum of solutions to three particular systems, namely

$$\begin{array}{llll}\nu\Delta\omega_\varepsilon^0 = 0, & \nu\Delta\omega_\varepsilon^1 = \mathbf{H}_1, & \nu\Delta\omega_\varepsilon^2 = \mathbf{H}_2 & \text{in } \Omega,\\ \omega_\varepsilon^0\cdot\boldsymbol{\tau}_1 = -\left(2\chi_2 - \frac{k}{\nu}\right)\mathbf{v}_\varepsilon\cdot\boldsymbol{\tau}_2, & \omega_\varepsilon^1\cdot\boldsymbol{\tau}_1 = 0, & \omega_\varepsilon^2\cdot\boldsymbol{\tau}_1 = 0 & \text{on } \partial\Omega,\\ \omega_\varepsilon^0\cdot\boldsymbol{\tau}_2 = -\left(2\chi_1 - \frac{k}{\nu}\right)\mathbf{v}_\varepsilon\cdot\boldsymbol{\tau}_1, & \omega_\varepsilon^1\cdot\boldsymbol{\tau}_2 = 0, & \omega_\varepsilon^2\cdot\boldsymbol{\tau}_2 = 0 & \text{on } \partial\Omega,\\ \operatorname{div}\omega_\varepsilon^0 = 0, & \operatorname{div}\omega_\varepsilon^1 = 0, & \operatorname{div}\omega_\varepsilon^2 = 0 & \text{on } \partial\Omega.\end{array} \tag{4.1}$$

Lemma 4.1. *If the vorticity $\omega_\varepsilon = \omega_\varepsilon^0 + \omega_\varepsilon^1 + \omega_\varepsilon^2$ solves (4.1), then we have*

$$\|\omega_\varepsilon^0\|_{W^{1,q}(\Omega)} + \|\omega_\varepsilon^1\|_{W^{1,q}(\Omega)} \leq C\big(1 + m^{1+\gamma(4/3-2/q)}\big),$$

for $q \in t[2, 6\gamma/(6+\gamma)]$, and

$$\|\omega_\varepsilon^2\|_{L^q(\Omega)} \leq C(m)\varepsilon^{1/2}, \quad \text{for } q \in [1, 2].$$

Proof. Following closely the corresponding considerations in [23], we deduce that (see [25])

$$\|\omega_\varepsilon^1\|_{W^{1,q}(\Omega)} \leq C\|\mathbf{H}_1\|_{W^{-1,q}(\Omega)} \quad \text{and} \quad \|\omega_\varepsilon^0\|_{W^{1,q}(\Omega)} \leq C\|\mathbf{v}_\varepsilon\|_{W^{1,q}(\Omega)},$$

but according to (3.19) for any q such that $2 \leq q \leq 6\gamma/(6+\gamma)$ we have

$$\|\mathbf{v}_\varepsilon\|_{W^{1,q}(\Omega)} \leq C\big(\|p_\varepsilon(\varrho_\varepsilon, c_\varepsilon)\|_{L^q(\Omega)} + \|\Delta c_\varepsilon\nabla c_\varepsilon\|_{L^q(\Omega)} + 1\big) \leq C\big(1 + m^{\gamma(1-2/q)}\big),$$

so we concentrate on ω_ε^1. If we denote $r = 6\gamma/(6+\gamma) > 3$ we get, by interpolation,

$$\begin{aligned}\|\omega_\varepsilon^1\|_{W^{1,q}(\Omega)} &\leq C\big(1 + \|K(\varrho_\varepsilon)\varrho_\varepsilon \mathbf{v}_\varepsilon \nabla \mathbf{v}_\varepsilon\|_{L^q(\Omega)} + \|\Delta c_\varepsilon \nabla c_\varepsilon\|_{L^q(\Omega)}\big) \\ &\leq C\big(1 + m\|\mathbf{v}_\varepsilon\|_{L^\infty(\Omega)}\|\nabla \mathbf{v}_\varepsilon\|_{L^q(\Omega)}\big) \\ &\leq C\Big(1 + m\|\mathbf{v}_\varepsilon\|_{L^6(\Omega)}^{(2r-6)/(3r-6)}\|\nabla \mathbf{v}_\varepsilon\|_{L^r(\Omega)}^{r/(3r-6)} \\ &\qquad \cdot \|\nabla \mathbf{v}_\varepsilon\|_{L^2(\Omega)}^{2(r-q)/(q(r-2))}\|\nabla \mathbf{v}_\varepsilon\|_{L^r(\Omega)}^{r(q-2)/(q(r-2))}\Big) \\ &\leq C\Big(1 + m\|\nabla \mathbf{v}_\varepsilon\|_{L^r(\Omega)}^{r/(3r-6)+r(q-2)/(q(r-2))}\Big) \\ &\leq C\Big(1 + m^{1+(4rq-6r)/(3q(r-2))\gamma(1-2/r)}\Big) \leq C\Big(1 + m^{1+\gamma(4/3-2/q)}\Big).\end{aligned}$$

Finally, for the last part we get, for $q \leq 2$,

$$\begin{aligned}\|\omega_\varepsilon^2\|_{L^q(\Omega)} &\leq C\|\varepsilon\Delta\varrho_\varepsilon \mathbf{v}_\varepsilon\|_{W^{-1,q}(\Omega)} \\ &\leq C\varepsilon\big(\|\nabla\varrho_\varepsilon \mathbf{v}_\varepsilon\|_{L^q(\Omega)} + \|\nabla\varrho_\varepsilon\nabla \mathbf{v}_\varepsilon\|_{L^{6/5}(\Omega)}\big) \leq C(m)\varepsilon^{1/2}. \qquad \square\end{aligned}$$

Now, we are approaching the key definition of the effective viscous flux. Inserting the Helmholtz decomposition into the approximative momentum equation yields

$$\begin{aligned}\nabla\big(-(2\nu+\eta)\Delta\varphi_\varepsilon + p_\varepsilon(\varrho_\varepsilon, c_\varepsilon)\big) &= \nu\Delta \operatorname{curl} \mathbf{A} + K(\varrho_\varepsilon)\varrho_\varepsilon \mathbf{F} \\ &\quad - K(\varrho_\varepsilon)\varrho_\varepsilon \mathbf{v}_\varepsilon \cdot \nabla \mathbf{v}_\varepsilon - \frac{1}{2}\varepsilon\Delta\varrho_\varepsilon \mathbf{v}_\varepsilon - \frac{1}{2}\varepsilon h K(\varrho_\varepsilon)\mathbf{v}_\varepsilon + \varepsilon\frac{1}{2}\varrho_\varepsilon \mathbf{v}_\varepsilon \\ &\quad + \operatorname{div}\left(\nabla c_\varepsilon \otimes \nabla c_\varepsilon - \frac{|\nabla c_\varepsilon|^2}{2}\mathbb{I}\right) - a\nabla c_\varepsilon \int_0^{\varrho_\varepsilon} tK'(t)\,dt,\end{aligned}$$

and we introduce the fundamental quantity

$$G_\varepsilon = -(2\nu+\eta)\Delta\varphi + p_\varepsilon(\varrho_\varepsilon, c_\varepsilon) = -(2\nu+\eta)\operatorname{div} \mathbf{v}_\varepsilon + p_\varepsilon(\varrho_\varepsilon, c_\varepsilon). \tag{4.2}$$

Similarly, inserting the Helmholtz decomposition into the limit momentum equation, we obtain (with usage of the fact that due to the continuity equation we have $\overline{K(\varrho)\varrho\nabla\mathbf{v}\mathbf{v}} = \overline{K(\varrho)\varrho}\mathbf{v}\cdot\nabla\mathbf{v}$) that

$$\begin{aligned}\nabla\big(-(2\nu+\eta)\Delta\varphi + \overline{p(\varrho,c)}\big) &= \nu\Delta \operatorname{curl} \mathbf{A} + \overline{K(\varrho)\varrho}\mathbf{F} - \overline{K(\varrho)\varrho}\mathbf{v}\cdot\nabla\mathbf{v} \\ &\quad + \operatorname{div}\left(\nabla c \otimes \nabla c - \frac{|\nabla c|^2}{2}\mathbb{I}\right) - a\nabla c\,\overline{\int_0^{\varrho} tK'(t)\,dt},\end{aligned}$$

hence we define the limit version of effective viscous flux by

$$G = -(2\nu+\eta)\Delta\varphi + \overline{p(\varrho,c)} = -(2\nu+\eta)\operatorname{div}\mathbf{v} + \overline{p(\varrho,c)}. \tag{4.3}$$

Note we have control of $\int_\Omega G_\varepsilon\,dx = \int_\Omega p_\varepsilon(\varrho_\varepsilon, c_\varepsilon)\,dx$ and $\int_\Omega G\,dx = \int_\Omega \overline{p(\varrho,c)}\,dx$. Further we state the most important features of the effective viscous flux.

Lemma 4.2. *There exists a subsequence such that* $G_\varepsilon \to G$ *(strongly) in* $L^2(\Omega)$, *and*

$$\|G\|_{L^\infty(\Omega)} \le C(\zeta)\big(1 + m^{1+2\gamma/3+\zeta}\big), \quad \textit{for any } \zeta \in (0, (\gamma-6)/3]. \tag{4.4}$$

Proof. Let us decompose G_ε to $G_\varepsilon = G_\varepsilon^1 + G_\varepsilon^2$, where G_ε^2 contains the "strongly ε-dependent" terms of the right-hand side of (4.2), namely

$$\nabla G_\varepsilon^2 = -\varepsilon \frac{1}{2}\Delta \varrho_\varepsilon \mathbf{v}_\varepsilon - \nu \operatorname{curl} \omega_\varepsilon^2 \quad \text{with} \int_\Omega G_\varepsilon^2 \, dx = 0,$$

so, for $q \in [1, 2]$,

$$\|G_\varepsilon^2\|_{L^q(\Omega)} \le C\big(\varepsilon \|\Delta \varrho_\varepsilon \mathbf{v}_\varepsilon\|_{W^{-1,q}(\Omega)} + \nu \|\operatorname{curl} \omega_\varepsilon^2\|_{W^{-1,q}(\Omega)}\big) \le C(m)\varepsilon^{1/2}.$$

Using once more the estimates from Lemma 4.1, we observe that

$$\left|\int_\Omega G_\varepsilon \, dx\right| \le C$$

and

$$\|G_\varepsilon^1\|_{W^{1,q}(\Omega)} \le C\big(1 + m^{1+\gamma(4/3-2/q)}\big), \quad \text{for } q \in [2, 6\gamma/(6+\gamma)]. \tag{4.5}$$

Therefore, since $\gamma > 6$ we have, at least for a suitably chosen subsequence,

$$G_\varepsilon^1 \to G^1 \quad \text{(strongly) in } L^\infty(\Omega) \quad \text{and} \quad G_\varepsilon^2 \to 0 \quad \text{(strongly) in } L^2(\Omega).$$

Thus, $G_\varepsilon = G_\varepsilon^1 + G_\varepsilon^2 \to G^1 = G$ (strongly) in $L^q(\Omega)$, for $q \in [1, 2]$.

Finally, setting $q = 3 + 3\zeta/(2\gamma - 3\zeta)$ in (4.5), we get the desired conclusion

$$\|G\|_{L^\infty(\Omega)} \le C(q)\|G\|_{W^{1,q}(\Omega)} \le C(\zeta)\big(1 + m^{1+(2\gamma)/3+\zeta}\big). \qquad \square$$

Now, we are ready to show that we are able to choose m in such a way that actually $\overline{K(\varrho)\varrho} = \varrho$ almost everywhere in Ω. This will be an immediate consequence of the following lemma.

Lemma 4.3. *There exists* m_0 *such that*

$$\frac{m-3}{m}(m-3)^\gamma - \|G\|_{L^\infty(\Omega)} \ge 1 \quad \textit{for } m > m_0, \tag{4.6}$$

and at least for a subsequence $\lim_{\varepsilon\to 0} |\{\varrho_\varepsilon > m-3\}| = 0$.

Proof. Let us define a smooth non-increasing function $N\colon [0,\infty) \to [0,1]$ such that

$$N(t) = \begin{cases} 1 & \text{for } t \in [0, m-3], \\ \in [0,1] & \text{for } t \in (m-3, m-2), \\ 0 & \text{for } t \in [m-2, \infty), \end{cases}$$

and multiply the approximative continuity equation by $N^l(\varrho_\varepsilon)$, for some suitable power $l \in \mathbb{N}$, in order to get after some manipulations

$$\int_\Omega \int_0^{\varrho_\varepsilon} t l N^{l-1}(t) N'(t) \, dt \operatorname{div} \mathbf{v}_\varepsilon \, dx \ge R_\varepsilon$$

with $R_\varepsilon = \varepsilon \int_\Omega \big(N^l(\varrho_\varepsilon)\Delta\varrho_\varepsilon + (h-\varrho_\varepsilon)N^l(\varrho_\varepsilon)\big)\,dx$, $R_\varepsilon \to 0$ as $\varepsilon \to 0$ ([7]). Further, by the definition of G_ε, we have

$$-(m-3)\int_\Omega \bigg(\int_0^{\varrho_\varepsilon} lN^{l-1}(t)N'(t)\,dt\bigg)p_\varepsilon(\varrho_\varepsilon,c_\varepsilon)\,dx \\ \leq m\bigg|\int_\Omega\bigg(\int_0^{\varrho_\varepsilon} -lN^{l-1}(t)N'(t)\,dt\bigg)G_\varepsilon\,dx\bigg| + R_\varepsilon,$$

and

$$\frac{m-3}{m}\int_{\{\varrho_\varepsilon>m-3\}}(1-N^l(\varrho_\varepsilon))p_\varepsilon(\varrho_\varepsilon,c_\varepsilon)\,dx \\ \leq \int_{\{\varrho_\varepsilon>m-3\}}(1-N^l(\varrho_\varepsilon))\,|G_\varepsilon|\,dx + |R_\varepsilon|\,.$$

Recalling the structure of the pressure, we have according to (3.1) and the fact that $c_\varepsilon \in [0,1]$

$$p_\varepsilon(\varrho_\varepsilon,c_\varepsilon) = P_b(\varrho_\varepsilon) + (ac_\varepsilon+d)\int_0^{\varrho_\varepsilon} K(t)\det(t^2 l'_\varepsilon(t))\,dt \geq P_b(\varrho_\varepsilon), \tag{4.7}$$

which yields

$$\frac{m-3}{m}(m-3)^\gamma\,|\{\varrho_\varepsilon > m-3\}| - \frac{m-3}{m}\|p_\varepsilon(\varrho_\varepsilon,c_\varepsilon)\|_{L^2(\Omega)}\|N^l(\varrho_\varepsilon)\|_{L^2(\Omega)} \\ \leq \|G\|_{L^\infty(\Omega)}\,|\{\varrho_\varepsilon > m-3\}| + \|G-G_\varepsilon\|_{L^1(\Omega)} + |R_\varepsilon|\,.$$

According to (4.4), we are able to choose m_0 satisfying (4.6), yielding

$$|\{\varrho_\varepsilon > m-3\}| \leq C\big(\|N^l(\varrho_\varepsilon)\|_{L^2(\{\varrho_\varepsilon>m-3\})} + \|G-G_\varepsilon\|_{L^1(\Omega)} + |R_\varepsilon|\big).$$

However, the last two terms tend to zero as $\varepsilon \to 0+$ and, having fixed $\varepsilon > 0$, $\|N^l(\varrho_\varepsilon)\|_{L^2(\{\varrho_\varepsilon>m-3\})}$ tends to zero as $l \to +\infty$ as well. Thus, Lemma 4.3 is proved. □

Finally, we deduce the pointwise convergence of densities. Our main aim is to show that

$$\overline{P_b(\varrho)\varrho} = \overline{P_b(\varrho)}\varrho,$$

which will further lead to $\varrho_\varepsilon \to \varrho$ strongly in $L^q(\Omega)$ for $q < \infty$.

Lemma 4.4. *The weak limits satisfy*

$$\int_\Omega \overline{p(\varrho,c)\varrho}\,dx \leq \int_\Omega G\varrho\,dx. \tag{4.8}$$

([7]) Note that $-\varepsilon l\int_\Omega N^{l-1}(\varrho_\varepsilon)N'(\varrho_\varepsilon)\,|\nabla\varrho_\varepsilon|^2\,dx \geq 0$.

Proof. Testing the approximative continuity equation by $\log m - \log(\varrho_\varepsilon + \delta)$ and taking the limit for $\delta \to 0+$, we get (see [22])

$$\int_\Omega K(\varrho_\varepsilon)\mathbf{v}_\varepsilon \cdot \nabla \varrho_\varepsilon \, dx \geq \varepsilon C(m)$$

and, by Lemma 4.3, $\int_\Omega \varrho_\varepsilon \operatorname{div} \mathbf{v}_\varepsilon \, dx \leq R_\varepsilon$, with $R_\varepsilon \to 0$ as $\varepsilon \to 0+$. Hence the definition of G yields

$$\int_\Omega p_\varepsilon(\varrho_\varepsilon, c_\varepsilon)\varrho_\varepsilon \, dx \leq \int_\Omega G_\varepsilon \varrho_\varepsilon \, dx - (2\nu + \eta)R_\varepsilon$$

and passing to the limit with ε we get (4.8), since, according to Lemma 4.2, $\overline{G\varrho} = G\varrho$. $\square$

Lemma 4.5.

$$\int_\Omega \overline{p(\varrho, c)}\varrho \, dx = \int_\Omega G\varrho \, dx. \tag{4.9}$$

Proof. First, with the usage of Fridrichs' commutator lemma, we are able to approximate ϱ by smooth bounded functions ϱ_n and deduce that

$$\int_\Omega \varrho \operatorname{div} \mathbf{v} + \mathbf{v} \cdot \nabla \varrho \, dx = 0.$$

Further, testing the continuity equation by $\log(\varrho_n + \delta) - \log \delta$ and then passing to the limit at first with $n \to \infty$ and then with $\delta \to 0+$ gives us $\int_\Omega \mathbf{v} \cdot \nabla \varrho \, dx = 0$, hence $\int_\Omega \varrho \operatorname{div} \mathbf{v} \, dx = 0$ as well. Thus, using again the properties of G, we obtain (4.9). $\square$

Further, the strong convergence of c and the convexity of $s \to s^\gamma$ and $s \to s^2 l_\varepsilon'(s)$ gives us

$$\overline{p(\varrho, c)}\varrho \leq \overline{p(\varrho, c)\varrho},$$

which combined with (4.8) and (4.9) yields

$$\overline{p(\varrho, c)\varrho} = \overline{p(\varrho, c)}\varrho \quad \text{a.e. in } \Omega.$$

Thus, $\overline{\varrho^{\gamma+1}} = \varrho\overline{\varrho^\gamma}$ and $\varrho_\varepsilon \to \varrho$ strongly in $L^\gamma(\Omega)$.

Now, we move our attention to the last two equations of system (1.2)–(1.5) and show that due to strong convergence ϱ_ε and c_ε all the remaining nonlinearities can be identified, so we have indeed obtained the solutions to our original system for $\gamma > 6$.

5. Existence for $\gamma > 3$

In order to prove the existence of only weak solutions of the system under consideration for $\gamma \in (3, 6]$ we will use the following idea. We modify the pressure $p_\delta(\varrho, c) = p(\varrho, c) + \delta\varrho^\Gamma$ with $\Gamma > 6$, for which the already proved result stays obviously valid and then, using the a priori estimates derived in Section 2, pass to the limit with $\delta \to 0+$. This limit passage can be performed in the same spirit as in the case of the Navier–Stokes system, using the techniques due to Lions and Feireisl, see e.g. [26] with θ replaced by c. The compactness of the additional stress in the momentum equation and in the additional equations will be just easy application of the Rellich–Kondrachov compactness theorem due to the uniform bound of Δc.

First, according to the already proven part, there exists a sequence of solutions satisfying the equations with the modified pressure p_δ denoted by $(\varrho_\delta, \mathbf{v}_\delta, c_\delta, \mu_\delta)$. Exactly with the same procedure as in Lemma 2.1 we can deduce that it satisfies

$$\begin{aligned}&\|\mathbf{v}_\delta\|_{W^{1,2}(\Omega)} + \|\varrho_\delta\|_{L^{3\gamma-6}(\Omega)} + \delta\|\varrho_\delta\|^{\Gamma+2\gamma-6}_{L^{\Gamma+2\gamma-6}(\Omega)} + \|\nabla c_\delta\|_{L^{(6\gamma-12)/\gamma}(\Omega)}\\ &\quad + \|\nabla^2 c_\delta\|_{L^{(6\gamma-12)/(3\gamma-4)}(\Omega)} + \|\mu_\delta\|_{L^2(\Omega)} + \|\varrho_\delta L'(c_\delta)\|_{L^{(6\gamma-12)/(3\gamma-4)}(\Omega)} \le C,\end{aligned}$$

with C independent of δ. So we can extract subsequences, denoted here in the same way, such that

$$\begin{aligned}
\mathbf{v}_\delta &\rightharpoonup \mathbf{v} && \text{in } W^{1,2}(\Omega),\\
\mathbf{v}_\delta &\to \mathbf{v} && \text{in } L^q(\Omega), \quad \text{for all } 1<q<6,\\
\varrho_\delta &\rightharpoonup \varrho && \text{in } L^{3\gamma-6}(\Omega),\\
\varrho_\delta^\gamma &\rightharpoonup \overline{\varrho^\gamma} && \text{in } L^r(\Omega), \quad \text{for some } r>1,\\
c_\delta &\rightharpoonup c && \text{in } W^{2,(6\gamma-12)/(3\gamma-4)}(\Omega),\\
\nabla c_\delta &\to \nabla c && \text{in } L^2(\Omega),\\
\varrho_\delta L'(c_\delta) &\rightharpoonup \varrho L'(c) && \text{in } L^{6/5}(\Omega),\\
\delta\varrho_\delta &\to 0 && \text{in } L^r(\Omega), \quad \text{for all } 1<r<\Gamma+3\gamma-6,\\
\mu_\delta &\rightharpoonup \mu && \text{in } L^2(\Omega),\\
\varrho_\delta\mu_\delta &\rightharpoonup \overline{\varrho\mu} && \text{in } L^2(\Omega).
\end{aligned}$$

Hence, consequently,

$$\begin{aligned}
\varrho_\delta\mathbf{v}_\delta &\rightharpoonup \varrho\mathbf{v} && \text{in } L^2(\Omega), & \varrho_\delta\mathbf{v}_\delta\otimes\mathbf{v}_\delta &\rightharpoonup \varrho\mathbf{v}\otimes\mathbf{v} && \text{in } L^{3/2}(\Omega),\\
\varrho_\delta\log\varrho_\delta &\rightharpoonup \overline{\varrho\log\varrho} && \text{in } L^3(\Omega) & \varrho_\delta c_\delta &\rightharpoonup \varrho c && \text{in } L^3(\Omega),\\
\nabla c_\delta\otimes\nabla c_\delta &\rightharpoonup \nabla c\otimes\nabla c && \text{in } L^1(\Omega), & |\nabla c_\delta|^2 &\rightharpoonup |\nabla c|^2 && \text{in } L^1(\Omega).
\end{aligned}$$

Note that c_δ are continuous and $c_\delta \in [0,1]$ almost everywhere in Ω, hence $\|c_\delta\|_{L^\infty} \le 1$ independently of $\delta > 0$. To summarize, we have shown that the limit solution satisfies, in particular, in the weak sense

$$\begin{gathered}
\operatorname{div}(\varrho\mathbf{v}) = 0,\\
\operatorname{div}(\varrho\mathbf{v}\otimes\mathbf{v}) - \operatorname{div}\mathbb{S}(\nabla\mathbf{v}) + \nabla\overline{p(\varrho,c)} = \varrho\mathbf{F} + \operatorname{div}\left(\nabla c\otimes\nabla c - \frac{|\nabla c|^2}{2}\,\mathbb{I}\right),\\
\mathbf{v}\cdot\mathbf{n} = 0,
\end{gathered}$$

$$\mathbf{n} \cdot \mathbb{T}(c, \mathbf{v}) \cdot \boldsymbol{\tau}_n + k\mathbf{v} \cdot \boldsymbol{\tau}_n = 0, \quad \text{on } \partial\Omega.$$

Thus, the main difficulty is to show that $\varrho_\delta \to \varrho$ in $L^s(\Omega)$ for some $s \geq 1$. To deal with this problem, we can proceed exactly in the same manner as in the standard case of the Navier–Stokes equations. Moreover, thanks to the constraint $\gamma > 3$, the renormalized continuity equation is satisfied, according to Fridrichs' commutator lemma, see [27, Section 3.1]. Thus, in order to prove the celebrated *effective viscous flux identity* discovered by Lions [20]

$$\overline{p(\varrho, c)\, T_k(\varrho)} - (2\nu + \eta)\overline{\operatorname{div} \mathbf{v}\, T_k(\varrho)} = \overline{p(\varrho, c)}\, \overline{T_k(\varrho)} - (2\nu + \eta) \operatorname{div} \mathbf{v}\, \overline{T_k(\varrho)}, \tag{5.1}$$

where

$$T_k(z) = kT\left(\frac{z}{k}\right), \qquad T(z) = \begin{cases} z & \text{for } 0 \leq z \leq 1, \\ \text{concave} & \text{on } (0, \infty), \\ 2 & \text{for } z \geq 3, \end{cases}$$

we will test for $k \in \mathbb{N}$ and $\zeta \in C_c^\infty(\Omega)$ the momentum equation and its approximate version by

$$\boldsymbol{\varphi} = \zeta \nabla \Delta^{-1}(1_\Omega \overline{T_k(\varrho)}), \quad \text{and} \quad \boldsymbol{\varphi} = \zeta \nabla \Delta^{-1}(1_\Omega T_k(\varrho_\delta)),$$

respectively. Equality (5.1) is then deduced from the above mentioned convergences and the properties of the Riesz transform, especially its commutators, see e.g. [10, Theorem 10.27]. Further, having (5.1) in hand together with the renormalized continuity equation, it is not difficult to show that, see [27],

$$\int_\Omega \overline{T_k(\varrho) \operatorname{div} \mathbf{v}}\, dx = 0, \qquad \int_\Omega T_k(\varrho) \operatorname{div} \mathbf{v}\, dx = 0,$$

and

$$\lim_{k \to \infty} \lim_{\delta \to 0^+} \|T_k(\varrho_\delta) - T_k(\varrho)\|_{L^{\gamma+1}(\Omega)} = 0.$$

This implies that $\|\varrho_\delta - \varrho\|_{L^1(\Omega)} \to 0$. Finally, we turn our attention back to the Allen–Cahn equation and realize that as soon as we have pointwise convergence of c and ϱ and weak convergence of μ, the fact that the limit of the sequence satisfies the original equations is immediate. This completes the proof of our main theorem. □

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Šimon Axmann
Mathematical Institute of Charles University
Prague, CZECH REPUBLIC
and
Department of Mathematics
University of Chemistry and Technology
Prague, CZECH REPUBLIC

E-mail address: axmas7am@karlin.mff.cuni.cz

Piotr Bogusław Mucha
Institute of Applied Mathematics and Mechanics
University of Warsaw
Warsaw, POLAND

E-mail address: p.mucha@mimuw.edu.pl