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MULTI-BUMP SOLUTIONS FOR A CLASS OF KIRCHHOFF TYPE PROBLEMS WITH CRITICAL GROWTH IN $\mathbb{R}^N$

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Abstract. Using variational methods, we establish existence of multi-bump solutions for a class of Kirchhoff type problems

$$-\bigg(1+b\int_{\mathbb{R}^N}|\nabla u|^p\,dx\bigg)\Delta_p u+(\lambda V(x)+Z(x))u^{p-1}=\alpha f(u)+u^{p^*-1},$$

where f is a continuous function, $V,Z\colon\mathbb{R}^N\to\mathbb{R}$ are continuous functions verifying some hypotheses. We show that if the zero set of V has several isolated connected components $\Omega_1,\ldots,\Omega_k$ such that the interior of Ω_i is not empty and $\partial\Omega_i$ is smooth, then for $\lambda>0$ large enough there exists, for any non-empty subset $\Gamma\subset\{1,\ldots,k\}$, a bump solution trapped in a neighbourhood of $\bigcup_{j\in\Gamma}\Omega_j$. The results are also new for the case $p=2$.

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1. Introduction

In this paper, we are concerned with the existence of multi-bump solutions of Kirchhoff type problems

$$
(1.1)\quad \begin{cases} -\left(1+b\displaystyle\int_{\mathbb{R}^N}\frac{1}{p}|\nabla u|^p\,dx\right)\Delta_p u+(\lambda V(x)+Z(x))u^{p-1}=\alpha f(u)+u^{p^*-1}, \\ \hfill x\in\mathbb{R}^N, \\ u\in W^{1,p}(\mathbb{R}^N),\quad u>0, \end{cases}
$$

where $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, $p^* = Np/(N-p)$ is the so-called Sobolev critical exponent, $2<p<N$, $N\geq 3$, b is a positive constant, $\lambda>0$ is a real parameter and f is a continuous function, $V,Z\colon\mathbb{R}^N\to\mathbb{R}$ are continuous functions with $V(x)\geq 0$ for all $x\in\mathbb{R}^N$, $\Omega=\operatorname{int}V^{-1}(0)$ has k connected components denoted by Ω_j, $j\in\{1,\ldots,k\}$, $V^{-1}(\{0\})=\overline{\Omega}$ and $\partial\Omega$ is smooth.

For the special case of problems (1.1), i.e. without $(\lambda V(x)+Z(x))u$, problem (1.1) reduces to the following Dirichlet problem of Kirchhoff type:

$$
(1.2)\quad \begin{cases} -\left(1+b\displaystyle\int_{\Omega}|\nabla u|^2\,dx\right)\Delta u=h(u),\quad x\in\Omega, \\ u|_{\partial\Omega}=0, \end{cases}
$$

where $\Omega\subset\mathbb{R}^N$ is a smooth bounded domain. Such problems are often referred to as being nonlocal because of the presence of the term $\int_\Omega|\nabla u|^2\,dx\Delta u$ which implies that the equation in (1.2) is no longer a pointwise identity. This phenomenon provokes some mathematical difficulties, which make the study of such a class of problems particularly interesting. On the other hand, we have its physical motivation. Indeed, this problem is a generalization of a model introduced by Kirchhoff [32]. More precisely, Kirchhoff proposed a model given by the equation

$$
(1.3)\quad \rho\frac{\partial^2 u}{\partial t^2}-\left(\frac{\rho_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2dx\right)\frac{\partial^2 u}{\partial x^2}=0,
$$

where ρ, ρ_0, h, E, L are constants, which extends the classical D'Alembert's wave equation, by considering the effects of changes in the length of strings during the vibrations. Equation (1.2) is related to the stationary analogue of problem (1.3). It received much attention only after Lions [33] proposed an abstract framework to the problem. Some important and interesting results can be found, for example, in [9], [16], [19], [20], [29], [37], [41]. In [9], Arosio and Panizzi studied the Cauchy–Dirichlet type problem related to (1.3) in the Hadamard sense as a special case of an abstract second-order Cauchy problem in a Hilbert space. Ma and Rivera [37] obtained positive solutions of such problems by using variational methods. Perera and Zhang [41] obtained a nontrivial solution of (1.2) via the Yang index and critical group. He and Zou [29] obtained infinitely many solutions by using the local minimum method and the fountain theorem. Recently, the paper [16] considered (1.2) with concave and convex nonlinearities by using the Nehari manifold and fibering map methods, and obtained the existence of multiple positive solutions.

For the case $p = 2$ and $b = 0$, problem (1.1) can be rewritten as follows:

$$\begin{cases} -\Delta u + (\lambda V(x) + Z(x))u = h(u), & x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \quad u > 0. \end{cases} \tag{1.4}$$

Different approaches have been taken to attack this problem under various hypotheses on the potential and the nonlinearity. For example, in the case when the potential function $\lambda V + Z$ is coercive, Miyagaki [38] proved some existence results for a positive solution to (1.4). For the case when the function $\lambda V + Z$ is 1-periodic, Alves et al. [2] showed the existence of positive solutions to (1.4). If $\lambda V + Z$ is radial, Alves et al. [3] also established the existence of a positive solution to (1.4). The papers cited above proved only the existence of positive solutions; the multiplicity of solutions was established in [10]–[12], [14], [15], [21], [22], [27], [28], [42], [43]. In the case when nonlinearities are assumed to be subcritical, there have been enormous investigations on problem (1.4). In [28], by using a Lyapunov–Schmidt reduction, Floer and Weinstein established the existence of a standing wave solutions of (1.4). Moreover they showed that u concentrates near the given non-degenerate critical point of the potential function. Their methods and results were later generalized by Oh [40] to the higher-dimensional case and the existence of multi-bump solutions concentrating near several non-degenerate critical points of the potential function was obtained. For more results, we refer to Ambrosetti, Badiale and Cingolani [7], Ambrosetti, Malchiodi and Secchi [8], Cingolani and Lazzo [17], Cingolani and Nolasco [18], Del Pino and Felmer [21], [22], Ding and Lin [24], Ding and Wei [25] and the references therein.

Recently, He and Zou [30] considered the following equation:

$$\begin{cases} -\left(\varepsilon^2 a + b\varepsilon \displaystyle\int_{\mathbb{R}^3} |\nabla u|^2\,dx\right)\Delta u + V(x)u = f(u), & x \in \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \quad u > 0, \end{cases}$$

where f is a C^1 and subcritical function. By using the Ljusternik–Schnirelmann theory (see [45]) and minimax methods, the author obtained the multiplicity of positive solutions, which concentrate on the minima of V as $\varepsilon \to 0$. This phenomenon of concentration is very interesting for both mathematicians and physicists. For the nonlinearity with critical growth, Wang, Tian, Xu and Zhang [44] obtained the multiplicity and concentration of positive solutions by using the minimax theorems and the Lusternik–Schnirelmann theory.

In [26], the authors studied problem (1.1) with $p = 2$, $b = 0$, Ding and Tanaka considered problem (1.1) without the critical term and assumed that $\alpha = 1$. Supposing that Ω has k connected components, the authors showed that, for this case, problem (1.1) has at least $2^k - 1$ solutions, for large enough λ, establishing the existence of solutions called multi-bumps. The same type of problems were considered by Alves et al. [6], [4] with critical growth respectively. Liang and Shi [36] showed the existence of multi-bump solutions for a class of Kirchhoff type problems by using variational methods.

In the present paper, we show the existence of multi-bump solutions to a class of Kirchhoff type problems (1.1) for the general case $p \geq 2$. We mainly follow the idea of [26], [30]. However, here we use a different approach in some estimates, because the p-Laplacian is not linear, and some properties that occur for the 2-Laplacian (Laplacian operator), in our opinion, not necessarily hold for the general case, $p \geq 2$, therefore, a careful analysis is needed. Moreover, our nonlinearity with critical growth and some arguments developed in [26] cannot be applied. So, we modify the sets that appear in the minimax arguments explored in [26]. The arguments developed in this paper are variational, and our main result completes the study made in [26], in the sense that we are working with the p-Laplacian and a general class of nonlinearities. As we shall see in the present paper, problem (1.1) can be viewed as a Schrödinger equation coupled with a nonlocal term. The competing effect of the nonlocal term with the critical nonlinearity and the lack of compactness of the embedding of $W^{1,p}(\mathbb{R}^N)$ into the space $L^{p^*}(\mathbb{R}^N)$ prevent us from using the variational methods in a standard way. Some new estimates for such Kirchhoff equation involving Palais–Smale sequences, which are key points to apply this kind of theory, should be reestablished. The Moser iterative method [39] has to be applied trickly. Let us point out that although the idea was used before for other problems, the adaptation to the procedure for our problem is not trivial, due to the appearance of the nonlocal term we must reconsider this problem and need more delicate estimates.

We make the following assumptions on V, Z and f throughout this paper:

- (V_1) $V \in C(\mathbb{R}^N, \mathbb{R})$ satisfies $V(x) \geq 0$.
- (V_2) The potential well $\Omega := \operatorname{int} V^{-1}(0)$ is a non-empty bounded open set with smooth boundary $\partial\Omega$ and $V^{-1}(0) = \overline{\Omega}$.
- (V_3) There exist two positive constants M_0 and M_1 such that the functions V and Z verify

$$0 < M_0 \leq V(x) + Z(x) \quad \text{for all } x \in \mathbb{R}^N, \tag{1.5}$$

$$|Z(x)| \leq M_1 \quad \text{for all } x \in \mathbb{R}^N. \tag{1.6}$$

- (V_4) The set Ω consists of k connected components denoted by Ω_j, $j \in \{1, \ldots, k\}$, satisfying $\operatorname{dist}(\Omega_i, \Omega_j) > 0$ for $i \neq j$, that is $\Omega = \Omega_1 \cup \ldots \cup \Omega_k$.

(f_1) There exists $p < q < p^*$ such that

$$\lim_{|t|\to\infty} \sup \frac{f(t)}{|t|^{q-1}} = m < +\infty.$$

(f_2) $\lim\limits_{s\to 0} f(s)/s^{p-1} = 0$.

(f_3) There is $2p < \theta < p^*$ verifying $0 < \theta F(s) \leq sf(s)$, for all $s \geq 0$, where $F(t) = \int_0^t f(\tau)\, d\tau$.

(f_4) The function $f(t)/t^{p-1}$ is increasing for $t \in [0, +\infty)$.

(f_5) There are $\varsigma > 0$ and $\iota \in (2p, p^*)$ such that $h(s) \geq \varsigma s^{\iota - 1}$ for all $s > 0$.

Our main results are the following.

Theorem 1.1. *Assume that* (V_1)–(V_4) *and* (f_1)–(f_5) *hold. Then, for any non-empty subset* Γ *of* $\{1, \ldots, k\}$, *there exist constants* $\alpha^* > 0$ *and* $\lambda^* = \lambda^*(\alpha^*)$ *such that, for all* $\alpha \geq \alpha^*$ *and* $\lambda \geq \lambda^*$, *problem* (1.1) *has a family* (u_λ) *of positive solutions which dependence on* α *verifies: for any sequence* $\lambda_n \to \infty$, *we can extract a subsequence* λ_{n_i} *such that* $u_{\lambda_{n_i}}$ *converges strongly in* $W^{1,p}(\mathbb{R}^N)$ *to a function* u *which satisfies* $u(x) = 0$ *for* $x \notin \Omega_\Gamma$ *and the restriction* $u|_{\Omega_j}$ *is a least energy solution of*

$$\begin{cases} -\left(1 + b\displaystyle\int_{\mathbb{R}^N} \frac{1}{p}|\nabla u|^p\, dx\right)\Delta_p u + Z(x)u^{p-1} = \alpha f(u) + u^{p^*-1}, & x \in \Omega_j,\ j \in \Gamma, \\ u > 0, & x \in \Omega_j, \\ u|_{\partial\Omega} = 0, \end{cases}$$

where $\Omega_\Gamma = \bigcup\limits_{j\in\Gamma} \Omega_j$.

Theorem 1.2. *Under the assumptions of Theorem* 1.1, *there exist constants* $\alpha^* > 0$ *and* $\lambda^* = \lambda^*(\alpha^*)$ *such that, for all* $\alpha \geq \alpha^*$ *and* $\lambda \geq \lambda^*$, *problem* (1.1) *has at least* $2^k - 1$ *positive solutions.*

2. Main results

In this section, we outline the variational framework for problem (1.1) and give some preliminary lemmas. Let us define the space of functions

$$E_\lambda := \left\{ u \in W^{1,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|u|^p\, dx < \infty \right\}$$

endowed with the norm

$$\|u\|_\lambda = \left(\int_{\mathbb{R}^N} (|\nabla u|^p + (\lambda V(x) + Z(x))|u|^p)\, dx \right)^{1/p}.$$

For $\lambda \geq 1$ it easy to see that $(E_\lambda, \|\cdot\|)$ is a Banach space and we have the following continuous imbedding: $E_\lambda \hookrightarrow W^{1,p}(\mathbb{R}^N)$.

We say that $u \in E_\lambda$ is a weak solution of (1.1) if, for $v \in E_\lambda$,

$$\int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \nabla v\,dx + b\bigg(\int_{\mathbb{R}^N} \frac{1}{p}|\nabla u|^p\,dx\bigg)\bigg(\int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u\nabla v\,dx\bigg) + \int_{\mathbb{R}^N} (\lambda V(x) + Z(x))|u|^{p-2}uv\,dx = \alpha\int_{\mathbb{R}^N} f(u)v\,dx + \int_{\mathbb{R}^N} u^{p^*-2}uv\,dx.$$

The nonnegative weak solutions of problem (1.1) are the critical points of $I\colon E_\lambda \to \mathbb{R}$ given by

$$I(u) := \frac{1}{p}\int_{\mathbb{R}^N} (|\nabla u|^p + (\lambda V(x) + Z(x))|u|^p)\,dx + \frac{b}{2}\bigg(\int_{\mathbb{R}^N} \frac{1}{p}|\nabla u|^p\,dx\bigg)^2 - \alpha\int_{\mathbb{R}^N} F(u_+)\,dx - \frac{1}{p^*}\int_{\mathbb{R}^N} u_+^{p^*}\,dx,$$

where $F(s) = \int_0^s f(\tau)\,d\tau$ and $u_+(x) = \max\{u(x),0\}$.

Moreover, let us assume that the set Ω consists of k connected components denoted by Ω_j, $j\in\{1,\ldots,k\}$, satisfying $\operatorname{dist}(\Omega_i,\Omega_j) > 0$ for $i\neq j$, that is $\Omega = \Omega_1\cup\ldots\cup\Omega_k$.

We also write for an open set $K\subset\mathbb{R}^N$,

$$E_\lambda(K) := \bigg\{u\in W^{1,p}(K) : \int_K V(x)|u|^p\,dx < \infty\bigg\}$$

and the norms

$$\|u\|_\lambda = \bigg(\int_K (|\nabla u|^p + (\lambda V(x) + Z(x))|u|^p)\,dx\bigg)^{1/p}.$$

In view of (V_3), we have

$$M_0|u|_{p,K}^p \leq \int_K (|\nabla u|^p + (\lambda V(x) + Z(x))|u|^p)\,dx$$

for all $u\in E_\lambda(K)$ and $\lambda\geq 1$ or equivalently

$$M_0|u|_{p,K}^p \leq \|u\|_{\lambda,K}^p \quad \text{for all } u\in E_\lambda(K) \text{ and } \lambda\geq 1,$$

where M_0 is a positive constant and $|u|_{p,K}^p = \int_K |u|^p\,dx$.

The next result is an immediate consequence of the above considerations.

Lemma 2.1. *There exist $\delta_0,\nu_0>0$ such that for all open sets $K\subset\mathbb{R}^N$*

$$\delta_0\|u\|_{\lambda,K}^p \leq \|u\|_{\lambda,K}^p - \nu_0|u|_{p,K}^p \quad \textit{for all } u\in E_\lambda(K) \textit{ and } \lambda\geq 1.$$

We recall the second concentration-compactness principle of Lions [34].

Lemma 2.2 ([34]). *Let $\{u_n\}\subset W^{1,p}(\mathbb{R}^N)$ be a bounded sequence which weakly converges to u in $L^{p^*}(\mathbb{R}^N)$. If $\{u_n\}$ is a subsequence such that $|u_n|^{p^*}\rightharpoonup\nu$ and $|\nabla u_n|^p\rightharpoonup\mu$ for some measures ν and μ, then there are sequences of points $\{x_n\}\subset\mathbb{R}^N$ and $\{\nu_n\}\subset[0,\infty)$ satisfying*

(a) $\nu = |u|^{p^*} + \sum_{i\in I}\delta_{x_i}\nu_i$, $\nu_i>0$,

(b) $\mu \geq |\nabla u|^p + \sum_{i\in I} \delta_{x_i}\mu_i,\ \mu_i > 0,$

(c) $\mu_i \geq S\nu_i^{p/p^*},$

where I is the index set and S is the best Sobolev constant, i.e.

$$S = \inf\bigg\{ \int_{\mathbb{R}^N} |\nabla u|^p \, dx : \int_{\mathbb{R}^N} |u|^{p^*} \, dx = 1\bigg\},$$

$x_j \in \mathbb{R}^N$, δ_{x_i} *are Dirac measures at* x_i *and* μ_i, ν_i *are constants.*

To finish this section, in what follows, for each $j \in \{1, \ldots, k\}$, we fix a bounded open subset Ω'_j with smooth boundary such that

(a) $\overline{\Omega_j} \subset \Omega'_j$;
(b) $\overline{\Omega'_j} \cap \overline{\Omega'_{\overline{j}}} = \emptyset$ for all $j \neq \overline{j}$.

3. Preliminaries

In this section, we adapt for our case some arguments developed in papers [23] and [26]. Let $\nu_0 > 0$ be the constant given in Lemma 2.1, $k > \theta(\theta - p)^{-1} > 1$, $a > 0$ verify $\alpha f(a) + a^{p^*-1} = k^{-1}a^{p-1}\nu_0$, and $\widetilde{f}, \widetilde{F} \colon \mathbb{R} \to \mathbb{R}$ be the functions

$$\widetilde{f}(s) = \begin{cases} \alpha f(s) + s^{p^*-1}, & \text{if } s \leq a, \\ k^{-1}\nu_0 s^{p-1}, & \text{if } s > a, \end{cases} \quad \text{and} \quad \widetilde{F}(s) = \int_0^s \widetilde{f}(\tau)\, d\tau.$$

From now on, we fix a non-empty subset $\Gamma \subset \{1, \ldots, k\}$ and

$$\Omega_\Gamma = \bigcup_{j\in\Gamma} \Omega_j, \qquad \Omega'_\Gamma = \bigcup_{j\in\Gamma} \Omega'_j, \qquad \chi_\Gamma(x) = \begin{cases} 1 & \text{for } x \in \Omega'_\Gamma, \\ 0 & \text{for } x \notin \Omega'_\Gamma, \end{cases}$$

and functions

$$g(x, s) = \chi_\Gamma(x)(\alpha f(s) + s^{p^*-1}) + (1 - \chi_\Gamma(x))\widetilde{f}(s), \tag{3.1}$$

$$G(x, s) = \int_0^s g(x, \tau)\, d\tau = \chi_\Gamma(x)F(s) + (1 - \chi_\Gamma(x))\widetilde{F}(s). \tag{3.2}$$

Notice that, using (f_1)–(f_4), it is easy to check that

(g_1) $g(z, s) = \alpha f(s) + s^{p^*-1} = o(s^{p-1})$, near the origin, uniformly in $z \in \mathbb{R}^N$.
(g_2) $g(z, s) \leq \alpha f(s) + s^{p^*-1}$ for all $s > 0$, $z \in \mathbb{R}^N$.
(g_3) (i) $0 < \theta G(x, s) \leq g(x, s)s$ for each $x \in \Omega'_\Gamma$, $s > 0$;
(ii) $0 \leq G(x, s) \leq k^{-1}p^{-1}\nu_0 s^p$ and $0 \leq g(x, s)s \leq k^{-1}\nu_0 s^p$ for each $x \in \mathbb{R}^N \setminus \Omega'_\Gamma$, $s > 0$.
(g_4) The function $g(z, s)/s^{p-1}$ is increasing for $s > 0$ for each fixed z.

Moreover, let $\Phi_\lambda\colon E_\lambda \to R$ denote the functional given by

$$\Phi_\lambda(u) := \frac{1}{p}\int_{\mathbb{R}^N} (|\nabla u|^p + (\lambda V(x) + Z(x))|u|^p)\,dx \\ + \frac{b}{2}\bigg(\int_{\mathbb{R}^N} \frac{1}{p}|\nabla u|^p\,dx\bigg)^2 - \int_{\mathbb{R}^N} G(x,u)\,dx.$$

Under conditions (V_1), (V_2), (f_1) and (f_2), $\Phi_\lambda \in C^1(E_\lambda, \mathbb{R})$ and its critical points are nonnegative weak solutions of

$$-\bigg(1+b\int_{\mathbb{R}^N} |\nabla u|^p\,dx\bigg)\Delta_p u + (\lambda V(x)+Z(x))|u|^{p-2}u = g(x,u), \quad x \in \mathbb{R}^N. \tag{3.3}$$

We remark that $\widetilde{f}(s) = \alpha f(s) + s^{p^*-1}$ for $s \in [0,a]$ and a critical point u_λ of $\Phi_\lambda(u)$ is a solution of problem (1.1) if and only if $u_\lambda(x) \le a$ in $\mathbb{R}^N \setminus \Omega'_\Gamma$.

4. Behavior of (PS)-sequences and the study of some energy levels

Recall that we say that a sequence (u_n) is a (PS)-sequence at level c ($(\mathrm{PS})_c$-sequence, for short) if $\Phi_\lambda(u_n) \to c$ and $\Phi'_\lambda(u_n) \to 0$. Φ_λ is said to satisfy the $(\mathrm{PS})_c$-condition if any $(\mathrm{PS})_c$-sequence contains a convergent subsequence.

Remark 4.1. By the definition of the functions f and F, the Palais–Smale sequences may be assumed to be nonnegative.

The next lemma establishes that all $(\mathrm{PS})_c$-sequences are bounded.

Lemma 4.2. *Assume that* (V_1)–(V_3) *and* (f_1)–(f_3) *are satisfied. Then there is a constant* $M(c) > 0$ *independent of* λ *such that, for any* $(\mathrm{PS})_c$*-sequence* (u_n) *for* $\Phi_\lambda(u)$*,*

$$\Phi_\lambda(u_n) \to c, \tag{4.1}$$

$$\Phi'_\lambda(u_n) \to 0 \quad \text{strongly in } E^*_\lambda, \tag{4.2}$$

$\|u_n\|^p_\lambda \le M(c)$ *for all* $n \in \mathbb{N}$.

Proof. It follows from (4.1)–(4.2) that

$$\Phi_\lambda(u_n) - \frac{1}{\theta}\Phi'_\lambda(u_n)u_n = c + o_n(1) + \varepsilon_n\|u_n\|_\lambda,$$

where $\varepsilon_n \to 0$ as $n \to \infty$. From (4.1)–(4.2), (g_3) (ii) and Lemma 2.1,

$$\begin{aligned}\Phi_\lambda(u_n) &- \frac{1}{\theta}\Phi'_\lambda(u_n)u_n \\ &= \bigg(\frac{1}{p} - \frac{1}{\theta}\bigg)\int_{\mathbb{R}^N} (|\nabla u_n|^p + (\lambda V(x) + Z(x))|u_n|^p)\,dx \\ &\quad + \frac{b}{p}\bigg(\frac{1}{2p} - \frac{1}{\theta}\bigg)\bigg(\int_{\mathbb{R}^N} |\nabla u_n|^p\,dx\bigg)^2 + \int_{\mathbb{R}^N} \bigg[\frac{1}{\theta}\,g(x,u_n)u_n - G(x,u_n)\bigg]\,dx \\ &\ge \bigg(\frac{1}{p} - \frac{1}{\theta}\bigg)\int_{\mathbb{R}^N} (|\nabla u_n|^p + (\lambda V(x) + Z(x))|u_n|^p)\,dx\end{aligned}$$

$$
\begin{aligned}
&+ \int_{\mathbb{R}^N\setminus\Omega'_\Gamma} \left[\frac{1}{\theta}\, g(x,u_n)u_n - G(x,u_n)\right] dx \\
&\geq \left(\frac{1}{p}-\frac{1}{\theta}\right)\int_{\mathbb{R}^N} (|\nabla u_n|^p + (\lambda V(x)+Z(x))|u_n|^p)\, x - \int_{\mathbb{R}^N\setminus\Omega'_\Gamma} G(x,u_n)\, dx \\
&\geq \left(\frac{1}{p}-\frac{1}{\theta}\right)\int_{\mathbb{R}^N} (|\nabla u_n|^p + (\lambda V(x)+Z(x))|u_n|^p)\, dx - \frac{1}{kp}\nu_0 \int_{\mathbb{R}^N} |u_n|^p\, dx \\
&\geq \left(\frac{1}{p}-\frac{1}{\theta}\right)(\|u_n\|_\lambda^p - \nu_0|u_n|_p^p) \geq \left(\frac{1}{p}-\frac{1}{\theta}\right)\delta_0\|u_n\|_\lambda^p.
\end{aligned}
$$

This fact implies that

$$
\left(\frac{1}{p}-\frac{1}{\theta}\right)\delta_0\|u_n\|_\lambda^p \leq c + o_n(1) + \varepsilon_n\|u_n\|_\lambda.
$$

Therefore

$$
\limsup_{n\to\infty} \|u_n\|_\lambda^p \leq \left(\frac{1}{p}-\frac{1}{\theta}\right)^{-1}\delta_0^{-1}c.
$$

From which it follows that there exists $M(c) > 0$ such that $\|u_n\|_\lambda^p \leq M(c)$ for all $n \in \mathbb{N}$. □

Next, for each fixed $j \in \Gamma$, let us denote by c_j the minimax level of the mountain pass theorem with the functional $I_j\colon W_0^{1,p}(\Omega_j) \to \mathbb{R}$, given by

$$
\begin{aligned}
I_j(u) = \frac{1}{p}\int_{\Omega_j} (|\nabla u|^p + Z(x)|u|^p)\, dx + \frac{b}{2}\bigg(\int_{\Omega_j} \frac{1}{p}|\nabla u|^p\, dx\bigg)^2 \\
- \alpha \int_{\Omega_j} F(u_+)\, dx - \frac{1}{p^*}\int_{\Omega_j} (u_+)^{p^*} dx.
\end{aligned}
$$

It is well-known that the critical points of I_j are weak solutions of the following problem:

$$
\begin{cases}
-\bigg(1 + b\displaystyle\int_{\Omega_j} |\nabla u|^p\, dx\bigg)\Delta u + Z(x)u^{p-1} = \alpha f(u) + u^{p^*-1}, \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x \in \Omega_j,\ j \in \Gamma, \\
u > 0, \qquad\qquad\qquad\qquad\qquad\qquad\qquad x \in \Omega_j, \\
u|_{\partial\Omega_j} = 0.
\end{cases}
\tag{4.3}
$$

The technique that we shall apply in order to prove Theorem 1.1 includes the comparison between some energy levels of the functional associated with (1.1) with the energy levels associated with other auxiliary problems related to (1.1), as well as the study of the behaviour of some $(\mathrm{PS})_c$-sequences. In this regard we prove the following results.

Lemma 4.3. *There exists $\alpha^* > 0$ such that, for all $\alpha \geq \alpha^*$, we have*

$$
c_j \in \left(0, \frac{1}{2p(k+1)}S^{N/p}\right) \quad \text{for all } j \in \{1,\dots,k\}.
$$

PROOF. We fix a nonnegative function $\varphi_j \in W_0^{1,p}(\Omega_j) \setminus \{0\}$ for each $j \in \{1,\ldots,k\}$. Observe that there exists $t_{\alpha,j} \in (0,+\infty)$ such that

$$c_j \leq I_j(t_{\alpha,j}\varphi_j) = \max_{t\geq 0} I_j(t\varphi_j)$$

and thus, by (f_5), we have

$$\begin{aligned} t^p_{\alpha,j} \int_{\Omega_j} (|\nabla\varphi_j|^p + Z(x)|\varphi_j|^p)\,dx + t^{2p}_{\alpha,j} b\bigg(\int_{\Omega_j} |\nabla\varphi_j|^p\,dx\bigg)^2 &= \alpha \int_{\Omega_j} f(t_{\alpha,j}\varphi_j)t_{\alpha,j}\varphi_j\,dx + t^{p^*}_{\alpha,j}\int_{\Omega_j}|\varphi_j|^{p^*}\,dx \\ &\geq \alpha \int_{\Omega_j} f(t_{\alpha,j}\varphi_j)t_{\alpha,j}\varphi_j\,dx \geq \alpha\varsigma t^{\iota}_{\alpha,j}\int_{\Omega_j}|\varphi_j|^{\iota}\,dx. \end{aligned} \tag{4.4}$$

If $|t_{\beta,j}| \leq 1$, by (4.4), we have

$$t^p_{\alpha,j} \int_{\Omega_j} (|\nabla\varphi_j|^p + Z(x)|\varphi_j|^p)\,dx + t^p_{\alpha,j} b\bigg(\int_{\Omega_j} |\nabla\varphi_j|^p dx\bigg)^2 \geq \alpha\varsigma t^{\iota}_{\alpha,j}\int_{\Omega_j}|\varphi_j|^{\iota}\,dx.$$

This fact implies that

$$t_{\alpha,j} \leq \left[\frac{\displaystyle\int_{\Omega_j} (|\nabla\varphi_j|^2 + Z(x)|\varphi_j|^2)\,dx + b\bigg(\int_{\Omega_j}|\nabla\varphi_j|^p\,dx\bigg)^2}{\displaystyle\alpha\varsigma\int_{\Omega_j}|\varphi_j|^{\iota}\,dx}\right]^{1/(\iota-p)}.$$

If $|t_{\beta,j}| \geq 1$, by (4.4), one has

$$t^{2p}_{\alpha,j} \int_{\Omega_j} (|\nabla\varphi_j|^p + Z(x)|\varphi_j|^p)\,dx + t^{2p}_{\alpha,j} b\bigg(\int_{\Omega_j} |\nabla\varphi_j|^p\,dx\bigg)^2 \geq \alpha\varsigma t^{\iota}_{\alpha,j}\int_{\Omega_j}|\varphi_j|^{\iota}\,dx.$$

This fact implies that

$$t_{\alpha,j} \leq \left[\frac{\displaystyle\int_{\Omega_j} (|\nabla\varphi_j|^2 + Z(x)|\varphi_j|^2)\,dx + b\bigg(\int_{\Omega_j}|\nabla\varphi_j|^p\,dx\bigg)^2}{\displaystyle\alpha\varsigma\int_{\Omega_j}|\varphi_j|^{\iota}\,dx}\right]^{1/(\iota-2p)}.$$

Using the above limit, we have $t_{\alpha,j} \to 0$ as $\alpha \to +\infty$. This fact implies that $I_j(t_{\alpha,j}\varphi_j) \to 0$ as $\alpha \to +\infty$, whence it follows that there exists $\alpha^* > 0$ such that

$$c_j \in \bigg(0, \frac{1}{2p(k+1)}S^{N/p}\bigg) \quad \text{for all } j \in \{1,\ldots,k\} \text{ and all } \alpha \in [\alpha^*,+\infty). \qquad \square$$

REMARK 4.4. In particular, Lemma 4.3 implies that

$$\sum_{j=1}^{k} c_j \in (0, S^{N/p}/(2p)). \tag{4.5}$$

The above result is very important, as we show in the following lemma.

LEMMA 4.5. *Assume that* (V_1)–(V_3) *and* (f_1)–(f_5) *are satisfied. For any* $\lambda \geq 1$, Φ_λ *satisfies the* $(\mathrm{PS})_c$*-condition, for all* $c \in (0, 1/2pS^{N/p})$, *that is any* $(\mathrm{PS})_c$*-sequence* $(u_n) \subset E_\lambda$ *has a strongly convergent subsequence in* E_λ.

PROOF. Let $(u_n) \subset E_\lambda$ be a $(\mathrm{PS})_c$-sequence, by Lemma 4.2, (u_n) is bounded in E_λ and we may assume the following facts:

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{weakly in } E_\lambda \text{ and } W^{1,p}(\mathbb{R}^N), \\ u_n &\to u \quad \text{strongly in } L^p_{\mathrm{loc}}(\mathbb{R}^N) \text{ for all } p \in [1, p^*), \\ |u_n|^{p^*} &\rightharpoonup \nu = |u|^{p^*} + \sum_{i \in I} \delta_{x_i} \nu_i, \qquad \nu_i > 0, \\ |\nabla u_n|^p &\rightharpoonup \mu \geq |\nabla u|^p + \sum_{i \in I} \delta_{x_j} \mu_i, \quad \mu_i > 0. \end{aligned}$$

First, we claim that

$$\int_{\mathbb{R}^N} g(x, u_n) u_n \, dx \to \int_{\mathbb{R}^N} g(x, u) u \, dx \quad \text{as } n \to \infty. \tag{4.6}$$

In fact, let x_i be a singular point of the measures μ and ν. We define a function $\phi \in C_0^\infty(\mathbb{R}^N)$ such that $\phi(x) = 1$ in $B(x_i, \varepsilon)$, $\phi(x) = 0$ in $\mathbb{R}^N \setminus B(x_i, 2\varepsilon)$ and $|\nabla \phi| \leq 2/\varepsilon$ in $\mathbb{R}^N$. Set $\psi_n = u_n \phi$, then $\{\psi_n\}$ is bounded in E_λ. Obviously, $\langle \Phi'_\lambda(u_n), \psi_n \rangle \to 0$, i.e.

$$\begin{aligned} -\lim_{n\to\infty} \bigg[&\int_{\mathbb{R}^N} u_n |\nabla u_n|^{p-2} \nabla u_n \nabla \phi \, dx \\ &+ b\bigg(\int_{\mathbb{R}^N} \frac{1}{p} |\nabla u_n|^p \, dx \bigg) \bigg(\int_{\mathbb{R}^N} u_n |\nabla u_n|^{p-2} \nabla u_n \nabla \phi \, dx \bigg) \bigg] \\ = \lim_{n\to\infty} \bigg\{ &\int_{\mathbb{R}^N} |\nabla u_n|^p \phi \, dx + \int_{\mathbb{R}^N} (\lambda V(x) + Z(x)) |u_n|^p \phi \, dx \\ &+ b\bigg(\int_{\mathbb{R}^N} \frac{1}{p} |\nabla u_n|^p \, dx \bigg) \bigg(\int_{\mathbb{R}^N} |\nabla u_n|^p \phi \, dx \bigg) - \int_{\mathbb{R}^N} g(x, u_n) u_n \phi \, dx \bigg\}. \end{aligned} \tag{4.7}$$

By definition of ϕ and due to strong convergence $u_n \to u$ in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$ for $p \in [1, p^*)$, one can obtain

$$\int_{\mathbb{R}^N} f(u_n) u_n \phi \, dx \to 0 \quad \text{as } n \to \infty.$$

From Lemma 4.2 and the Hölder inequality, we have that

$$\begin{aligned} 0 &\leq \lim_{\varepsilon\to 0} \lim_{n\to\infty} \bigg| \int_{\mathbb{R}^N} u_n |\nabla u_n|^{p-2} \nabla u_n \nabla \phi \, dx \bigg| \\ &\leq C \lim_{\varepsilon\to 0} \lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n |\nabla u_n|^{p-1} \nabla \phi| \, dx \end{aligned} \tag{4.8}$$

$$\le C \lim_{\varepsilon\to 0}\lim_{n\to\infty}\bigg[\bigg(\int_{\mathbb{R}^N}|\nabla u_n|^p\,dx\bigg)^{(p-1)/p}\bigg(\int_{\mathbb{R}^N}|u_n\nabla\phi|^p\,dx\bigg)^{1/p}\bigg]$$
$$\le C \lim_{\varepsilon\to 0}\bigg(\int_{B(x_i,2\varepsilon)}|u|^{p^*}\,dx\bigg)^{1/p^*} = 0$$

and

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\bigg[b\bigg(\int_{\mathbb{R}^N}\frac{1}{p}|\nabla u_n|^p\,dx\bigg)\bigg(\int_{\mathbb{R}^N}u_n|\nabla u_n|^{p-2}\nabla u_n\nabla\phi\,dx\bigg)\bigg] = 0. \tag{4.9}$$

From inequalities (4.7)–(4.9) and condition (g_2), we get

$$\begin{aligned}0 \ge \lim_{\varepsilon\to 0}\lim_{n\to\infty}\bigg\{&\int_{\mathbb{R}^N}|\nabla u_n|^p\phi\,dx + \int_{\mathbb{R}^N}(\lambda V(x)+Z(x))|u_n|^p\phi\,dx\\ &+ b\bigg(\int_{\mathbb{R}^N}\frac{1}{p}|\nabla u_n|^p\,dx\bigg)\bigg(\int_{\mathbb{R}^N}|\nabla u_n|^p\phi\,dx\bigg)\\ &- \alpha\int_{\mathbb{R}^N}f(u_n)u_n\phi\,dx - \int_{\mathbb{R}^N}u_n^{p^*}\phi\,dx\bigg\}\\ \ge \lim_{\varepsilon\to 0}\lim_{n\to\infty}\bigg[&\int_{\mathbb{R}^N}|\nabla u_n|^p\phi\,dx - \int_{\mathbb{R}^N}u_n^{p^*}\phi\,dx\bigg] = \mu_i - \nu_i.\end{aligned} \tag{4.10}$$

Combining this with Lemma 2.2, we obtain $\nu_i \ge S\nu_i^{p/p^*}$. This result implies that

(I) $\nu_i = 0$ or
(II) $\nu_i \ge S^{N/p}$.

If the second case $\nu_i \ge S^{N/p}$ holds, for some $i \in I$, then by using Lemma 2.2 and the Hölder inequality, we have

$$\begin{aligned}c &= \lim_{n\to\infty}\bigg(\Phi_\lambda(u_n) - \frac{1}{2p}\langle\Phi'_\lambda(u_n),u_n\rangle\bigg)\\ &= \frac{1}{2p}\int_{\mathbb{R}^N}(|\nabla u_n|^p + (\lambda V(x)+Z(x))|u_n|^p)\,dx + \int_{\mathbb{R}^N}\bigg(\frac{1}{2p}g(x,u_n) - G(x,u_n)\bigg)\,dx\\ &\ge \frac{1}{2p}\int_{\mathbb{R}^N}|\nabla u_n|^p dx = \frac{1}{2p}\int_{\mathbb{R}^N}d\mu\\ &\ge \frac{1}{2p}\int_{\mathbb{R}^N}|\nabla u|^p\,dx + \frac{1}{2p}\mu_i \ge \frac{1}{2p}S\nu_i^{p/p^*} \ge \frac{1}{2p}S^{N/p}.\end{aligned}$$

This is impossible. Consequently, $\nu_i = 0$ for all $i \in I$ and hence

$$u_n \to u \quad \text{in } L^{p^*}_{\text{loc}}(\mathbb{R}^N). \tag{4.11}$$

In order to prove (4.6), we will prove that the possible concentration of mass at infinity. By hypothesis, for any bounded sequence $(\varphi_n) \subset E_\lambda$, we have $\Phi'_\lambda(u_n)\varphi_n = o_n(1)$. Let us choose a special φ_n for our purposes:

$$\varphi_n(x) = \eta(x)u_n(x),$$

where $\eta \in C^\infty(\mathbb{R}^N)$ is given by

$$\eta(x) = \begin{cases} 1 & \text{for } x \in B^c_R(0), \\ 0 & \text{for } x \in B_{R/2}(0), \end{cases}$$

$$\eta(x) \in [0,1] \quad \text{with } \Omega'_\Gamma \subset B_{R/2}(0).$$

Here and below $B^c_R(0) = \{x \in \mathbb{R}^N : |x| \geq R\}$. Using the above data and adapting arguments used in [23, Lemma 1.1] one proves that, for each $\varepsilon > 0$, there exists $R > 0$ such that

$$\int_{\{x\in\mathbb{R}^N:|x|\geq R\}} (|\nabla u_n|^p + (\lambda V(x) + Z(x))|u_n|^p)\, dx < \varepsilon \tag{4.12}$$

and

$$\bigg(\int_{\{x\in\mathbb{R}^N:|x|\geq R\}} |\nabla u_n|^p dx\bigg)^2 < \varepsilon, \tag{4.13}$$

for large enough $n \in \mathbb{N}$. From (4.11)–(4.13), we obtain (4.6) holds. Thus, from the Brezis–Lieb lemma [13] and the weak lower semicontinuity of the norm we have

$$\begin{aligned} o(1)\|v_n\| = \langle \Phi'_\lambda(u_n), u_n\rangle = &\int_{\mathbb{R}^N} (|\nabla u_n|^p + (\lambda V(x) + Z(x))|u_n|^p)\, dx \\ &+ \frac{b}{p}\bigg(\int_{\mathbb{R}^N} |\nabla u_n|^p\, dx\bigg)^2 - \int_{\mathbb{R}^N} g(x,u_n)\, dx \\ \geq &\|u_n - u\|^p_\lambda + \|u\|^p_\lambda + \frac{b}{p}\bigg(\int_{\mathbb{R}^N} |\nabla u|^p\, dx\bigg)^2 - \int_{\mathbb{R}^N} g(x,u)\, dx \\ = &\|u_n - u\|^p_\lambda + o(1)\|u\|, \end{aligned}$$

since $\Phi'_\lambda(u) = 0$. Thus we proved that $\{u_n\}$ strongly converges to u in E_λ. This completes the proof of Lemma 4.5. □

Remark 4.6. Since Φ_λ verifies the mountain pass geometry, the above results imply the existence of a nontrivial critical point to Φ_λ.

Our next step is to study the behavior of a $(\mathrm{PS})_{\infty,c}$-sequence, that is, a sequence $(u_n) \subset W^{1,p}(\mathbb{R}^N)$ that satisfies

$$u_n \subset E_\lambda \quad \text{and} \quad \lambda_n \to \infty, \qquad \Phi_{\lambda_n}(u_n) \to c, \qquad \|\Phi'_{\lambda_n}(u_n)\|^*_{\lambda_n} \to 0.$$

Lemma 4.7. *Let a sequence (u_n) be a $(\mathrm{PS})_{\infty,c}$-sequence with $c \in (0, 1/2pS^{N/p})$. Then, for some subsequence, still denoted by (u_n), there exists $u \in W^{1,p}(\mathbb{R}^N)$ such that*

$$u_n \rightharpoonup u \quad \textit{weakly in } E_\lambda \quad \textit{and} \quad W^{1,p}(\mathbb{R}^N) \quad \textit{as } n \to \infty.$$

Moreover,

(a) $u \equiv 0$ *in* $\mathbb{R}^N \setminus \Omega_\Gamma$ *and* u *is a nonnegative solution of*

$$\begin{cases} -\Big(1 + b\displaystyle\int_{\mathbb{R}^N} |\nabla u|^p\,dx\Big)\Delta_p u + Z(x)|u|^{p-2}u = \alpha f(u) + |u|^{p^*-2}u & \\ & \text{in } \in \Omega_j, \\ u = 0 & \text{on } \partial\Omega_j, \end{cases} \tag{4.14}$$

for each $j \in \Gamma$;

(b) $\|u_n - u\|_{\lambda_n} \to 0$;

(c) u_n *also satisfies*

$$\lambda_n \int_{\mathbb{R}^N} V(x)|u_n|^p dx \to 0, \qquad \|u_n - u\|^p_{\lambda_n, \mathbb{R}^N \setminus \Omega_\Gamma} \to 0,$$
$$\|u_n\|^p_{\lambda_n, \Omega'_j} \to \int_{\Omega_j} (|\nabla u|^p + Z(x)|u|^p)dx \quad \text{for all } j \in \Gamma.$$

Proof. Similarly to the proof of Lemma 4.2, we can prove that there exists $M(c) > 0$ such that

$$\|u_n\|^p_\lambda \le M(c) \quad \text{for all } n \in \mathbb{N}.$$

Thus (u_n) is a bounded sequence in $W^{1,p}(\mathbb{R}^N)$ and, for some subsequence, still denoted by (u_n), we may assume that for some $u \in W^{1,p}(\mathbb{R}^N)$

$$\begin{aligned} u_n &\rightharpoonup u && \text{weakly in } E_\lambda \text{ and } W^{1,p}(\mathbb{R}^N), \\ u_n &\to u && \text{strongly in } L^p_{\rm loc}(\mathbb{R}^N) \text{ for all } p \in [1, p^*), \\ u_n(x) &\to u(x) && \text{a.e. in } \mathbb{R}^N. \end{aligned}$$

Using once more similar arguments explored in Lemma 4.5, we get

$$u_n \to u \quad \text{in } E_\lambda. \tag{4.15}$$

To show (a), we fix the set $C_m = \{x \in \mathbb{R}^N : V(x) \ge 1/m\}$. Then

$$\begin{aligned} \int_{C_m} |u_n|^p\,dx &\le \frac{m}{\lambda_n} \int_{\mathbb{R}^N} \lambda_n V(x)|u_n|^p\,dx \\ &\le \frac{m}{\lambda_n} \int_{\mathbb{R}^N} (|\nabla u_n|^p + (\lambda_n V(x) + Z(x))|u_n|^p)\,dx = \frac{m}{\lambda_n}\|u_n\|^p_{\lambda_n}. \end{aligned}$$

The last inequality together with Fatou's lemma imply

$$\int_{C_m} |u|^p dx = 0 \quad \text{for all } m \in \mathbb{N}.$$

Thus $u(x) = 0$ on $\bigcup_{m=1}^{+\infty} C_m = \mathbb{R}^N \setminus \overline{\Omega}$ and we can assert that $u|_{\Omega_j} \in W^{1,p}_0(\Omega_j)$ for all $j \in \{1, \ldots, k\}$.

Once we have proved that $\Phi'_{\lambda_n}(u_n)\varphi \to 0$ for $\varphi \in C_0^\infty(\Omega_j)$ ($j \in \{1,\dots,k\}$), it follows from (4.15) that

$$\begin{aligned}0 &= \int_{\Omega_j} (|\nabla u|^{p-2}\nabla u \nabla \varphi + Z(x)|u|^{p-2}u\varphi)\,dx \\ &\quad + b\bigg(\int_{\mathbb{R}^N} \frac{1}{p}|\nabla u|^p\,dx\bigg)\int_{\Omega_j} |\nabla u|^{p-2}\nabla u\nabla\varphi\,dx - \int_{\Omega_j} g(x,u)\varphi\,dx,\end{aligned}$$

showing that u is a solution to (4.14) for each $j \in \Gamma$.

For $j \in \{1,\dots,k\}\setminus\Gamma$, setting $\varphi(x) = u(x)$, we have

$$\int_{\Omega_j} (|\nabla u|^p + Z(x)|u|^p)\,dx + \frac{b}{p}\bigg(\int_{\Omega_j} |\nabla u|^p\,dx\bigg)^2 - \int_{\Omega_j} \widetilde{f}(u)u\,dx = 0.$$

From Lemma 2.1 and the fact that $\widetilde{f}(s)s \le \nu_0 s^p$ for all $s \in \mathbb{R}$, we have

$$\begin{aligned}\delta_0\|u\|^p_{\lambda,\Omega_j} &\le \|u\|^p_{\lambda,\Omega_j} - \nu_0|u|^p_{p,\Omega'_j} \\ &\le \int_{\Omega_j} (|\nabla u|^p + Z(x)|u|^p)\,dx + \frac{b}{p}\bigg(\int_{\Omega_j} |\nabla u|^p\,dx\bigg)^2 - \int_{\Omega_j} \widetilde{f}(u)u\,dx = 0.\end{aligned}$$

Thus, $u = 0$ in Ω_j for $j \in \{1,\dots,k\}\setminus\Gamma$ showing that (a) holds.

For (b), we use [46, Lemma 4.2] to get the following inequality:

$$\begin{aligned}&\|u_n - u\|^p_{\lambda_n} + b\bigg(\int_{\mathbb{R}^N} \frac{1}{p}|\nabla u_n|^p\,dx\bigg)\int_{\mathbb{R}^N} |\nabla u_n|^{p-2}\nabla u_n(\nabla u_n - \nabla u)\,dx \\ &\quad - b\bigg(\int_{\mathbb{R}^N} \frac{1}{p}|\nabla u|^p\,dx\bigg)\int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u(\nabla u - \nabla u_n)\,dx \\ &\quad - \int_{\mathbb{R}^N\setminus\Omega'_\Gamma} (\widetilde{f}(u_n) - \widetilde{f}(u))(u_n - u)\,dx - \int_{\Omega'_\Gamma} (f(u_n) - f(u))(u_n - u)\,dx \\ &\le \Phi'_{\lambda_n}(u_n)(u_n - u) - \Phi'_{\lambda_n}(u)(u_n - u).\end{aligned}$$

Since $u_n \to u$ in $L^p(\Omega'_J)$ and due to assumptions (f_1)–(f_2), we have

$$\int_{\Omega'_\Gamma} (f(u_n) - f(u))(u_n - u)\,dx = o_n(1). \tag{4.16}$$

On the other hand,

$$|\Phi'_{\lambda_n}(u_n)(u_n - u)| \le \|\Phi'_{\lambda_n}(u_n)\|^*_{\lambda_n}(\|u_n\|_{\lambda_n} + \|u\|_{\lambda_n}) = o_n(1). \tag{4.17}$$

By inequalities (4.16) and (4.17), we have

$$\|u_n - u\|^p_{\lambda_n} + \int_{\mathbb{R}^N\setminus\Omega'_\Gamma} (\widetilde{f}(u_n) - \widetilde{f}(u))(u_n - u)\,dx \to 0.$$

By a similar argument as in the proof of Lemma 4.3, we obtain

$$\begin{aligned}\delta_0\|u_n-u\|_{\lambda_n}^p &\le \|u_n-u\|_{\lambda_n}^p-\nu_0|u_n-u|_p^p\\ &\le \int_{\mathbb{R}^N}(|\nabla u_n-\nabla u|^p+\lambda V(x)|u_n-u|^p)\,dx\\ &\quad+\int_{\mathbb{R}^N\setminus\Omega'_\Gamma}(\widetilde{f}(u_n)-\widetilde{f}(u))(u_n-u)\,dx\le 0.\end{aligned}$$

Thus (b) is obtained.

Now we show (c). Indeed,

$$\begin{aligned}\int_{\mathbb{R}^N}\lambda_n V(x)|u_n|^p\,dx &= \int_{\mathbb{R}^N\setminus\Omega_\Gamma}\lambda_n V(x)|u_n|^p\,dx\\ &= \int_{\mathbb{R}^N\setminus\Omega_\Gamma}\lambda_n V(x)|u_n-u|^p\,dx\le \|u_n-u\|_{\lambda_n}^p\to 0 \quad\text{as } n\to\infty.\end{aligned}$$

This completes the proof of Lemma 4.7. □

The following lemma plays a fundamental role in the study of solutions to problem (1.1). We sketch its proof by using some arguments explored in [5] and [35], which are initiated in the Moser iterative method [39].

Lemma 4.8. *Let u_λ be a family of positive solutions of* (3.3) *satisfying*

$$\sup_{\lambda\ge 1}\{\Phi_\lambda(u_\lambda)\}<\frac{1}{2p}S^{N/p}.$$

Then, there exists $\Lambda_0>0$ such that $|u_\lambda|_{L^\infty(\mathbb{R}^N)\setminus\Omega'_\Gamma}\le a$ for $\lambda\ge\Lambda_0$. In particular, u_λ solves the original problem (1.1).

Proof. In this proof we adapt some arguments developed in Li [35, Theorem 1.11] (see also [1] and [31]). Let (λ_n) be a sequence with $\lambda_n\to\infty$ and define $u_n(x)=u_{\lambda_n}(x)$. For any $R>0$, $0<r\le R/2$, let $\eta\in C^\infty(\mathbb{R}^N)$, $0\le\eta\le 1$ with $\eta(x)=1$ if $|x|\ge R$ and $\eta(x)=0$ if $|x|\le R-r$ and $|\nabla\eta|\le 2/r$. For each $n\in\mathbb{N}$ and $\rho>0$, put

$$u_{l,n}(x):=\begin{cases}u_n(x) & \text{if } u_n(x)\le\rho,\\ \rho & \text{if } u_n(x)\ge\rho,\end{cases}$$

and $z_{\rho,n}:=\eta^p u_{\rho,n}^{p(\sigma-1)}u_n$, $w_{\rho,n}:=\eta u_n u_{\rho,n}^{\sigma-1}$ with $\sigma>1$ to be determined later. Taking $z_{\rho,n}$ as the test function, we have

$$\begin{aligned}0=&\int_{\mathbb{R}^N}\big(|\nabla u_n|^{p-2}\nabla u_n\nabla z_{\rho,n}+(\lambda V(x)+Z(x))|u_n|^{p-2}u_n z_{\rho,n}\big)\,dx\\ &+b\bigg(\int_{\mathbb{R}^N}\frac{1}{p}|\nabla u_n|^p\,dx\bigg)\int_{\mathbb{R}^N}|\nabla u_n|^{p-2}\nabla u_n\nabla z_{\rho,n}\,dx-\int_{\mathbb{R}^N}g(x,u_n)z_{\rho,n}\,dx,\end{aligned}$$

i.e.

$$\begin{aligned}0 &= \left(1 + b\int_{\mathbb{R}^N} \frac{1}{p}|\nabla u_n|^p\,dx\right)\\ &\quad\cdot \int_{\mathbb{R}^N} \big(|\nabla u_n|^p \eta^p u_{\rho,n}^{p(\sigma-1)} + p|\nabla u_n|^{p-2}\nabla u_n \nabla\eta \eta^{p-1} u_n u_{\rho,n}^{p(\sigma-1)}\\ &\quad + p(\sigma-1)|\nabla u_n|^{p-2}\nabla u_n \nabla u_{\rho,n} u_n u_{\rho,n}^{p(\sigma-1)-1}\eta^p\big)\,dx\\ &\quad + \int_{\mathbb{R}^N} (\lambda V(x) + Z(x))|u_n|^p \eta^p u_{\rho,n}^{p(\sigma-1)}\,dx - \int_{\mathbb{R}^N} g(x,u_n)\eta^p u_n u_{\rho,n}^{p(\sigma-1)}\,dx.\end{aligned}$$

For simplicity, we denote

$$A_n := 1 + b\int_{\mathbb{R}^N} \frac{1}{p}|\nabla u_n|^p\,dx.$$

Since $u_n \to u$ in $W^{1,p}(\mathbb{R}^N)$ with $u \neq 0$, we see that $a \le A_n \le a^*$ for some constant $a^* > 0$. Therefore, we can rewrite the above equality as

$$\begin{aligned}A_n \int_{\mathbb{R}^N} \eta^p u_{\rho,n}^{p(\sigma-1)}|\nabla u_n|^p\,dx &= \int_{\mathbb{R}^N} g(x,u_n)\eta^p u_n u_{\rho,n}^{p(\sigma-1)}\,dx\\ &\quad - pA_n \int_{\mathbb{R}^N} \eta^{p-1} u_n u_{\rho,n}^{p(\sigma-1)}|\nabla u_n|^{p-2}\nabla u_n \nabla\eta\,dx\\ &\quad - pA_n(\sigma-1)\int_{\mathbb{R}^N} u_n u_{\rho,n}^{p(\sigma-1)-1}\eta^p |\nabla u_n|^{p-2}\nabla u_n \nabla u_{\rho,n}\,dx\\ &\quad - \int_{\mathbb{R}^N} (\lambda V(x) + Z(x))|u_n|^p \eta^p u_{\rho,n}^{p(\sigma-1)}\,dx.\end{aligned}$$

By conditions (f_1)–(f_2), we have that for any $\tau > 0$, there exists $D_\tau > 0$ such that $g(x,s) \le \tau s + D_\tau s^{p^*-1}$ for all $(x,s) \in \mathbb{R}^N \times [0,+\infty)$. Since τ can be sufficiently small, we have the following inequality:

$$\begin{aligned}A_n \int_{\mathbb{R}^N} \eta^p u_{\rho,n}^{p(\sigma-1)}|\nabla u_n|^p\,dx &\le D_\tau \int_{\mathbb{R}^N} u_n^{p^*}\eta^p u_{\rho,n}^{p(\sigma-1)}\,dx\\ &\quad - pA_n \int_{\mathbb{R}^N} \eta^{p-1} u_n u_{\rho,n}^{p(\sigma-1)}|\nabla u_n|^{p-2}\nabla u_n \nabla\eta\,dx.\end{aligned}$$

For each $\delta > 0$, using Young's inequality, we get

$$\begin{aligned}A_n \int_{\mathbb{R}^N} \eta^p u_{\rho,n}^{p(\sigma-1)}|\nabla u_n|^p\,dx &= D_\tau \int_{\mathbb{R}^N} u_n^{p^*}\eta^p u_{\rho,n}^{p(\sigma-1)}\,dx\\ &\quad + pA_n\delta \int_{\mathbb{R}^N} |\nabla u_n|^p u_{\rho,n}^{p(\sigma-1)}\eta^p\,dx + pA_n C_\delta \int_{\mathbb{R}^N} |u_n|^p |\nabla\eta|^p u_{\rho,n}^{p(\sigma-1)}\,dx.\end{aligned}$$

Choosing $\delta \in (0,1/p)$, we have

$$\begin{aligned}\int_{\mathbb{R}^N} \eta^p u_{\rho,n}^{p(\sigma-1)}|\nabla u_n|^p\,dx &\le C\int_{\mathbb{R}^N} u_n^{p^*}\eta^p u_{\rho,n}^{p(\sigma-1)}\,dx\\ &\quad + C\int_{\mathbb{R}^N} |u_n|^p |\nabla\eta|^p u_{\rho,n}^{p(\sigma-1)}\,dx.\end{aligned} \tag{4.18}$$

On the other hand, by the Sobolev and Hölder inequalities we have

$$
\begin{aligned}
(4.19)\qquad |w_{\rho,n}|_{p^*}^p &\le C\int_{\mathbb{R}^N}|\nabla w_{\rho,n}|^p\,dx = C\int_{\mathbb{R}^N}|\nabla(\eta u_n u_{\rho,n}^{\sigma-1})|^p\,dx\\
&\le C\sigma^p\bigg(\int_{\mathbb{R}^N}|\nabla\eta|^p u_n^p u_{\rho,n}^{p(\sigma-1)}\,dx + \int_{\mathbb{R}^N}\eta^p u_{\rho,n}^{p(\sigma-1)}|\nabla u_n|^p\,dx\bigg).
\end{aligned}
$$

Combining (4.18)–(4.19), we have

$$
(4.20)\qquad |w_{\rho,n}|_{p^*}^p \le C\sigma^p\bigg(\int_{\mathbb{R}^N}|\nabla\eta|^p|u_n|^p u_{\rho,n}^{p(\sigma-1)}dx + \int_{\mathbb{R}^N}u_n^{p^*}\eta^p u_{\rho,n}^{p(\sigma-1)}dx\bigg).
$$

Choosing $\sigma = p^*/p$, by the definition of $w_{\rho,n}$ and (4.20), we rewrite the last inequality as

$$
\begin{aligned}
&\bigg(\int_{\mathbb{R}^N}\big(\eta u_n u_{\rho,n}^{(p^*-p)/p}\big)^{p^*}dx\bigg)^{p/p^*}\\
&\quad\le C(N,p)\bigg\{\bigg[\int_{\mathbb{R}^N}\big(\eta u_n u_{\rho,n}^{(p^*-p)/p}\big)^{p^*}dx\bigg]^{p/p^*}\bigg(\int_{|x|>R-r}u_n^{p^*}\,dx\bigg)^{p^*-p/p}\\
&\qquad+\int_{\mathbb{R}^N}|\nabla\eta|^p|u_n|^p u_{\rho,n}^{p^*-p}\,dx\bigg\}\\
&\quad\le C(N,p)\bigg\{\bigg[\int_{\mathbb{R}^N}\big(\eta u_n u_{\rho,n}^{(p^*-p)/p}\big)^{p^*}dx\bigg]^{p/p^*}\|u_n\|_{p^*(|x|\ge R/2)}^{p^*-p}\\
&\qquad+\int_{\mathbb{R}^N}|\nabla\eta|^p|u_n|^p u_{\rho,n}^{p^*-p}\,dx\bigg\}.
\end{aligned}
$$

In view of $u_n\to u$ in $W^{1,p}(\mathbb{R}^N)$, for R large enough, we conclude that

$$
\|u_n\|_{p^*(|x|\ge R/2)}^{p^*-p} \le \frac{1}{C(N,p)}\quad\text{uniformly in } n.
$$

Hence we obtain

$$
\begin{aligned}
\bigg(\int_{|x|\ge R}(u_n u_{\rho,n}^{(p^*-p)/p})^{p^*}dx\bigg)^{p/p^*} &\le \bigg(\int_{\mathbb{R}^N}(\eta u_n u_{\rho,n}^{(p^*-p)/p})^{p^*}dx\bigg)^{p/p^*}\\
&\le C(N,p)\int_{\mathbb{R}^N}|\nabla\eta|^p|u_n|^p u_{\rho,n}^{p^*-p}\,dx \le \frac{C}{r^p}\int_{\mathbb{R}^N}|u_n|^{p^*}dx.
\end{aligned}
$$

Using Fatou's lemma in the variable ρ, we have

$$
(4.21)\qquad u_n\in L^{{p^*}^2/p}(|x|\ge R)\quad\text{for } R \text{ large enough.}
$$

Next, if we put $\sigma = p^*(t-1)/(pt)$ with $t = {p^*}^2/((p^*-p)p)$, then $\sigma>1$, $pt/(t-1)<p^*$. Now suppose that $u_n\in L^{p\sigma t/(t-1)}$ $(|x|\ge R-r)$ for some $\sigma\ge 1$. Returning to (4.20) and using the Hölder inequality with exponent $t/(t-1)$

and t, we have

$$
\begin{aligned}
(4.22)\qquad |w_{\rho,n}|_{p^*}^p \leq & C\sigma^p\bigg\{\bigg[\int_{|x|\geq R-r}(\eta^p u_n^{p\sigma})^{t/(t-1)}\,dx\bigg]^{1-1/t}\\
&\cdot\bigg(\int_{|x|>R-r}u_n^{(p^*-p)t}\,dx\bigg)^{1/t}\\
&+\frac{[R^N-(R-r)^N]^{1/t}}{r^p}\bigg(\int_{|x|\geq R-r}u_n^{p\sigma t/(t-1)}\,dx\bigg)^{1-1/t}\bigg\}\\
\leq & C\sigma^p\bigg(1+\frac{R^{N/t}}{r^p}\bigg)\bigg(\int_{|x|\geq R-r}u_n^{p\sigma t/(t-1)}\,dx\bigg)^{1-1/t}.
\end{aligned}
$$

Letting $\rho\to+\infty$ in (4.22), we obtain

$$\|u_n\|_{p^*\sigma(|x|\geq R)}^{p\sigma}\leq C\sigma^p\bigg(1+\frac{R^{N/t}}{r^p}\bigg)\|u_n\|_{p\sigma t/(t-1)(|x|\geq R-r)}^{p\sigma}.$$

If we set $\chi:=p^*(t-1)/(pt)$, $s:=pt/(t-1)$, then

$$(4.23)\qquad \|u_n\|_{\sigma\chi s,(|x|\geq R)}\leq C^{1/\sigma}\sigma^{1/\sigma}\bigg(1+\frac{R^{N/t}}{r^p}\bigg)^{1/(p\sigma)}\|u_n\|_{\sigma s(|x|\geq R-r)}.$$

Let $\sigma=\chi^m$ $(m=1,2,\ldots)$, we obtain

$$\|u_n\|_{\chi^{m+1}s(|x|\geq R)}\leq C^{\chi^{-m}}\chi^{m\chi^{-m}}\bigg(1+\frac{R^{N/t}}{r^p}\bigg)^{1/(p\sigma)}\|u_n\|_{\chi^m s(|x|\geq R-r)}.$$

It is clear that $p>N/t$. Therefore, if we take $r_m:=2^{-(m+1)}R$, then inequality (4.23) implies

$$
\begin{aligned}
\|u_n\|_{\chi^{m+1}s(|x|\geq R)}\leq & |u_n|_{\chi^{m+1}s(|x|\geq R-r_{m+1})}\\
\leq & C^{\sum_{i=1}^m\chi^{-i}}\chi^{\sum_{i=1}^m i\chi^{-i}}\exp\bigg(\sum_{i=1}^m\frac{\ln(1+2^{p(i+1)})}{p\chi^i}\bigg)\|u_n\|_{\chi s,(|x|\geq R-r_1)}\\
\leq & C\|u_n\|_{p^*(|x|\geq R/2)}.
\end{aligned}
$$

Letting $m\to\infty$ in the last inequality, we get $\|u_n\|_{L^\infty(|x|\geq R)}\leq C|u_n|_{p^*(|x|\geq R/2)}$. By the convergence of $\{u_n\}$ to u in $W^{1,p}(\mathbb{R}^N)$, for any fixed $a>0$, there exists $R>0$ such that $\|u_n\|_{L^\infty(|x|\geq R)}\leq a$ for all $n\in\mathbb{N}$. Therefore, $\lim_{|x|\to\infty}u_n(x)=0$ uniformly in n. □

5. The existence of multi-bump positive solutions

For $j\in\Gamma$, we consider the following two functionals:

$$
\begin{aligned}
(5.1)\qquad I_j(u):=&\frac{1}{p}\int_{\Omega_j}(|\nabla u|^p+Z(x)|u|^p)\,dx\\
&+\frac{b}{2}\bigg(\int_{\Omega_j}\frac{1}{p}|\nabla u|^p\,dx\bigg)^2-\alpha\int_{\Omega_j}F(u)\,dx-\frac{1}{p^*}\int_{\Omega_j}(u_+)^{p^*}\,dx
\end{aligned}
$$

and

$$\Phi_{\lambda,j}(u) := \frac{1}{p}\int_{\Omega'_j} (|\nabla u|^p + (\lambda V(x) + Z(x))|u|^p)\,dx + \frac{b}{2}\bigg(\int_{\Omega'_j}\frac{1}{p}|\nabla u|^p\,dx\bigg)^2 - \alpha\int_{\Omega'_j}F(u)\,dx - \frac{1}{p^*}\int_{\Omega'_j}(u_+)^{p^*}\,dx.$$

We know that the critical points of the above functions are related with the weak solutions to the problem

$$\begin{cases} -\bigg(1+b\displaystyle\int_{\mathbb{R}^N}|\nabla u|^p\,dx\bigg)\Delta_p u + Z(x)u^{p-1} = \alpha f(u) + u^{p^*-1} & \text{in } \Omega_j, \\ u = 0 & \text{on } \partial\Omega_j, \end{cases} \tag{5.2}$$

and

$$\begin{cases} -\bigg(1+b\displaystyle\int_{\mathbb{R}^N}|\nabla u|^p\,dx\bigg)\Delta_p u + (\lambda V(x) + Z(x))u^{p-1} \\ \qquad\qquad = \alpha f(u) + u^{p^*-1} & \text{in } \Omega'_j, \\ \dfrac{\partial u}{\partial \eta} = 0 & \text{on } \partial\Omega'_j. \end{cases} \tag{5.3}$$

It is easy to check that functionals I_j and $\Phi_{\lambda,j}$ satisfy the mountain pass geometry. That is,

(i) $I_j(0) = \Phi_{\lambda,j}(0) = 0$.

(ii) There exist $\rho_0 > 0$ and ρ_1 independent of $\lambda \geq 0$ such that

$$\|u\|_{0,\Omega_j} \leq \rho_0 \Rightarrow I_j(u) \geq 0, \tag{5.4}$$

$$\|u\|_{0,\Omega_j} = \rho_0 \Rightarrow I_j(u) \geq \rho_1, \tag{5.5}$$

$$\|u\|_{0,\Omega_j} \leq \rho_0 \Rightarrow \Phi_{\lambda,j}(u) \geq 0, \tag{5.6}$$

$$\|u\|_{0,\Omega_j} = \rho_0 \Rightarrow \Phi_{\lambda,j}(u) \geq \rho_1. \tag{5.7}$$

Here we use the notation:

$$\|u\|_{0,\Omega_j} = \int_{\Omega_j}(|\nabla u|^p + Z(x)|u|^p)\,dx \quad \text{for } u \in W_0^{1,p}(\Omega_j).$$

(iii) There exists $\varphi_j(x) \in C_0^\infty(\Omega_j)$ such that

$$\|\varphi_j(x)\|_{\lambda,\Omega'_j} = \|\varphi_j(x)\|_{0,\Omega_j} \geq \rho_1, \qquad \Phi_{\lambda,\Omega'_j}(\varphi_j) = I_j(\varphi_j) < 0.$$

In what follows, we denote by c_j and $c_{\lambda,j}$ the minimax level related to the above functions defined by

$$c_j := \inf_{\gamma\in\Lambda_j}\max_{t\in[0,1]} I_j(\gamma(t)) \quad \text{and} \quad c_{\lambda,j} := \inf_{\gamma\in\Lambda_{\lambda,j}}\max_{t\in[0,1]}\Phi_{\lambda,j}(\gamma(t)),$$

where

$$\Lambda_j := \{\gamma \in C([0,1], W_0^{1,p}(\Omega_j)) : \gamma(0) = 0,\ I_j(\gamma(1)) < 0\},$$
$$\Lambda_{\lambda,j} := \{\gamma \in C([0,1], W_0^{1,p}(\Omega'_j)) : \gamma(0) = 0,\ \Phi_{\lambda,j}(\gamma(1)) < 0\}.$$

Moreover, the $(\mathrm{PS})_c$-condition implies that there exist two nonnegative functions $w_j \in W_0^{1,p}(\Omega_j)$ and $w_{\lambda,j} \in W_0^{1,p}(\Omega_j')$ satisfying

$$\begin{aligned} I_j(w_j) &= c_j \quad \text{and} \quad I_j'(w_j) = 0, \\ \Phi_{\lambda,j}(w_{\lambda,j}) &= c_{\lambda,j} \quad \text{and} \quad \Phi_{\lambda,j}'(w_{\lambda,j}) = 0. \end{aligned}$$

We have the following lemma.

LEMMA 5.1.

(a) $0 < \rho_1 \le c_{\lambda,j} \le c_j$ *for all* $\lambda \ge 0$.

(b) c_j ($c_{\lambda,j}$, *respectively*) *is a least energy level for* $I_j(u)$ ($\Phi_{\lambda,j}(u)$, *respectively*), *that is*

$$\begin{aligned} c_j &= \inf\{I_j(u) : u \in W_0^{1,p}(\Omega_j) \setminus \{0\}, I_j'(u) \cdot u = 0\}, \\ c_{\lambda,j} &= \inf\{\Phi_{\lambda,j}(u) : u \in W_0^{1,p}(\Omega_j') \setminus \{0\}, \Phi_{\lambda,j}'(u) \cdot u = 0\}. \end{aligned}$$

(c) $c_j = \max_{r>0} I_j(rw_j)$, $c_{\lambda,j} = \max_{r>0} I_{\lambda,j}(rw_{\lambda,j})$.

(d) $c_{\lambda,j} \to c_j$ *as* $j \to \infty$.

PROOF. From (5.7), it is easy to see that $c_{\lambda,j} \ge \rho_1$. On the other hand, for any $u \in W_0^{1,p}(\Omega_j)$, we may extend u to $\widetilde{u} \in W_0^{1,p}(\Omega_j')$ by

$$\widetilde{u}(x) = \begin{cases} u(x) & \text{in } \Omega_j, \\ 0 & \text{in } \Omega_j' \setminus \overline{\Omega}_j, \end{cases}$$

we regard $W_0^{1,p}(\Omega_j) \subset W_0^{1,p}(\Omega_j')$. Thus we have $\Lambda_j \subset \Lambda_{\lambda,j}$ and

$$\begin{aligned} c_{\lambda,j} &= \inf_{\gamma \in \Lambda_{\lambda,j}} \max_{t \in [0,1]} \Phi_{\lambda,j}(\gamma(t)) \\ &\le \inf_{\gamma \in \Lambda_j} \max_{t \in [0,1]} \Phi_{\lambda,j}(\gamma(t)) \le \inf_{\gamma \in \Lambda_j} \max_{t \in [0,1]} I_j(\gamma(t)) = c_j. \end{aligned} \tag{5.8}$$

Thus we have (a).

(b)–(c) are standard consequences which can be obtained from the monotonicity of the nonlinearity $g(x,s)/s^{p-1}$ (see Chapter 4 in [45]).

Now we show (d). Using Lemma 4.7, we may extract a subsequence $\lambda_n \to \infty$ such that $w_{\lambda,j} \to u_0$ strongly in $W_0^{1,p}(\Omega_j')$, where $u_0 \in W_0^{1,p}(\Omega_j)$ is a solution of (5.1) and $\Phi_{\lambda,j}(w_{\lambda,j}) \to I_j(u_0)$. By the definition of c_j, we have

$$\limsup_{\lambda \to \infty} c_{\lambda,j} = \limsup_{\lambda \to \infty} \Phi_{\lambda,j}(w_{\lambda,j}) \ge I_j(u_0) \ge c_j.$$

Compare with (5.8), we get (iv) and this completes the proof of Lemma 5.1. □

In what follows, let us fix $R > 1$ such that

$$\left| I_j\left(\frac{1}{R} w_j\right) \right| < \frac{1}{2} c_j \quad \text{for all } j \in \Gamma \tag{5.9}$$

and

$$|I_j(Rw_j) - c_j| \ge 1 \quad \text{for all } j \in \Gamma.$$

From the definition of c_j, the equality below is standard:

$$\max_{s\in[1/R^2,1]} I_j(sRw_j) = c_j \quad \text{for all } j \in \Gamma,$$

where the interval $[1/R^2, 1]$ is chosen conveniently for our purposes.

Reordering the set Γ, we may consider $\Gamma = \{1, \dots, l\}$ $(l \le k)$,

$$\gamma_0(s_1, \dots, s_l)(x) = \sum_{j=1}^{l} s_j R w_j(x) \quad \text{for all } (s_1, \dots, s_l) \in \left[\frac{1}{R^2}, 1\right]^l, \tag{5.10}$$

$$\Lambda_* := \left\{\gamma \in C\left(\left[\frac{1}{R^2}, 1\right]^l, E_\lambda \setminus \{0\}\right) : \gamma = \gamma_0 \text{ on } \partial\left(\left[\frac{1}{R^2}, 1\right]^l\right)\right\}$$

and

$$b_{\lambda,\Gamma} = \inf_{\gamma\in\Lambda_*} \max_{(s_1,\dots,s_l)\in[1/R^2,1]^l} \Phi_\lambda(\gamma(s_1, \dots, s_l)).$$

Remark 5.2. As $\gamma_0 \in \Lambda_*$, $\Lambda_* \ne \emptyset$ and $b_{\lambda,\Gamma}$ is well-defined.

Lemma 5.3. *For any $\gamma \in \Lambda_*$, there exists $(t_1, \dots, t_l) \in [1/R^2, 1]^l$ such that*

$$\Phi'_{\lambda,j}(\gamma(t_1, \dots, t_l))(\gamma(t_1, \dots, t_l)) = 0 \quad \textit{for all } j \in \{1, \dots, l\}.$$

Proof. For a given $\gamma \in \Lambda_*$, let us consider the map $\widetilde{\gamma}\colon [1/R^2, 1]^l \to R^l$ defined by

$$\widetilde{\gamma}(s_1, \dots, s_l) = (\Phi'_{\lambda,1}(\gamma)(\gamma), \dots, \Phi'_{\lambda,l}(\gamma)(\gamma)),$$

where $\Phi'_{\lambda,j}(\gamma)(\gamma) = \Phi'_{\lambda,j}(\gamma(s_1, \dots, s_l))\gamma(s_1, \dots, s_l)$ for all $j \in \Gamma$.

For any $(s_1, \dots, s_l) \in \partial([1/R^2, 1]^l)$, it follows that

$$\gamma(s_1, \dots, s_l) = \gamma_0(s_1, \dots, s_l),$$
$$\Phi'_{\lambda,j}(\gamma_0(s_1, \dots, s_l))\gamma_0(s_1, \dots, s_l) = 0.$$

From this fact, we have $s_j \notin \{1/R^2, 1\}$ for $j \in \Gamma$. Thus $(0, \dots, 0) \notin \widetilde{\gamma}(\partial([1/R^2, 1]^l))$. Hence, it follows from the topological degree that

$$\deg\left(\widetilde{\gamma}, \left(\frac{1}{R^2}, 1\right)^l, (0, \dots, 0)\right) = (-1)^l \ne 0.$$

Thus, there exists $(t_1, \dots, t_l) \in (1/R^2, 1)^l$ satisfying

$$\Phi'_{\lambda,j}(\gamma(t_1, \dots, t_l))(\gamma(t_1, \dots, t_l)) = 0 \quad \text{for } j \in \{1, \dots, l\}.$$

This completes the proof of Lemma 5.3. □

In the sequel, the number $c_\Gamma = \sum_{j=1}^{l} c_j$ is very important in the proof of Theorem 1.1. Let us analyse the interaction between $\sum_{j=1}^{l} c_{\lambda,j}$, $b_{\lambda,\Gamma}$ and c_Γ, using the fact that $c_\Gamma \in (0, S^{N/p}/(2p))$ (see Remark 4.4).

Lemma 5.4. *We have the following facts:*

(a) $\sum_{j=1}^{l} c_{\lambda,j} \le b_{\lambda,\Gamma} \le c_\Gamma$ *for all* $\lambda \ge 1$.

(b) $\Phi_\lambda(\gamma(s_1,\dots,s_l)) < c_\Gamma$ *for all* $\lambda \ge 1$, $\gamma \in \Lambda_*$ *and* $(s_1,\dots,s_l) \in \partial([1/R^2,1]^l)$.

Proof. Since γ_0 defined in (5.10) belongs to Λ_*, we have

$$\begin{aligned} b_{\lambda,\Gamma} &\le \max_{(s_1,\dots,s_l)\in[1/R^2,1]^l} \Phi_\lambda(\gamma_0(s_1,\dots,s_l)) \\ &= \max_{(s_1,\dots,s_l)\in[1/R^2,1]^l} \sum_{j=1}^{l} I_j(sRw_j) = \sum_{j=1}^{l} c_j = c_\Gamma. \end{aligned}$$

Fix $(t_1,\dots,t_l) \in [1/R^2,1]^l$ given in Lemma 5.3 and recall that $c_{\lambda,j}$ is given by

$$c_{\lambda,j} = \inf\{\Phi_{\lambda,j}(u) : u \in W_0^{1,p}(\Omega_j') \setminus \{0\}, \Phi_{\lambda,j}'(u)\cdot u = 0\}.$$

It follows that $\Phi_{\lambda,j}(\gamma(t_1,\dots,t_l)) \ge c_{\lambda,j}$ for all $j \in \Gamma$. On the other hand,

$$\Phi_\lambda(u) = \Phi_{\lambda,\mathbb{R}^N\setminus\Omega_\Gamma'}(u) + \sum_{j=1}^{l} \Phi_{\lambda,\Omega_j'}(u),$$

where

$$\begin{aligned} \Phi_{\lambda,\mathbb{R}^N\setminus\Omega_\Gamma'}(u) &:= \frac{1}{p}\int_{\lambda,\mathbb{R}^N\setminus\Omega_\Gamma'} (|\nabla u|^p + (\lambda V(x) + Z(x))|u|^p)\,dx \\ &\quad + \frac{b}{2}\bigg(\int_{\lambda,\mathbb{R}^N\setminus\Omega_\Gamma'} \frac{1}{p}|\nabla u|^p\,dx\bigg)^2 - \int_{\lambda,\mathbb{R}^N\setminus\Omega_\Gamma'} \widetilde{F}(u)\,dx. \end{aligned}$$

If we recall the definition of $\widetilde{F}(u)$, we have $\widetilde{F}(u) \le \nu_0|s|^p/p$. This fact together with Lemma 2.1 imply that

$$\begin{aligned} \Phi_{\lambda,\mathbb{R}^N\setminus\Omega_\Gamma'}(u) &:= \frac{1}{p}\int_{\lambda,\mathbb{R}^N\setminus\Omega_\Gamma'} (|\nabla u|^p + (\lambda V(x) + Z(x))|u|^p)\,dx \\ &\quad + \frac{b}{2}\bigg(\int_{\lambda,\mathbb{R}^N\setminus\Omega_\Gamma'} \frac{1}{p}|\nabla u|^p\,dx\bigg)^2 - \int_{\lambda,\mathbb{R}^N\setminus\Omega_\Gamma'} \widetilde{F}(u)\,dx \\ &\ge \frac{1}{p}(\|u\|_{\lambda,\mathbb{R}^N\setminus\Omega_\Gamma'}^p - \nu_0|u|_{p,\mathbb{R}^N\setminus\Omega_\Gamma'}^p) \ge \frac{1}{p}\delta_0\|u\|_{\lambda,\mathbb{R}^N\setminus\Omega_\Gamma'}^p \ge 0. \end{aligned}$$

Thus

$$\begin{aligned} \Phi_\lambda(u) &= \Phi_{\lambda,\mathbb{R}^N\setminus\Omega_\Gamma'}(u) + \sum_{j=1}^{l} \Phi_{\lambda,\Omega_j'}(u) \ge \sum_{j=1}^{l} \Phi_{\lambda,\Omega_j'}(u) \\ &\ge \sum_{j=1}^{l} \inf\{\Phi_{\lambda,j}(u) : u \in E_\lambda \setminus \{0\}, \Phi_{\lambda,j}'(u)\cdot u = 0\} = \sum_{j=1}^{l} c_{\lambda,j}. \end{aligned}$$

From definition of $b_{\lambda,\Gamma}$, we conclude that

$$b_{\lambda,\Gamma} \geq \sum_{j=1}^{l} c_{\lambda,j}.$$

This completes the proof of Lemma 5.4 (a).

Since $\gamma(s_1,\dots,s_l) = \gamma_0(s_1,\dots,s_l)$ on $\partial([1/R^2,1]^l)$ we have

$$\Phi_\lambda(\gamma_0(s_1,\dots,s_l)) = \sum_{j=1}^{l} I_j(s_jRw_j).$$

Moreover, $I_j(s_jRw_j) \leq c_j$ for all $j \in \Gamma$ and for some $j_0 \in \Gamma$, $s_{j_0} \in \{1/R^2, 1\}$ and $I_{j_0}(s_{j_0}Rw_{j0}) \leq c_{j_0}/2$. Therefore, $\Phi_\lambda(\gamma_0(s_1,\dots,s_l)) \leq c_\Gamma - \varepsilon$, for some $\varepsilon > 0$. This completes the proof of Lemma 5.4 (b). □

Corollary 5.5.

(a) *$b_{\lambda,\Gamma} \to c_\Gamma$ as $\lambda \to \infty$.*

(b) *$b_{\lambda,j}$ is a critical value of Φ_λ for large λ.*

Proof. (a) For all $\lambda \geq 1$ and for each j we have $0 < c_{\lambda,j} \leq c_j$. Using the same type of idea as explored in the proof of Lemma 4.7, we can prove that $c_{\lambda,j} \to c_j$ as $\lambda \to \infty$ and thus, from Lemma 5.4, $b_{\lambda,\Gamma} \to c_\Gamma$ as $\lambda \to \infty$.

(b) By Corollary 5.5 (a) and (4.5), we may choose λ large enough such that $b_{\lambda,\Gamma},\ c_\Gamma \in (0, S^{N/p}/(2p))$. Lemma 4.5 implies that any $(\mathrm{PS})_{b_{\lambda,\Gamma}}$-sequence of the functional Φ_λ has a strongly convergent subsequence in E_λ. Employing this fact, we can use the well-known arguments involving the deformation lemma to conclude that $b_{\lambda,\Gamma}$ is a critical level of Φ_λ for $\lambda \geq 1$. □

In order to prove Theorem 1.1, we need to find a positive solution u_λ for λ large enough, which approaches a least-energy solution in each Ω_j $(j \in \Gamma)$ and vanishes elsewhere as $\lambda \to \infty$. To this end, we will prove two lemmas that, together with the estimates made in the above section, imply that Theorem 1.1 holds.

Hereafter

$$M := 1 + \sum_{j=1}^{k} \sqrt{\left(\frac{1}{p} - \frac{1}{\theta}\right)^{-1} c_j}, \qquad \overline{B}_{M+1}(0) := \{u \in E_\lambda : \|u\|_\lambda \leq M+1\}$$

and we choose

$$0 < \mu < \frac{1}{3} \min_{j \in \{1,\dots,l\}} c_j. \tag{5.11}$$

We define

$$D^\mu_\lambda := \{u \in \overline{B}_{M+1}(0) : \|u\|_{\lambda,\mathbb{R}^N \setminus \Omega'_j} \leq \mu, |\Phi_{\lambda,\Omega'_j}(u) - c_j| \leq \mu \text{ for all } j \in \Gamma\}.$$

We also use the notation $\Phi_\lambda^{c_\Gamma} := \{u \in E_\lambda : \Phi_\lambda(u) \le c_\Gamma\}$. We remark that $D_\mu^\lambda \cap \Phi_\lambda^{c_\Gamma}$ contains all functions of the form

$$w(x) = \begin{cases} w_j(x) & \text{in } \Omega_j, \\ 0 & \text{in } \mathbb{R}^N \setminus \overline{\Omega}_\Gamma. \end{cases}$$

We have the following lemma.

Lemma 5.6. *There exist $\sigma_0 > 0$ and $\Lambda_0 \ge 0$ independent of λ such that*

$$\|\Phi_\lambda'(u)\|_\lambda \ge \sigma_0 \quad \text{for } \lambda_0 \ge \Lambda_0 \text{ and for all } u \in (D_\lambda^{2\mu} \setminus D_\lambda^\mu) \cap \Phi_\lambda^{c_\Gamma}. \tag{5.12}$$

Proof. We prove it by contradiction. Suppose that there exist $\lambda_n \to \infty$ and $u_n \in (D_\lambda^{2\mu} \setminus D_\lambda^\mu) \cap \Phi_\lambda^{c_\Gamma}$ such that $\|\Phi_\lambda'(u_n)\|_\lambda \to 0$. Since $u_n \in D_{\lambda_n}^{2\mu}$, thus u_n is bounded in E_λ and $\Phi_{\lambda_n}(u_n)$ stays bounded as $n \to \infty$. We may assume that $\Phi_{\lambda_n}(u_n) \to c \le c_\Gamma$ up to a subsequence.

Applying Lemma 4.7, we can extract a subsequence $u_n \to u$ in $W^{1,p}(\mathbb{R}^N)$ and

$$\lim_{n\to\infty} \Phi_{\lambda_n}(u_n) = \sum_{j=1}^{l} I_j(u) \le c_\Gamma, \tag{5.13}$$

$$u_n \to u \quad \text{in } W^{1,p}(\mathbb{R}^N), \tag{5.14}$$

$$\int_{\mathbb{R}^N} \lambda_n V(x)|u_n|^p \, dx \to 0, \tag{5.15}$$

$$\|u_n\|_{\lambda_n, \mathbb{R}^N \setminus \Omega_\Gamma} \to 0. \tag{5.16}$$

Since $c_\Gamma = \sum_{j=1}^{l} c_j$ and c_j is the least energy level for $I_j(u)$, we have the following two possibilities:

(1) $I_j(u|_{\Omega_j}) = c_j$ for all $j \in \Gamma$.
(2) $I_{j_0}(u|_{\Omega_{j_0}}) = 0$, that is $u|_{\Omega_{j_0}} = 0$ for some $j_0 \in \Gamma$.

If (1) occurs, we have

$$\frac{1}{p}\int_{\Omega_j} (|\nabla u|^p + Z(x)|u|^p)\,dx + \frac{b}{2}\bigg(\int_{\Omega_j} \frac{1}{p}|\nabla u|^p dx\bigg)^2 - \alpha \int_{\Omega_j} f(u)\,dx - \frac{1}{p^*}\int_{\Omega_j} u^{p^*}dx = c_j$$

for all $j \in \Gamma$. It follows from (5.13)–(5.16) and together with the fact that $u_n \in D_{\lambda_n}^\mu$ for large n which is a contradiction to $u_n \in (D_\lambda^{2\mu} \setminus D_\lambda^\mu)$.

If (2) occurs, from (5.14) and (5.15) it follows that $|\Phi_{\lambda,j}(u_n) - c_{j_0}| \to c_{j_0} \ge 3\mu$. This is also a contraction to $u_n \in (D_\lambda^{2\mu} \setminus D_\lambda^\mu)$ and we complete the proof. □

The following lemma is the key of the proof of our main result.

Lemma 5.7. *Let μ satisfy* (5.11) *and let Λ_0 be the constant given in Lemma* 5.6. *Then for any $\lambda \geq \Lambda_0$, there exists a solution u_λ of problem* (3.3) *satisfying $u_\lambda \in D_\lambda^\mu \cap \Phi_\lambda^{c_\Gamma}$.*

Proof. Assume by contradiction that there are no critical points in $D_\lambda^\mu \cap \Phi_\lambda^{c_\Gamma}$. Since the Palais–Smale condition holds for $\Phi_\lambda(u)$ in $(0, (aS)^{N/p}/(2p))$, there exists a constant $d_\lambda > 0$ such that $\|\Phi'_\lambda(u)\| \geq d_\lambda$ for all $u \in D_\lambda^\mu \cap \Phi_\lambda^{c_\Gamma}$ and, from Lemma 5.6, we have $\|\Phi'_\lambda(u)\| \geq \sigma_0$ for all $u \in (D_\lambda^{2\mu} \setminus D_\lambda^\mu) \cap \Phi_\lambda^{c_\Gamma}$.

Let $\varphi\colon E_\lambda \to \mathbb{R}$ be a Lipschitz continuous function such that

$$\varphi(u) = \begin{cases} 1 & \text{for } u \in D_\lambda^{3\mu/2}, \\ 0 & \text{for } u \notin D_\lambda^{2\mu}, \end{cases}$$

and $0 \leq \varphi(u) \leq 1$ for any $u \in E_\lambda$. For any $u \in \Phi_\lambda^{c_\Gamma}$, we define

$$W(u) := -\varphi(u) \frac{\Phi'_\lambda(u)}{\|\Phi'_\lambda(u)\|_\lambda} \colon \Phi_\lambda^{c_\Gamma} \to E_\lambda.$$

Here we identify E_λ^* and E_λ by the Riesz representation theorem. We consider the deformation flow $\eta\colon [0,\infty) \times \Phi_\lambda^{c_\Gamma} \to \Phi_\lambda^{c_\Gamma}$ defined by

$$\frac{d\eta}{dt} = W(\eta(t,u)), \quad \eta(0,u) = u \in \Phi_\lambda^{c_\Gamma}.$$

Thus η has the following properties:

$$\frac{d}{dt}\Phi_\lambda(\eta(t,u)) = -\varphi(\eta(t,u))\|\Phi'_\lambda(\eta(t,u))\|_\lambda \leq 0, \tag{5.17}$$

$$\left\|\frac{d\eta}{dt}\right\|_\lambda \leq 1 \quad \text{for all } t, u, \tag{5.18}$$

$$\eta(t,u) = u \quad \text{for all } t \geq 0 \text{ and } u \in \Phi_\lambda^{c_\Gamma} \setminus D_\lambda^{2\mu}. \tag{5.19}$$

Let $\gamma_0(s_1,\ldots,s_l) \in \Lambda_*$ be a path defined in (5.10). We consider $\eta(t,\gamma_0(s_1,\ldots,s_l))$ for large enough t. Since for all $(s_1,\ldots,s_l) \in \partial([1/R^2,1]^l)$, $\gamma_0(s_1,\ldots,s_l) \notin D_\lambda^{2\mu}$, thus we have by (5.19) that

$$\eta(t,\gamma_0(s_1,\ldots,s_l)) = \gamma_0(s_1,\ldots,s_l) \quad \text{for all } (s_1,\ldots,s_l) \in \partial([1/R^2,1]^l)$$

and $\eta(t,\gamma_0(s_1,\ldots,s_l)) \in \Lambda_*$ for all $t \geq 0$.

Since $\operatorname{supp}\gamma_0((s_1,\ldots,s_l)(x)) \subset \overline{\Omega}_\Gamma$ for all $(s_1,\ldots,s_l) \in \partial([1/R^2,1]^l)$ and hence $\Phi_\lambda(\gamma_0(s_1,\ldots,s_l)(x))$ and $\|\gamma_0(s_1,\ldots,s_l)(x)\|_{\lambda,j}$, etc. do not depend on $\lambda \geq 0$. On the other hand, $\Phi_\lambda(\gamma_0(s_1,\ldots,s_l)(x)) \leq c_\Gamma$ for all $(s_1,\ldots,s_l) \in [1/R^2,1]^l$ and $\Phi_\lambda(\gamma_0(s_1,\ldots,s_l)(x)) = c_\Gamma$ if and only if $s_j = 1/R$, that is $\gamma_0(s_1,\ldots,s_l)(x)|_{\Omega_j} = w_j$ for $j \in \Gamma$. Thus we have

$$m_0 := \max\{\Phi_\lambda(u) : u \in \gamma_0([1/R^2,1]^l) \setminus D_\lambda^\mu\} \tag{5.20}$$

is dependent of λ and $m_0 < c_\Gamma$.

From (5.19), it is easy to see that for any $t > 0$

$$\|\eta(0, \gamma_0(s_1, \dots, s_l)) - \eta(t, \gamma_0(s_1, \dots, s_l))\|_\lambda \leq t.$$

Since $\Phi_{\lambda,j}(u) \in C^1(E_\lambda)$ for all $j = 1, \dots, l$, and due to assumptions (f_1)–(f_4), it is easy to see that for large enough T, there exists a positive number $p_0 > 0$ which is independent of λ such that for all $j = 1, \dots, l$ and $t \in [0, T]$,

$$\|\Phi'_{\lambda,j}(\eta(t, \gamma_0(s_1, \dots, s_l)))\|_\lambda \leq p_0. \tag{5.21}$$

We claim that for large enough T,

$$\max_{(s_1,\dots,s_l)\in[1/R^2,1]^l} \Phi_\lambda(\eta(T, \gamma_0(s_1, \dots, s_l)(x))) \leq \max\{m_0,\ c_\Gamma - \tau_0\mu/2\}, \tag{5.22}$$

where m_0 is given in (5.20), $\tau_0 = \max\{\sigma_0, \sigma_0/p_0\}$. In fact, if $\gamma_0(s_1, \dots, s_l)(x) \notin D^\mu_\lambda$, then by (5.21) we have $\Phi_\lambda(\eta(T, \gamma_0(s_1, \dots, s_l)(x))) \leq m_0$ and thus (5.22) holds. Now we consider the case $\gamma_0(s_1, \dots, s_l)(x) \in D^\mu_\lambda$, we consider the behavior of $\widetilde{\eta}(t) := \eta(t, \gamma_0(s_1, \dots, s_l))$.

We set $\widetilde{d}_\lambda := \min\{d_\lambda, \sigma_0\}$ and $T = \sigma_0\mu/(2\widetilde{d}_\lambda)$. We consider two cases:

(1) $\widetilde{\eta}(t) \in D^{3\mu/2}_\lambda$ for all $t \in [0, T]$.

(2) $\widetilde{\eta}(t_0) \in \partial D^{3\mu/2}_\lambda$ for some $t_0 \in [0, T]$.

When (1) holds, we have $\varphi(\widetilde{\eta}(t)) \equiv 1$ and $\|\Phi'_\lambda(\widetilde{\eta}(t))\|_\lambda \geq \widetilde{d}_\lambda$ for all $t \in [0, T]$. Thus by (5.17), we have

$$\begin{aligned}
\Phi_\lambda(\widetilde{\eta}(T)) &= \Phi_\lambda(\gamma_0(s_1, \dots, s_l)) + \int_0^T \frac{d}{ds}\Phi_\lambda(\widetilde{\eta}(t)) \\
&= \Phi_\lambda(\gamma_0(s_1, \dots, s_l)) - \int_0^T \varphi(\widetilde{\eta}(s))\|\Phi'_\lambda(\widetilde{\eta}(s))\|_\lambda \, ds \\
&\leq c_\Gamma - \int_0^T \widetilde{d}_\lambda ds = c_\Gamma - \widetilde{d}_\lambda T = c_\Gamma - \frac{1}{2}\sigma_0\mu \leq c_\Gamma - \frac{1}{2}\tau_0\mu.
\end{aligned}$$

When (2) holds, there exists $0 \leq t_1 < t_2 \leq T$ such that

$$\widetilde{\eta}(t_1) \in \partial D^\mu_\lambda, \tag{5.23}$$

$$\widetilde{\eta}(t_2) \in \partial D^{3\mu/2}_\lambda, \tag{5.24}$$

$$\widetilde{\eta}(t) \in \partial D^{3\mu/2}_\lambda \setminus D^\mu_\lambda \quad \text{for all } t \in [t_1, t_2]. \tag{5.25}$$

It follows (5.24) that, for some $j_0 \in \Gamma$,

$$\|\widetilde{\eta}(t_2)\|_{\lambda,\mathbb{R}^N\setminus\Omega'_\Gamma} = \frac{3\mu}{2} \quad \text{or} \quad |\Phi_{\lambda,\Omega'_{j_0}}(\widetilde{\eta}(t_2)) - c_{j_0}| = \frac{3\mu}{2}.$$

We only consider the latter case, the former case can be obtained in a similar way. By (5.23), $|\Phi_{\lambda,\Omega'_{j_0}}(\widetilde{\eta}(t_1)) - c_{j_0}| \leq \mu$. Thus we have

$$|\Phi_{\lambda,\Omega'_{j_0}}(\widetilde{\eta}(t_2)) - \Phi_{\lambda,\Omega'_{j_0}}(\widetilde{\eta}(t_1))| \geq |\Phi_{\lambda,\Omega'_{j_0}}(\widetilde{\eta}(t_2)) - c_{j_0}| - |\Phi_{\lambda,\Omega'_{j_0}}(\widetilde{\eta}(t_1)) - c_{j_0}| \geq \frac{1}{2}\mu.$$

On the other hand, by the mean value theorem, there exists $t' \in (t_1, t_2)$ such that

$$|\Phi_{\lambda,\Omega'_{j_0}}(\widetilde{\eta}(t_2)) - \Phi_{\lambda,\Omega'_{j_0}}(\widetilde{\eta}(t_1))| = \bigg|\Phi_{\lambda,\Omega'_{j_0}} \cdot \frac{d\widetilde{\eta}}{dt}(t')\bigg|(t_2 - t_1).$$

From (5.18) and (5.21) we have that $t_2 - t_1 \geq \mu/2p_0$. Thus we have

$$\begin{aligned}\Phi_\lambda(\widetilde{\eta}(T)) &= \Phi_\lambda(\gamma_0(s_1, \ldots, s_l)) + \int_0^T \frac{d}{ds}\Phi_\lambda(\widetilde{\eta}(t)) \\ &= \Phi_\lambda(\gamma_0(s_1, \ldots, s_l)) - \int_0^T \varphi(\widetilde{\eta}(s))\|\Phi'_\lambda(\widetilde{\eta}(s))\|_\lambda \, ds \\ &\leq c_\Gamma - \int_{t_1}^{t_2} \varphi(\widetilde{\eta}(s))\|\Phi'_\lambda(\widetilde{\eta}(s))\|_\lambda \, ds \\ &= c_\Gamma - \sigma_0(t_2 - t_1) = c_\Gamma - \frac{1}{2}\sigma_0\mu \leq c_\Gamma - \frac{1}{2}\tau_0\mu\end{aligned}$$

and so (5.22) is proved. We recall that $\widetilde{\eta}(T) = \eta(T, \gamma_0(s_1, \ldots, s_l)) \in \Lambda_*$. Thus

$$b_{\lambda,\Gamma} \leq \Phi_\lambda(\widetilde{\eta}(T)) \leq \max\{m_0,\ c_\Gamma - \tau_0\mu/2\}. \tag{5.26}$$

However, by Corollary 5.5, we have $b_{\lambda,\Gamma} \to c_\Gamma$. This is a contradiction with (5.26), thus $\Phi_\lambda(u)$ has a critical point $u_\lambda \in D^\mu_\lambda$ for large enough λ and we have completed the proof of Lemma 5.7. □

Now we give the proof of main results.

Proof of Theorem 1.1. From Lemma 5.7 there exists a family (u_λ) of positive solutions to problem (3.3) verifying the following properties:

(I) For fixed $\mu > 0$ there exists Λ_0 such that $\|u_\lambda\|_{\lambda,\mathbb{R}^N\setminus\Omega'_\Gamma} \leq \mu$ for all $\lambda \geq \Lambda_0$. Thus, from the proof of Lemma 4.8, for fixed sufficiently small μ, we conclude that $|u_\lambda|_{\infty,\mathbb{R}^N\setminus\Omega'_\Gamma} \leq a$ for all $\lambda \geq \Lambda_0$ showing that u_λ is a positive solution to problem (1.1).

(II) Fix $\lambda_n \to \infty$ and $\mu_n \to 0$, the sequence (u_n) satisfies

- $\Phi'_{\lambda_n}(u_{\lambda_n}) = 0$ for all $n \in \mathbb{N}$,
- $\|u_{\lambda_n}\|_{\lambda_n,\mathbb{R}^N\setminus\Omega'_\Gamma} \to 0$,
- $\Phi_{\lambda_n,j}(u_{\lambda_n}) \to c_j$ for all $j \in \Gamma$, and
- $u_{\lambda_n} \to u$ in $W^{1,p}(\mathbb{R}^N)$ with $u \in W^{1,p}_0(\Omega_\Gamma)$.

The proof of Theorem 1.1 follows. □

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