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BIFURCATION AND MULTIPLICITY RESULTS FOR CLASSES OF p, q-LAPLACIAN SYSTEMS

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Abstract. We study positive solutions to boundary value problems of the form

$$\begin{cases} -\Delta_p u = \lambda\{u^{p-1-\alpha} + f(v)\} & \text{in } \Omega, \\ -\Delta_q v = \lambda\{v^{q-1-\beta} + g(u)\} & \text{in } \Omega, \\ u = 0 = v & \text{on } \partial\Omega, \end{cases}$$

where $\Delta_m u := \operatorname{div}(|\nabla u|^{m-2}\nabla u)$, $m > 1$, is the m-Laplacian operator of u, $\lambda > 0$, $p, q > 1$, $\alpha \in (0, p-1)$, $\beta \in (0, q-1)$ and Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 1$, with smooth boundary $\partial\Omega$. Here $f, g\colon [0,\infty) \to \mathbb{R}$ are nondecreasing continuous functions with $f(0) = 0 = g(0)$. We first establish that for $\lambda \approx 0$ there exist positive solutions bifurcating from the trivial branch $(\lambda, u \equiv 0, v \equiv 0)$ at $(0, 0, 0)$. We further discuss an existence result for all $\lambda > 0$ and a multiplicity result for a certain range of λ under additional assumptions on f and g. We employ the method of sub-super solutions to establish our results.

1. Introduction

Consider boundary value problems of the form

$$\begin{cases} -\Delta_p u = \lambda \widetilde{f}(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

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where $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $p > 1$, is the p-Laplacian operator of u, $\lambda > 0$ and Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 1$, with smooth boundary $\partial\Omega$. Here $\widetilde{f}\colon [0,\infty) \to \mathbb{R}$ is a nondecreasing continuous function. When $\widetilde{f}(0) > 0$, there is a rich history on the study of positive solutions. The authors in [5] have considered such problems in the Laplacian case ($p = 2$) and established an existence result for all $\lambda > 0$ and a multiplicity result for a certain range of λ under additional assumptions on $\widetilde{f}$. Later in [7], these results were extended to the p-Laplacian case ($p > 1$). In particular, the authors in [7] proved the existence of a positive solution for all $\lambda > 0$ when $\widetilde{f}$ is p sublinear at ∞, and multiplicity results for a certain range of λ when there exist a and b such that $0 < a < b$ and $(a^{p-1}/\widetilde{f}(a))/(b^{p-1}/\widetilde{f}(b))$ is sufficiently large. See also [1], [2] and [6] for related results in the case $\widetilde{f}(0) > 0$. Here, we focus on the case $\widetilde{f}(0) = 0$. If $\widetilde{f}(0) > 0$, then $u \equiv 0$ is a very useful nonnegative strict subsolution to help with the study of establishing positive solutions. In this paper, $u \equiv 0$ is a solution for each $\lambda > 0$ and hence we lack the presence of this trivial nonnegative strict subsolution. However, we use the presence of the term $u^{p-1-\alpha}$ as our advantage to overcome this difficulty and show that positive solutions bifurcate at $(0,0)$ from the trivial branch $(\lambda, u \equiv 0)$. Under additional properties on $\widetilde{f}$, we establish further existence and multiplicity results. We also extend these results to classes of p,q-Laplacian systems. In particular, we consider boundary value problems of the form

$$
\begin{cases}
-\Delta_p u = \lambda\{u^{p-1-\alpha} + f(v)\} & \text{in } \Omega,\\
-\Delta_q v = \lambda\{v^{q-1-\beta} + g(u)\} & \text{in } \Omega,\\
u = 0 = v & \text{on } \partial\Omega,
\end{cases}
\tag{1.1}
$$

where $\Delta_m u := \operatorname{div}(|\nabla u|^{m-2}\nabla u)$, $m > 1$, is the m-Laplacian operator of u, $\lambda > 0$, $p,q > 1$, $\alpha \in (0, p-1)$, $\beta \in (0, q-1)$ are parameters and Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 1$, with smooth boundary $\partial\Omega$. Here $f,g\colon [0,\infty) \to \mathbb{R}$ are nondecreasing continuous functions with $f(0) = 0 = g(0)$. Clearly for all λ, $(u \equiv 0,\ v \equiv 0)$ is a solution of (1.1). In this paper, we are interested in the study of solution $(u,v) \in W^{1,p}(\Omega) \cap C(\overline{\Omega}) \times W^{1,q}(\Omega) \cap C(\overline{\Omega})$ with $u, v > 0$ in Ω. We first establish:

Theorem 1.1. *There exists $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0)$, (1.1) has a positive solution (u,v) such that $\|u\|_\infty \to 0$, $\|v\|_\infty \to 0$ as $\lambda \to 0$ (see Figure 1).*

Next we consider the case when f,g satisfy the following combined p,q sublinear condition at ∞:

(H$_1$) $\displaystyle\lim_{s\to\infty} \frac{f(Mg(s)^{1/(q-1)})}{s^{p-1}} = 0$, for all $M > 0$,

and establish:

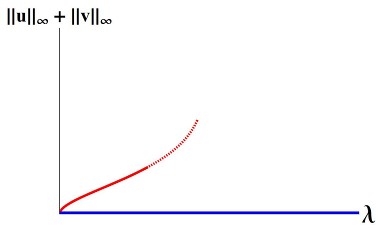

FIGURE 1

THEOREM 1.2. *Let* (H_1) *hold. Then* (1.1) *has a positive solution* (u, v) *for all* $\lambda > 0$ (*see Figure* 2).

FIGURE 2

If in addition, f, g satisfy:

(H_2) There exist positive constants a_1 and $a_2 (> a_1)$ such that

$$\frac{\min[\min\{a_1^\alpha, a_1^{p-1}/f(a_1)\}, \min\{a_1^\beta, a_1^{q-1}/g(a_1)\}]}{\max\{a_2^{p-1}/f(a_2), a_2^{q-1}/g(a_2)\}} \geq C(\Omega),$$

where

$$C(\Omega) = 2\max\{\|e_p\|_\infty^{p-1}, \|e_q\|_\infty^{q-1}\} \min\big[\max\{A_{p,p}, A_{p,q}\}, \max\{A_{q,p}, A_{q,q}\}\big],$$

e_m is the solution of $-\Delta_m e = 1;\ \Omega,\ e = 0;\ \partial\Omega$,

$$A_{m,n} = \frac{(N+m-1)^{N+n-1}}{R^n N^{N-1}(m-1)^{n-1}}$$

and R is the radius of the largest inscribed ball B_R in Ω,

we prove:

THEOREM 1.3. *Let* (H_1)–(H_2) *hold. Then* (1.1) *has at least three positive solutions for* $\lambda \in (\lambda_*, \lambda^*)$, *where* (*see Figure* 3)

$$\lambda_* = \min\left[\max\left\{\frac{a_2^{p-1}}{f(a_2)}A_{p,p}, \frac{a_2^{q-1}}{g(a_2)}A_{p,q}\right\}, \max\left\{\frac{a_2^{p-1}}{f(a_2)}A_{q,p}, \frac{a_2^{q-1}}{g(a_2)}A_{q,q}\right\}\right],$$

$$\lambda^* = \min\left[\frac{1}{2\|e_p\|_\infty^{p-1}}\min\left\{a_1^\alpha, \frac{a_1^{p-1}}{f(a_1)}\right\}, \frac{1}{2\|e_q\|_\infty^{q-1}}\min\left\{a_1^\beta, \frac{a_1^{q-1}}{g(a_1)}\right\}\right].$$

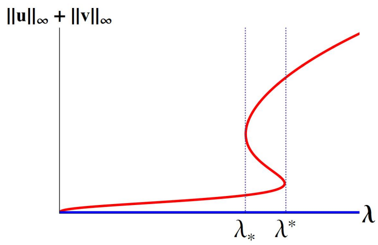

Figure 3

In the case of single equations, namely equation of the form

$$\begin{cases} -\Delta_p u = \lambda\{u^{p-1-\alpha} + f(u)\} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

our results easily reduce to:

Theorem 1.4. *There exists $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0)$, (1.2) has a positive solution u such that $\|u\|_\infty \to 0$ as $\lambda \to 0$.*

Theorem 1.5. *Assume $f(s)/s^{p-1} \to 0$ as $s \to \infty$. Then (1.2) has a positive solution u for all $\lambda > 0$.*

Theorem 1.6. *Assume $f(s)/s^{p-1} \to 0$ as $s \to \infty$ and there exist positive constants a_1 and $a_2(> a_1)$ such that*

$$\frac{\min\{a_1^\alpha, a_1^{p-1}/f(a_1)\}}{a_2^{p-1}/f(a_2)} > \widetilde{C}(\Omega), \quad \textit{where } \widetilde{C}(\Omega) = 2\|e_p\|_\infty^{p-1} A_{p,p},$$

e_p is a solution of $-\Delta_p e = 1$; Ω, $e = 0$; $\partial\Omega$,

$$A_{p,p} = \frac{(N+p-1)^{N+p-1}}{R^p N^{N-1}(p-1)^{p-1}}$$

and R is the radius of the largest inscribed ball B_R in Ω. Then (1.2) has a positive solution for all λ, and at least three positive solutions for $\lambda \in (\lambda_*, \lambda^*)$, where

$$\lambda_* = \frac{a_2^{p-1}}{f(a_2)} A_{p,p}, \qquad \lambda^* = \frac{1}{2\|e_p\|_\infty^{p-1}} \min\left\{a_1^\alpha, \frac{a_1^{p-1}}{f(a_1)}\right\}.$$

Remark 1.7. If $s^{p-1}/(s^{p-1-\alpha} + f(s))$ is strictly increasing on $(0, \infty)$, which will be the case if $s^{p-1}/f(s)$ is increasing on $(0, \infty)$, and there exists $c > 0$ such that $s^{p-1-\alpha} + f(s) \le c(s^{p-1} + 1)$ for $s \in [0, \infty)$, which will be satisfied if $f(s)/s^{p-1} \to 0$ as $s \to \infty$, then (1.2) has at most one positive solution for all $\lambda > 0$ (see [4]). Note that the hypotheses in Theorem 1.6 do not allow $s^{p-1}/f(s)$ to be increasing for all $s \in (0, \infty)$.

We establish Theorems 1.1–1.3 by the method of sub-super solutions. By a subsolution of (1.1) we mean a pair of functions $(\psi,\overline{\psi}) \in W^{1,p}(\Omega)\cap C(\overline{\Omega}) \times W^{1,q}(\Omega)\cap C(\overline{\Omega})$ such that $(\psi,\overline{\psi}) = (0,0)$ on $\partial\Omega$ and

$$\int_\Omega |\nabla\psi|^{p-2}\nabla\psi\cdot\nabla\zeta \le \int_\Omega \lambda(\psi^{p-1-\alpha} + f(\overline{\psi}))\zeta \quad \text{for all } \zeta\in W,$$
$$\int_\Omega |\nabla\overline{\psi}|^{q-2}\nabla\overline{\psi}\cdot\nabla\zeta \le \int_\Omega \lambda(\overline{\psi}^{q-1-\beta} + g(\psi))\zeta \quad \text{for all } \zeta\in W,$$

where $W = \{h\in C_0^\infty(\Omega) : h \ge 0 \text{ in } \Omega\}$. By a supersolution of (1.1) we mean a pair of functions $(\phi,\overline{\phi}) \in W^{1,p}(\Omega)\cap C(\overline{\Omega}) \times W^{1,q}(\Omega)\cap C(\overline{\Omega})$ such that $(\phi,\overline{\phi}) = (0,0)$ on $\partial\Omega$ and

$$\int_\Omega |\nabla\phi|^{p-2}\nabla\phi\cdot\nabla\zeta \ge \int_\Omega \lambda(\phi^{p-1-\alpha} + f(\overline{\phi}))\zeta \quad \text{for all } \zeta\in W,$$
$$\int_\Omega |\nabla\overline{\phi}|^{q-2}\nabla\overline{\phi}\cdot\nabla\zeta \ge \int_\Omega \lambda(\overline{\phi}^{q-1-\beta} + g(\phi))\zeta \quad \text{for all } \zeta\in W.$$

By a strict subsolution of (1.1) we mean a subsolution which is not a solution. By a strict supersolution of (1.1) we mean a supersolution which is not a solution. Then the following results are well-known (see [3], [6] and [8]).

PROPOSITION 1.8. *If there exist a subsolution* $(\psi,\overline{\psi})$ *and a supersolution* $(\phi,\overline{\phi})$ *of* (1.1) *such that* $(\psi,\overline{\psi}) \le (\phi,\overline{\phi})$, *then* (1.1) *has at least one solution* $(u,v) \in W^{1,p}(\Omega)\cap C(\overline{\Omega}) \times W^{1,q}(\Omega)\cap C(\overline{\Omega})$ *satisfying* $(\psi,\overline{\psi}) \le (u,v) \le (\phi,\overline{\phi})$.

PROPOSITION 1.9. *Let f and g be nonnegative and nondecreasing, and suppose there exist a subsolution* $(\psi_1,\overline{\psi}_1)$, *a strict supersolution* $(\phi_1,\overline{\phi}_1)$, *a strict subsolution* $(\psi_2,\overline{\psi}_2)$, *and a supersolution* $(\phi_2,\overline{\phi}_2)$ *for* (1.1) *such that* $(\psi_1,\overline{\psi}_1) \le (\phi_1,\overline{\phi}_1) \le (\phi_2,\overline{\phi}_2)$, $(\psi_1,\overline{\psi}_1) \le (\psi_2,\overline{\psi}_2) \le (\phi_2,\overline{\phi}_2)$, *and* $(\psi_2,\overline{\psi}_2) \not\le (\phi_1,\overline{\phi}_1)$. *Then* (1.1) *has at least three distinct solutions* (u_i,v_i), $i=1,2,3$, *such that*

$$(u_1,v_1)\in[(\psi_1,\overline{\psi}_1),(\phi_1,\overline{\phi}_1)], \qquad (u_2,v_2)\in[(\psi_2,\overline{\psi}_2),(\phi_2,\overline{\phi}_2)],$$
$$(u_3,v_3)\in[(\psi_1,\overline{\psi}_1),(\phi_2,\overline{\phi}_2)]\setminus([(\psi_1,\overline{\psi}_1),(\phi_1,\overline{\phi}_1)]\cup[(\psi_2,\overline{\psi}_2),(\phi_2,\overline{\phi}_2)]).$$

We will establish Theorem 1.1 in Section 2 and Theorems 1.2–1.3 in Section 3. Finally in Section 4, we discuss simple examples satisfying the hypotheses of Theorems 1.3 and 1.6.

2. Proof of Theorem 1.1

PROOF OF THEOREM 1.1. Let $\gamma > 0$ be such that $\gamma\alpha < 1$, $\gamma\beta < 1$, $\gamma(p-1) < 1$ and $\gamma(q-1) < 1$. For sufficiently small λ, we have

$$1 \ge \lambda^{1-\gamma\alpha}\|e_p\|_\infty^{p-1-\alpha} + \lambda^{1-\gamma(p-1)} f(\lambda^\gamma \|e_q\|_\infty),$$
$$1 \ge \lambda^{1-\gamma\beta}\|e_q\|_\infty^{q-1-\beta} + \lambda^{1-\gamma(q-1)} g(\lambda^\gamma \|e_p\|_\infty).$$

Let $(w_1, \overline{w}_1) = (\lambda^\gamma e_p, \lambda^\gamma e_q)$. Then

$$\begin{aligned} -\Delta_p w_1 &= \lambda^{\gamma(p-1)} \geq \lambda((\lambda^\gamma \|e_p\|_\infty)^{p-1-\alpha} + f(\lambda^\gamma \|e_q\|_\infty)) \\ &\geq \lambda((\lambda^\gamma e_p)^{p-1-\alpha} + f(\lambda^\gamma e_q)) = \lambda(w_1^{p-1-\alpha} + f(\overline{w}_1)), \\ -\Delta_q \overline{w}_1 &= \lambda^{\gamma(q-1)} \geq \lambda((\lambda^\gamma \|e_q\|_\infty)^{q-1-\beta} + g(\lambda^\gamma \|e_p\|_\infty)) \\ &\geq \lambda((\lambda^\gamma e_q)^{q-1-\beta} + g(\lambda^\gamma e_p)) = \lambda(\overline{w}_1^{q-1-\beta} + g(w_1)). \end{aligned}$$

Thus, $(w_1, \overline{w}_1)$ is a supersolution of (1.1) for sufficiently small λ. Next, we construct a positive subsolution of (1.1). Let $z_m > 0$; Ω be the eigenfunction with $\|z_m\|_\infty = 1$ corresponding the principal eigenvalue $\lambda_{1,m}$ of the problem

$$\begin{cases} -\Delta_m z = \lambda |z|^{m-2} z & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega. \end{cases}$$

Let $m_\lambda > 0$ be sufficiently small such that $\lambda_{1,p} m_\lambda^\alpha \leq \lambda$ and $\lambda_{1,q} m_\lambda^\beta \leq \lambda$. Let $(\psi_1, \overline{\psi}_1) = (m_\lambda z_p, m_\lambda z_q)$. Then

$$\begin{aligned} -\Delta_p \psi_1 &= \lambda_{1,p} (m_\lambda z_p)^{p-1} \leq \lambda (m_\lambda z_p)^{p-1-\alpha} \leq \lambda(\psi_1^{p-1-\alpha} + f(\overline{\psi}_1)), \\ -\Delta_q \overline{\psi}_1 &= \lambda_{1,q} (m_\lambda z_q)^{q-1} \leq \lambda (m_\lambda z_q)^{q-1-\beta} \leq \lambda(\overline{\psi}_1^{q-1-\beta} + g(\psi_1)). \end{aligned}$$

Thus, $(\psi_1, \overline{\psi}_1)$ is a subsolution of (1.1) for all $\lambda > 0$. Further, we can choose m_λ sufficiently small such that $(\psi_1, \overline{\psi}_1) \leq (w_1, \overline{w}_1)$. By Proposition 1.8, there exists a positive solution (u, v) such that $(\psi_1, \overline{\psi}_1) \leq (u, v) \leq (w_1, \overline{w}_1)$ for sufficiently small λ, and note that $\|w_1\|_\infty \to 0$ and $\|\overline{w}_1\|_\infty \to 0$ as $\lambda \to 0$. □

3. Proofs of Theorems 1.2 and 1.3

Proof of Theorem 1.2. Let $(\psi_1, \overline{\psi}_1)$ be as before in the proof of Theorem 1.1. Then $(\psi_1, \overline{\psi}_1)$ is a subsolution for all $\lambda > 0$. Next, we construct a positive supersolution $(\phi_1, \overline{\phi}_1)$ of (1.1). If f and g are bounded, choose $M_\lambda \gg 1$ such that $M_\lambda^\alpha \geq 2\lambda^{(p-1-\alpha)/(p-1)} \|e_p\|_\infty^{p-1-\alpha}$, $M_\lambda^\beta \geq 2\lambda^{(q-1-\beta)/(q-1)} \|e_q\|_\infty^{q-1-\beta}$, $M_\lambda^{p-1} \geq 2\|f\|_\infty$ and $M_\lambda^{q-1} \geq 2\|g\|_\infty$.

Let $(\phi_1, \overline{\phi}_1) = (\lambda^{1/(p-1)} M_\lambda e_p, \lambda^{1/(q-1)} M_\lambda e_q)$. Then

$$\begin{aligned} -\Delta_p \phi_1 &= \lambda M_\lambda^{p-1} \geq \lambda(\lambda^{(p-1-\alpha)/(p-1)} M_\lambda^{p-1-\alpha} \|e_p\|_\infty^{p-1-\alpha} + \|f\|_\infty) \\ &\geq \lambda((\lambda^{1/(p-1)} M_\lambda e_p)^{p-1-\alpha} + f(\lambda^{1/(q-1)} M_\lambda e_q)) = \lambda(\phi_1^{p-1-\alpha} + f(\overline{\phi}_1)), \\ -\Delta_q \overline{\phi}_1 &= \lambda M_\lambda^{q-1} \geq \lambda(\lambda^{(q-1-\beta)/(q-1)} M_\lambda^{q-1-\beta} \|e_q\|_\infty^{q-1-\beta} + \|g\|_\infty) \\ &\geq \lambda((\lambda^{1/(q-1)} M_\lambda e_q)^{q-1-\beta} + g(\lambda^{1/(p-1)} M_\lambda e_p)) = \lambda(\overline{\phi}_1^{q-1-\beta} + g(\phi_1)). \end{aligned}$$

Thus, $(\phi_1, \overline{\phi}_1)$ is a supersolution of (1.1) for all $\lambda > 0$.

Suppose $g(s) \to \infty$ as $s \to \infty$. Choose $M_\lambda \gg 1$ such that

$$\frac{1}{\|e_p\|_\infty^{p-1}} \geq \lambda \bigg(\frac{1}{M_\lambda^\alpha \|e_p\|_\infty^\alpha} + \frac{f((2\lambda)^{1/(q-1)} \|e_q\|_\infty g(M_\lambda \|e_p\|_\infty)^{1/(q-1)})}{(M_\lambda \|e_p\|_\infty)^{p-1}} \bigg)$$

and

$$g(M_\lambda\|e_p\|_\infty)^{\beta/(q-1)} \geq (2\lambda)^{(q-1-\beta)/(q-1)}\|e_q\|_\infty^{q-1-\beta}.$$

Let $(\phi_1,\overline{\phi}_1) = (M_\lambda e_p, (2\lambda)^{1/(q-1)}g(M_\lambda\|e_p\|_\infty)^{1/(q-1)}e_q)$. Then

$$\begin{aligned}-\Delta_p\phi_1 &= M_\lambda^{p-1}\\ &\geq \lambda(M_\lambda^{p-1-\alpha}\|e_p\|_\infty^{p-1-\alpha} + f((2\lambda)^{1/(q-1)}g(M_\lambda\|e_p\|_\infty)^{1/(q-1)}\|e_q\|_\infty))\\ &\geq \lambda((M_\lambda e_p)^{p-1-\alpha} + f((2\lambda)^{1/(q-1)}g(M_\lambda\|e_p\|_\infty)^{1/(q-1)}e_q))\\ &= \lambda(\phi_1^{p-1-\alpha} + f(\overline{\phi}_1))\end{aligned}$$

and

$$\begin{aligned}-\Delta_q\overline{\phi}_1 &= 2\lambda g(M_\lambda\|e_p\|_\infty)\\ &\geq \lambda((2\lambda)^{(q-1-\beta)/(q-1)}g(M_\lambda\|e_p\|_\infty)^{(q-1-\beta)/(q-1)}\|e_q\|_\infty^{q-1-\beta} + g(M_\lambda\|e_p\|_\infty))\\ &\geq \lambda(((2\lambda)^{1/(q-1)}g(M_\lambda\|e_p\|_\infty)^{1/(q-1)}e_q)^{q-1-\beta} + g(M_\lambda e_p))\\ &= \lambda(\overline{\phi}_1^{q-1-\beta} + g(\phi_1)).\end{aligned}$$

Thus, $(\phi_1,\overline{\phi}_1)$ is a supersolution of (1.1) for all $\lambda > 0$. If g is bounded and $f(s) \to \infty$ as $s \to \infty$, then $\lim_{s\to\infty} g(Mf(s)^{1/(p-1)})/s^{q-1} = 0$ for every $M > 0$. Then $(\phi_1,\overline{\phi}_1) = ((2\lambda)^{1/(p-1)}f(M_\lambda\|e_q\|_\infty)^{1/(p-1)}e_p, M_\lambda e_q)$ is a supersolution of (1.1) for all $\lambda > 0$ by arguments similar to the previous case with the roles of f and g interchanged. Also, in each case if M_λ is sufficiently large, $(\psi_1,\overline{\psi}_1) \leq (\phi_1,\overline{\phi}_1)$. Hence, by Proposition 1.8, there exists a positive solution (u,v) such that $(\psi_1,\overline{\psi}_1) \leq (u,v) \leq (\phi_1,\overline{\phi}_1)$ for each $\lambda > 0$. □

PROOF OF THEOREM 1.3. We first establish this result when Ω is a ball of radius R. Let $(\phi_2,\overline{\phi}_2) = (a_1e_p/\|e_p\|_\infty, a_1e_q/\|e_q\|_\infty)$. For

$$\lambda < \min\left[\frac{1}{2\|e_p\|_\infty^{p-1}}\min\left\{a_1^\alpha, \frac{a_1^{p-1}}{f(a_1)}\right\}, \frac{1}{2\|e_q\|_\infty^{q-1}}\min\left\{a_1^\beta, \frac{a_1^{q-1}}{g(a_1)}\right\}\right],$$

we have

$$\begin{aligned}-\Delta_p\phi_2 &= \frac{a_1^{p-1}}{\|e_p\|_\infty^{p-1}} = \frac{a_1^{p-1}}{2\|e_p\|_\infty^{p-1}} + \frac{a_1^{p-1}}{2\|e_p\|_\infty^{p-1}} > \lambda a_1^{p-1-\alpha} + \lambda f(a_1)\\ &\geq \lambda\left(\left(\frac{a_1}{\|e_p\|_\infty}e_p\right)^{p-1-\alpha} + f\left(\frac{a_1}{\|e_q\|_\infty}e_q\right)\right) = \lambda(\phi_2^{p-1-\alpha} + f(\overline{\phi}_2))\end{aligned}$$

and

$$\begin{aligned}-\Delta_q\overline{\phi}_2 &= \frac{a_1^{q-1}}{\|e_q\|_\infty^{q-1}} = \frac{a_1^{q-1}}{2\|e_q\|_\infty^{q-1}} + \frac{a_1^{q-1}}{2\|e_q\|_\infty^{q-1}} > \lambda a_1^{q-1-\beta} + \lambda g(a_1)\\ &\geq \lambda\left(\left(\frac{a_1}{\|e_q\|_\infty}e_q\right)^{q-1-\beta} + g\left(\frac{a_1}{\|e_p\|_\infty}e_p\right)\right) = \lambda(\overline{\phi}_2^{q-1-\beta} + g(\phi_2)).\end{aligned}$$

Hence $(\phi_2,\overline{\phi}_2)$ is a strict supersolution with $\|\phi_2\|_\infty = \|\overline{\phi}_2\|_\infty = a_1$.

We construct a positive strict subsolution $(\psi_2, \overline{\psi}_2)$ of (1.1) for

$$\lambda > \lambda_* = \min\left[\max\left\{\frac{a_2^{p-1}}{f(a_2)}A_{p,p}, \frac{a_2^{q-1}}{g(a_2)}A_{p,q}\right\}, \max\left\{\frac{a_2^{p-1}}{f(a_2)}A_{q,p}, \frac{a_2^{q-1}}{g(a_2)}A_{q,q}\right\}\right].$$

Consider the following boundary value problem:

$$\begin{cases} -\Delta_p u = \lambda f(v) & \text{in } \Omega, \\ -\Delta_q v = \lambda g(u) & \text{in } \Omega, \\ u = 0 = v & \text{on } \partial\Omega. \end{cases}$$

For $0 < \varepsilon < R$ and $\delta, \eta > 1$, define $\rho\colon [0,R] \to [0,1]$ by

$$\rho(t) = \begin{cases} 1 & \text{for } 0 \le t \le \varepsilon, \\ 1 - \left(1 - \left(\dfrac{R-t}{R-\varepsilon}\right)^{\eta}\right)^{\delta} & \text{for } \varepsilon < t \le R. \end{cases}$$

Then

$$\rho'(t) = \begin{cases} 0 & \text{for } 0 \le t \le \varepsilon, \\ -\dfrac{\delta\eta}{R-\varepsilon}\left(\dfrac{R-t}{R-\varepsilon}\right)^{\eta-1}\left(1 - \left(\dfrac{R-t}{R-\epsilon}\right)^{\eta}\right)^{\delta-1} & \text{for } \varepsilon < t \le R. \end{cases}$$

Let $d(t) = a_2\rho(t)$. Define $(\psi_2, \overline{\psi}_2)$ as the positive radially symmetric and decreasing solution of

$$\begin{cases} -\Delta_p \psi_2 = \lambda f(d) & \text{in } B_R, \\ -\Delta_q \overline{\psi}_2 = \lambda g(d) & \text{in } B_R, \\ \psi_2 = 0 = \overline{\psi}_2 & \text{on } \partial B_R. \end{cases}$$

Then ψ_2 and $\overline{\psi}_2$ satisfy

$$\begin{cases} -(t^{N-1}\varphi_p(\psi_2'(t)))' = \lambda t^{N-1} f(d(t)) & \text{for } t \in (0,R), \\ -(t^{N-1}\varphi_q(\overline{\psi}_2'(t)))' = \lambda t^{N-1} g(d(t)) & \text{for } t \in (0,R), \\ \psi_2'(0) = 0, \quad \psi_2(R) = 0, \quad \overline{\psi}_2'(0) = 0, \quad \overline{\psi}_2(R) = 0, \end{cases}$$

where $\varphi_m(t) = |t|^{m-2}t$ for all $t \in \mathbb{R}$. Integrating once, we get for $0 < t < R$,

$$-\varphi_p(\psi_2'(t)) = \frac{\lambda}{t^{N-1}}\int_0^t s^{N-1} f(d(s))\,ds,$$
$$-\varphi_q(\overline{\psi}_2'(t)) = \frac{\lambda}{t^{N-1}}\int_0^t s^{N-1} g(d(s))\,ds.$$

Since φ_m is monotone, φ_m^{-1} is also continuous and monotone. Hence, we have

$$-\psi_2'(t) = \varphi_p^{-1}\left(\frac{\lambda}{t^{N-1}}\int_0^t s^{N-1} f(d(s))\,ds\right),$$
$$-\overline{\psi}_2'(t) = \varphi_q^{-1}\left(\frac{\lambda}{t^{N-1}}\int_0^t s^{N-1} g(d(s))\,ds\right).$$

For $\lambda > \lambda_*$, we claim that

$$\psi_2(t) > d(t) \quad \text{and} \quad \overline{\psi}_2(t) > d(t), \quad 0 \le t < R. \tag{3.1}$$

If our claim is true, $(\psi_2, \overline{\psi}_2)$ is a strict subsolution of the boundary value problem (1.1) since

$$\begin{aligned} -\Delta_p \psi_2 &= \lambda f(d) < \lambda(\psi_2^{p-1-\alpha} + f(\overline{\psi}_2)) \quad \text{in } B_R, \\ -\Delta_q \overline{\psi}_2 &= \lambda g(d) < \lambda(\overline{\psi}_2^{q-1-\beta} + g(\psi_2)) \quad \text{in } B_R. \end{aligned}$$

It suffices to prove that $\psi_2'(t) < d'(t)$ and $\overline{\psi}_2'(t) < d'(t)$ on $(0,R]$ in order to show (3.1) since $\psi_2(R) = \overline{\psi}_2(R) = 0 = d(R)$. It is obvious on $(0,\varepsilon]$ because $\psi_2'(t) < 0 = d'(t)$ and $\overline{\psi}_2'(t) < 0 = d'(t)$. For $t > \varepsilon$, we have

$$\begin{aligned} -\psi_2'(t) &= \varphi_p^{-1}\bigg(\frac{\lambda}{t^{N-1}} \int_0^t s^{N-1} f(d(s))\,ds\bigg) \ge \varphi_p^{-1}\bigg(\frac{\lambda}{R^{N-1}} \int_0^\varepsilon s^{N-1} f(d(s))\,ds\bigg) \\ &= \varphi_p^{-1}\bigg(\frac{\lambda}{R^{N-1}} \int_0^\varepsilon s^{N-1} f(a_2)\,ds\bigg) = \varphi_p^{-1}\bigg(\frac{\lambda}{R^{N-1}} \frac{\varepsilon^N}{N} f(a_2)\bigg) \end{aligned}$$

and

$$\begin{aligned} -\overline{\psi}_2'(t) &= \varphi_q^{-1}\bigg(\frac{\lambda}{t^{N-1}} \int_0^t s^{N-1} g(d(s))\,ds\bigg) \ge \varphi_q^{-1}\bigg(\frac{\lambda}{R^{N-1}} \int_0^\varepsilon s^{N-1} g(d(s))\,ds\bigg) \\ &= \varphi_q^{-1}\bigg(\frac{\lambda}{R^{N-1}} \int_0^\varepsilon s^{N-1} g(a_2)\,ds\bigg) = \varphi_q^{-1}\bigg(\frac{\lambda}{R^{N-1}} \frac{\varepsilon^N}{N} g(a_2)\bigg). \end{aligned}$$

Noting that $|d'(t)| \le a_2\delta\eta/(R-\varepsilon)$ on (ε, R), it is easy to see that $\psi_2'(t) < d'(t)$ and $\overline{\psi}_2'(t) < d'(t)$ on (ε, R) provided

$$\varphi_p^{-1}\bigg(\frac{\lambda}{R^{N-1}} \frac{\varepsilon^N}{N} f(a_2)\bigg) > a_2 \frac{\delta\eta}{R-\varepsilon} \quad \text{and} \quad \varphi_q^{-1}\bigg(\frac{\lambda}{R^{N-1}} \frac{\varepsilon^N}{N} g(a_2)\bigg) > a_2 \frac{\delta\eta}{R-\varepsilon}.$$

Equivalently, if

$$\lambda > \max\bigg\{(\delta\eta)^{p-1} \frac{a_2^{p-1}}{f(a_2)} \frac{R^{N-1}N}{\varepsilon^N (R-\varepsilon)^{p-1}}, (\delta\eta)^{q-1} \frac{a_2^{q-1}}{g(a_2)} \frac{R^{N-1}N}{\varepsilon^N (R-\varepsilon)^{q-1}}\bigg\}. \tag{3.2}$$

Now, if

$$\lambda_* = \max\bigg\{\frac{a_2^{p-1}}{f(a_2)} A_{p,p}, \frac{a_2^{q-1}}{g(a_2)} A_{p,q}\bigg\},$$

choosing $\varepsilon = NR/(N+p-1)$ in the definition of ρ, (3.2) reduces to showing

$$\lambda > \max\bigg\{(\delta\eta)^{p-1} \frac{a_2^{p-1}}{f(a_2)} A_{p,p}, (\delta\eta)^{q-1} \frac{a_2^{q-1}}{g(a_2)} A_{p,q}\bigg\}. \tag{3.3}$$

But $\lambda > \lambda_*$. Hence we can choose $\delta(>1)$ and $\eta(>1)$ such that (3.3) is satisfied. Next, if

$$\lambda_* = \max\bigg\{\frac{a_2^{p-1}}{f(a_2)} A_{q,p}, \frac{a_2^{q-1}}{g(a_2)} A_{q,q}\bigg\},$$

choosing $\varepsilon = NR/(N+q-1)$ in the definition of ρ, (3.2) reduces to showing

$$\lambda > \max\bigg\{(\delta\eta)^{p-1}\frac{a_2^{p-1}}{f(a_2)}A_{q,p}, (\delta\eta)^{q-1}\frac{a_2^{q-1}}{g(a_2)}A_{q,q}\bigg\}. \tag{3.4}$$

Again, since $\lambda > \lambda_*$, we can choose $\delta(> 1)$ and $\eta(> 1)$ such that (3.4) is satisfied. Hence, (3.1) holds for $\lambda > \lambda_*$. Thus, $(\psi_2, \overline{\psi}_2)$ is a strict subsolution of (1.1). From the proof of Theorem 1.2, we have a sufficiently small positive subsolution $(\psi_1, \overline{\psi}_1)$ and a sufficiently large positive supersolution $(\phi_1, \overline{\phi}_1)$ such that $(\psi_1, \overline{\psi}_1) \le (\phi_2, \overline{\phi}_2) \le (\phi_1, \overline{\phi}_1)$ and $(\psi_1, \overline{\psi}_1) \le (\psi_2, \overline{\psi}_2) \le (\phi_1, \overline{\phi}_1)$. Since $\|\psi_2\|_\infty \ge \|d\|_\infty = a_2$ and $\|\phi_2\|_\infty = a_1$, we have $(\psi_2, \overline{\psi}_2) \not\le (\phi_2, \overline{\phi}_2)$. By Proposition 1.9, (1.1) has at least three distinct solutions for $\lambda \in (\lambda_*, \lambda^*)$.

Next, when Ω is a general bounded domain, let B_R be the largest inscribed ball in Ω. Define

$$\chi(x) = \begin{cases} \psi_2 & \text{for } x \in B_R, \\ 0 & \text{for } x \in \Omega - B_R, \end{cases} \quad \text{and} \quad \overline{\chi}(x) = \begin{cases} \overline{\psi}_2 & \text{for } x \in B_R, \\ 0 & \text{for } x \in \Omega - B_R, \end{cases}$$

where $(\psi_2, \overline{\psi}_2)$ is a second subsolution of (1.1) constructed above for $\Omega = B_R$. Then $\chi \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ and $\overline{\chi} \in W^{1,q}(\Omega) \cap C(\overline{\Omega})$. Further, on B_R we have

$$\begin{aligned} -\Delta_p \chi &= -\Delta_p \psi_2 < \lambda(\psi_2^{p-1-\alpha} + f(\overline{\psi}_2)) = \lambda(\chi^{p-1-\alpha} + f(\overline{\chi})), \\ -\Delta_q \overline{\chi} &= -\Delta_q \overline{\psi}_2 < \lambda(\overline{\psi}_2^{q-1-\beta} + g(\psi_2)) = \lambda(\overline{\chi}^{q-1-\beta} + g(\chi)), \end{aligned}$$

while outside B_R we have

$$-\Delta_p \chi = 0 = \lambda(\chi^{p-1-\alpha} + f(\overline{\chi})) \quad \text{and} \quad -\Delta_q \overline{\chi} = 0 = \lambda(\overline{\chi}^{q-1-\beta} + g(\chi)).$$

Hence, $(\chi, \overline{\chi})$ is a strict subsolution of (1.1) in Ω. The rest of the proof is identical to the previous case except that here for the second subsolution we will use $(\chi, \overline{\chi})$ described above. □

4. Examples

We illustrate in this section simple examples that satisfy the hypotheses in Theorems 1.3 and 1.6.

Example 4.1. Consider the boundary value problem

$$\begin{cases} -\Delta_p u = \lambda\{u^{p-1-\alpha} + e^{\tau u/(\tau+u)} - 1\} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where $f(s) = e^{\tau s/(\tau+s)} - 1$ with $\tau > 0$ and Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. Clearly, $f(0) = 0$, and $f(s)/s^{p-1} \to 0$ as $s \to \infty$ since f is bounded for each $\tau > 0$.

Next, choosing $a_1 = 1$ and $a_2 = \tau$, we have

$$A = \frac{\min\{a_1^\alpha, a_1^{p-1}/f(a_1)\}}{a_2^{p-1}/f(a_2)} = \frac{1}{\tau^{p-1}} \frac{e^{\tau/2} - 1}{e^{\tau/(1+\tau)} - 1} \quad \text{for } \tau \gg 1$$

and hence $A \to \infty$ as $\tau \to \infty$. Hence, the hypotheses in Theorem 1.6 are satisfied. In particular, for any $\tau > 0$, (4.1) has a positive solution for all $\lambda > 0$, and for sufficiently large τ, (4.1) has at least three positive solutions for $\lambda \in (\lambda_*, \lambda^*)$ where

$$\lambda_* = \frac{\tau^{p-1}}{e^{\tau/2} - 1} A_{p,p} \quad \text{and} \quad \lambda^* = \frac{1}{2\|e_p\|_\infty^{p-1}(e^{\tau/(1+\tau)} - 1)}.$$

In fact, given $\lambda \in (0, 1/(2\|e_p\|_\infty^{p-1}(e-1)))$, there exists $\tau_0 > 0$ such that (4.1) has at least three positive solutions for $\tau > \tau_0$.

EXAMPLE 4.2. Consider the system

$$\begin{cases} -\Delta_p u = \lambda\{u^{p-1-\alpha} + e^{\tau v/(\tau+v)} - 1\} & \text{in } \Omega, \\ -\Delta_q v = \lambda\{v^{q-1-\beta} + u^\xi\} & \text{in } \Omega, \\ u = 0 = v & \text{on } \partial\Omega, \end{cases} \tag{4.2}$$

where $f(s) = e^{\tau s/(\tau+s)} - 1$ with $\tau > 0$, $g(s) = s^\xi$ with $\xi > 0$ and Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. Clearly, $f(0) = g(0) = 0$ and (H_1) is satisfied since f is bounded for each $\tau > 0$. Hence, Theorems 1.1 and 1.2 hold for all $\tau > 0$ and $\xi > 0$. Next, choosing $a_1 = 1$ and $a_2 = \tau$,

$$A = \frac{\min[\min\{a_1^\alpha, a_1^{p-1}/f(a_1)\}, \min\{a_1^\beta, a_1^{q-1}/g(a_1)\}]}{\max\{a_2^{p-1}/f(a_2), a_2^{q-1}/g(a_2)\}} = \frac{\tau^{\xi-q+1}}{e^{\tau/(1+\tau)} - 1} \quad \text{for } \tau \gg 1$$

and hence (H_2) is satisfied for $\tau \gg 1$ and $\xi > q - 1$ since $A \to \infty$ as $\tau \to \infty$. For sufficiently large τ, there exist at least three positive solutions of (4.2) for $\lambda \in (\lambda_*, \lambda^*)$ where $\lambda_* = \tau^{q-1-\xi} \min\{A_{p,q}, A_{q,q}\}$ and

$$\lambda^* = \min\left\{\frac{1}{2\|e_p\|_\infty^{p-1}(e^{\tau/(1+\tau)} - 1)}, \frac{1}{2\|e_q\|_\infty^{q-1}}\right\}.$$

In fact, given

$$\lambda \in \left(0, \min\left\{\frac{1}{2\|e_p\|_\infty^{p-1}(e-1)}, \frac{1}{2\|e_q\|_\infty^{q-1}}\right\}\right),$$

there exists $\tau_0 > 0$ such that (4.2) has at least three positive solutions for $\tau > \tau_0$.

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