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MOTION PLANNING ALGORITHMS FOR CONFIGURATION SPACES IN THE HIGHER DIMENSIONAL CASE

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Abstract. The aim of this paper is to give an explicit motion planning algorithm for configuration spaces in the higher dimensional case.

1. Introduction

The topological approach to the motion planning problem was introduced by Farber in [2] and [3]. A motion planning problem is a rule assigning a continuous path to given two configurations – initial point and desired final point of a robot. Farber introduced the notion of topological complexity which measures the discontinuity of any motion planner in a configuration space. In [6], Rudyak introduced higher topological complexity, the concept fully developed in [1]. Higher topological complexity is related to motion planning problem which assigns a continuous path (with n-legs) to given n configurations. More precisely, it can be understood as a motion planning algorithm when a robot travels from the initial point A_1 to A_2, then from A_2 to A_3, and this keeps going until it reaches at the desired final point A_n.

This paper is based on the work of Mas–Ku and Torres–Giese who gave an explicit motion planning algorithm for configuration spaces $F(\mathbb{R}^2, k)$ and $F(\mathbb{R}^n, k)$, in [5]. In the last section, we will consider the higher dimensional case

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in the sense of Rudyak in [6], and give an explicit motion planning algorithm for this case.

2. Preliminaries

In this section, we will re-phrase the definitions and propositions for $F(\mathbb{R}^n, k)$ which are given in [5].

A vector $A = (a_1, \ldots, a_l)$ (where a_i is a positive integer for $i = 1, \ldots, l$) which satisfies $\sum a_i = k$ is called a partition of k. Here, the number $|A| = l$ is called the number of levels of A.

Recall the reverse lexicographic order on $\mathbb{R}^n$: $(b_1, \ldots, b_n) \leq (c_1, \ldots, c_n)$ if there is an index $k \in \{1, \ldots, n\}$ such that $b_i = c_i$ for $k < i \leq n$ and $b_k < c_k$.

As stated in [5], if $x = (x_1, \ldots, x_k) \in F(\mathbb{R}^2, k)$, then there is a unique permutation $\sigma \in \Sigma_k$ such that $x_{\sigma(1)} < \ldots < x_{\sigma(k)}$. Such a permutation is denoted by σ_x. A similar argument can be stated for $F(\mathbb{R}^n, k)$, namely, if $x = (x_1, \ldots, x_k) \in F(\mathbb{R}^n, k)$, then there is a unique permutation $\sigma \in \Sigma_k$ such that $x_{\sigma(1)} < \ldots < x_{\sigma(k)}$.

Let $\pi_n \colon \mathbb{R}^n \to \mathbb{R}$, given by $\pi_n(x_1, \ldots, x_n) = x_n$, be the projection to the n-th factor. For the configuration $x = (x_1, \ldots, x_k) \in F(\mathbb{R}^n, k)$ which is reverse lexicographically ordered, we can find positive integers $a_1, \ldots, a_l$ as follows:

$$\begin{aligned}
\pi_n(x_1) &= \ldots = \pi_n(x_{a_1}) < \pi_n(x_{a_1+1}),\\
\pi_n(x_{a_1+1}) &= \ldots = \pi_n(x_{a_1+a_2}) < \pi_n(x_{a_1+a_2+1}),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
\pi_n(x_{a_1+\ldots+a_{l-2}+1}) &= \ldots = \pi_n(x_{a_1+\ldots+a_{l-1}}) < \pi_n(x_{a_1+\ldots+a_{l-1}+1}),\\
\pi_n(x_{a_1+\ldots+a_{l-1}+1}) &= \ldots = \pi_n(x_{a_1+\ldots+a_l}) = \pi_n(x_k).
\end{aligned}$$

Since $a_1 + \ldots + a_l = k$, $(a_1, \ldots, a_l)$ is a partition of k. This partition is denoted by A_x. If A is obtained from the configuration x as in the above paragraph, then x is called an A-configuration.

Let $x = (x_1, \ldots, x_k) \in F(\mathbb{R}^n, k)$ be an A-configuration. Then x has $|A|$ levels. Moreover, x_i and x_j are said to have the same level if $\pi_n(x_i) = \pi_n(x_j)$. Given a partition A of k and a permutation $\sigma \in \Sigma_k$, let

$$F_{A,\sigma} = \{x = (x_1, \ldots, x_k) \in F(\mathbb{R}^n, k) : \sigma_x = \sigma \text{ and } x \text{ is an } A\text{-configuration}\}.$$

Define

$$F_A = \bigcup_{\sigma \in \Sigma_k} F_{A,\sigma}.$$

In fact, F_A denotes the set consisting of configurations x which produce A. Moreover, notice that $F(\mathbb{R}^n, k) = \bigcup\limits_{A} F_A$.

3. m-dimensional motion planners on $F(\mathbb{R}^n, k)$

Definition 3.1 ([6, 3.1. Definition]). Let J_m ($m \in \mathbb{N}$) be the wedge sum of m closed intervals $[0,1]_i$ for $i = 1, \dots, m$, where the zeros 0_i are identified. Let X be a path-connected space and X^{J_m} denote the set of paths with m-legs. Then there is a fibration $e_m \colon X^{J_m} \to X^m$ given by $e_m(f) = (f(1_1), \dots, f(1_m))$. The higher topological complexity $TC_m(X)$ is defined to be the Schwarz genus of e_m.

For $i \in \{m, m+1, \dots, mk\}$, let us define

$$F_i = \bigcup_{|A_1|+\ldots+|A_m|=i} F_{A_1} \times \ldots \times F_{A_m}.$$

Notice that F_i's are disjoint and they cover $F((R)^n, k)^m$. The ideas in Lemmas 13 and 14 in [5] tells that:

(1) F_i's are ENR (Euclidean Neighbourhood Retract).
(2) The expression for F_i (as a union) in te formula in display above, is in fact a topological disjoint union, so that a function defined on F_i which is continuous on each of the products $F_{A_1} \times \ldots \times F_{A_m}$ must be necessarily be continuous on the whole of F_i.

Higher dimensional analog of motion planner can be defined as follows:

Definition 3.2. Let X be a path-connected space and let $e_m \colon X^{J_m} \to X^m$ be the fibration as in 3.1. A motion planner in X is given by finitely many subsets $U_1, \dots, U_n \subset X^m$ and by continuous maps $s_i \colon U_i \to X^{J_m}$ where $i = 1, \dots, n$ such that the following is satisfied:

(a) Sets U_i are disjoint and they cover X^m.
(b) $e_m \circ s_i = \mathrm{id}_{U_i}$ for any $i = 1, \dots, n$.
(c) Each U_i is an ENR.

We will call such motion planners a m-dimensional motion planner, in order to indicate that it is related to the m-dimensional topological complexity.

A construction of motion planners. Let us denote the coordinates of $\mathbb{R}^n$ by $y_1, \dots, y_n$ to avoid any confusion. Let $\pi_1 \colon \mathbb{R}^n \to \mathbb{R}$ be the projection to the first factor. Let $\bar{p} \colon (\mathbb{R}^n)^{mk} \to \mathbb{R}$ be given by $(x_1, \dots, x_{mk}) \mapsto \max_{1 \le j \le mk} \{\pi_1(x_j)\}$, where $x_i \in \mathbb{R}^n$ for $i = 1, \dots, mk$. The map $\bar{p}$ is continuous [5, Lemma 16].

Take $x = (x^1, \dots, x^m) \in F_{A_1,1} \times \ldots \times F_{A_m,1} \subset F_q \subset F(\mathbb{R}^n, k)^m$, where $q = |A_1| + \ldots + |A_m|$. Notice that each $x^i \in F(\mathbb{R}^n, k)$ can be written as $(x^i_1, \dots, x^i_k)$, where $x^i_j = (x^i_{j1}, \dots, x^i_{jn})$ and $x^i_{js} \in \mathbb{R}$ for $s = 1, \dots, n$.

Define $p\colon F(\mathbb{R}^n,k)^m \to \mathbb{R}$ by

$$(x^1,\dots,x^m) = ((x^1_1,\dots,x^1_k),\dots,(x^m_1,\dots,x^m_k)) \mapsto \max_{1\le j_1,\dots,j_m\le k}\{\pi_1(x^1_{j_1}),\dots,\pi_1(x^m_{j_m})\}.$$

Since the map p is the restriction of the map $\bar{p}$ to $F(\mathbb{R}^n,k)^m$, p is continuous.

A_i-configuration $x^i \in F_{A_i,1}$ is mapped to a configuration by means of straight lines to the line L_{x^i} which is parallel to the y_n-axis and which intersects the y_1-axis at the point $(p(x^1,\dots,x^m)+i,0,\dots,0)$. The set of these lines (paths) determines a path Q_{x^i} in $F(\mathbb{R}^n,k)$.

Take a fixed configuration $x^0 \in F_{A_0,1}$ for a vector of positive integers A_0 which lies on the y_n-axis. Let $\alpha(x^0,x^i)$ be the path from Q_{x^0} to Q_{x^i} that connects by means of straight lines. The path from x^0 to x^i is given by

$$Q_{x^0}\cdot\alpha(x^0,x^i)\cdot Q^{-1}_{x^i}.$$

Since the path Q_{x^0} is constant, it is the path $\alpha(x^0,x^i)\cdot Q^{-1}_{x^i}\colon [0,1]_i \to F(\mathbb{R}^n,k)$, where $[0,1]_i$ is a notation to emphasize that it is the interval $[0,1]$ corresponding to x^i. Here, we have m different paths. Let us consider the wedge sum of the images of these paths, namely, $\mathrm{Im}(\alpha(x^0,x^1)\cdot Q^{-1}_{x^1})\vee\dots\vee\mathrm{Im}(\alpha(x^0,x^m)\cdot Q^{-1}_{x^m})$, where $(\alpha(x^0,x^i)\cdot Q^{-1}_{x^i})(0_i)$ are identified for $i=1,2,\dots,m$ and 0_i is the zero of the interval $[0,1]_i$. In fact, $\mathrm{Im}(\alpha(x^0,x^1)\cdot Q^{-1}_{x^1})\vee\dots\vee\mathrm{Im}(\alpha(x^0,x^m)\cdot Q^{-1}_{x^m})$ is a path with m-legs in $F(\mathbb{R}^n,k)^m$. Let us denote the corresponding path (with m-legs) by $\beta_{x^0,\dots,x^m}\colon J^m \to F(\mathbb{R}^n,k)$. Then, for a fixed A_0-configuration x^0, the motion planner $s_{A_1,\dots,A_m}$ is determined by the formula

$$(x^1,\dots,x^m)\mapsto \beta_{x^0,\dots,x^m}.$$

In the above calculation, we considered the case $F_{A_1,1}\times\dots\times F_{A_m,1}$. Without loss of generality, it can be extended to the case $F_{A_1,\sigma_1}\times\dots\times F_{A_m,\sigma_m}$.

Theorem 3.3. *The collection of pairs* (F_q,s_q) (*where* s_q *is given by means of motion planners on each* $F_{A_1,\sigma_1}\times\dots\times F_{A_m,\sigma_m}\subset F_q$ *for* $q=|A_1|+\dots+|A_m|$) *forms* m*-dimensional motion planning algorithm for* $m\le q\le mk$. *Consequently,* $TC_m(F(\mathbb{R}^n,k))\le m(k-1)+1$.

In view of Theorem 1.3 in [4], the m-dimensional motion planner described in Theorem 3.3 is optimal when n is odd, while the motion planner is within 1 unit from being optimal when n is even.

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