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THERMO-VISCO-ELASTICITY FOR MODELS WITH GROWTH CONDITIONS IN ORLICZ SPACES

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Abstract. We study a quasi-static evolution of the thermo-visco-elastic model. We act with external forces on a non-homogeneous material body, which is a subject of our research. Such action may cause deformation of this body and may change its temperature. Mechanical part of the model contains two kinds of deformation: elastic and visco-elastic. The mechanical deformation is coupled with temperature and both of them may influence each other. Since the constitutive function on evolution of the visco-elastic deformation depends on temperature, the visco-elastic properties of material also depend on temperature. We consider the thermodynamically complete model related to a hardening rule with growth condition in generalized Orlicz spaces. We provide the proof of existence of solutions for such class of models.

1. Introduction

The objective of this paper is to show the existence of solutions to a special class of thermo-visco-elastic models. We consider the reaction of a material body treated by external forces and heat flux through the boundary. In the case of

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ideal elastic deformations, the body should return to its initial state after termination of external forces activity. However, if deformations are not elastic, i.e. there is a loss of potential energy, we deal with a special kind of inelastic deformations. The potential energy lost during the process may be transformed into the thermal energy. We focus on the visco-elastic type of deformations, which for instance may be observed in polymers. Both deformations are coupled in physical phenomena and they may be observed at the same time. Consequently, these two types of deformations appear in the models considered in this paper. The elastic deformation is reversible, whereas the visco-elastic one is irreversible.

The thermo-visco-elastic system of equations, as a consequence of balance of momentum and balance of energy, cf. [19], [33], see also [21], captures displacement, temperature and visco-elastic strain. Since these two principles do not take into account the material properties of considered body, we may complement it by adding constitutive relations which complete missing information. The standard technique in the solid body deformation is to work with two constitutive relations. The first one describes the dependency between stress and strains, i.e. this is an equation for the Cauchy stress tensor. The second one is a constitutive equation which is characterized by the evolution of visco-elastic strain tensor.

We assume that the body $\Omega \subset \mathbb{R}^3$ is an open bounded set with a C^2 boundary. Then the quasi-static evolution problem is formulated by the following system of equations:

$$
\begin{cases}
-\operatorname{div} \boldsymbol{T} = \boldsymbol{f} & \text{in } \Omega \times (0,T), \\
\boldsymbol{T} = \boldsymbol{D}(\varepsilon(\boldsymbol{u}) - \varepsilon^{\mathbf{p}}) & \text{in } \Omega \times (0,T), \\
\varepsilon^{\mathbf{p}}_t = \boldsymbol{G}(\theta, \boldsymbol{T}^d) & \text{in } \Omega \times (0,T), \\
\theta_t - \Delta\theta = \boldsymbol{T}^d : \boldsymbol{G}(\theta, \boldsymbol{T}^d) & \text{in } \Omega \times (0,T).
\end{cases}
\tag{1.1}
$$

By the solution of this system we understand finding the displacement of material $\boldsymbol{u}\colon \Omega \times \mathbb{R}_+ \to \mathbb{R}^3$, the temperature of material $\theta\colon \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ and the visco-elastic strain tensor $\boldsymbol{\varepsilon}^{\mathbf{p}}\colon \Omega \times \mathbb{R}_+ \to \mathcal{S}^3_d$. We denote by $\mathcal{S}^3$ the set of symmetric 3×3-matrices with real entries and by $\mathcal{S}^3_d$ a subset of $\mathcal{S}^3$ which contains traceless matrices. The function $\boldsymbol{T}\colon \Omega \times \mathbb{R}_+ \to \mathcal{S}^3$ stays for the Cauchy stress tensor. By $\boldsymbol{I}$ we mean the identity matrix from $\mathcal{S}^3$, thus $\boldsymbol{T}^d$ is the deviatoric (traceless) part of the tensor $\boldsymbol{T}$, i.e. $\boldsymbol{T}^d = \boldsymbol{T} - \operatorname{tr}(\boldsymbol{T})\boldsymbol{I}/3$. Additionally, we denote by $\boldsymbol{\varepsilon}(\boldsymbol{u})$ the deformation tensor associated to $\boldsymbol{u}$, i.e. $\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\nabla \boldsymbol{u} + \nabla^T \boldsymbol{u})/2$.

The motivation for the current paper is to extend results presented in [20], where we proved the existence of solutions to the Norton–Hoff model, i.e. the model with the growth condition on the visco-elastic strain tensor in Lebesgue spaces. The model with the growth condition in generalized Orlicz spaces is a natural extension of the Norton–Hoff model as a next step to make an approximation of the Prandtl–Reuss model. The use of generalized Orlicz spaces takes into consideration more rapid growth than in the case of growth condition in Lebesgue spaces. Furthermore, the choice of generalized Orlicz spaces allows us to consider non-homogeneous materials. Since an N-function depends on the spatial variable x, different regions of Ω may have different growth condition. Consideration of non-homogeneous materials implies that the operator $\boldsymbol{D}$ may also depend on the spatial variable x. In previous papers, see [21], [20], we considered only homogeneous materials.

Studying mechanical problems in Orlicz spaces is not an isolated issue. In the case of visco-elastic deformation, the problem involving Orlicz spaces was considered in [14], but only in the case of N-function independent of the spatial variable x. In the case of N-function depending on the spatial variable x some accurate assumptions must be done. There are two possible ways to make it. Firstly, we may assume the regularity with respect to x, e.g. the log-Hölder continuity in [42], [41], where the author considers abstract parabolic problems. Secondly, the lower growth condition of N-function with respect to the last variable can be considered, e.g. see [45], [25], [26], [24], [11], where authors consider models of non-Newtonian flows. There are no results for thermo-visco-elastic problems without any upper and lower growth condition on the N-function with respect to the last variable.

System of equations (1.1) is a mathematical simplification of a more general model. We consider the quasi-static evolution with small displacement. It means that we omit the acceleration term in the momentum equation as a consequence of long-term character of external forces. Small displacement allows us to use the Hooke law in the definition of the Cauchy stress tensor $(1.1)_{(2)}$. Moreover, the material does not change its volume with temperature, i.e. there is no thermal expansion of the body, thus the Cauchy stress tensor does not depend on temperature explicitly.

System (1.1) may be completed by formulating the initial conditions

$$\begin{cases} \theta(x,0) = \theta_0(x), \\ \boldsymbol{\varepsilon}^{\mathbf{p}}(x,0) = \boldsymbol{\varepsilon}_0^{\mathbf{p}}(x), \end{cases} \tag{1.2}$$

in Ω and boundary conditions

$$\begin{cases} \boldsymbol{u} = \boldsymbol{g}, \\ \dfrac{\partial\theta}{\partial\boldsymbol{n}} = g_\theta, \end{cases} \tag{1.3}$$

on $\partial\Omega\times(0,T)$. We control the shape of Ω and the heat flux through the boundary.

To discuss two other equations and to formulate the statement of this paper, we need to use some definitions which are mentioned below for better readability of the paper. Let us begin with presenting the notion of generalized Orlicz spaces. For a more general concept of Orlicz space we refer the reader to [1], [37], [38] and [32]. We start with defining the notion of N-function.

Definition 1.1. Let Ω be a bounded open domain in $\mathbb{R}^3$. A function $M\colon \Omega\times \mathcal{S}^3 \to \mathbb{R}_+$ is said to be an N-function if it satisfies the following conditions:

(a) M is a Carathéodory function (measurable with respect to x and continuous with respect to $\boldsymbol{\xi}$) such that $M(x,\boldsymbol{\xi})=0$ if and only if $\boldsymbol{\xi}=\mathbf{0}$;
(b) $M(x,\boldsymbol{\xi})=M(x,-\boldsymbol{\xi})$ almost everywhere in Ω;
(c) $M(x,\boldsymbol{\xi})$ is a convex function with respect to $\boldsymbol{\xi}$;
(d) $\lim_{|\boldsymbol{\xi}|\to 0} M(x,\boldsymbol{\xi})/|\boldsymbol{\xi}|=0$ for almost all $x\in\Omega$;
(e) $\lim_{|\boldsymbol{\xi}|\to \infty} M(x,\boldsymbol{\xi})/|\boldsymbol{\xi}|=\infty$ for almost all $x\in\Omega$.

Definition 1.2. The function M^* which is complementary to a function M is defined by

$$M^*(x,\boldsymbol{\eta}) = \sup_{\boldsymbol{\xi}\in\mathcal{S}^3} (\boldsymbol{\xi}:\boldsymbol{\eta} - M(x,\boldsymbol{\xi})), \quad \text{for } \boldsymbol{\eta}\in\mathcal{S}^3,\ x\in\Omega.$$

Remark 1.3. The complementary function M^* to an N-function M is also an N-function.

Let us denote $Q=\Omega\times(0,T)$. The generalized Orlicz class $\mathcal{L}_M(Q)$ is the set of all measurable functions $\boldsymbol{\xi}\colon Q\to\mathcal{S}^3$ such that

$$\int_Q M(x,\boldsymbol{\xi}(x,t))\,dx\,dt<\infty.$$

The generalized Orlicz space $L_M(Q)$ can be defined as the smallest linear space containing $\mathcal{L}_M(Q)$. By $E_M(Q)$ we denote the closure of the set of bounded functions in the L_M-norm. The generalized Orlicz space $L_M(Q)$ is a Banach space with respect to the Orlicz norm

$$\|\boldsymbol{\xi}\|_{O,M} = \sup\bigg\{\int_Q \boldsymbol{\xi}:\boldsymbol{\eta}\,dx\,dt : \boldsymbol{\eta}\in L_{M^*}(Q),\ \int_Q M^*(x,\boldsymbol{\eta})\,dx\,dt\le 1\bigg\},$$

or equivalently with respect to the Luxemburg norm

$$\|\boldsymbol{\xi}\|_{L,M} = \inf\bigg\{\lambda>0 : \int_Q M\bigg(x,\frac{\boldsymbol{\xi}}{\lambda}\bigg)\,dx\,dt\le 1\bigg\}.$$

Definition 1.4. We say that an N-function M satisfies the Δ_2-condition if for almost all $x\in\Omega$ and for all $\boldsymbol{\xi}\in\mathcal{S}^3$, there exist a constant c and a nonnegative integrable function $h\colon\Omega\to\mathbb{R}$ such that

$$M(x,2\boldsymbol{\xi})\le cM(x,\boldsymbol{\xi})+h(x). \tag{1.4}$$

Remark 1.5. For every M the following inclusions hold:

$$E_M(Q)\subseteq\mathcal{L}_M(Q)\subseteq L_M(Q).$$

In particular, if M satisfies the Δ_2-condition, $E_M(Q) = L_M(Q)$. If the Δ_2-condition fails, we lose numerous properties of the space $L_M(Q)$, like separability, reflexivity and many others, cf. [1], [37] and in particular [22] for generalized Orlicz spaces.

The space $L_{M^*}(Q)$ is the dual space of $E_M(Q)$. The functional

$$\rho(\boldsymbol{\xi}) = \int_Q M(x, \boldsymbol{\xi})\, dx\, dt$$

is a modular.

Definition 1.6. We say that a sequence $\{\boldsymbol{\xi}_i\}_{i=1}^{\infty}$ converges modularly to $\boldsymbol{\xi}$ in $L_M(Q)$ if there exists $\lambda > 0$ such that

$$\int_Q M\left(x, \frac{\boldsymbol{\xi}_i - \boldsymbol{\xi}}{\lambda}\right) dx\, dt \to 0,$$

as i tends to ∞. We use the notation $\boldsymbol{\xi}_i \xrightarrow{M} \boldsymbol{\xi}$ for the modular convergence in $L_M(Q)$.

In Appendix B we present several lemmas related to Orlicz spaces. We use these lemmas to prove the existence of thermo-visco-elasticity model solutions.

Now we may discuss the constitutive relations used to complement system (1.1) with initial and boundary conditions (1.2)–(1.3). The relation between the Cauchy stress tensor and the strain tensor is defined by the operator $\boldsymbol{D}\colon \mathcal{S}^3 \to \mathcal{S}^3$, which is linear, positively definite and bounded. Moreover, $\boldsymbol{D}$ is a four-index matrix, i.e. $\boldsymbol{D} = \boldsymbol{D}(x) = \{d_{i,j,k,l}(x)\}_{i,j,k,l=1}^3$ and the following equalities hold:

$$d_{i,j,k,l}(x) = d_{j,i,k,l}(x), \quad d_{i,j,k,l}(x) = d_{i,j,l,k}(x), \quad d_{i,j,k,l}(x) = d_{k,l,i,j}(x),$$

for all $i,j,k,l = 1,2,3$ and for every $x \in \Omega$. Additionally, for each $i,j,k,l = 1,2,3$ the function $d_{i,j,k,l}$ is Lipschitz continuous.

The second constitutive relation is an evolutionary equation for the visco-elastic strain tensor. The function $\boldsymbol{G}\colon \Omega \times \mathbb{R}_+ \times \mathcal{S}^3_d \to \mathcal{S}^3_d$ is a function of temperature and deviatoric part of Cauchy stress tensor. We discussed more precisely the concept of such choice in [20]. The properties of the considered material imply the choice of a specific function. Various other models were also considered, e.g. the Bodner–Partom model [6], [13], [12], the Mróz model [21], [10], [30], the Norton–Hoff model [20], [2], the Prandtl–Reuss model with linear kinematic hardening [15].

Assumption 1.7. We assume that the function $\boldsymbol{G}(x, \theta, \boldsymbol{T}^d)$ is a Carethéodory function, i.e. it is measurable with respect to x and continuous with respect to θ and $\boldsymbol{T}^d$, and satisfies the following conditions:

(a) $(\boldsymbol{G}(x,\theta,\boldsymbol{T}^d_1) - \boldsymbol{G}(x,\theta,\boldsymbol{T}^d_2)) : (\boldsymbol{T}^d_1 - \boldsymbol{T}^d_2) \geq 0$, for all $\boldsymbol{T}^d_1, \boldsymbol{T}^d_2 \in \mathcal{S}^3_d$ and $\theta \in \mathbb{R}_+$;

(b) $\boldsymbol{G}(x,\theta,\boldsymbol{T}^d) : \boldsymbol{T}^d \geq c(M(x,\boldsymbol{T}^d) + M^*(x,\boldsymbol{G}(\theta,\boldsymbol{T}^d)))$, where $\boldsymbol{T}^d \in \mathcal{S}^3_d$, $\theta \in \mathbb{R}_+$ and c is a positive constant independent of temperature θ;

(c) $\boldsymbol{G}(x,\theta,\mathbf{0}) = \mathbf{0}$ for almost all $x \in \Omega$.

Moreover, we assume that the generalized Orlicz spaces fulfill:

$$\int_Q M^*(x,\boldsymbol{A}(x,t))\,dx\,dt \leq \int_Q |\boldsymbol{A}|^2\,dx\,dt \quad \text{for all } \boldsymbol{A} \in L_{M^*}(Q)$$

and M^* satisfies the Δ_2-condition.

Further, we write $\boldsymbol{G}(\theta,\boldsymbol{T}^d)$ instead of $\boldsymbol{G}(x,\theta,\boldsymbol{T}^d)$. We keep in mind that one of variables of the function $\boldsymbol{G}$ is x but we omit repetitions in order to make the content more clear for the reader. Dealing with such assumption on the function $\boldsymbol{G}(\,\cdot\,,\,\cdot\,)$ implies the displacement space.

DEFINITION 1.8. Let us define the space $BD_{M^*}(Q,\mathbb{R}^3)$ by the formula

$$BD_{M^*}(Q,\mathbb{R}^3) = \{\boldsymbol{u} \in L^1(\Omega,\mathbb{R}^3) : \boldsymbol{\varepsilon}(\boldsymbol{u}) \in L_{M^*}(\Omega,\mathcal{S}^3)\}.$$

The space $BD_{M^*}(Q,\mathbb{R}^3)$ is a Banach space with the norm

$$\|\boldsymbol{u}\|_{BD_M(Q)} = \|\boldsymbol{u}\|_{L^1(Q)} + \|\boldsymbol{\varepsilon}(\boldsymbol{u})\|_{M^*}.$$

The space $BD_{M^*}(Q,\mathbb{R}^3)$ is a subspace of the space of bounded deformations $BD(Q,\mathbb{R}^3)$

$$BD(Q,\mathbb{R}^3) = \{\boldsymbol{u} \in L^1(\Omega,\mathbb{R}^3) : [\boldsymbol{\varepsilon}(\boldsymbol{u})]_{i,j} \in \mathcal{M}(Q)\},$$

where $\mathcal{M}(Q)$ is a space of bounded measures on Q and

$$[\boldsymbol{\varepsilon}(\boldsymbol{u})]_{i,j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),$$

cf. [27]. According to [43, Theorem 1.1], there exists a unique continuous operator γ_0 from $BD_{M^*}(Q)$ onto $L^1((0,T)\times\partial\Omega)$ such that the generalized Green formula

$$2\int_Q \phi\boldsymbol{\varepsilon}_{i,j}(\boldsymbol{u})\,dx\,dt$$
$$= -\int_Q \left(u_i\frac{\partial\phi}{\partial x_i} + u_j\frac{\partial\phi}{\partial x_j}\right) + \int_0^T\int_{\partial\Omega}\phi(\gamma_0(u_i)n_j + \gamma_0(u_j)n_i)\,d\mathcal{H}^2\,dt$$

holds for every $\phi \in C^1(\overline{Q})$, $i,j = 1,2,3$, and where $\boldsymbol{n} = (n_1,n_2,n_3)^T$ is the unit outward normal vector on $\partial\Omega$ and H^2 is the 2-Hausdorff measure. Moreover, $BD(Q,\mathbb{R}^3)$ is compactly embedded in $L^q(Q,\mathbb{R}^3)$ for every $1 \leq q < 3/2$, see [43, Remark 2.3].

Furthermore, we understand $\boldsymbol{v} \in BD_{M^*}(Q,\mathbb{R}^3) + L^\infty(0,T,W^{2,p}(\Omega,\mathbb{R}^3))$ in the following way: There exists a decomposition $\boldsymbol{v} = \boldsymbol{v}_1 + \boldsymbol{v}_2$, where $\boldsymbol{v}_1 \in BD_{M^*}(Q,\mathbb{R}^3)$ and $\boldsymbol{v}_2 \in L^\infty(0,T,W^{2,p}(\Omega,\mathbb{R}^3))$.

In contrast to [20] or [30], we use another approach to the heat equation. By Assumption 1.7, we know that the right-hand side function $\boldsymbol{G}(\theta,\boldsymbol{T}^d) : \boldsymbol{T}^d$ is only integrable. Following Boccardo and Gallouët, cf. [9], we proved in [20] that solutions to the heat equation belong to $L^q(0,T,W^{1,q}(\Omega))$ for all $q \in (1,5/4)$. A weak point of this approach is lack of uniqueness. Hence, we change the approach and, following Blanchard and Murat, prove the existence of a renormalised solution. The concept of renormalised solutions to parabolic equation with Dirichlet boundary condition was presented in [7], [8]. In Appendix A we prove existence of a renormalised solution in the case of the Neumann boundary condition.

While modelling physical phenomena we should not forget about their physical properties. Losses of energy or admission of negative temperature causes that the mathematical result has no physical interpretation. In the case of solid mechanics, the model should fulfill the principle of thermodynamics. The considered model is thermodynamically complete, i.e. the principle of thermodynamics is fulfilled. In [20], [21], we discussed conservation of energy, positivity of temperature and existence of entropy, which has a positive rate of production. Considering the quasi-static evolution causes that the energy of system consists of internal (thermal) and potential energy. Lack of acceleration term in balance of momentum implies that the kinetic energy of Ω fails.

DEFINITION 1.9 (Weak-renormalised solution to system (1.1)). The triple of functions $\boldsymbol{u} \in BD_{M^*}(Q,\mathbb{R}^3) + L^\infty(0,T,W^{2,p}(\Omega,\mathbb{R}^3))$, $\boldsymbol{T} \in L^2(0,T,L^2(\Omega,\mathcal{S}^3))$ and $\theta \in C([0,T],L^1(\Omega))$ such that for every $K \in \mathbb{N}$, $\mathcal{T}_K(\theta) \in L^2(0,T,W^{1,2}(\Omega))$ is a weak-renormalised solution to system (1.1) when

$$\int_0^T \int_\Omega \boldsymbol{T} : \nabla \boldsymbol{\varphi}\, dx\, dt = \int_0^T \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi}\, dx\, dt,$$

where $\boldsymbol{T} = \boldsymbol{D}(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}^{\mathbf{p}})$, holds for every test function $\boldsymbol{\varphi} \in C^\infty([0,T],C_c^\infty(\Omega,\mathbb{R}^3))$ and

$$\begin{aligned}-\int_Q S(\theta-\widetilde{\theta})\frac{\partial\phi}{\partial t}\, dx\, dt - \int_\Omega S(\theta_0-\widetilde{\theta}_0)\phi(x,0)\, dx + \int_Q S'(\theta-\widetilde{\theta})\nabla(\theta-\widetilde{\theta})\cdot\nabla\phi\, dx\, dt \\ + \int_Q S''(\theta-\widetilde{\theta})|\nabla(\theta-\widetilde{\theta})|^2\phi\, dx\, dt = \int_Q \boldsymbol{G}(\theta,\boldsymbol{T}^d) : \boldsymbol{T}^d S'(\theta-\widetilde{\theta})\phi\, dx\, dt\end{aligned}$$

holds for every test function $\phi \in C_c^\infty([-\infty,T),C^\infty(\Omega))$, for every function $S \in C^\infty(\mathbb{R})$ such that $S' \in C_0^\infty(\mathbb{R})$ and for $\widetilde{\theta}$ which is a solution of the problem

$$\begin{cases} \widetilde{\theta}_t - \Delta\widetilde{\theta} = 0 & \text{in } \Omega\times(0,T), \\ \dfrac{\partial\widetilde{\theta}}{\partial\boldsymbol{n}} = g_\theta & \text{on } \partial\Omega\times(0,T), \\ \widetilde{\theta}(x,0) = \widetilde{\theta}_0 & \text{in } \Omega. \end{cases} \tag{1.5}$$

Furthermore, the visco-elastic strain tensor can be recovered from the equation on its evolution, i.e.

$$\boldsymbol{\varepsilon}^{\mathbf{p}}(x,t) = \boldsymbol{\varepsilon}_0^{\mathbf{p}}(x) + \int_0^t \boldsymbol{G}(\theta(x,\tau),\boldsymbol{T}^d(x,\tau))\, d\tau,$$

for almost every $x \in \Omega$ and $t \in [0,T)$. Moreover, $\boldsymbol{\varepsilon}^{\mathbf{p}}, \boldsymbol{\varepsilon}^{\mathbf{p}}_t \in L_{M^*}(Q)$.

Theorem 1.10. *Let initial conditions satisfy* $\theta_0 \in L^1(\Omega)$, $\boldsymbol{\varepsilon}^{\mathbf{p}}_0 \in L_{M^*}(\Omega, \mathcal{S}^3_d)$, *boundary conditions satisfy* $g_\theta \in L^2(0,T,L^2(\partial\Omega))$, *for* $p > 3$ *the function* $\widetilde{\boldsymbol{g}}$ *be defined on* $\partial\Omega \times (0,T)$, *so that there exists its extension on* Q *such that* $\boldsymbol{g} \in L^\infty(0,T,W^{2,p}(\Omega,\mathbb{R}^3))$, *the volume force* $\boldsymbol{f} \in L^\infty(0,T,L^p(\Omega,\mathbb{R}^3))$, *and also the function* $\boldsymbol{G}(\,\cdot\,,\,\cdot\,)$ *satisfy the same conditions as in Assumption* 1.7. *Then there exists a weak solution to system* (1.1).

The idea of proof is similar to the one in [20]. We use Galerkin approximations. Usage of the growth condition in Orlicz spaces instead of the growth condition in Lebesgue spaces implies utilization of different analytic tools, e.g. the Minty–Browder trick for Orlicz spaces, which appear here to be non-reflexive, to identify the weak limit of nonlinear term and biting limit to show convergences in $L^1(Q)$ of right-hand side in the heat equation. Moreover, the Young measures tools are used and some important lemmas for the Young measure are presented in Appendix C.

2. Proof of Theorem 1.10

The proof of Theorem 1.10 consists of several steps. Each step is presented in a separate subsection.

2.1. Transformation to a homogeneous boundary-value problem. The first step of the proof is to transform the system into a homogeneous boundary-value problem. The construction of solution is more clear in this case. Moreover, we also cut off the right-hand side function in the elastic problems. Thereby, instead of the volume force and boundary values we receive the same influence of exterior by using the shifts of solutions. It allows us to focus on the important issues instead of calculation difficulties.

Let us consider two independent systems of equations with given initial conditions and boundary data. The boundary conditions are the same as in (1.3).

$$\begin{cases} -\operatorname{div}\widetilde{\boldsymbol{T}} = \boldsymbol{f} & \text{in } \Omega \times (0,T), \\ \widetilde{\boldsymbol{T}} = \boldsymbol{D}\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}) & \text{in } \Omega \times (0,T), \\ \widetilde{\boldsymbol{u}} = \widetilde{\boldsymbol{g}} & \text{on } \partial\Omega \times (0,T), \end{cases} \tag{2.1}$$

and

$$\begin{cases} \widetilde{\theta}_t - \Delta\widetilde{\theta} = 0 & \text{in } \Omega \times (0,T), \\ \dfrac{\partial\widetilde{\theta}}{\partial\boldsymbol{n}} = g_\theta & \text{on } \partial\Omega \times (0,T), \\ \widetilde{\theta}(x,0) = \widetilde{\theta}_0 & \text{in } \Omega. \end{cases} \tag{2.2}$$

Lemma 2.1. *For $p > 3$, let $\widetilde{\theta}_0 \in L^2(\Omega)$, the function $\widetilde{\boldsymbol{g}}$ be defined on $\partial\Omega \times (0,T)$, so that there exists its extension $\boldsymbol{g} \in L^\infty(0,T,W^{2,p}(\Omega,\mathbb{R}^3))$, $g_\theta \in L^2(0,T,L^2(\partial\Omega))$ and $\boldsymbol{f} \in L^\infty(0,T,L^p(\Omega,\mathbb{R}^3))$. Then there exists a solution to systems* (2.1) *and* (2.2). *Additionally, the following estimates hold:*

$$\|\widetilde{\boldsymbol{u}}\|_{L^\infty(0,T,W^{2,p}(\Omega))} \le C_1\big(\|\boldsymbol{g}\|_{L^\infty(0,T,W^{2,p}(\Omega))} + \|\boldsymbol{f}\|_{L^\infty(0,T,L^p(\Omega))}\big),$$
$$\|\widetilde{\theta}\|_{L^\infty(0,T,L^1(\Omega))} + \|\widetilde{\theta}\|_{L^2(0,T,W^{1,2}(\Omega))} \le C_2\big(\|g_\theta\|_{L^2(0,T,L^2(\partial\Omega))} + \|\widetilde{\theta}_0\|_{L^2(\Omega)}\big).$$

Moreover, the following estimate holds for the Cauchy stress tensor:

$$\|\widetilde{\boldsymbol{T}}\|_{L^\infty(Q)} \le C_3\big(\|\boldsymbol{g}\|_{L^\infty(0,T,W^{2,p}(\Omega))} + \|\boldsymbol{f}\|_{L^\infty(0,T,L^{,p}(\Omega))}\big). \tag{2.3}$$

Results for temperature are straightforward, hence let us discuss only existence of solution to the elastic system of equations.

Proof. Rewriting the solution in the form $\widetilde{\boldsymbol{u}} = \widetilde{\boldsymbol{u}}_1 + \boldsymbol{g}$, instead of looking for $\widetilde{\boldsymbol{u}}$ we may search for $\widetilde{\boldsymbol{u}}_1$, where $\widetilde{\boldsymbol{u}}_1$ is a solution to the system

$$\begin{cases} -\operatorname{div} \boldsymbol{D}\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}_1) = \boldsymbol{f} + \operatorname{div} \boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{g}) & \text{in } \Omega \times (0,T), \\ \widetilde{\boldsymbol{u}}_1 = 0 & \text{on } \partial\Omega \times (0,T), \end{cases} \tag{2.4}$$

where the function $\boldsymbol{f} + \operatorname{div} \boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{g})$ belongs to $L^\infty(0,T,L^p(\Omega,\mathbb{R}^3))$. By [44, Theorem 7.1], we know that there exists a unique solution to the elasticity problem. As $p > 3$ and by using the general Sobolev inequalities [17, Theorem 6, p. 270], we obtain inequality (2.3). This estimate is crucial in the next steps of the proof. □

Instead of finding $(\widehat{\boldsymbol{u}}, \widehat{\theta})$, the solution to problem (1.1)–(1.3), we shall search for $(\boldsymbol{u}, \theta)$, where $\boldsymbol{u} = \widehat{\boldsymbol{u}} - \widetilde{\boldsymbol{u}}$ and $\theta = \widehat{\theta} - \widetilde{\theta}$ where $(\widetilde{\boldsymbol{u}}, \widetilde{\theta})$ solves (2.1) and (2.2). Furthermore, we consider the following system of equations:

$$\begin{cases} -\operatorname{div} \boldsymbol{T} = 0, \\ \boldsymbol{T} = \boldsymbol{D}(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}^{\mathbf{p}}), \\ \boldsymbol{\varepsilon}^{\mathbf{p}}_t = \boldsymbol{G}(\theta + \widetilde{\theta}, \boldsymbol{T}^d + \widetilde{\boldsymbol{T}}^d), \\ \theta_t - \Delta\theta = \big(\boldsymbol{T}^d + \widetilde{\boldsymbol{T}}^d\big) : \boldsymbol{G}(\theta + \widetilde{\theta}, \boldsymbol{T}^d + \widetilde{\boldsymbol{T}}^d), \end{cases} \tag{2.5}$$

with boundary and initial conditions

$$\begin{cases} \boldsymbol{u} = 0 & \text{on } \partial\Omega \times (0,T), \\ \dfrac{\partial\theta}{\partial\boldsymbol{n}} = 0 & \text{on } \partial\Omega \times (0,T), \\ \theta(\,\cdot\,,0) = \widehat{\theta}_0 - \widetilde{\theta}_0 \equiv \theta_0 & \text{in } \Omega, \\ \boldsymbol{\varepsilon}^{\mathbf{p}}(\,\cdot\,,0) = \boldsymbol{\varepsilon}^{\mathbf{p}}_0 & \text{in } \Omega, \end{cases} \tag{2.6}$$

where $\widehat{\theta}_0$ is an initial condition for the whole temperature and $\widetilde{\theta}_0$ is the initial condition for system (2.2).

2.2. Approximate solution. The construction of approximate solutions does not differ from the one presented in [20]. Let us define the standard truncation operator $\mathcal{T}_k(\,\cdot\,)$ by

$$\mathcal{T}_k(x) = \begin{cases} k & \text{if } x > k, \\ x & \text{if } |x| \le k, \\ -k & \text{if } x < -k, \end{cases} \tag{2.7}$$

for $k \in \mathbb{N}$. The use of truncation is implied only by integrability of the right-hand side of the heat equation and initial condition for temperature. In the proof of solutions existence we use the truncation of solution as a test function. This truncation does not need to be a linear combination of basis functions. Thus, we use two level approximation, i.e. independent approximation parameters in the displacement and temperature. Due to this construction the limit passage in each approximation level may be done independently. As the first step we pass to the limit with temperature approximations parameter, i.e. as $l \to \infty$, and next we pass to the limit with the displacement approximation parameter. Moreover, we construct the approximate solution for visco-elastic strain tensor. After the first limit passage the visco-elastic strain tensor is an infinite dimensional approximation. The low regularity of data implies that the second limit passage requires a closer attention.

The construction of approximate solution requires usage of three different bases, i.e. bases for temperature, displacement and visco-elastic strain. Moreover, let $\{v_i\}_{i=1}^{\infty}$ be the subset of $W^{1,2}(\Omega)$ such that

$$\int_\Omega (\nabla v_i \cdot \nabla \phi - \mu_i v_i \phi)\, dx = 0$$

holds for every function $\phi \in C^\infty(\overline{\Omega})$, see [3], [39]. Let $\{\mu_i\}$ be the set of the corresponding eigenvalues. We may assume that $\{v_i\}$ is orthogonal in $W^{1,2}(\Omega)$ and orthonormal in $L^2(\Omega)$.

To construct the basis functions for approximation let us start from considering the space $L^2(\Omega, \mathcal{S}^3)$ with the scalar product defined by

$$(\boldsymbol{\xi}, \boldsymbol{\eta})_{\boldsymbol{D}} := \int_\Omega \boldsymbol{D}^{1/2}\boldsymbol{\xi} : \boldsymbol{D}^{1/2}\boldsymbol{\eta}\, dx \quad \text{for } \boldsymbol{\xi}, \boldsymbol{\eta} \in L^2(\Omega, \mathcal{S}^3),$$

where $\boldsymbol{D}^{1/2} \circ \boldsymbol{D}^{1/2} = \boldsymbol{D}$. Moreover, let $\{\boldsymbol{w}_i\}_{i=1}^{\infty}$ be the set of eigenfunctions of the elasto-static operator $-\mathrm{div}\, \boldsymbol{D}\boldsymbol{\varepsilon}(\,\cdot\,)$ with the domain $W_0^{1,2}(\Omega, \mathbb{R}^3)$ and $\{\lambda_i\}$ be the corresponding eigenvalues such that $\{\boldsymbol{w}_i\}$ is orthogonal in $W_0^{1,2}(\Omega, \mathbb{R}^3)$ with the inner product

$$(\boldsymbol{w}, \boldsymbol{v})_{W_0^{1,2}(\Omega)} = (\boldsymbol{\varepsilon}(\boldsymbol{w}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\boldsymbol{D}}$$

and orthonormal in $L^2(\Omega, \mathbb{R}^3)$. Since functions $\{d_{i,j,k,l}\}$ are Lipschitz continuous, then eigenfunctions $\{\boldsymbol{w}_i\}$ are smooth, see [18]. Moreover, $\|\cdot\|_{\boldsymbol{D}}$ is a norm of

$L^2(\Omega,\mathcal{S}^3)$, i.e. $\|\boldsymbol{\varepsilon}(\boldsymbol{w})\|^2_{\boldsymbol{D}} = (\boldsymbol{\varepsilon}(\boldsymbol{w}),\boldsymbol{\varepsilon}(\boldsymbol{w}))_{\boldsymbol{D}}$. Furthermore, by using the symmetry of operator $\boldsymbol{D}$ the following equality holds for basis functions $\boldsymbol{w}_i,\boldsymbol{w}_j$:

$$\int_\Omega \boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{w}_i) : \boldsymbol{\varepsilon}(\boldsymbol{w}_j)\,dx = \lambda_i \int_\Omega \boldsymbol{w}_i\cdot\boldsymbol{w}_j\,dx = 0,$$

when $i\neq j$.

The idea of constructing the visco-elastic strain approximations was presented in [20] and we hereby refer the reader to this paper for more details. We observe that $\boldsymbol{\varepsilon}(\boldsymbol{w}_i)$ are elements of $H^s(\Omega,\mathcal{S}^3)$ by regularity of eigenfunctions, where $H^s(\Omega,\mathcal{S}^3)$ is a fractional Sobolev space with the scalar product denoted by $(\!(\,\cdot\,,\,\cdot\,)\!)_s$ and $3/2 < s \leq 2$. Let us define the orthogonal complement in $L^2(\Omega,\mathcal{S}^3)$ as

$$V_k := (\mathrm{span}\{\boldsymbol{\varepsilon}(\boldsymbol{w}_1),\dots,\boldsymbol{\varepsilon}(\boldsymbol{w}_k)\})^\perp, \tag{2.8}$$

taken with respect to the scalar product $(\,\cdot\,,\,\cdot\,)_{\boldsymbol{D}}$. Moreover, let us define

$$V_k^s := V_k \cap H^s(\Omega,\mathcal{S}^3). \tag{2.9}$$

Due to [34, Theorem 4.11, Appendix], which was also used in [20], there exists an orthonormal basis $\{\boldsymbol{\zeta}^k_n\}_{n=1}^\infty$ of V_k which is also an orthogonal basis of V_k^s. The basis for the visco-elastic strain consists of two subsets. The first subset is a set of the first k symmetric gradients from the basis $\{\boldsymbol{w}_i\}_{i=1}^\infty$. The second subset consists of the first l functions from $\{\boldsymbol{\zeta}^k_n\}_{n=1}^\infty$. Thus, for each step of approximation we use $k+l$ functions to construct the visco-elastic strain.

For $k,l\in\mathbb{N}$ we define

$$\begin{aligned} \boldsymbol{u}_{k,l} &= \sum_{n=1}^k \alpha^n_{k,l}(t)\boldsymbol{w}_n, \qquad \theta_{k,l} = \sum_{m=1}^l \beta^m_{k,l}(t)v_m,\\ \boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l} &= \sum_{n=1}^k \gamma^n_{k,l}(t)\boldsymbol{\varepsilon}(\boldsymbol{w}_n) + \sum_{m=1}^l \delta^m_{k,l}(t)\boldsymbol{\zeta}^k_m, \end{aligned} \tag{2.10}$$

such that $\boldsymbol{u}_{k,l}$, $\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l}$ and $\theta_{k,l}$ solve the system of equations

$$\int_\Omega \boldsymbol{T}_{k,l} : \boldsymbol{\varepsilon}(\boldsymbol{w}_n)\,dx = 0, \quad n=1,\dots,k, \tag{2.11}$$

$$\boldsymbol{T}_{k,l} = \boldsymbol{D}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}) - \boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l}), \tag{2.12}$$

$$\begin{aligned} &\int_\Omega (\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t : \boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{w}_n)\,dx \\ &\qquad = \int_\Omega \boldsymbol{G}(\widetilde{\theta}+\theta_{k,l}, \widetilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}) : \boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{w}_n)\,dx, \quad n=1,\dots,k, \end{aligned} \tag{2.13}$$

$$\int_\Omega (\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t : \boldsymbol{D}\boldsymbol{\zeta}^k_m \, dx = \int_\Omega \boldsymbol{G}(\widetilde{\theta}+\theta_{k,l}, \widetilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}) : \boldsymbol{D}\boldsymbol{\zeta}^k_m \, dx, \quad m = 1,\dots,l, \tag{2.14}$$

$$\int_\Omega (\theta_{k,l})_t v_m \, dx + \int_\Omega \nabla\theta_{k,l}\cdot\nabla v_m \, dx = \int_\Omega \mathcal{T}_k\big((\boldsymbol{T}^d_{k,l}+\widetilde{\boldsymbol{T}}^d) : \boldsymbol{G}(\widetilde{\theta}+\theta_{k,l}, \widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l})\big) v_m \, dx, \quad m = 1,\dots,l, \tag{2.15}$$

for almost all $t \in (0,T)$. Moreover, the solutions fulfill initial conditions in the following form:

$$\begin{cases} (\theta_{k,l}(x,0), v_m) = (\mathcal{T}_k(\theta_0), v_m) & \text{for } m = 1,\dots,l, \\ (\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l}(x,0), \boldsymbol{\varepsilon}(\boldsymbol{w}_n))_{\boldsymbol{D}} = (\boldsymbol{\varepsilon}^{\mathbf{p}}_0, \boldsymbol{\varepsilon}(\boldsymbol{w}_n))_{\boldsymbol{D}} & \text{for } n = 1,\dots,k, \\ (\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l}(x,0), \boldsymbol{\zeta}^k_m)_{\boldsymbol{D}} = (\boldsymbol{\varepsilon}^{\mathbf{p}}_0, \boldsymbol{\zeta}^k_m)_{\boldsymbol{D}} & \text{for } m = 1,\dots,l, \end{cases} \tag{2.16}$$

where $(\,\cdot\,,\,\cdot\,)$ denotes the inner product in $L^2(\Omega)$ and $(\,\cdot\,,\,\cdot\,)_{\boldsymbol{D}}$ denotes the inner product in $L^2(\Omega,\mathcal{S}^3)$.

Let us notice that $\alpha^n_{k,l}(t) = \gamma^n_{k,l}(t)$ by the selection of Galerkin bases and representation of approximate solution (2.10). To present it more clearly let

$$\boldsymbol{\xi}(t) = \big(\beta^1_{k,l}(t),\dots,\beta^l_{k,l}(t),\gamma^1_{k,l}(t),\dots,\gamma^k_{k,l}(t),\delta^1_{k,l}(t),\dots,\delta^l_{k,l}(t)\big)^T.$$

Then system of equations (2.11)–(2.15) may be rewritten in the form of ODE's system

$$\begin{cases} (\gamma^n_{k,l}(t))_t = \dfrac{1}{\lambda_n}\displaystyle\int_\Omega \widetilde{\boldsymbol{G}}(x,t,\boldsymbol{\xi}(t)) : \boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{w}_n)\, dx, \\ (\delta^m_{k,l}(t))_t = \displaystyle\int_\Omega \widetilde{\boldsymbol{G}}(x,t,\boldsymbol{\xi}(t)) : \boldsymbol{D}\boldsymbol{\zeta}^k_m \, dx, \\ (\beta^m_{k,l}(t))_t = \displaystyle\int_\Omega \mathcal{T}_k\Bigg(\bigg(-\Big(\boldsymbol{D}\sum_{m=1}^l \delta^m_{k,l}(t)\boldsymbol{\zeta}^k_m\Big)^d + \widetilde{\boldsymbol{T}}^d\bigg) : \\ \qquad\qquad \widetilde{\boldsymbol{G}}(x,t,\boldsymbol{\xi}(t))\Bigg) v_m \, dx + \mu_m \beta^m_{k,l}(t), \end{cases} \tag{2.17}$$

for $n = 1,\dots,k$ and $m = 1,\dots,l$, where

$$\begin{aligned} \widetilde{\boldsymbol{G}}(x,t,\boldsymbol{\xi}(t)) &:= \boldsymbol{G}(\widetilde{\theta}+\theta_{k,l}, \widetilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}) \\ &= \boldsymbol{G}\bigg(\sum_{m=1}^l \beta^m_{k,l}(t) v_j(x) + \widetilde{\theta}, -\boldsymbol{D}\Big(\sum_{m=1}^l \delta^m_{k,l}(t)\boldsymbol{\zeta}^k_m\Big)^d + \widetilde{\boldsymbol{T}}^d\bigg). \end{aligned}$$

Hence, the existence of solution to the approximate system is equivalent to the existence of solution to the following ODE's system:

$$\begin{aligned} \frac{d\boldsymbol{\xi}}{dt} &= \boldsymbol{F}(\boldsymbol{\xi}(t),t), \quad t\in[0,T), \\ \boldsymbol{\xi}(0) &= \boldsymbol{\xi}_0, \end{aligned} \tag{2.18}$$

where $\boldsymbol{\xi}_0$ is a vector of initial conditions obtained from (2.16).

LEMMA 2.2 (Existence of approximate solution). *For initial condition satisfying $\boldsymbol{\varepsilon}_0^{\mathbf{p}} \in L_{M^*}(\Omega, \mathcal{S}^3_d)$ and $\theta_0 \in L^1(\Omega)$ there exists a local solution to* (2.18) *which is absolutely continuous in time.*

The proof of Lemma 2.2 is a consequence of application of the Carathéodory Theorem, see [34, Theorem 3.4, Appendix] or [47, Appendix (61)]. We obtain the existence of unique absolutely continuous solution for some time interval $[0, t^*]$.

2.3. Boundedness of energy. Since we consider the physical model, the total energy of the system should be finite. Omission of the kinetic effect implies that the total energy consists of thermal energy and potential energy. We start with consideration related to potential energy. The part related to thermal energy estimates is similar to the one presented in [20], hence we recall the lemmas without proofs.

DEFINITION 2.3. We say that $\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}^{\mathbf{p}})$ is the potential energy if

$$\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}^{\mathbf{p}}) := \frac{1}{2}\int_\Omega \boldsymbol{D}(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}^{\mathbf{p}}) : (\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}^{\mathbf{p}})\, dx.$$

LEMMA 2.4. *There exists a constant C (uniform with respect to k and l) such that*

$$\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l})\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})(t) + \frac{2c-d}{2}\int_Q M^*\big(x, \boldsymbol{G}(\widetilde{\theta} + \theta_{k,l}, \widetilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})\big)\, dx\, dt + c\int_Q M(x, \widetilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})\, dx\, dt \le C,$$

where c is a constant from Assumption 1.7 *and $d = \min(1, c)$. Moreover, the constant C depends on the solution of additional problem* (2.1) *and potential energy at the initial time*

$$C = \int_Q M\left(x, \frac{2}{d}\widetilde{\boldsymbol{T}}^d\right) dx\, dt + \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}), \boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})(0).$$

PROOF. Let us start with calculating the time derivative of the potential energy $\mathcal{E}(t)$. For almost all $t \in [0, T]$ we obtain

$$\frac{d}{dt}\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}), \boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l}) = \int_\Omega \boldsymbol{D}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}) - \boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l}) : (\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}))_t\, dx - \int_\Omega \boldsymbol{D}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}) - \boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l}) : (\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t\, dx.$$

The terms on the right-hand side of above equation may be rewritten with usage of approximate system of equations (2.11)–(2.15). Firstly, for each $n \le k$ let us multiply (2.11) by $(\alpha^n_{k,l})_t$. After summing over $n = 1, \ldots, k$ we get

$$\int_\Omega \boldsymbol{D}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}) - \boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l}) : (\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}))_t\, dx = 0. \tag{2.19}$$

Then, for each $n \le k$, let us multiply (2.13) by $\gamma^n_{k,l}$ and, for each $m \le l$, let us multiply (2.14) by $\delta^n_{k,l}$. Summing over $n = 1, \dots, k$ and $m = 1, \dots, l$, we obtain

$$\int_\Omega (\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t : \boldsymbol{D}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}) - \boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})\, dx = \int_\Omega \boldsymbol{G}(\widetilde{\theta} + \theta_{k,l}, \widetilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}) : \boldsymbol{T}_{k,l}\, dx. \tag{2.20}$$

Hence

$$\frac{d}{dt}\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}), \boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l}) = -\int_\Omega \boldsymbol{G}(\widetilde{\theta} + \theta_{k,l}, \widetilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}) : \boldsymbol{T}_{k,l}\, dx, \tag{2.21}$$

and then using the property of traceless matrices we get

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}), \boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l}) = &-\int_\Omega \boldsymbol{G}(\widetilde{\theta} + \theta_{k,l}, \widetilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}) : (\widetilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})\, dx \\ &+ \int_\Omega \boldsymbol{G}(\widetilde{\theta} + \theta_{k,l}, \widetilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}) : \widetilde{\boldsymbol{T}}^d\, dx.\end{aligned}$$

Thus, using Assumption 1.7 and the Fenchel–Young inequality, we estimate changes of potential energy as

$$\begin{aligned}&\frac{d}{dt}\,\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}), \boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l}) \\ &\le -c\bigg(\int_\Omega M(x, \widetilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})\, dx + \int_\Omega M^*\big(x, \boldsymbol{G}(\widetilde{\theta} + \theta_{k,l}, \widetilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})\big)\, dx\bigg) \\ &\quad + \int_\Omega M\bigg(x, \frac{2}{d}\widetilde{\boldsymbol{T}}^d\bigg)\, dx + \int_\Omega M^*\bigg(x, \frac{d}{2}\boldsymbol{G}(\widetilde{\theta} + \theta_{k,l}, \widetilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})\bigg)\, dx,\end{aligned}$$

where $d = \min(1, c)$. Then due to convexity of N-function we obtain

$$\begin{aligned}&\frac{d}{dt}\,\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}), \boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l}) \\ &\le -c\bigg(\int_\Omega M(x, \widetilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})\, dx + \int_\Omega M^*\big(x, \boldsymbol{G}(\widetilde{\theta} + \theta_{k,l}, \widetilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})\big)\, dx\bigg) \\ &\quad + \int_\Omega M\bigg(x, \frac{2}{d}\widetilde{\boldsymbol{T}}^d\bigg)\, dx + \frac{d}{2}\int_\Omega M^*\big(x, \boldsymbol{G}(\widetilde{\theta} + \theta_{k,l}, \widetilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})\big)\, dx.\end{aligned}$$

Finally, after integration over the time interval $(0, t)$, with $0 \le t \le T$, we obtain

$$\begin{aligned}\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}), \boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})(t) &+ c\int_Q M(x, \widetilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})\, dx\, dt \\ &+ \frac{2c-d}{2}\int_Q M^*\big(x, \boldsymbol{G}(\widetilde{\theta} + \theta_{k,l}, \widetilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})\big)\, dx\, dt \\ &\le \int_Q M\bigg(x, \frac{2}{d}\widetilde{\boldsymbol{T}}^d\bigg)\, dx\, dt + \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}), \boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})(0),\end{aligned}$$

which completes the proof. □

REMARK 2.5. From Lemma 2.4 we know that the sequence $\{\boldsymbol{T}^d_{k,l}\}$ is uniformly bounded in $L_M(Q, \mathcal{S}^3)$ with respect to k and l, also the sequence $\{\boldsymbol{G}(\widetilde{\theta} + \theta_{k,l}, \widetilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})\}$ is uniformly bounded in the space $L_{M^*}(Q, \mathcal{S}^3)$ with respect to

k and l. Hence using the Fenchel–Young inequality, the sequence $\{(\widetilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}) : \boldsymbol{G}(\widetilde{\theta} + \theta_{k,l}, \widetilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})\}$ is uniformly bounded in $L^1(Q)$.

REMARK 2.6. From Lemma 2.4 we know that the sequence $\{\boldsymbol{T}_{k,l}\}$ is uniformly bounded in $L^\infty(0,T,L^2(\Omega,\mathcal{S}^3))$ and in particular in $L^2(0,T,L^2(\Omega,\mathcal{S}^3))$.

To prove the uniform estimates for the sequence $\{(\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t\}$ let us recall that $\{\boldsymbol{\zeta}^k_n\}_{n=1}^\infty$ is an orthogonal basis of V^s_k and an orthonormal basis of V_s, defined by (2.9) and (2.8), respectively. Moreover, the basis $\{\boldsymbol{\zeta}^k_n\}_{n=1}^\infty$ contains the eigenvalue of the problem

$$(2.22)\qquad (\!(\boldsymbol{\zeta}_i, \boldsymbol{\Phi})\!)_s = \lambda_i(\boldsymbol{\zeta}_i, \boldsymbol{\Phi})_{\boldsymbol{D}} \quad \text{for all } \boldsymbol{\Phi} \in V^s_k,$$

where by $(\!(\,\cdot\,,\,\cdot\,)\!)_s$ we denote the scalar product in $H^s(\Omega,\mathcal{S}^3)$ and $(\,\cdot\,,\,\cdot\,)_{\boldsymbol{D}}$ is the previously defined scalar product in $L^2(\Omega,\mathcal{S}^3)$. We define the following projections:

$$P^l_{H^s}\colon H^s \to \operatorname{lin}\{\boldsymbol{\zeta}^k_1,\ldots,\boldsymbol{\zeta}^k_l\}, \qquad P^l_{H^s}\boldsymbol{v} := \sum_{i=1}^{l} \left(\!\!\left(\boldsymbol{v}, \frac{\boldsymbol{\zeta}^k_i}{\sqrt{\lambda_i}}\right)\!\!\right)_s \frac{\boldsymbol{\zeta}^k_i}{\sqrt{\lambda_i}},$$

$$P^l_{L^2}\colon L^2 \to \operatorname{lin}\{\boldsymbol{\zeta}^k_1,\ldots,\boldsymbol{\zeta}^k_l\}, \qquad P^l_{L^2}\boldsymbol{v} := \sum_{i=1}^{l} (\boldsymbol{v}, \boldsymbol{\zeta}^k_i)_{\boldsymbol{D}}\boldsymbol{\zeta}^k_i.$$

Therefore, for $\boldsymbol{\varphi} \in V^s_k$ we obtain

$$P^l_{L^2}\boldsymbol{\varphi} = \sum_{i=1}^{l} (\boldsymbol{\varphi}, \boldsymbol{\zeta}^k_i)_{\boldsymbol{D}}\boldsymbol{\zeta}^k_i = \sum_{i=1}^{l} \left(\!\!\left(\boldsymbol{\varphi}, \frac{\boldsymbol{\zeta}^k_i}{\sqrt{\lambda_i}}\right)\!\!\right)_s \frac{\boldsymbol{\zeta}^k_i}{\sqrt{\lambda_i}} = P^l_{H^s}\boldsymbol{\varphi},$$

where the second equality is a condition on eigenvalues. This implies that $P^l_{L^2}\big|_{V^s_k} = P^l_{H^s}\big|_{V^s_k}$. Moreover, the norms $\|P^l_{H^s}\|_{\mathcal{L}(H^s)}$ and $\|P^l_{L^2}\|_{\mathcal{L}(L^2)}$ are equal to 1. Now, let us define the projection

$$P^k\colon L^2 \to \operatorname{lin}\{\boldsymbol{\varepsilon}(\boldsymbol{w}_1),\ldots,\boldsymbol{\varepsilon}(\boldsymbol{w}_k)\}, \qquad P^k\boldsymbol{v} := \sum_{i=1}^{k} (\boldsymbol{v}, \boldsymbol{\varepsilon}(\boldsymbol{w}_i))_{\boldsymbol{D}}\boldsymbol{\varepsilon}(\boldsymbol{w}_i).$$

Thus, we may observe that for $\boldsymbol{v} \in H^s$ it holds that

$$\begin{aligned} P^l_{H^s}(\mathrm{Id} - P^k)\boldsymbol{v} &= \sum_{i=1}^{l} \left(\!\!\left((\mathrm{Id} - P^k)\boldsymbol{v}, \frac{\boldsymbol{\zeta}^k_i}{\sqrt{\lambda_i}}\right)\!\!\right)_s \frac{\boldsymbol{\zeta}^k_i}{\sqrt{\lambda_i}} \\ &= \sum_{i=1}^{l} ((\mathrm{Id} - P^k)\boldsymbol{v}, \boldsymbol{\zeta}_i)_{\boldsymbol{D}}\boldsymbol{\zeta}^k_i = \sum_{i=1}^{l} (\boldsymbol{v}, \boldsymbol{\zeta}_i)_{\boldsymbol{D}}\boldsymbol{\zeta}^k_i = P^l_{L^2}\boldsymbol{v}. \end{aligned}$$

In the following lemma we obtain the estimates independent of l. Let us observe that since P^k is a projection which does not dependent on l, then there exists $c(k)$ (depending only on k) such that for every $\boldsymbol{\varphi} \in H^s(\Omega,\mathcal{S}^3)$ it holds

$$(2.23)\qquad \max\big(\|P^k\boldsymbol{\varphi}\|_{H^s}, \|(\mathrm{Id} - P^k)\boldsymbol{\varphi}\|_{H^s}\big) \le c(k)\|\boldsymbol{\varphi}\|_{H^s}.$$

Lemma 2.7. *The sequence* $\{(\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t\}$ *is, with respect to* l, *uniformly bounded in* $L^1(0,T,(H^s(\Omega,\mathcal{S}^3))')$.

Proof. Let $\boldsymbol{\varphi} \in L^\infty(0,T,H^s(\Omega,\mathcal{S}^3))$. We have the following estimate:

$$
\begin{aligned}
\int_0^T |((\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t,\boldsymbol{\varphi})_{\boldsymbol{D}}|\,dt &= \int_0^T |((\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t,(P^k+P^l_{L^2})\boldsymbol{\varphi})_{\boldsymbol{D}}|\,dt \\
&\le \int_0^T |((\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t,P^k\boldsymbol{\varphi})_{\boldsymbol{D}}|\,dt + \int_0^T |((\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t,P^l_{L^2}\boldsymbol{\varphi})_{\boldsymbol{D}}|\,dt,
\end{aligned}
\tag{2.24}
$$

where the equality results from orthogonality of subspaces $\mathrm{lin}\{\boldsymbol{\varepsilon}(\boldsymbol{w}_1),\ldots,\boldsymbol{\varepsilon}(\boldsymbol{w}_k)\}$ and $\mathrm{lin}\{\boldsymbol{\zeta}^k_1,\ldots,\boldsymbol{\zeta}^k_l\}$. Then

$$
\begin{aligned}
\int_0^T |((\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t,\boldsymbol{\varphi})_{\boldsymbol{D}}|\,dt &\le \int_0^T \left|\int_\Omega \boldsymbol{D}\boldsymbol{G}(\widetilde{\theta}+\theta_{k,l},\widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}) : P^k\boldsymbol{\varphi}\,dx\right| dt \\
&\quad + \int_0^T \left|\int_\Omega \boldsymbol{D}\boldsymbol{G}(\widetilde{\theta}+\theta_{k,l},\widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}) : P^l_{L^2}\boldsymbol{\varphi}\,dx\right| dt \\
&\le d\int_0^T \left|\int_\Omega \boldsymbol{G}(\widetilde{\theta}+\theta_{k,l},\widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}) : P^k\boldsymbol{\varphi}\,dx\right| dt \\
&\quad + d\int_0^T \left|\int_\Omega \boldsymbol{G}(\widetilde{\theta}+\theta_{k,l},\widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}) : (P^l_{H^s}\circ(\mathrm{Id}-P^k))\boldsymbol{\varphi}\,dx\right| dt \\
&\le d\int_0^T \|\boldsymbol{G}(\widetilde{\theta}+\theta_{k,l},\widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l})\|_{L^1(\Omega)}\|P^k\boldsymbol{\varphi}\|_{L^\infty(\Omega)}\,dt \\
&\quad + d\int_0^T \|\boldsymbol{G}(\widetilde{\theta}+\theta_{k,l},\widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l})\|_{L^1(\Omega)}\|(P^l_{H^s}\circ(\mathrm{Id}-P^k))\boldsymbol{\varphi}\|_{L^\infty(\Omega)}\,dt.
\end{aligned}
$$

Hence $s > 3/2$ and by the Sobolev inequality, $\|(P^l_{H^s}\circ(\mathrm{Id}-P^k))\boldsymbol{\varphi}\|_{L^\infty(\Omega)} \le \widetilde{c}\|(P^l_{H^s}\circ(\mathrm{Id}-P^k))\boldsymbol{\varphi}\|_{H^s(\Omega)}$ and $\|P^k\boldsymbol{\varphi}\|_{L^\infty(\Omega)} \le \widetilde{c}\|P^k\boldsymbol{\varphi}\|_{H^s(\Omega)}$, where $\widetilde{c}$ is an optimal embedding constant. Then

$$
\begin{aligned}
\int_0^T |((\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t,\boldsymbol{\varphi})_{\boldsymbol{D}}|\,dt &\le d\widetilde{c}\int_0^T \|\boldsymbol{G}(\widetilde{\theta}+\theta_{k,l},\widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l})\|_{L^1(\Omega)}\|P^k\boldsymbol{\varphi}\|_{H^s(\Omega)}\,dt \\
&\quad + d\widetilde{c}\int_0^T \|\boldsymbol{G}(\widetilde{\theta}+\theta_{k,l},\widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l})\|_{L^1(\Omega)}\|(P^l_{H^s}\circ(\mathrm{Id}-P^k))\boldsymbol{\varphi}\|_{H^s(\Omega)}\,dt \\
&\le dc(k)\widetilde{c}\int_0^T \|\boldsymbol{G}(\widetilde{\theta}+\theta_{k,l},\widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l})\|_{L^1(\Omega)}\|\boldsymbol{\varphi}\|_{H^s(\Omega)}\,dt \\
&\quad + dc(k)\widetilde{c}\int_0^T \|\boldsymbol{G}(\widetilde{\theta}+\theta_{k,l},\widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l})\|_{L^1(\Omega)}\|\boldsymbol{\varphi}\|_{H^s(\Omega)}\,dt \\
&\le 2c(k)d\widetilde{c}\|\boldsymbol{G}(\widetilde{\theta}+\theta_{k,l},\widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l})\|_{L^1(Q)}\|\boldsymbol{\varphi}\|_{L^\infty(0,T,H^s(\Omega))}.
\end{aligned}
$$

It is obvious that $\|\boldsymbol{G}(\widetilde{\theta}+\theta_{k,l},\widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l})\|_{L^1(Q)}$ is uniformly bounded. Hence there exists $C>0$ such that

$$\sup_{\substack{\boldsymbol{\varphi}\in L^\infty(0,T,H^s(\Omega,\mathcal{S}^3))\\ \|\boldsymbol{\varphi}\|_{L^\infty(0,T,H^s(\Omega,\mathcal{S}^3))}\le 1}} \int_0^T |((\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t,\boldsymbol{\varphi})_{\boldsymbol{D}}|\,dt \le C$$

and hence the sequence $\{(\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t\}$ is uniformly bounded in $L^1(0,T,(H^s(\Omega,\mathcal{S}^3))')$.□

The remaining part is related with considering the internal energy of the system. Two following lemmas can be found in [20].

Lemma 2.8. *The sequence $\{\theta_{k,l}\}$ is uniformly bounded in $L^\infty(0,T;L^1(\Omega))$ with respect to k and l.*

Lemma 2.9. *There exists a constant C, depending on the domain Ω and time interval $(0,T)$, such that for every $k\in\mathbb{N}$*

$$\sup_{0\le t\le T}\|\theta_{k,l}(t)\|^2_{L^2(\Omega)}+\|\theta_{k,l}\|^2_{L^2(0,T,W^{1,2}(\Omega))}+\|(\theta_{k,l})_t\|^2_{L^2(0,T,W^{-1,2}(\Omega))} \tag{2.25}$$
$$\le C\big(\|\mathcal{T}_k\big((\widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}):\boldsymbol{G}(\widetilde{\theta}+\theta_{k,l},\widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l})\big)\|^2_{L^2(0,T,L^2(\Omega))}+\|\mathcal{T}_k(\theta_0)\|^2_{L^2(\Omega)}\big).$$

The estimates in Lemma 2.9 depend on k. To complete this section we observe that the uniform boundedness of solutions implies the global existence of approximate solutions. For each $n=1,\dots,k$ and $m=1,\dots,l$ the solutions $\{\alpha^n_{k,l}(t),\beta^m_{k,l}(t),\gamma^n_{k,l}(t),\delta^m_{k,l}(t)\}$ exist on the whole time interval $[0,T]$.

2.4. Limit passage $l\to\infty$ and uniform estimates. Multiplying (2.11), (2.13)–(2.15) by time dependent test functions $\varphi_1(t),\varphi_2(t),\varphi_3(t)\in C^\infty([0,T])$ and $\varphi_4(t)\in C^\infty_c([-\infty,T])$ and after integration over the time interval $[0,T]$, we obtain the following system of equations:

$$\int_0^T\int_\Omega \boldsymbol{T}_{k,l}:\boldsymbol{\varepsilon}(\boldsymbol{w}_n)\varphi_1(t)\,dx\,dt=0, \tag{2.26}$$
$$\boldsymbol{T}_{k,l}=\boldsymbol{D}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l})-\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l}), \tag{2.27}$$

$$\int_0^T\int_\Omega(\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t:\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{w}_n)\varphi_2(t)\,dx\,dt = \int_0^T\int_\Omega \boldsymbol{G}(\widetilde{\theta}+\theta_{k,l},\widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}):\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{w}_n)\varphi_2(t)\,dx\,dt, \tag{2.28}$$

$$\int_0^T\int_\Omega(\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t:\boldsymbol{D}\boldsymbol{\zeta}^k_m\varphi_3(t)\,dx\,dt = \int_0^T\int_\Omega \boldsymbol{G}(\widetilde{\theta}+\theta_{k,l},\widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}):\boldsymbol{D}\boldsymbol{\zeta}^k_m\varphi_3(t)\,dx\,dt, \tag{2.29}$$

$$(2.30)\quad -\int_0^T\int_\Omega \theta_{k,l} v_m \varphi_4'(t)\,dx\,dt - \int_\Omega \theta_{0,k,l}(x) v_m\varphi_4(0)\,dx + \int_0^T\int_\Omega \nabla\theta_{k,l}\cdot\nabla v_m\varphi_4(t)\,dx\,dt = \int_0^T\int_\Omega \mathcal{T}_k\big((\widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}):\boldsymbol{G}(\widetilde{\theta}+\theta_{k,l},\widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l})\big)v_m\varphi_4(t)\,dx\,dt,$$

where equations (2.26) and (2.28) hold for $n=1,\dots,k$ and (2.29) and (2.30) hold for $m=1,\dots,l$.

Firstly, we pass to the limit as $l\to\infty$ – the Galerkin approximation of temperature. From the previous section we get the uniform boundedness with respect to l for appropriate sequences. Using the appropriate subsequence, but still denoted by the indexes k and l, we get the following convergences:

$$(2.31)\quad \begin{aligned} \boldsymbol{T}_{k,l} &\rightharpoonup \boldsymbol{T}_k && \text{weakly in } L^2(Q,\mathcal{S}^3),\\ \boldsymbol{T}^d_{k,l} &\rightharpoonup^* \boldsymbol{T}^d_k && \text{weakly* in } L_M(Q,\mathcal{S}^3_d),\\ \boldsymbol{G}(\widetilde{\theta}+\theta_{k,l},\widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}) &\rightharpoonup^* \boldsymbol{\chi}_k && \text{weakly* in } L_{M^*}(Q,\mathcal{S}^3_d),\\ \theta_{k,l} &\rightharpoonup \theta_k && \text{weakly in } L^2(0,T,W^{1,2}(\Omega)),\\ \theta_{k,l} &\to \theta_k && \text{a.e. in } \Omega\times(0,T),\\ (\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t &\rightharpoonup (\boldsymbol{\varepsilon}^{\mathbf{p}}_k)_t && \text{weakly in } L^1(0,T,(H^s(\Omega,\mathcal{S}^3))'). \end{aligned}$$

We pass to the limit as $l\to\infty$ in (2.26), (2.28)–(2.30), using convergences from (2.31). For $n=1,\dots,k$ and $m\in\mathbb{N}$ the following equations hold:

$$(2.32)\quad \int_0^T\int_\Omega \boldsymbol{T}_k:\boldsymbol{\varepsilon}(\boldsymbol{w}_n)\varphi_1(t)\,dx\,dt=0,$$

$$(2.33)\quad \int_0^T\int_\Omega (\boldsymbol{\varepsilon}^{\mathbf{p}}_k)_t:\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{w}_m)\varphi_2(t)\,dx\,dt = \int_0^T\int_\Omega \boldsymbol{\chi}_k:\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{w}_m)\varphi_2(t)\,dx\,dt,$$

$$(2.34)\quad \int_0^T\int_\Omega (\boldsymbol{\varepsilon}^{\mathbf{p}}_k)_t:\boldsymbol{D}\boldsymbol{\zeta}^k_m\varphi_3(t)\,dx\,dt = \int_0^T\int_\Omega \boldsymbol{\chi}_k:\boldsymbol{D}\boldsymbol{\zeta}^k_m\varphi_3(t)\,dx\,dt.$$

Moreover, $\{\boldsymbol{\varepsilon}(\boldsymbol{w}_n),\boldsymbol{\zeta}_m\}_{n=1,\dots,k;m=1,\dots,\infty}$ is a base of the whole space $H^s(\Omega,\mathcal{S}^3)$ hence equations (2.33) and (2.34) can be replaced by

$$(2.35)\quad \int_0^T\int_\Omega (\boldsymbol{\varepsilon}^{\mathbf{p}}_k)_t:\boldsymbol{\zeta}\,dx\,dt = \int_0^T\int_\Omega \boldsymbol{\chi}_k:\boldsymbol{\zeta}\,dx\,dt$$

for $\boldsymbol{\zeta}\in L^\infty(0,T,H^s(\Omega,\mathcal{S}^3))$. To show that (2.35) holds also for all $\boldsymbol{\zeta}\in L_M(Q,\mathcal{S}^3)$, we proceed similarly as in [24] and [26].

To complete the limit passage in heat equation (2.30) we encounter the same problem as in [20], but here we should use a different technique. It holds due to the use of generalized Orlicz spaces instead of Lebesgue spaces. As previously, we may precisely consider the right-hand side of (2.30) and this reasoning consists of three steps. The first step is to show the inequality in Lemma 2.10. The second step is to identify the weak limit of the nonlinear term by using the Minty–Browder trick. For the Minty–Browder trick in non-reflexive spaces we refer the reader to [45]. And finally, the last step is to show the convergence of the right-hand side of heat equation.

STEP 1. *Limiting inequality.*

LEMMA 2.10. *The following inequality holds for the solution of approximate systems:*

$$\limsup_{l\to\infty}\int_0^t\int_\Omega \boldsymbol{G}(\widetilde{\theta}+\theta_{k,l},\widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}):\boldsymbol{T}^d_{k,l}\,dx\,dt\le\int_0^t\int_\Omega \boldsymbol{\chi}_k:\boldsymbol{T}^d_k\,dx\,dt. \tag{2.36}$$

PROOF. Let us start with the definition of the function $\psi_{\mu,\tau}:\mathbb{R}_+\to\mathbb{R}_+$. For each $\mu>0$, $\tau\le T-\mu$, $s\ge 0$, the function $\psi_{\mu,\tau}$ is defined by

$$\psi_{\mu,\tau}(s)=\begin{cases}1 & \text{for } s\in[0,\tau),\\ -\dfrac{1}{\mu}s+\dfrac{1}{\mu}\tau+1 & \text{for } s\in[\tau,\tau+\mu),\\ 0 & \text{for } s\ge\tau+\mu.\end{cases} \tag{2.37}$$

We use $\psi_{\mu,\tau}(t)$ as a test function in (2.21), then after integration over the time interval $(0,T)$ we get

$$\begin{aligned}\int_0^T \frac{d}{dt}\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}),\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})\,\psi_{\mu,\tau}\,dt&\\ =-\int_0^T\int_\Omega \boldsymbol{G}(\widetilde{\theta}+\theta_{k,l},\widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}):\boldsymbol{T}^d_{k,l}\,\psi_{\mu,\tau}\,dx\,dt.&\end{aligned} \tag{2.38}$$

Integrating by parts the left-hand side of (2.38), we obtain

$$\begin{aligned}\int_0^T \frac{d}{dt}\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}),\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})\,\psi_{\mu,\tau}\,dt&\\ =\frac{1}{\mu}\int_\tau^{\tau+\mu}\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}),\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})(t)\,dt-\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}),\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})(0).&\end{aligned} \tag{2.39}$$

Passing to the limit as $l\to\infty$ and using the lower semicontinuity in $L^2(0,T,L^2(\Omega;\mathcal{S}^3))$, we get

$$\begin{aligned}\liminf_{l\to\infty}\int_0^T \frac{d}{d\tau}\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}),\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})\,\psi_{\mu,\tau}\,dt&\\ =\liminf_{l\to\infty}\frac{1}{\mu}\int_\tau^{\tau+\mu}\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}),\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})(t)\,dt-\lim_{l\to\infty}\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}),\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})(0)&\\ \ge\frac{1}{\mu}\int_\tau^{\tau+\mu}\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_k),\boldsymbol{\varepsilon}^{\mathbf{p}}_k)(t)\,dt-\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_k),\boldsymbol{\varepsilon}^{\mathbf{p}}_k)(0).&\end{aligned} \tag{2.40}$$

Comparing (2.40) and (2.38) we obtain

$$\liminf_{l\to\infty}\Big(-\int_0^T\int_\Omega \boldsymbol{G}(\widetilde{\theta}+\theta_{k,l},\widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}):\boldsymbol{T}^d_{k,l}\,\psi_{\mu,\tau}\,dx\,dt\Big) \tag{2.41}$$

$$
\begin{aligned}
&= \liminf_{l\to\infty} \int_0^T \frac{d}{dt}\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}), \boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})\, \psi_{\mu,\tau}\, dt \\
&\geq \frac{1}{\mu}\int_\tau^{\tau+\mu} \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_k), \boldsymbol{\varepsilon}^{\mathbf{p}}_k)(t)\, dt - \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_k), \boldsymbol{\varepsilon}^{\mathbf{p}}_k)(0),
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
\limsup_{l\to\infty} \int_0^T \int_\Omega \boldsymbol{G}(\widetilde{\theta} + \theta_{k,l}, \widetilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}) : \boldsymbol{T}^d_{k,l}\ \psi_{\mu,\tau}\, dx\, dt& \\
\leq -\frac{1}{\mu}\int_\tau^{\tau+\mu} \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_k), \boldsymbol{\varepsilon}^{\mathbf{p}}_k)(t)\, dt + \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_k), \boldsymbol{\varepsilon}^{\mathbf{p}}_k)(0).&
\end{aligned}
\tag{2.42}
$$

Since $(\alpha^n_k)_t$ is not regular enough to use as a test function in (2.11) we may use the mollifier to improve its regularity. Thus let η_ϵ be a standard mollifier and we mollify with respect to time. Then let us choose $\varphi_1(t) = ((\alpha^n_k)_t * \eta_\epsilon \mathbf{1}_{(t_1,t_2)}) * \eta_\epsilon$, where $0 < t_1 < t_2 < T$ and ϵ is sufficiently small ($\epsilon < \min(t_1, T - t_2)$), as a test function in (2.32) and $\boldsymbol{\zeta} = (\boldsymbol{T}^d_k * \eta_\epsilon \mathbf{1}_{(t_1,t_2)}) * \eta_\epsilon$ as a test function in (2.35), then

$$
\int_Q \boldsymbol{T}_k : \boldsymbol{\varepsilon}(((\alpha^n_k)_t * \eta_\epsilon \mathbf{1}_{(t_1,t_2)}) * \eta_\epsilon \boldsymbol{w}_n)\, dx\, dt = 0,
\tag{2.43}
$$

$$
\begin{aligned}
\int_Q (\boldsymbol{\varepsilon}^{\mathbf{p}}_k)_t : (\boldsymbol{T}^d_k * \eta_\epsilon \mathbf{1}_{(t_1,t_2)}) * \eta_\epsilon\, dx\, dt& \\
= \int_Q \boldsymbol{\chi}_k : (\boldsymbol{T}^d_k * \eta_\epsilon \mathbf{1}_{(t_1,t_2)}) * \eta_\epsilon\, dx\, dt,&
\end{aligned}
\tag{2.44}
$$

for $n = 1, \ldots, k$. Summing (2.43) over $n = 1, \ldots, k$, we obtain

$$
\int_{t_1}^{t_2}\int_\Omega \boldsymbol{D}\,(\boldsymbol{\varepsilon}(\boldsymbol{u}_k) - \boldsymbol{\varepsilon}^{\mathbf{p}}_k) * \eta_\epsilon : (\boldsymbol{\varepsilon}(\boldsymbol{u}_k) * \eta_\epsilon)_t\, dx\, dt = 0.
\tag{2.45}
$$

Moreover, using properties of traceless matrices,

$$
\int_{t_1}^{t_2}\int_\Omega (\boldsymbol{\varepsilon}^{\mathbf{p}}_k * \eta_\epsilon)_t : \boldsymbol{T}_k * \eta_\epsilon\, dx\, dt = \int_{t_1}^{t_2}\int_\Omega \boldsymbol{\chi}_k * \eta_\epsilon : \boldsymbol{T}_k * \eta_\epsilon\, dx\, dt,
\tag{2.46}
$$

and products in (2.46) are well defined. For the matrices $\boldsymbol{A} \in \mathcal{S}^3_d$ and $\boldsymbol{B} \in \mathcal{S}^3$ the equivalence $\boldsymbol{A} : \boldsymbol{B}^d = \boldsymbol{A} : \boldsymbol{B}$ holds and the sequence $\{\boldsymbol{T}^d_k\}$ is uniformly bounded in $L_M(Q, \mathcal{S}^3_d)$. Subtracting (2.45) from (2.46), we get

$$
\begin{aligned}
\int_{t_1}^{t_2}\int_\Omega \boldsymbol{D}(\boldsymbol{\varepsilon}(\boldsymbol{u}_k) - \boldsymbol{\varepsilon}^{\mathbf{p}}_k) * \eta_\epsilon : ((\boldsymbol{\varepsilon}(\boldsymbol{u}_k) - \boldsymbol{\varepsilon}^{\mathbf{p}}_k) * \eta_\epsilon)_t\, dx\, dt& \\
= -\int_{t_1}^{t_2}\int_\Omega \boldsymbol{\chi}_k * \eta_\epsilon : \boldsymbol{T}^d_k * \eta_\epsilon\, dx\, dt.&
\end{aligned}
\tag{2.47}
$$

Since $\boldsymbol{\varepsilon}(\boldsymbol{u}_k) - \boldsymbol{\varepsilon}^{\mathbf{p}}_k$ belongs to $L^2(Q, \mathcal{S}^3)$, we may pass to the limit as $\varepsilon \to 0$ in the left-hand side of equation (2.47).

To make a limit passage as $\varepsilon \to 0$ on the right-hand side of (2.47) we use the lemmas presented in Appendix B. From Lemma B.7, we know that sequences

$\{M(x,\boldsymbol{T}_k^d*\eta_\epsilon)\}$ and $\{M^*(x,\boldsymbol{\chi}_k*\eta_\epsilon)\}$ are uniformly integrable with respect to ϵ. Moreover, $\{\boldsymbol{T}_k^d*\eta_\epsilon\}_\epsilon$ converges in measure to $\boldsymbol{T}_k^d$ and $\{\boldsymbol{\chi}_k*\eta_\epsilon\}_\epsilon$ converges in measure to $\boldsymbol{\chi}_k$ (by Lemma B.6) as ϵ goes to 0. Uniform integrability of the sequence and convergence in measure of this sequence imply (by Lemma B.3) that

$$\begin{aligned}\boldsymbol{T}_k^d*\eta_\epsilon &\xrightarrow{M} \boldsymbol{T}_k^d \quad \text{modularly in } L_M(Q,\mathcal{S}^3_d),\\ \boldsymbol{\chi}_k*\eta_\epsilon &\xrightarrow{M^*} \boldsymbol{\chi}_k \quad \text{modularly in } L_{M^*}(Q,\mathcal{S}^3_d),\end{aligned}$$

as $\epsilon\to 0$. On the basis of Lemma B.5 we complete the limit passage on the right-hand side of (2.47). Then we obtain the following equality:

$$\frac{1}{2}\int_\Omega \boldsymbol{D}(\boldsymbol{\varepsilon}(\boldsymbol{u}_k)-\boldsymbol{\varepsilon}_k^{\mathbf{p}}):(\boldsymbol{\varepsilon}(\boldsymbol{u}_k)-\boldsymbol{\varepsilon}_k^{\mathbf{p}})\,dx\bigg|_{t_1}^{t_2} = -\int_{t_1}^{t_2}\int_\Omega \boldsymbol{\chi}_k:\boldsymbol{T}_k^d\,dx\,dt. \tag{2.48}$$

Since $\boldsymbol{\varepsilon}(\boldsymbol{u}_k),\boldsymbol{\varepsilon}_k^{\mathbf{p}}\in C_w([0,T],L^2(\Omega,\mathcal{S}^3))$ (where by $C_w([0,T],\cdot)$ we denote the space of functions which are weakly continuous with respect to time), we may pass to the limit as $t_1\to 0$ and conclude

$$\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_k),\boldsymbol{\varepsilon}_k^{\mathbf{p}})(t_2)-\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_k),\boldsymbol{\varepsilon}_k^{\mathbf{p}})(0) = -\int_0^{t_2}\int_\Omega \boldsymbol{\chi}_k:\boldsymbol{T}_k^d\,dx\,dt. \tag{2.49}$$

To complete the proof let us multiply (2.49) by $1/\mu$ and integrate over the interval $(\tau,\tau+\mu)$,

$$\begin{aligned}\frac{1}{\mu}\int_\tau^{\tau+\mu}\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_k),\boldsymbol{\varepsilon}_k^{\mathbf{p}})(s)\,ds-\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_k),\boldsymbol{\varepsilon}_k^{\mathbf{p}})(0) &=\\ -\frac{1}{\mu}\int_\tau^{\tau+\mu}\int_0^s\int_\Omega \boldsymbol{\chi}_k:\boldsymbol{T}_k^d\,dx\,dt\,ds.\end{aligned} \tag{2.50}$$

For conciseness let us define the function

$$F(t):=\int_\Omega \boldsymbol{\chi}_k:\boldsymbol{T}_k^d\,dx$$

which belongs to $L^1(0,T)$. Then, applying the Fubini theorem, we obtain

$$\begin{aligned}\frac{1}{\mu}\int_\tau^{\tau+\mu}\int_0^s F(t)\,dt\,ds &= \frac{1}{\mu}\int_{\mathbb{R}^2}\mathbf{1}_{\{0\le t\le s\}}(t)\mathbf{1}_{\{\tau\le s\le\tau+\mu\}}(s)F(t)\,dt\,ds\\ &=\int_{\mathbb{R}}\left(\frac{1}{\mu}\int_{\mathbb{R}}\mathbf{1}_{\{0\le t\le s\}}(t)\mathbf{1}_{\{\tau\le s\le\tau+\mu\}}(s)\,ds\right)F(t)\,dt.\end{aligned} \tag{2.51}$$

Using the definition of the function $\psi_{\mu,\tau}$, we observe that

$$\psi_{\mu,\tau}(t)=\frac{1}{\mu}\int_{\mathbb{R}}\mathbf{1}_{\{0\le t\le s\}}(t)\mathbf{1}_{\{\tau\le s\le\tau+\mu\}}(s)\,ds. \tag{2.52}$$

Hence, comparing (2.42) and (2.50), we obtain

$$\limsup_{l\to\infty}\int_0^T\int_\Omega \boldsymbol{G}(\widetilde{\theta}+\theta_{k,l},\widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}):\boldsymbol{T}^d_{k,l}\ \psi_{\mu,\tau}(t)\,dx\,dt \leq \int_0^T\int_\Omega \boldsymbol{\chi}_k:\boldsymbol{T}^d_k\psi_{\mu,\tau}(t)\,dx\,dt. \tag{2.53}$$

To complete the proof of Lemma 2.10 let us show the following estimates:

$$\begin{aligned}
\limsup_{l\to\infty}&\int_0^{t_2}\int_\Omega \boldsymbol{G}(\widetilde{\theta}+\theta_{k,l},\widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}):\boldsymbol{T}^d_{k,l}\,dx\,dt\\
&\leq \limsup_{l\to\infty}\int_0^{t_2}\int_\Omega \boldsymbol{G}(\widetilde{\theta}+\theta_{k,l},\widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}):(\widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l})\,dx\,dt\\
&\quad-\lim_{l\to\infty}\int_0^{t_2}\int_\Omega \boldsymbol{G}(\widetilde{\theta}+\theta_{k,l},\widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}):\widetilde{\boldsymbol{T}}^d\,dx\,dt\\
&\leq \limsup_{l\to\infty}\int_0^{t_2+\mu}\int_\Omega \boldsymbol{G}(\widetilde{\theta}+\theta_{k,l},\widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}):(\widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l})\psi_{\mu,t_2}(t)\,dx\,dt\\
&\quad-\lim_{l\to\infty}\int_0^{t_2}\int_\Omega \boldsymbol{G}(\widetilde{\theta}+\theta_{k,l},\widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}):\widetilde{\boldsymbol{T}}^d\,dx\,dt,
\end{aligned}$$

where the last inequality follows from the definition of $\psi_{\mu,t_2}(t)$. Then, using (2.53), we obtain

$$\begin{aligned}
\limsup_{l\to\infty}&\int_0^{t_2}\int_\Omega \boldsymbol{G}(\widetilde{\theta}+\theta_{k,l},\widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}):\boldsymbol{T}^d_{k,l}\,dx\,dt\\
&\leq \limsup_{l\to\infty}\int_0^{t_2+\mu}\int_\Omega \boldsymbol{G}(\widetilde{\theta}+\theta_{k,l},\widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}):\boldsymbol{T}^d_{k,l}\ \psi_{\mu,t_2}(t)\,dx\,dt\\
&\quad+\lim_{l\to\infty}\int_0^{t_2+\mu}\int_\Omega \boldsymbol{G}(\widetilde{\theta}+\theta_{k,l},\widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}):\widetilde{\boldsymbol{T}}^d\ \psi_{\mu,t_2}(t)\,dx\,dt\\
&\quad-\lim_{l\to\infty}\int_0^{t_2}\int_\Omega \boldsymbol{G}(\widetilde{\theta}+\theta_{k,l},\widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}):\widetilde{\boldsymbol{T}}^d\,dx\,dt\\
&\leq \int_0^{t_2+\mu}\int_\Omega \boldsymbol{\chi}_k:\boldsymbol{T}^d_k\psi_{\mu,t_2}(t)\,dx\,dt\\
&\quad+\lim_{l\to\infty}\int_{t_2}^{t_2+\mu}\int_\Omega \boldsymbol{G}(\widetilde{\theta}+\theta_{k,l},\widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}):\widetilde{\boldsymbol{T}}^d\psi_{\mu,t_2}(t)\,dx\,dt\\
&\leq \int_0^{t_2+\mu}\int_\Omega \boldsymbol{\chi}_k:\boldsymbol{T}^d_k\ \psi_{\mu,t_2}(t)\,dx\,dt\\
&\quad+\lim_{l\to\infty}\int_{t_2}^{t_2+\mu}\int_\Omega \boldsymbol{G}(\widetilde{\theta}+\theta_{k,l},\widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}):\widetilde{\boldsymbol{T}}^d_k\,dx\,dt.
\end{aligned}$$

Passing to the limit as $\mu\to 0$ yields (2.36). The proof is complete. $\square$

Step 2. *Minty–Browder trick.*

We use the Minty–Browder trick to identify the weak limits $\boldsymbol{\chi}_k$. For $s \in (0,T]$ let us define $Q^s = \Omega \times (0,s)$. From the monotonicity of the function $\boldsymbol{G}(\theta, \cdot)$ we obtain

$$\int_{Q^s} \big(\boldsymbol{G}(\widetilde{\theta}+\theta_{k,l}, \widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l})-\boldsymbol{G}(\widetilde{\theta}+\theta_{k,l}, \widetilde{\boldsymbol{T}}^d+\boldsymbol{W}^d)\big) : (\boldsymbol{T}^d_{k,l}-\boldsymbol{W}^d)\, dx\, dt \geq 0 \tag{2.54}$$

for all $\boldsymbol{W}^d \in L^\infty(Q, \mathcal{S}^3_d)$. Applying Lemma 2.10, we obtain

$$\limsup_{l\to\infty} \int_{Q^s} \boldsymbol{G}\big(\widetilde{\theta}+\theta_{k,l}, \widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}\big) : \boldsymbol{T}^d_{k,l}\, dx\, dt \leq \int_{Q^s} \boldsymbol{\chi}_k : \boldsymbol{T}^d_k\, dx\, dt.$$

Moreover, using (2.31), we get

$$\lim_{l\to\infty} \int_{Q^s} \boldsymbol{G}(\widetilde{\theta}+\theta_{k,l}, \widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}) : \boldsymbol{W}^d\, dx\, dt = \int_{Q^s} \boldsymbol{\chi}_k : \boldsymbol{W}^d\, dx\, dt.$$

The pointwise convergence of $\{\theta_{k,l}\}$ implies the pointwise convergence of $\{\boldsymbol{G}(\widetilde{\theta}+\theta_{k,l}, \widetilde{\boldsymbol{T}}^d+\boldsymbol{W}^d)\}$. Furthermore, from Assumption 1.7 and non-negativity of N-functions we get

$$|\widetilde{\boldsymbol{T}}^d+\boldsymbol{W}^d| \geq c\frac{M^*(x, \boldsymbol{G}(\widetilde{\theta}+\theta_{k,l}, \widetilde{\boldsymbol{T}}^d+\boldsymbol{W}^d))}{|\boldsymbol{G}(\widetilde{\theta}+\theta_{k,l}, \widetilde{\boldsymbol{T}}^d+\boldsymbol{W}^d)|}.$$

Since $\widetilde{\boldsymbol{T}}^d+\boldsymbol{W}^d$ belongs to $L^\infty(Q, \mathcal{S}^3_d)$ and M^* is an N-function, the sequence $\{\boldsymbol{G}(\widetilde{\theta}+\theta_{k,l}, \widetilde{\boldsymbol{T}}^d+\boldsymbol{W}^d)\}$ belongs to $L^\infty(Q, \mathcal{S}^3_d)$. By Lemma B.3, we obtain

$$\boldsymbol{G}(\widetilde{\theta}+\theta_{k,l}, \widetilde{\boldsymbol{T}}^d+\boldsymbol{W}^d) \xrightarrow{M^*} \boldsymbol{G}(\widetilde{\theta}+\theta_k, \widetilde{\boldsymbol{T}}^d+\boldsymbol{W}^d),$$

modularly in $L_{M^*}(Q)$. Then

$$\begin{aligned}
&\int_Q \big|\boldsymbol{G}(\widetilde{\theta}+\theta_{k,l}, \widetilde{\boldsymbol{T}}^d+\boldsymbol{W}^d) : (\boldsymbol{T}^d_{k,l}-\boldsymbol{W}) - \boldsymbol{G}(\widetilde{\theta}+\theta_k, \widetilde{\boldsymbol{T}}^d+\boldsymbol{W}^d) : (\boldsymbol{T}^d_k-\boldsymbol{W}^d)\big|\, dx\, dt \\
&\leq \int_Q \big|\big(\boldsymbol{G}(\widetilde{\theta}+\theta_{k,l}, \widetilde{\boldsymbol{T}}^d+\boldsymbol{W}^d) - \boldsymbol{G}(\widetilde{\theta}+\theta_k, \widetilde{\boldsymbol{T}}^d+\boldsymbol{W}^d)\big) : (\boldsymbol{T}^d_{k,l}-\boldsymbol{W}^d)\big|\, dx\, dt \\
&\quad + \int_Q \big|\boldsymbol{G}(\widetilde{\theta}+\theta_k, \widetilde{\boldsymbol{T}}^d+\boldsymbol{W}^d) : (\boldsymbol{T}^d_{k,l}-\boldsymbol{T}^d_k)\big|\, dx\, dt.
\end{aligned}$$

Using the Hölder inequality (Lemma B.2), we get

$$\begin{aligned}
\int_Q \big|\boldsymbol{G}(\widetilde{\theta}+\theta_{k,l}, &\widetilde{\boldsymbol{T}}^d+\boldsymbol{W}^d) : (\boldsymbol{T}^d_{k,l}-\boldsymbol{W}) \\
&- \boldsymbol{G}(\widetilde{\theta}+\theta_k, \widetilde{\boldsymbol{T}}^d+\boldsymbol{W}^d) : (\boldsymbol{T}^d_k-\boldsymbol{W}^d)\big|\, dx\, dt \\
\leq 2\big\|&\boldsymbol{G}(\widetilde{\theta}+\theta_{k,l}, \widetilde{\boldsymbol{T}}^d+\boldsymbol{W}^d) \\
&- \boldsymbol{G}(\widetilde{\theta}+\theta_k, \widetilde{\boldsymbol{T}}^d+\boldsymbol{W}^d)\big\|_{L,M^*} \|\boldsymbol{T}^d_{k,l}-\boldsymbol{W}^d\|_{L,M} \\
&+ \int_Q \big|\boldsymbol{G}(\widetilde{\theta}+\theta_k, \widetilde{\boldsymbol{T}}^d+\boldsymbol{W}^d) : (\boldsymbol{T}^d_{k,l}-\boldsymbol{T}^d_k)\big|\, dx\, dt.
\end{aligned} \tag{2.55}$$

Since $\|\boldsymbol{T}^d_{k,l}-\boldsymbol{W}^d\|_{L,M}$ is uniformly bounded, $\|\boldsymbol{G}(\widetilde{\theta}+\theta_{k,l},\widetilde{\boldsymbol{T}}^d+\boldsymbol{W}^d)-\boldsymbol{G}(\widetilde{\theta}+\theta_k,\widetilde{\boldsymbol{T}}^d+\boldsymbol{W}^d)\|_{L,M^*}\to 0$ and $\boldsymbol{T}^d_{k,l}-\boldsymbol{T}^d_k\rightharpoonup 0$ in $L_M(Q,\mathcal{S}^3_d)$ as l goes to ∞, then the right-hand side of (2.55) goes to zero as l goes to ∞. Hence

$$\lim_{l\to\infty}\int_{Q^s}\boldsymbol{G}(\widetilde{\theta}+\theta_{k,l},\widetilde{\boldsymbol{T}}^d+\boldsymbol{W}^d):(\boldsymbol{T}^d_{k,l}-\boldsymbol{W}^d)\,dx\,dt = \int_{Q^s}\boldsymbol{G}(\widetilde{\theta}+\theta_k,\widetilde{\boldsymbol{T}}^d+\boldsymbol{W}^d):(\boldsymbol{T}^d_k-\boldsymbol{W}^d)\,dx\,dt. \tag{2.56}$$

Summing up, passing to the limit as $l\to\infty$ in (2.54), we get

$$\int_{Q^s}\big(\boldsymbol{\chi}_k-\boldsymbol{G}(\widetilde{\theta}+\theta_k,\widetilde{\boldsymbol{T}}^d+\boldsymbol{W}^d)\big):(\boldsymbol{T}^d_k-\boldsymbol{W}^d)\,dx\,dt\geq 0 \tag{2.57}$$

for all $\boldsymbol{W}^d\in L^\infty(Q^s,\mathcal{S}^3_d)$. For $i>0$ let us define the set

$$Q_i=\{(t,x)\in Q^s:|\boldsymbol{T}^d_k|\leq i \text{ a.e. in } Q^s\}.$$

Then for $0<j<i$ and for arbitrary $h>0$ we define the function

$$\boldsymbol{W}^d=-\widetilde{\boldsymbol{T}}^d\mathbf{1}_{Q^s\setminus Q_i}+\boldsymbol{T}^d_k\mathbf{1}_{Q_i}+h\boldsymbol{U}^d\mathbf{1}_{Q_j}, \tag{2.58}$$

where $\boldsymbol{U}^d\in L^\infty(Q,\mathcal{S}^3_d)$ and $\mathbf{1}_H$ is a characteristic function of the set H. As we may observe $\boldsymbol{W}^d$ belongs to $L^\infty(\Omega,\mathcal{S}^3_d)$ due to (2.3). Using the function defined in (2.58) as a test function in (2.57), we obtain

$$\int_{Q^s}\big(\boldsymbol{\chi}_k-\boldsymbol{G}(\widetilde{\theta}+\theta_k,\widetilde{\boldsymbol{T}}^d-\widetilde{\boldsymbol{T}}^d\mathbf{1}_{Q^s\setminus Q_i}+\boldsymbol{T}^d_k\mathbf{1}_{Q_i}+h\boldsymbol{U}^d\mathbf{1}_{Q_j})\big): (\boldsymbol{T}^d_k+\widetilde{\boldsymbol{T}}^d\mathbf{1}_{Q^s\setminus Q_i}-\boldsymbol{T}^d_k\mathbf{1}_{Q_i}-h\boldsymbol{U}^d\mathbf{1}_{Q_j})\,dx\,dt\geq 0.$$

Since $Q_j\subset Q_i\subset Q^s$ we get

$$-h\int_{Q_j}\big(\boldsymbol{\chi}_k-\boldsymbol{G}(\widetilde{\theta}+\theta_k,\widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k+h\boldsymbol{U}^d)\big):\boldsymbol{U}^d\,dx\,dt + \int_{Q_i\setminus Q_j}\big(\boldsymbol{\chi}_k-\boldsymbol{G}(\widetilde{\theta}+\theta_k,\widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)\big):(\boldsymbol{T}^d_k-\boldsymbol{T}^d_k)\,dx\,dt + \int_{Q^s\setminus Q_i}\big(\boldsymbol{\chi}_k-\boldsymbol{G}(\widetilde{\theta}+\theta_k,\mathbf{0})\big):(\widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)\,dx\,dt\geq 0.$$

Using Assumption 1.7, we obtain $M^*(x,\boldsymbol{G}(\widetilde{\theta}+\theta_k,\mathbf{0}))=0$ and then, using Definition 1.1, we get that $\boldsymbol{G}(\widetilde{\theta}+\theta_k,\mathbf{0})=\mathbf{0}$. Hence

$$-h\int_{Q_j}\big(\boldsymbol{\chi}_k-\boldsymbol{G}(\widetilde{\theta}+\theta_k,\widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k+h\boldsymbol{U}^d)\big):\boldsymbol{U}^d\,dx\,dt + \int_{Q^s\setminus Q_i}\boldsymbol{\chi}_k:(\widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)\,dx\,dt\geq 0. \tag{2.59}$$

Moreover, from the definition of characteristic function

$$\int_{Q^s\setminus Q_i} \boldsymbol{\chi}_k : \big(\boldsymbol{T}_k^d + \widetilde{\boldsymbol{T}}^d\big)\,dx\,dt = \int_Q \big(\boldsymbol{\chi}_k : (\widetilde{\boldsymbol{T}}^d + \boldsymbol{T}_k^d)\big)\mathbf{1}_{Q^s\setminus Q_i}\,dx\,dt. \tag{2.60}$$

Since $\int_Q \boldsymbol{\chi}_k : \big(\widetilde{\boldsymbol{T}}^d + \boldsymbol{T}_k^d\big) < \infty$ and $\big(\boldsymbol{\chi}_k : (\widetilde{\boldsymbol{T}}^d + \boldsymbol{T}_k^d)\big)\mathbf{1}_{Q^s\setminus Q_i} \to 0$ almost everywhere in Q as i goes to ∞, the Lebesgue dominated convergence theorem implies that

$$\lim_{i\to\infty}\int_{Q^s\setminus Q_i} \boldsymbol{\chi}_k : \big(\widetilde{\boldsymbol{T}}^d + \boldsymbol{T}_k^d\big)\,dx\,dt = 0.$$

Passing to the limit as i goes to ∞ in (2.59) and dividing by h, we obtain

$$\int_{Q_j} \big(\boldsymbol{\chi}_k - \boldsymbol{G}\big(\widetilde{\theta} + \theta_k, \widetilde{\boldsymbol{T}}^d + \boldsymbol{T}_k^d + h\boldsymbol{U}^d\big)\big) : \boldsymbol{U}^d\,dx\,dt \le 0. \tag{2.61}$$

Since $\widetilde{\boldsymbol{T}}^d + \boldsymbol{T}_k^d + h\boldsymbol{U}^d$ goes to $\widetilde{\boldsymbol{T}}^d + \boldsymbol{T}_k^d$ almost everywhere in Q as $h \to 0^+$, $\{\boldsymbol{G}(\widetilde{\theta}+\theta_k, \widetilde{\boldsymbol{T}}^d+\boldsymbol{T}_k^d+h\boldsymbol{U}^d)\}_{h>0}$ is uniformly bounded in $L_{M^*}(Q_j, \mathcal{S}^3)$, we conclude that

$$\boldsymbol{G}\big(\widetilde{\theta} + \theta_k, \widetilde{\boldsymbol{T}}^d + \boldsymbol{T}_k^d + h\boldsymbol{U}^d\big) \rightharpoonup^* \boldsymbol{G}\big(\widetilde{\theta} + \theta_k, \widetilde{\boldsymbol{T}}^d + \boldsymbol{T}_k^d\big)$$

in $L_{M^*}(Q_j, \mathcal{S}^3)$ as h goes to 0^+. Consequently, passing to the limit as h goes to ∞ in (2.61), we obtain

$$\int_{Q_j} \big(\boldsymbol{\chi}_k - \boldsymbol{G}\big(\widetilde{\theta} + \theta_k, \widetilde{\boldsymbol{T}}^d + \boldsymbol{T}_k^d\big)\big) : \boldsymbol{U}^d\,dx\,dt \le 0,$$

for all $\boldsymbol{U}^d \in L^\infty(Q, \mathcal{S}^3_d)$, so taking

$$\boldsymbol{U}^d = \begin{cases} \dfrac{\boldsymbol{\chi}_k - \boldsymbol{G}(\widetilde{\theta} + \theta_k, \widetilde{\boldsymbol{T}}^d + \boldsymbol{T}_k^d)}{|\boldsymbol{\chi}_k - \boldsymbol{G}(\widetilde{\theta} + \theta_k, \widetilde{\boldsymbol{T}}^d + \boldsymbol{T}_k^d)|} & \text{when } \boldsymbol{\chi}_k \neq \boldsymbol{G}(\widetilde{\theta} + \theta_k, \widetilde{\boldsymbol{T}}^d + \boldsymbol{T}_k^d), \\ 0 & \text{when } \boldsymbol{\chi}_k = \boldsymbol{G}(\widetilde{\theta} + \theta_k, \widetilde{\boldsymbol{T}}^d + \boldsymbol{T}_k^d), \end{cases}$$

we obtain

$$\int_{Q_j} \big|\boldsymbol{\chi}_k - \boldsymbol{G}(\widetilde{\theta} + \theta_k, \widetilde{\boldsymbol{T}}^d + \boldsymbol{T}_k^d)\big|\,dx\,dt \le 0,$$

i.e. $\boldsymbol{\chi}_k = \boldsymbol{G}(\widetilde{\theta}+\theta_k, \widetilde{\boldsymbol{T}}^d+\boldsymbol{T}_k^d)$ almost everywhere in Q^s. From the arbitrary choices of $j > 0$ and $0 \le s \le T$ we get $\boldsymbol{\chi}_k = \boldsymbol{G}(\widetilde{\theta}+\theta_k, \widetilde{\boldsymbol{T}}^d+\boldsymbol{T}_k^d)$ almost everywhere in Q.

STEP 3. *Limit of the right-hand side of heat equations.*

The idea for the third step came from the paper by Gwiazda et al. [29]. Let us start from the formulation of auxiliary lemmas which may be found with proofs in [29]. We denote by $\xrightarrow{b}$ the biting limit used in Chacon's, cf. [5].

DEFINITION 2.11 (Biting limit). Let $\{f^\nu\}$ be a bounded sequence in $L^1(Q)$. We say that $f \in L^1(Q)$ is a biting limit of $\{f^\nu\}$ if there exists a nonincreasing sequence $\{E_k\}$ with $E_k \subset Q$ and $\lim_{k\to\infty} |E_k| = 0$, such that f^ν converges weakly to f in $L^1(Q \setminus E_k)$ for every fixed k.

LEMMA 2.12. *Let $a_n \in L^1(Q)$ and $0 \le a_0 \in L^1(Q)$, and*

$$a_n \ge -a_0, \quad a_n \xrightarrow{b} a \quad \text{and} \quad \limsup_{n\to\infty} \int_Q a_n \,dx\,dt \le \int_Q a \,dx\,dt$$

then $a_n \rightharpoonup a$ weakly in $L^1(Q)$.

LEMMA 2.13. *The sequence $\{\boldsymbol{G}(\widetilde{\theta}+\theta_{k,l}, \widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}) : (\widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l})\}_{l=1}^\infty$ converges weakly to $\boldsymbol{G}(\widetilde{\theta}+\theta_k, \widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k) : (\widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)$ in $L^1(Q)$, for each $k \in \mathbb{N}$.*

PROOF. To characterize the limit of the right-hand side we use the same argumentation as in [29]. Using Assumption 1.7, the Frechet–Young inequality and convexity of N-functions, we get

$$\begin{aligned}
c\big(M(x,\widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k) &+ M^*(x, \boldsymbol{G}(\widetilde{\theta}+\theta_{k,l}, \widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k))\big) \\
&\le \boldsymbol{G}(\widetilde{\theta}+\theta_{k,l}, \widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k) : (\widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k) \\
&\le M\left(x, \frac{2}{d}(\widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)\right) + M^*(x, \frac{d}{2}\boldsymbol{G}(\widetilde{\theta}+\theta_{k,l}, \widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)) \\
&\le M\left(x, \frac{2}{d}(\widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)\right) + \frac{d}{2}M^*(x, \boldsymbol{G}(\widetilde{\theta}+\theta_{k,l}, \widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)),
\end{aligned}$$

where $d = \min(c, 1)$. And finally

$$cM(x,\widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k) + \frac{2c-d}{2}M^*(x, \boldsymbol{G}(\widetilde{\theta}+\theta_{k,l}, \widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)) \le M\left(x, \frac{2}{d}(\widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)\right).$$

Hence the sequence $\{\boldsymbol{G}(\widetilde{\theta}+\theta_{k,l}, \widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)\}_{l=1}^\infty$ is uniformly bounded in $L_{M^*}(Q)$. Using the monotonicity of the function $\boldsymbol{G}(\cdot,\cdot)$ with respect to the second variable, we get

$$0 \le \big(\boldsymbol{G}(\widetilde{\theta}+\theta_{k,l}, \widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}) - \boldsymbol{G}(\widetilde{\theta}+\theta_{k,l}, \widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)\big) : (\boldsymbol{T}^d_{k,l} - \boldsymbol{T}^d_k). \tag{2.62}$$

The right-hand side of above inequality is uniformly bounded in $L^1(Q)$. Thus, there exists a Young measure denoted by $\mu_{x,t}(\,\cdot\,,\,\cdot\,)$, see [36, Theorem 3.1], such that the following convergence holds:

$$\begin{aligned}
&\big(\boldsymbol{G}(\widetilde{\theta}+\theta_{k,l}, \widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}) - \boldsymbol{G}(\widetilde{\theta}+\theta_{k,l}, \widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)\big) : (\boldsymbol{T}^d_{k,l} - \boldsymbol{T}^d_k) \\
&\quad\xrightarrow{b} \int_{\mathbb{R}\times\mathbb{R}^{3\times3}} \big(\boldsymbol{G}(s,\boldsymbol{\lambda}) - \boldsymbol{G}(s, \widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)\big) : \big(\boldsymbol{\lambda} - (\widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)\big)\, d\mu_{x,t}(s,\boldsymbol{\lambda})
\end{aligned} \tag{2.63}$$

as $l \to \infty$. Using Lemma C.2, we obtain that the measure $\mu_{x,t}(s,\boldsymbol{\lambda})$ can be represented in the form of $\delta_{\widetilde{\theta}+\theta_k}(s) \otimes \nu_{x,t}(\boldsymbol{\lambda})$. Then

$$\begin{aligned}
\int_{\mathbb{R}\times\mathbb{R}^{3\times3}} &\big(\boldsymbol{G}(s,\boldsymbol{\lambda}) - \boldsymbol{G}(s, \widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)\big) : \big(\boldsymbol{\lambda} - (\widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)\big)\, d\mu_{x,t}(s,\boldsymbol{\lambda}) \\
&= \int_{\mathbb{R}^{3\times3}} \big(\boldsymbol{G}(\widetilde{\theta}+\theta_k,\boldsymbol{\lambda}) - \boldsymbol{G}(\widetilde{\theta}+\theta_k, \widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)\big) : \big(\boldsymbol{\lambda} - (\widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)\big)\, d\nu_{x,t}(\boldsymbol{\lambda})
\end{aligned}$$

$$= \int_{\mathbb{R}^{3\times 3}} \boldsymbol{G}(\widetilde{\theta}+\theta_k,\boldsymbol{\lambda}) : \big(\boldsymbol{\lambda} - (\widetilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_k)\big)\, d\nu_{x,t}(\boldsymbol{\lambda})$$
$$- \int_{\mathbb{R}^{3\times 3}} \boldsymbol{G}\big(\widetilde{\theta}+\theta_k,\widetilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_k\big) : \big(\boldsymbol{\lambda} - (\widetilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_k)\big)\, d\nu_{x,t}(\boldsymbol{\lambda}).$$

Since the sequence $\{\widetilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}\}$ generates the measure $d\nu_{x,t}(\boldsymbol{\lambda})$,

$$\int_{\mathbb{R}^{3\times 3}} \boldsymbol{\lambda}\, d\nu_{x,t}(\boldsymbol{\lambda}) = \widetilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_k \quad \text{a.e.}$$

Thus, the second term in above equation disappears. Indeed,

$$- \int_{\mathbb{R}^{3\times 3}} \boldsymbol{G}\big(\widetilde{\theta}+\theta_k,\widetilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_k\big) : \big(\boldsymbol{\lambda} - (\widetilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_k)\big)\, d\nu_{x,t}(\boldsymbol{\lambda})$$
$$= -\boldsymbol{G}\big(\widetilde{\theta}+\theta_k,\widetilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_k\big) : \bigg(\int_{\mathbb{R}^{3\times 3}} \boldsymbol{\lambda}\, d\nu_{x,t}(\boldsymbol{\lambda}) - (\widetilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_k)\bigg).$$

Moreover, the uniform boundedness of the sequence $\{\boldsymbol{G}(\widetilde{\theta}+\theta_{k,l},\widetilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}) : (\widetilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})\}_{l=1}^{\infty}$ in $L^1(Q)$ implies that

$$\boldsymbol{G}(\widetilde{\theta}+\theta_{k,l},\widetilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}) : (\widetilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}) \xrightarrow{b} \int_{\mathbb{R}\times\mathbb{R}^{3\times 3}} \boldsymbol{G}(s,\boldsymbol{\lambda}) : \boldsymbol{\lambda}\, d\mu_{x,t}(s,\boldsymbol{\lambda})$$
$$= \int_{\mathbb{R}^{3\times 3}} \boldsymbol{G}(\widetilde{\theta}+\theta_k,\boldsymbol{\lambda}) : \boldsymbol{\lambda}\, d\nu_{x,t}(\boldsymbol{\lambda}).$$

Hence, by the positivity of $\boldsymbol{G}(\widetilde{\theta}+\cdot,\widetilde{\boldsymbol{T}}^d + \cdot\,) : (\widetilde{\boldsymbol{T}}^d + \cdot\,)$ and using Lemma C.1, we get

$$\liminf_{l\to\infty} \int_Q \boldsymbol{G}(\widetilde{\theta}+\theta_{k,l},\widetilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}) : (\widetilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})\, dx\, dt$$
$$\geq \int_Q \int_{\mathbb{R}^{3\times 3}} \boldsymbol{G}(\widetilde{\theta}+\theta_k,\boldsymbol{\lambda}) : \boldsymbol{\lambda}\, d\nu_{x,t}(\boldsymbol{\lambda})\, dx\, dt.$$

Using Lemma 2.10 and knowing that $\boldsymbol{\chi}_k = \boldsymbol{G}(\widetilde{\theta}+\theta_k,\widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)$ almost everywhere in Q, we get

$$\int_Q \boldsymbol{G}(\widetilde{\theta}+\theta_k,\widetilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_k) : (\widetilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_k)\, dx\, dt \geq \int_Q \int_{\mathbb{R}^{3\times 3}} \boldsymbol{G}(\widetilde{\theta}+\theta_k,\boldsymbol{\lambda}) : \boldsymbol{\lambda}\, d\nu_{x,t}(\boldsymbol{\lambda})\, dx\, dt. \tag{2.64}$$

Since

$$\boldsymbol{G}(\widetilde{\theta}+\theta_k,\widetilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_k) = \int_{\mathbb{R}^{3\times 3}} \boldsymbol{G}(\widetilde{\theta}+\theta_k,\boldsymbol{\lambda})\, d\nu_{x,t}(\boldsymbol{\lambda})$$

and (2.64) holds, we obtain that the right-hand side of (2.63) is not positive. Hence

$$\big(\boldsymbol{G}(\widetilde{\theta}+\theta_{k,l},\widetilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}) - \boldsymbol{G}(\widetilde{\theta}+\theta_{k,l},\widetilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_k)\big) : (\boldsymbol{T}^d_{k,l} - \boldsymbol{T}^d_k) \xrightarrow{b} 0.$$

Using again the biting limit, we get

$$\boldsymbol{G}(\widetilde{\theta}+\theta_{k,l},\widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k):(\boldsymbol{T}^d_{k,l}-\boldsymbol{T}^d_k)\xrightarrow{b}0.$$

Hence

$$\boldsymbol{G}(\widetilde{\theta}+\theta_{k,l},\widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}):(\boldsymbol{T}^d_k+\widetilde{\boldsymbol{T}}^d)\xrightarrow{b}\boldsymbol{G}(\widetilde{\theta}+\theta_k,\widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k):(\boldsymbol{T}^d_k+\widetilde{\boldsymbol{T}}^d).$$

We use Lemma 2.12 to complete the proof. □

Now, with $\varphi_4(t)\in C^\infty([0,T]\times\Omega)$, we can pass to the limit as $l\to\infty$ in (2.26).

$$\begin{aligned}(2.65)\qquad &-\int_0^T\int_\Omega\theta_k(\varphi_4(t))_t\,dx\,dt\\ &-\int_\Omega\theta_k(x,0)\varphi_4(x,0)\,dx+\int_0^T\int_\Omega\nabla\theta_k\cdot\nabla v_m\varphi_4(t)\,dx\,dt\\ &=\int_0^T\int_\Omega\mathcal{T}_k\big((\widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k):\boldsymbol{G}(\theta_k+\widetilde{\theta},\widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)\big)v_m\varphi_4(t)\,dx\,dt\end{aligned}$$

We finish this section with two lemmas. We prove the uniform boundedness of the sequences $\{\boldsymbol{\varepsilon}^{\mathbf{p}}_k\}$ and $\{\boldsymbol{u}_k\}$ in proper spaces. This allows us to make the limit passage with the second parameter in the next section.

Lemma 2.14. *The sequence $\{\boldsymbol{\varepsilon}^{\mathbf{p}}_k\}$ is uniformly bounded in $L_{M^*}(Q,\mathcal{S}^3_d)$. Moreover, the sequence $\{(\boldsymbol{\varepsilon}^{\mathbf{p}}_k)_t\}$ is also uniformly bounded in $L_{M^*}(Q,\mathcal{S}^3_d)$.*

Proof. Let us consider the equation for the evolution of the visco-elastic strain tensor

$$(\boldsymbol{\varepsilon}^{\mathbf{p}}_k)_t=\boldsymbol{G}(\widetilde{\theta}+\theta_k,\widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k).$$

Moreover,

$$\boldsymbol{\varepsilon}^{\mathbf{p}}_k(x,t)=\boldsymbol{\varepsilon}^{\mathbf{p}}_k(x,0)+\int_0^t(\boldsymbol{\varepsilon}^{\mathbf{p}}_k(x,s))_s\,ds.$$

Integrating $M^*(x,\boldsymbol{\varepsilon}^{\mathbf{p}}_k(x,t))$ over the cylinder Q and using the Δ_2-condition of the N-function M^* (1.4), we get

$$\begin{aligned}\int_Q M^*(x,\boldsymbol{\varepsilon}^{\mathbf{p}}_k(x,t))\,dx\,dt&\le c\int_Q M^*\Big(x,\frac12\boldsymbol{\varepsilon}^{\mathbf{p}}_k(x,t)\Big)\,dx\,dt+T\int_\Omega h(x)\,dx\\ &=c\int_Q M^*\Big(x,\frac12\boldsymbol{\varepsilon}^{\mathbf{p}}_k(x,0)+\frac12\int_0^t(\boldsymbol{\varepsilon}^{\mathbf{p}}_k(x,s))_s\,ds\Big)\,dx\,dt+T\int_\Omega h(x)\,dx.\end{aligned}$$

Using the convexity of M^*, we obtain

$$\begin{aligned}(2.66)\qquad &\int_Q M^*(x,\boldsymbol{\varepsilon}^{\mathbf{p}}_k(x,t))\,dx\,dt\le\frac{c}{2}\int_Q M^*(x,\boldsymbol{\varepsilon}^{\mathbf{p}}_k(x,0))\,dx\,dt\\ &+\frac{c}{2}\int_Q M^*\Big(x,\int_0^t\boldsymbol{G}(\widetilde{\theta}+\theta_k,\widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)(x,s)\,ds\Big)\,dx\,dt+T\int_\Omega h(x)\,dx.\end{aligned}$$

Let us focus on the middle term on the right-hand side of above equation. Changing the variable $\tau = t/T$, we obtain

$$\int_0^T \int_\Omega M^*\Big(x, \int_0^t \boldsymbol{G}(\widetilde{\theta}+\theta_k, \widetilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_k)(x,s)\,ds\Big)\,dx\,dt$$
$$= T\int_0^1 \int_\Omega M^*\Big(x, \int_0^{\tau T} \boldsymbol{G}(\widetilde{\theta}+\theta_k, \widetilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_k)(x,s)\,ds\Big)\,dx\,d\tau.$$

By the Jensen inequality, we get

$$\begin{aligned}
T\int_0^1 &\int_\Omega M^*\Big(x, \int_0^t \boldsymbol{G}(\widetilde{\theta}+\theta_k, \widetilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_k)(x,s)\,ds\Big)\,dx\,dt\\
&\le T\int_0^1 \int_\Omega \frac{1}{\tau T}\int_0^{\tau T} M^*\big(x, \tau T\boldsymbol{G}(\widetilde{\theta}+\theta_k, \widetilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_k)\big)\,ds\,dx\,d\tau\\
&\le T\int_0^1 \int_\Omega \frac{1}{\tau T}\int_0^{\tau T} \tau M^*\big(x, T\boldsymbol{G}(\widetilde{\theta}+\theta_k, \widetilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_k)\big)\,ds\,dx\,d\tau\\
&= \int_0^1 \int_\Omega \int_0^{\tau T} M^*\big(x, T\boldsymbol{G}(\widetilde{\theta}+\theta_k, \widetilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_k)\big)\,ds\,dx\,d\tau.
\end{aligned}$$

There exists $d \in \mathbb{R}$ such that $2^d \ge T$. Then, using the Δ_2-condition, coming back to the original variable and using the Fubini theorem, we get

$$\begin{aligned}
\int_0^1 &\int_\Omega \int_0^{\tau T} M^*\big(x, T\boldsymbol{G}(\widetilde{\theta}+\theta_k, \widetilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_k)\big)\,ds\,dx\,d\tau\\
&\le \int_0^1 \int_\Omega \int_0^{\tau T} M^*\big(x, 2^d\boldsymbol{G}(\widetilde{\theta}+\theta_k, \widetilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_k)\big)\,ds\,dx\,d\tau\\
&\le c^d\int_0^1 \int_\Omega \int_0^{\tau T} M^*\big(x, \boldsymbol{G}(\widetilde{\theta}+\theta_k, \widetilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_k)\big)\,ds\,dx\,d\tau + C(d)\int_\Omega h(x)\,dx\\
&= \frac{c^d}{T}\int_0^T \int_\Omega \int_0^{t} M^*\big(x, \boldsymbol{G}(\widetilde{\theta}+\theta_k, \widetilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_k)\big)\,ds\,dx\,dt + C(d)\int_\Omega h(x)\,dx\\
&\le c^d\int_0^T \int_\Omega M^*\big(x, \boldsymbol{G}(\widetilde{\theta}+\theta_k, \widetilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_k)\big)\,dx\,dt + C(d)\int_\Omega h(x)\,dx.
\end{aligned}$$

Coming back to (2.66), we get

$$\int_Q M^*(x, \boldsymbol{\varepsilon}^{\mathbf{p}}_k(x,t))\,dx\,dt \le \frac{cT}{2}\int_\Omega M^*(x, \boldsymbol{\varepsilon}^{\mathbf{p}}_k(x,0))\,dx$$
$$+ c^d\int_0^T \int_\Omega M^*\big(x, \boldsymbol{G}(\widetilde{\theta}+\theta_k, \widetilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_k)\big)\,dx\,dt + C(d)\int_\Omega h(x)\,dx.$$

Lemma 2.4 and the initial condition in $L_{M^*}(\Omega, \mathcal{S}^3_d)$ complete the proof. □

Lemma 2.15. *The sequence* $\{\boldsymbol{u}_k\}$ *is uniformly bounded in* $BD_{M^*}(\Omega, \mathbb{R}^3)$.

Proof. Let us start with showing the uniform boundedness of the sequence $\{\boldsymbol{\varepsilon}(\boldsymbol{u}_k)\}$ in the space $L_{M^*}(Q)$. Using the Δ_2-condition, convexity of N-function and Assumption 1.7, we obtain

$$\begin{aligned}
\int_Q M^*(x,\boldsymbol{\varepsilon}(\boldsymbol{u}_k))\,dx\,dt &\le c\int_Q M^*\Big(x,\frac{1}{2}\boldsymbol{\varepsilon}(\boldsymbol{u}_k)\Big)\,dx\,dt + \int_Q h(x)\,dx\,dt\\
&= c\int_Q M^*\Big(x,\frac{1}{2}(\boldsymbol{\varepsilon}(\boldsymbol{u}_k)-\boldsymbol{\varepsilon}^{\mathbf{p}}_k)+\frac{1}{2}\boldsymbol{\varepsilon}^{\mathbf{p}}_k\Big)\,dx\,dt + T\int_\Omega h(x)\,dx\\
&\le \frac{c}{2}\int_Q M^*(x,\boldsymbol{\varepsilon}(\boldsymbol{u}_k)-\boldsymbol{\varepsilon}^{\mathbf{p}}_k)\,dx\,dt + \frac{c}{2}\int_Q M^*(x,\boldsymbol{\varepsilon}^{\mathbf{p}}_k)\,dx\,dt + T\int_\Omega h(x)\,dx\\
&\le \frac{c}{2}\int_Q |\boldsymbol{\varepsilon}(\boldsymbol{u}_k)-\boldsymbol{\varepsilon}^{\mathbf{p}}_k|^2\,dx\,dt + \frac{c}{2}\int_Q M^*(x,\boldsymbol{\varepsilon}^{\mathbf{p}}_k)\,dx\,dt + T\int_\Omega h(x)\,dx\\
&\le \frac{c}{2}\int_Q |\boldsymbol{T}_k|^2\,dx\,dt + \frac{c}{2}\int_Q M^*(x,\boldsymbol{\varepsilon}^{\mathbf{p}}_k)\,dx\,dt + T\int_\Omega h(x)\,dx.
\end{aligned}$$

Following Anzellotti and Giaquinta [4, Proposition 1.2 a)], we get the inequality

$$\|\boldsymbol{u}_k\|_{L^1(Q)} \le C\|\boldsymbol{\varepsilon}(\boldsymbol{u}_k)\|_{L^1(Q)},$$

where C is a constant depending on Ω. Finally, by the Fenchel–Young inequality, we get the estimate

$$\|\boldsymbol{u}_k\|_{L^1(Q)} \le C_{Q,M}\int_Q M^*(x,\boldsymbol{\varepsilon}(\boldsymbol{u}_k))\,dx\,dt,$$

where the constant $C_{Q,M}$ depends on the N-function M and the space-time cylinder Q. This completes the proof. □

2.5. Limit passage as $k\to\infty$. The considerations over the second limit passage we start from discussing the existence of heat equation solution. In Appendix A, we prove the existence of renormalised solution to the parabolic equation with Neumann boundary condition, which is an extension of results presented by Blanchard and Murat in a series of papers. In [7], [8], the existence and uniqueness of renormalised solution is proved in the case of Dirichlet boundary condition. Repeating their reasoning, we are able to prove that there exists $\theta\in C(0,T,L^1(\Omega))$ such that $\theta_k\to\theta$ almost everywhere in Q.

The uniform boundedness presented in the previous sections gives us the following convergences:

$$\begin{aligned}
\boldsymbol{u}_k &\rightharpoonup \boldsymbol{u} && \text{weakly in } L^1(Q,\mathbb{R}^3),\\
\boldsymbol{\varepsilon}(\boldsymbol{u}_k) &\rightharpoonup^* \boldsymbol{\varepsilon}(\boldsymbol{u}) && \text{weakly* in } L_{M^*}(Q,\mathbb{R}^3),\\
\boldsymbol{T}_k &\rightharpoonup \boldsymbol{T} && \text{weakly in } L^2(Q,\mathcal{S}^3),\\
\boldsymbol{T}^d_k &\rightharpoonup^* \boldsymbol{T}^d && \text{weakly* in } L_M(Q,\mathcal{S}^3_d),\\
\boldsymbol{G}(\widetilde{\theta}+\theta_k,\widetilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k) &\rightharpoonup^* \boldsymbol{\chi} && \text{weakly* in } L_{M^*}(Q,\mathcal{S}^3_d),\\
(\boldsymbol{\varepsilon}^{\mathbf{p}}_k)_t &\rightharpoonup^* (\boldsymbol{\varepsilon}^{\mathbf{p}})_t && \text{weakly* in } L_{M^*}(Q,\mathcal{S}^3_d).
\end{aligned}\tag{2.67}$$

Using these convergences in (2.32) and (2.35), we get

$$\int_Q \boldsymbol{T} : \nabla\boldsymbol{\varphi}\,dx\,dt = 0, \qquad \int_Q (\boldsymbol{\varepsilon}^{\mathbf{p}})_t : \boldsymbol{\psi}\,dx\,dt = \int_Q \boldsymbol{\chi} : \boldsymbol{\psi}\,dx\,dt \tag{2.68}$$

for $\boldsymbol{\varphi} \in C^\infty([0,T], L^2(\Omega,\mathbb{R}^3))$ and $\boldsymbol{\psi} \in L_M(Q,\mathcal{S}^3)$. To complete the limit passage we deal with the same problem as in the previous step, i.e. we have to identify the limit of the right-hand side of heat equation. Once again, the identification of this limit cosists of three steps.

In the proof of the following lemma we proceed similarly as in the proof of Lemma 2.10.

Lemma 2.16. *The inequality*

$$\limsup_{k\to\infty} \int_0^{t_2}\int_\Omega \boldsymbol{G}(\widetilde{\theta}+\theta_k, \widetilde{\boldsymbol{T}}^d + \boldsymbol{T}_k^d) : \boldsymbol{T}_k^d\,dx\,dt \le \int_0^{t_2}\int_\Omega \boldsymbol{\chi} : \boldsymbol{T}^d\,dx\,dt \tag{2.69}$$

holds for the solution of approximate systems.

Proof. Using the lower semicontinuity in $L^2(Q)$, we get

$$\begin{aligned}
\liminf_{k\to\infty} \int_0^T \frac{d}{dt}\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_k), \boldsymbol{\varepsilon}_k^{\mathbf{p}})\,\psi_{\mu,\tau}\,dt
&= \liminf_{k\to\infty} \frac{1}{\mu}\int_\tau^{\tau+\mu} \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_k), \boldsymbol{\varepsilon}_k^{\mathbf{p}})(t)\,dt - \lim_{k\to\infty}\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_k), \boldsymbol{\varepsilon}_k^{\mathbf{p}})(0)\\
&\ge \frac{1}{\mu}\int_\tau^{\tau+\mu} \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_k), \boldsymbol{\varepsilon}_k^{\mathbf{p}})(t)\,dt - \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}^{\mathbf{p}})(0).
\end{aligned}\tag{2.70}$$

We use $\boldsymbol{\varphi}_1 = ((\boldsymbol{\varepsilon}(\boldsymbol{u}_k) * \eta_\epsilon)_t \mathbf{1}_{(t_1,t_2)}) * \eta_\epsilon$, where η_ϵ is a standard mollifier with respect to time, $0 < t_1 < t_2 < T$, and ϵ is sufficiently small ($\epsilon < \min(t_1, T-t_2)$), as a test function in (2.68), then

$$\int_{t_1}^{t_2}\int_\Omega \boldsymbol{D}(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}^{\mathbf{p}}) * \eta_\epsilon : (\boldsymbol{\varepsilon}(\boldsymbol{u}_k) * \eta_\epsilon)_t\,dx\,dt = 0. \tag{2.71}$$

Moreover, we use $\boldsymbol{\psi} = (\boldsymbol{T}^d * \eta_\epsilon \mathbf{1}_{(t_1,t_2)}) * \eta_\epsilon$ as a test function in (2.35). Then

$$\begin{aligned}
\int_{t_1}^{t_2}\int_\Omega (\boldsymbol{\varepsilon}_k^{\mathbf{p}} * \eta_\epsilon)_t : \boldsymbol{T} * \eta_\epsilon\,dx\,dt
&= \int_{t_1}^{t_2}\int_\Omega \boldsymbol{G}(\widetilde{\theta}+\theta_k, \widetilde{\boldsymbol{T}}^d + \boldsymbol{T}_k^d) * \eta_\epsilon : \boldsymbol{T} * \eta_\epsilon\,dx\,dt.
\end{aligned}\tag{2.72}$$

Products in (2.72) are well defined. Subtracting these two equations, we get

$$\begin{aligned}
\int_{t_1}^{t_2}\int_\Omega \boldsymbol{T} * \eta_\epsilon : (\boldsymbol{\varepsilon}(\boldsymbol{u}_k) - \boldsymbol{\varepsilon}_k^{\mathbf{p}})_t * \eta_\epsilon\,dx\,dt
&= -\int_{t_1}^{t_2}\int_\Omega \boldsymbol{G}(\widetilde{\theta}+\theta_k, \widetilde{\boldsymbol{T}}^d + \boldsymbol{T}_k^d) * \eta_\epsilon : \boldsymbol{T}^d * \eta_\epsilon\,dx\,dt.
\end{aligned}\tag{2.73}$$

For every $\varepsilon > 0$ the sequence $\{(\boldsymbol{\varepsilon}(\boldsymbol{u}_k) - \boldsymbol{\varepsilon}^{\mathbf{p}}_k)_t * \eta_\epsilon\}$ belongs to $L^2(Q, \mathcal{S}^3)$ and is uniformly bounded in $L^2(Q, \mathcal{S}^3)$ with respect to k, hence we pass to the limit as $k \to \infty$ and we obtain

$$\int_{t_1}^{t_2} \int_\Omega \boldsymbol{T} * \eta_\epsilon : (\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}^{\mathbf{p}})_t * \eta_\epsilon \, dx \, dt = - \int_{t_1}^{t_2} \int_\Omega \boldsymbol{\chi} * \eta_\epsilon : \boldsymbol{T}^d * \eta_\epsilon \, dx \, dt.$$

Using the properties of convolution, we get

$$\int_\Omega \boldsymbol{T} * \eta_\epsilon : (\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}^{\mathbf{p}}) * \eta_\epsilon \, dx \Big|_{t_1}^{t_2} = - \int_{t_1}^{t_2} \int_\Omega \boldsymbol{\chi} * \eta_\epsilon : \boldsymbol{T}^d * \eta_\epsilon \, dx \, dt.$$

In the same way as in the previous section we pass to the limit as $\epsilon \to 0$ and next as $t_1 \to 0$

$$\int_\Omega \boldsymbol{D}(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}^{\mathbf{p}}) : (\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}^{\mathbf{p}}) \, dx \Big|_0^{t_2} = - \int_0^{t_2} \int_\Omega \boldsymbol{\chi} : \boldsymbol{T}^d \, dx \, dt. \tag{2.74}$$

We multiply (2.74) by $1/\mu$ and integrate over $(\tau, \tau + \mu)$ and proceed further in the same manner as in the proof of Lemma 2.10. □

The second and the third steps are conducted in the same way as in the previous limit passage, hence we omit this calculation. Using the Minty–Browder trick, we show that

$$\boldsymbol{\chi} = \boldsymbol{G}(\widetilde{\theta} + \theta, \widetilde{\boldsymbol{T}}^d + \boldsymbol{T}^d)$$

almost everywhere in Q. Moreover, using the Young measures tools we may pass to the limit in the right-hand side term of heat equation. Repeating the procedure from the previous limit passage, we obtain

$$(\boldsymbol{T}^d_k + \widetilde{\boldsymbol{T}}^d) : \boldsymbol{G}(\theta_k + \widetilde{\theta}, \boldsymbol{T}^d_k + \widetilde{\boldsymbol{T}}^d) \rightharpoonup (\boldsymbol{T}^d + \widetilde{\boldsymbol{T}}^d) : \boldsymbol{G}(\theta + \widetilde{\theta}, \boldsymbol{T}^d + \widetilde{\boldsymbol{T}}^d)$$

in $L^1(Q)$. Since the sequence regarding truncations of the integrable function $\{\mathcal{T}_k(\,\cdot\,)\}$ converges strongly to this function in $L^1(Q)$ as $k \to \infty$, we observe

$$\mathcal{T}_k\big((\boldsymbol{T}^d_k + \widetilde{\boldsymbol{T}}^d) : \boldsymbol{G}(\theta_k + \widetilde{\theta}, \boldsymbol{T}^d_k + \widetilde{\boldsymbol{T}}^d)\big) \rightharpoonup (\boldsymbol{T}^d + \widetilde{\boldsymbol{T}}^d) : \boldsymbol{G}(\theta + \widetilde{\theta}, \boldsymbol{T}^d + \widetilde{\boldsymbol{T}}^d)$$

weakly in $L^1(Q)$. This information, see Appendix A, provides that we also have the convergence

$$\mathcal{T}_K(\theta_k) \to \mathcal{T}_K(\theta) \quad \text{in } L^2(0, T, W^{1,2}(\Omega)), \tag{2.75}$$

for every $K > 0$. Using the solution to problem (2.1), we obtain

$$\int_0^T \int_\Omega (\widetilde{\boldsymbol{T}} + \boldsymbol{T}) : \nabla \boldsymbol{\varphi} \, dx \, dt = \int_0^T \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi} \, dx \, dt, \tag{2.76}$$

where $\boldsymbol{T} = \boldsymbol{D}(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}^{\mathbf{p}})$, and (2.76) holds for every test function $\boldsymbol{\varphi} \in C^\infty([0, T], C^\infty_c(\Omega, \mathbb{R}^3))$. To get the renormalised solution to the heat equation let us take $S'(\theta)\phi$ as a test function in (2.65), where S is a $C^\infty(\mathbb{R})$ function such that S' has a compact support. Then, by Appendix A, the limit passage in the heat equation is clear and

$$
\begin{aligned}
-\int_Q S(\theta)\frac{\partial\phi}{\partial t}\,dx\,dt - \int_\Omega S(\theta_0)\phi(x,0)\,dx + \int_Q S'(\theta)\nabla\theta\cdot\nabla\phi\,dx\,dt \\
+ \int_Q S''(\theta)|\nabla(\theta)|^2\phi\,dx\,dt = \int_Q \boldsymbol{G}(\theta,\boldsymbol{T}^d):\boldsymbol{T}^d S'(\theta)\phi\,dx\,dt
\end{aligned}
$$

holds for every test function $\phi \in C_c^\infty([-\infty,T), C^\infty(\Omega))$ and for every function $S \in C^\infty(\mathbb{R})$ such that $S' \in C_0^\infty(\mathbb{R})$, which completes the proof of Theorem 1.10.

Appendix A. Renormalised solutions to the heat equation

To deal with heat equations we introduce the renormalised solutions. The renormalised solution to the parabolic equation was presented in [7], [8], but only for Dirichlet boundary conditions. Some proofs from [7], [8] need a modification for the case of Neumann boundary conditions.

Let us consider the system of equations

$$
\begin{cases}
\dfrac{\partial\theta^\varepsilon}{\partial t} - \Delta\theta^\varepsilon = f^\varepsilon & \text{in } Q,\\
\dfrac{\partial\theta^\varepsilon}{\partial\boldsymbol{n}} = 0 & \text{on } \partial\Omega\times(0,T),\\
\theta^\varepsilon(t=0) = \theta_0^\varepsilon, & \text{in } \Omega,
\end{cases}
\tag{A.1}
$$

where for every positive ε the function f^ε belongs to $L^2(Q)$, the sequence $\{f^\varepsilon\}$ is uniformly bounded in $L^1(Q)$, θ_0^ε belongs to $L^2(\Omega)$ and converges strongly to θ_0 in $L^1(\Omega)$ as ε tends to 0.

In our case $1/\varepsilon = k$ and

$$
\begin{cases}
f^\varepsilon = \mathcal{T}_k\big((\widetilde{\boldsymbol{T}}^d + \boldsymbol{T}_k^d) : \boldsymbol{G}(\widetilde{\theta} + \theta_k, \widetilde{\boldsymbol{T}}^d + \boldsymbol{T}_k^d)\big) & \text{in } Q,\\
\theta^\varepsilon(x,0) = \mathcal{T}_k(\theta_0) & \text{in } \Omega,
\end{cases}
\tag{A.2}
$$

and moreover, we know that the sequence $\{(\widetilde{\boldsymbol{T}}^d + \boldsymbol{T}_k^d) : \boldsymbol{G}(\widetilde{\theta} + \theta_k, \widetilde{\boldsymbol{T}}^d + \boldsymbol{T}_k^d)\}$ is uniformly bounded in $L^1(Q)$. Hence, there exists a weak limit of this sequence. Identification of this weak limit is discussed in Section 2.5.

Definition A.1 (Renormalised solution to the heat equation [8, Definition 2.2]). Let f belong to $L^1(Q)$ and θ_0 belong to $L^1(\Omega)$. A real-valued function θ defined on Q is a renormalised solution to the heat equation if

(a) θ belongs to $C([0,T], L^1(\Omega))$ and $\mathcal{T}_K(\theta)$ belongs to $L^2(0,T,W^{1,2}(\Omega))$ for all positive K;

(b) for all positive c, $\mathcal{T}_{K+c}(\theta) - \mathcal{T}_k(\theta) \to 0$ in $L^2(0,T,W^{1,2}(\Omega))$ as K goes to ∞; and

(c) $\theta(t=0) = \theta_0$.

Moreover, for all functions $S \in C^\infty(\mathbb{R})$, such that S' belongs to $C_0^\infty(\mathbb{R})$ (S' has a compact support), the equality

$$-\int_Q S(\theta)\frac{\partial\phi}{\partial t}\,dx\,dt - \int_\Omega S(\theta_0)\phi(x,0)\,dx + \int_Q S'(\theta)\nabla\theta\cdot\nabla\phi\,dx\,dt$$
$$+ \int_Q S''(\theta)|\nabla\theta|^2\phi\,dx\,dt = \int_Q fS'(\theta)\phi\,dx\,dt$$

holds for all $\phi \in C_0^\infty(Q)$.

We use the notation $\lim\limits_{\eta,\varepsilon\to 0}$ when the order in the passing to the limit is not relevant, i.e.

$$\lim_{\eta,\varepsilon\to 0} F_{\eta,\varepsilon} = \lim_{\eta\to 0}\lim_{\varepsilon\to 0} F_{\eta,\varepsilon} = \lim_{\varepsilon\to 0}\lim_{\eta\to 0} F_{\eta,\varepsilon}.$$

Lemma A.2. *Let us assume that the sequence $\{f_k\}$ is uniformly bounded in $L^1(Q)$. Then there exist a subsequence of the sequence $\{\theta^\varepsilon\}_\varepsilon$ (still denoted by ε) and a measurable function θ such that as ε tends to 0 and for any fixed positive real number K the following conditions are satisfied*:

(a) *θ^ε converges almost everywhere in Q to a measurable function θ*;
(b) *$\mathcal{T}_K(\theta^\varepsilon)$ converges weakly to $\mathcal{T}_K(\theta)$ in $L^2(0,T,W^{1,2}(\Omega))$.*

Proof. The pointwise convergence of temperature is obtained by use of the same argumentations as in the Boccardo and Gallouët approach, see [9] for Dirichlet boundary condition and in [20] for Neumann boundary condition.

Let us take $\mathcal{T}_K(\theta^\varepsilon)$ as a test function in (A.1). Then for $t\in(0,T)$

$$\int_0^t\int_\Omega \frac{\partial\theta^\varepsilon}{\partial t}\mathcal{T}_K(\theta^\varepsilon)\,dx\,dt + \int_0^t\int_\Omega |\nabla\mathcal{T}_K(\theta^\varepsilon)|^2\,dx\,dt = \int_0^t\int_\Omega f^\varepsilon\mathcal{T}_k(\theta^\varepsilon)\,dx\,dt,$$

and

$$\int_\Omega \widetilde{\mathcal{T}}_K(\theta^\varepsilon)(t)\,dx + \int_0^t\int_\Omega |\nabla\mathcal{T}_K(\theta^\varepsilon)|^2\,dx\,dt = \int_0^t\int_\Omega f^\varepsilon\mathcal{T}_k(\theta^\varepsilon)\,dx\,dt + \int_\Omega \widetilde{\mathcal{T}}_K(\theta_0^\varepsilon)\,dx,$$

where $\widetilde{\mathcal{T}}_K(r) = \int_0^r \mathcal{T}_K(z)\,dz$ is a positive real valued function. Using the definition of the truncation and linear growth of the function $\widetilde{\mathcal{T}}_K(r)$ at infinity, the following estimate holds:

$$\int_\Omega \widetilde{\mathcal{T}}_K(\theta^\varepsilon)(t)\,dx + \int_0^t\int_\Omega |\nabla\mathcal{T}_K(\theta^\varepsilon)|^2\,dx\,dt \le K\|f\|_{L^1(Q)} + C(K)\|\theta_0^\varepsilon\|_{L^1(\Omega)}.$$

It is enough to estimate $\|\mathcal{T}_K(\theta^\varepsilon)\|_{L^2(Q)}$ by $\|\widetilde{\mathcal{T}}_K(\theta^\varepsilon)\|_{L^1(Q)}$ and $\|\nabla\mathcal{T}_K(\theta^\varepsilon)\|_{L^2(Q)}$ to show that the sequence $\{\mathcal{T}_K(\theta^\varepsilon)\}_{\varepsilon>0}$ is uniformly bounded in $L^2(0,T,W^{1,2}(\Omega))$. By the Poincaré inequality, we get

$$\begin{aligned}\|\mathcal{T}_K(\theta^\varepsilon)\|_{L^2(Q)} &\le \|\mathcal{T}_K(\theta^\varepsilon) - (\mathcal{T}_K(\theta^\varepsilon))_\Omega\|_{L^2(Q)} + \|(\mathcal{T}_K(\theta^\varepsilon))_\Omega\|_{L^2(Q)}\\ &\le \|\nabla\mathcal{T}_K(\theta^\varepsilon)\|_{L^2(Q)} + \|(\mathcal{T}_K(\theta^\varepsilon))_\Omega\|_{L^2(Q)},\end{aligned}$$

where by $(\mathcal{T}_K(\theta^\varepsilon))_\Omega$ we denote the mean value. Using the definition of the truncation operator, we obtain

$$\widetilde{\mathcal{T}}_K(\theta^\varepsilon) = \begin{cases} \dfrac{1}{2}(\theta^\varepsilon)^2 & \text{if } |\theta^\varepsilon| \le K, \\ \dfrac{1}{2}K^2 + K(|\theta^\varepsilon| - K) & \text{if } |\theta^\varepsilon| > K, \end{cases} \tag{A.3}$$

and then it remains to show the estimates for $(\mathcal{T}_K(\theta^\varepsilon))_\Omega$

$$\int_\Omega |\mathcal{T}_K(\theta^\varepsilon)|^2\,dx = \int_{\{x\in\Omega:|\theta^\varepsilon|\le K\}} |\theta^\varepsilon|^2\,dx + \int_{\{x\in\Omega:|\theta^\varepsilon|>K\}} K^2\,dx \le 2\int_\Omega \widetilde{\mathcal{T}}_K(\theta^\varepsilon)\,dx.$$

The finite measure of Q implies that the sequence $\{\mathcal{T}_K(\theta^\varepsilon)\}_{\varepsilon>0}$ is uniformly bounded in $L^2(0,T,W^{1,2}(\Omega))$. □

Lemma A.3. *Let us assume that the sequence f_k converges weakly to f in $L^1(Q)$. Then the sequence $\{\theta^\varepsilon\}_\varepsilon$ converges to θ in $C([0,T],L^1(\Omega))$. Moreover, for any fixed positive real number K there exists the following limit:*

$$\lim_{\eta,\varepsilon\to\infty} \int_Q |\nabla \mathcal{T}_k(\theta^\varepsilon - \theta^\eta)|\,dx\,dt = 0.$$

Proof. Let us test the difference of two approximate equations (A.1):

$$\frac{\partial}{\partial t}(\theta^\varepsilon - \theta^\eta) - \Delta(\theta^\varepsilon - \theta^\eta) = f^\varepsilon - f^\eta, \tag{A.4}$$

by $\mathcal{T}_K(\theta^\varepsilon - \theta^\eta)$. Then, integrating over Ω and the time interval $(0,t)$, where $t \le T$, we obtain

$$\begin{aligned} \int_\Omega \widetilde{\mathcal{T}}_K(\theta^\varepsilon - \theta^\eta)(t)\,dx &+ \int_\Omega\int_0^t |\nabla\mathcal{T}_K(\theta^\varepsilon - \theta^\eta)|^2\,dx\,d\tau \\ &= \int_\Omega\int_0^t (f^\varepsilon - f^\eta)\mathcal{T}_K(\theta^\varepsilon - \theta^\eta)\,dx\,d\tau + \int_\Omega \widetilde{\mathcal{T}}_K(\theta_0^\varepsilon - \theta_0^\eta)\,dx. \end{aligned} \tag{A.5}$$

The second term in the right-hand side converges to zero as $\varepsilon,\eta \to 0$. To prove that the first term also converges to zero let us observe that, dividing Q into sufficiently small set B and the rest $Q \setminus B$, we may write

$$\begin{aligned} &\int_Q (f^\varepsilon - f^\eta)\mathcal{T}_K(\theta^\varepsilon - \theta^\eta)\,dx\,dt \\ &= \int_{Q\setminus B} (f^\varepsilon - f^\eta)\mathcal{T}_K(\theta^\varepsilon - \theta^\eta)\,dx\,dt + \int_B (f^\varepsilon - f^\eta)\mathcal{T}_K(\theta^\varepsilon - \theta^\eta)\,dx\,dt \\ &\le \int_{Q\setminus B} (f^\varepsilon - f^\eta)\mathcal{T}_K(\theta^\varepsilon - \theta^\eta)\,dx\,dt + 2K\int_B |f^\varepsilon|\,dx\,dt. \end{aligned}$$

By the Dunford–Pettis theorem [35, Theorem T23], the sequence $\{f^\varepsilon\}$ is also uniformly integrable. Thus, using the Egorov theorem, we obtain that for every positive ε there exists $\delta > 0$ such that for every $B \subset Q$ with $\mathrm{meas}(B) < \delta$, $\mathcal{T}_K(\theta^\varepsilon - \theta^\eta)$ converges uniformly to 0 on $Q \setminus B$ and $\int_B |f^\varepsilon|\,dx\,dt < \varepsilon$. Passing to the limit as $\eta, \varepsilon \to 0$, we obtain that the right-hand side of (A.5) goes to 0 and we obtain

$$
\begin{aligned}
&\lim_{\varepsilon,\eta\to 0}\sup_{t\in[0,T]}\int_\Omega \widetilde{\mathcal{T}}_K(\theta^\varepsilon-\theta^\eta)(t)\,dx = 0,\\
&\lim_{\varepsilon,\eta\to 0}\int_Q |\nabla \mathcal{T}_K(\theta^\varepsilon-\theta^\eta)|^2\,dx\,dt = 0.
\end{aligned}\tag{A.6}
$$

To complete the proof let us observe that, using (A.3) and $(\text{A.6})_{(1)}$, for every $\varepsilon > 0$ there exists γ such that for every $0 \le \eta, \varepsilon \le \gamma$ it holds

$$
\begin{aligned}
0 \le \sup_{t\in[0,T]}\biggl(\frac12\int_{\{x\in\{|\theta^\varepsilon-\theta^\eta|<K\}}&|\theta^\varepsilon-\theta^\eta|^2(t)\,dx\\
&+K\int_{\{x\in\{|\theta^\varepsilon-\theta^\eta|\ge K\}}\Bigl(|\theta^\varepsilon-\theta^\eta|-\frac12 K\Bigr)\,dx\biggr)\le\varepsilon.
\end{aligned}
$$

Thus, we may observe that

$$
0\le \int_{\{x\in\{|\theta^\varepsilon-\theta^\eta|\ge K\}}|\theta^\varepsilon-\theta^\eta|\,dx-\frac12\int_{\{x\in\{|\theta^\varepsilon-\theta^\eta|\ge K\}}K\,dx\le\frac{\varepsilon}{K}
$$

and then

$$
0\le\frac12\int_{\{x\in\{|\theta^\varepsilon-\theta^\eta|\ge K\}}|\theta^\varepsilon-\theta^\eta|\,dx\le\frac{\varepsilon}{K}.
$$

Passing to the limit as $\varepsilon, \eta \to 0$, we get

$$
\lim_{\varepsilon,\eta\to 0}\sup_{t\in[0,T]}\int_\Omega|\theta^\varepsilon-\theta^\eta|(t)\,dx=0
$$

which completes the proof. □

LEMMA A.4. *Let K be a fixed positive real number. The sequence $\{\mathcal{T}_K(\theta^\varepsilon)\}$ converges strongly to $\mathcal{T}_K(\theta)$ in $L^2(0,T,W^{1,2}(\Omega))$.*

The proof of this lemma can be found in [7], [31].

PROOF. Multiplying (A.1) by $S'(\theta^\varepsilon)\phi$, where $S \in C^\infty(\mathbb{R})$ and S' has a compact support and $\phi \in C_0^\infty(Q)$, we get

$$
\begin{aligned}
-\int_Q S(\theta^\varepsilon)\frac{\partial\phi}{\partial t}\,dx\,dt-\int_\Omega S(\theta_0^\varepsilon)\phi(x,0)\,dx+\int_Q S'(\theta^\varepsilon)\nabla\theta^\varepsilon\cdot\nabla\phi\,dx\,dt&\\
+\int_Q S''(\theta^\varepsilon)|\nabla\theta^\varepsilon|^2\phi\,dx\,dt=\int_Q f^\varepsilon S'(\theta^\varepsilon)\phi\,dx\,dt.&
\end{aligned}\tag{A.7}
$$

S' has a compact support, hence there exists $0 < M < \infty$ such that $\operatorname{supp}(S') \subset [-M, M]$. This allows us to enter into equation (A.7) the truncations operator

$$\tag{A.8} \begin{aligned} -\int_Q S(\theta^\varepsilon)\frac{\partial\phi}{\partial t}\,dx\,dt - \int_\Omega S(\theta_0^\varepsilon)\phi(x,0)\,dx \\ + \int_Q S'(\mathcal{T}_M(\theta^\varepsilon))\nabla\mathcal{T}_M(\theta^\varepsilon)\cdot\nabla\phi\,dx\,dt + \int_Q S''(\mathcal{T}_M(\theta^\varepsilon))|\nabla\mathcal{T}_M(\theta^\varepsilon)|^2\phi\,dx\,dt \\ = \int_Q f^\varepsilon S'(\mathcal{T}_M(\theta^\varepsilon))\phi\,dx\,dt. \end{aligned}$$

Using the Egorov theorem applied to $S'(\theta^\varepsilon)$ or to $S''(\theta^\varepsilon)$ and using the bounded character of the remaining terms, we can pass to the limit as ε goes to 0 in (A.8) and obtain

$$\tag{A.9} \begin{aligned} -\int_Q S(\theta)\frac{\partial\phi}{\partial t}\,dx\,dt - \int_\Omega S(\theta_0)\phi(x,0)\,dx \\ + \int_Q S'(\mathcal{T}_M(\theta))\nabla\mathcal{T}_M(\theta)\cdot\nabla\phi\,dx\,dt + \int_Q S''(\mathcal{T}_M(\theta))|\nabla\mathcal{T}_M(\theta)|^2\phi\,dx\,dt \\ = \int_Q fS'(\mathcal{T}_M(\theta))\phi\,dx\,dt. \end{aligned}$$

And finally, using the compact support of S', we can omit the truncations in (A.9)

$$\tag{A.10} \begin{aligned} -\int_Q S(\theta)\frac{\partial\phi}{\partial t}\,dx\,dt - \int_\Omega S(\theta_0)\phi(x,0)\,dx + \int_Q S'(\theta)\nabla\theta\cdot\nabla\phi\,dx\,dt \\ + \int_Q S''(\theta)|\nabla\theta|^2\phi\,dx\,dt = \int_Q fS'(\theta)\phi\,dx\,dt, \end{aligned}$$

which completes the proof of existence regarding the renormalised solution to the parabolic equation with Neumann boundary condition. □

Lemma A.5. *Assume that $\theta_{0,1}$ and $\theta_{0,2}$ lie in $L^1(\Omega)$, f_1 and f_2 lie in $L^1(Q)$ and satisfy*

$$\begin{cases} \theta_{0,1} \le \theta_{0,2}, \\ f_1 \le f_2. \end{cases}$$

Then, if θ_1, θ_2 are two renormalised solutions respectively for the data $(\theta_{0,1}, f_1)$ and $(\theta_{0,2}, f_2)$, we have $\theta_1 \le \theta_2$ almost everywhere in Q.

The proof of this lemma can be found in [8], [31].

Remark A.6. As a consequence of Lemma A.5, the renormalised solution is unique.

Appendix B. Orlicz spaces tools

Assumption 1.7 requires the use of basic tools regarding generalized Orlicz spaces. Here we present some basic lemmas, which have been used to prove the existence of thermo-visco-elastic model solution. The following lemmas with proofs can be found in [26], [28], [24], [16], [46].

Lemma B.1 (Fenchel–Young inequality). *Let M be an N-function and M^* be complementary to M. Then the inequality*

$$|\boldsymbol{\xi} : \boldsymbol{\eta}| \le M(x, \boldsymbol{\xi}) + M^*(x, \boldsymbol{\eta})$$

is satisfied for all $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathcal{S}^3$ and for almost all $x \in \Omega$.

Lemma B.2 (Hölder inequality). *Let M be an N-function and M^* be complementary to M. Then the following inequality is satisfied:*

$$\left| \int_Q \boldsymbol{\xi} : \boldsymbol{\eta} \, dx \, dt \right| \le 2 \|\boldsymbol{\xi}\|_{L,M} \|\boldsymbol{\eta}\|_{L,M^*}. \tag{B.1}$$

Lemma B.3. *Let $\boldsymbol{\xi}_i \colon Q \to \mathbb{R}^d$ be a measurable sequence. Then $\boldsymbol{\xi}_i \xrightarrow{M} \boldsymbol{\xi}$ in $L_M(Q)$ modularity if and only if $\boldsymbol{\xi}_i \to \boldsymbol{\xi}$ in measure and there exists some $\lambda > 0$ such that the sequence $\{M(\,\cdot\,, \lambda \boldsymbol{\xi}_i)\}$ is uniformly integrable, i.e.*

$$\lim_{R \to \infty} \left(\sup_{i \in \mathbb{N}} \int_{\{(t,x) : |M(x, \lambda \boldsymbol{\xi}_i)| \ge R\}} M(x, \lambda \boldsymbol{\xi}_i) \, dx \, dt \right) = 0.$$

Lemma B.4. *Let M be an N-function and for all $i \in \mathbb{N}$, let*

$$\int_Q M(x, \boldsymbol{\xi}_i) \, dx \, dt \le c.$$

Then the sequence $\{\boldsymbol{\xi}_i\}$ is uniformly integrable.

Lemma B.5. *Let M be an N-function and M^* its complementary function. Suppose that the sequences $\boldsymbol{\Phi}_i \colon Q \to \mathcal{S}^3$ and $\boldsymbol{\Psi}_i \colon Q \to \mathcal{S}^3$ are uniformly bounded in $L_M(Q)$ and $L_{M^*}(Q)$, respectively. Moreover, $\boldsymbol{\Phi}_i \xrightarrow{M} \boldsymbol{\Phi}$ modularly in $L_M(Q)$ and $\boldsymbol{\Phi}_i \xrightarrow{M^*} \boldsymbol{\Phi}$ modularly in $L_{M^*}(Q)$. Then, $\boldsymbol{\Phi}_i : \boldsymbol{\Psi}_i \to \boldsymbol{\Phi} : \boldsymbol{\Psi}$ strongly in L^1.*

Lemma B.6. *Let ρ_i be a standard mollifier, i.e. $\rho \in C^\infty(\mathbb{R})$, ρ has a compact support and $\int_{\mathbb{R}} \rho(\tau) \, d\tau = 1$, $\rho(\tau) = \rho(-\tau)$. We define $\rho_i(\tau) = i\rho(i\tau)$. Moreover, let $*$ denote a convolution in the variable τ. Then for any function $\boldsymbol{\Phi} \colon Q \to \mathcal{S}^3$, such that $\boldsymbol{\Phi} \in L^1(Q, \mathcal{S}^3)$, it holds*

$$\rho_i * \boldsymbol{\Phi} \to \boldsymbol{\Phi} \quad \text{in measure.}$$

Lemma B.7. *Let ρ_i be a standard mollifier. Given an N-function M and a function $\boldsymbol{\Phi} \colon Q \to \mathcal{S}^3$ such that $\boldsymbol{\Phi} \in \mathcal{L}_M(Q)$, the sequence $\{M(x, \rho_i * \boldsymbol{\Phi})\}$ is uniformly integrable.*

Appendix C. Young measures tools

The right-hand side term in the approximated heat equation is a product of elements of two sequences which converge weakly. To characterize the limit of this term we use the Young measure theory. In this section, we present necessary lemmas. They come from [36, Corollaries 3.2–3.4]. A similar technique was also used in [23], [40].

Lemma C.1. *Suppose that the sequence of maps $z_j\colon Q \to \mathbb{R}^d$ generates the Young measure $\nu\colon Q \to \mathcal{M}(\mathbb{R}^d)$. Let $F\colon \Omega\times\mathbb{R}^d \to \mathbb{R}^d$ be a Carathéodory function (i.e. measurable in the first argument and continuous in the second). Let us also assume that the negative part $F^-(x, z_j(x, t))$ is weakly relatively compact in $L^1(Q)$. Then*

$$\liminf_{j\to\infty} \int_E F(x, z_j(x, t))\, dx\, dt \geq \int_E \int_{\mathbb{R}^d} F(x, \lambda)\, d\nu_x(\lambda)\, dx\, dt.$$

If, in addition, the sequence of functions $x \mapsto |F|(x, z_j(x, t))$ is weakly relatively compact in $L^1(Q)$, then

$$F(\,\cdot\,, z_j(\,\cdot\,,\,\cdot\,)) \rightharpoonup \int_{\mathbb{R}^d} F(\,\cdot\,, \lambda)\, d\nu_x(\lambda) \quad \text{in } L^1(Q).$$

Lemma C.2. *Let $u_j\colon Q \to \mathbb{R}^d$, $v_j\colon Q \to \mathbb{R}^{d'}$ be measurable and suppose that $u_j \to u$ almost everywhere, while v_j generates the Young measure ν. Then the sequence of pairs $(u_j, v_j)\colon Q \to \mathbb{R}^{d+d'}$ generates the Young measure $x \mapsto \delta_{u(x)} \otimes \nu_x$.*

Lemma C.3. *Suppose that a sequence z_j of measurable functions from Q to $\mathbb{R}^d$ generates the Young measure $\nu\colon Q \to \mathcal{M}(\mathbb{R}^d)$. Then $z_j \to z$ in measure if and only if $\nu_x = \delta_{z(x)}$ almost everywhere.*

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