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THREE SOLUTIONS FOR SECOND-ORDER IMPULSIVE DIFFERENTIAL INCLUSIONS WITH STURM–LIOUVILLE BOUNDARY CONDITIONS VIA NONSMOOTH CRITICAL POINT THEORY

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Abstract. A second-order impulsive differential inclusion with Sturm–Liouville boundary conditions is studied. By using a nonsmooth version of a three critical point theorem of Ricceri, the existence of three solutions is obtained.

1. Introduction

In this paper, we will study a second-order impulsive differential inclusion subject to Sturm–Liouville boundary conditions

$$
\begin{cases}
-(\rho(x)\Phi_p(u'(x)))' + s(x)\Phi_p(u(x)) \in \lambda F(u(x)) + \mu G(x,u(x)) & \\
& \text{in } [a,b]\setminus\{x_1,\ldots,x_l\}, \\
-\Delta(\rho(x_k)\Phi_p(u'(x_k))) = I_k(u(x_k)), & k=1,\ldots,l, \\
\alpha u'(a) - \beta u(a) = 0, \quad \gamma u'(b) + \sigma u(b) = 0, &
\end{cases}
\tag{1.1}
$$

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where $p > 1$, $\Phi_p(u) := |u|^{p-2}u$, $\rho, s \in C[a,b]$, $\rho(x) > 0$, $s(x) > 0$, $\alpha, \beta, \gamma, \sigma > 0$, and $a = x_0 < x_1 < \ldots < x_l < x_{l+1} = b$.

Here,

$$\Delta(\rho(x_i)\Phi_p(u'(x_i))) = \rho(x_i^+)\Phi_p(u'(x_i^+)) - \rho(x_i^-)\Phi_p(u'(x_i^-)),$$

where $u'(x_i^+)$ (respectively, $u'(x_i^-)$) denotes the right hand limit (respectively, left hand limit) of $u'(x)$ at $x = x_i$, $I_i \in C(R, R)$, $i = 1, \ldots, l$, λ, μ are positive parameters, F is a multifunction defined on R satisfying:

(F_1) $F\colon R \to 2^R$ is upper semicontinuous (u.s.c.) with compact convex values;
(F_2) $\min F, \max F\colon R \to R$ are $\mathcal{L} \times \mathcal{B}$-measurable;
(F_3) $|\xi| \le \delta(1 + |s|^{p-1})$ for all $s \in R$ and $\xi \in F(s)$, for some $\delta > 0$;

and G is a multifunction defined on $[a,b] \times R$ satisfying:

(G_1) $G(x,\cdot)\colon R \to 2^R$ is u.s.c. with compact convex values for almost every $x \in [a,b] \setminus \{x_1, \ldots, x_l\}$;
(G_2) $\min G, \max G\colon [a,b] \setminus \{x_1, \ldots, x_l\} \times R \to R$ are $\mathcal{L} \times \mathcal{B}$-measurable;
(G_3) $|\xi| \le \delta(1 + |s|^{p-1})$ for almost every $x \in [a,b]$, $s \in R$ and $\xi \in G(x,s)$.

We shall apply a nonsmooth version of the critical point theory of Ricceri to prove that, if λ is large enough and μ is small enough, then (1.1) admits at least three solutions. Moreover, we obtain estimates of the solutions' norms that are independent of G, λ, and μ.

The study of impulsive differential equations and inclusions is linked to their utility in simulating processes and phenomena subject to short-time perturbations during their evolution. The perturbations are considered to take place in the form of impulses since the perturbations are performed discretely and their durations are negligible in comparison with the total duration of the processes and phenomena (see [9], [14]). In recent years, there has been an increasing interest in the study of differential inclusions and impulsive differential inclusions due to the fact that they often arise in models for control systems, mechanical systems, economical systems, game theory, and biological systems to name a few (see [1]–[5], [7], [11]–[13], [15], [19]). The first work dealing with partial differential inclusions with a general set-valued right hand side via variational methods was, to the best of our knowledge, that of Frigon [10]. Ribarska et al. [17] defined a single-valued energy functional that was locally Lipschitz and proved that its critical points were just the solutions of the original problem. With this, nonsmooth variational methods can be applied to differential inclusions.

In papers [20] and [21], the impulsive differential equations with Sturm–Liouville boundary conditions are studied by variational methods.

In this paper, we apply this approach to impulsive differential inclusion with Sturm–Liouville boundary conditions. We need to overcome certain difficulties such as: how to define the weak solutions; how to prove that a weak solution is a classical solution of the original problem; and how to verify the regularity assumptions in view of the presence of impulsive terms. This paper is a generalization of the results on impulsive differential equations with Sturm–Liouville boundary conditions found in [20] and [21]. We note that the nonsmooth version of the critical point theory of Ricceri has been studied and applied extensively, e.g. [11] and [12]. This paper is the further application of nonsmooth version of the critical point theory.

Our paper is organized as follows. In Section 2, we recall some basic concepts from nonsmooth analysis and the abstract result we are going to apply. We also present our main result. In Section 3, we introduce a variational method for problem (1.1), and in Section 4, we prove our main result and illustrate it with an example.

2. Some nonsmooth analysis and the main result

We begin by collecting some basic notions and results on nonsmooth analysis, namely, the calculus for locally Lipschitz functionals developed by Clarke [6], and Motreanu and Panagiotopoulos [16].

Let $(X, \|\cdot\|_X)$ be a Banach space, $(X^*, \|\cdot\|_{X^*})$ be its topological dual, and $\varphi\colon X \to R$ be a functional. We recall that φ is *locally Lipschitz* (*l.L.*) if, for all $u \in X$, there exist a neighborhood U of u and a real number $L > 0$ such that

$$|\varphi(v) - \varphi(w)| \leq L\|v - w\|_X \quad \text{for all } v, w \in U.$$

If φ is l.L. and $u \in X$, the *generalized directional derivative* of φ at u along the direction $v \in X$ is

$$\varphi^0(u; v) = \limsup_{w \to u,\, \tau \to 0^+} \frac{\varphi(w + \tau v) - \varphi(w)}{\tau}.$$

The *generalized gradient* of φ at u is the set

$$\partial\varphi(u) = \{u^* \in X^* : \langle u^*, v\rangle \leq \varphi^0(u; v) \text{ for all } v \in X\}.$$

Thus, $\partial\varphi\colon X \to 2^{X^*}$ is a multifunction. We say that φ has a *compact gradient* if $\partial\varphi$ maps bounded subsets of X into relatively compact subsets of X^*.

Lemma 2.1 ([16, Proposition 1.1]). *Let $\varphi \in C^1(X)$ be a functional. Then φ is l.L. and*

$$\varphi^0(u; v) = \langle \varphi'(u), v\rangle \quad \text{for all } u, v \in X; \tag{2.1}$$

$$\partial\varphi(u) = \{\varphi'(u)\} \quad \text{for all } u \in X. \tag{2.2}$$

Lemma 2.2 ([16, Proposition 1.3]). *Let $\varphi\colon X \to R$ be an l.L. functional. Then*

(2.3) $\varphi^0(u; \cdot\,)$ *is subadditive and positively homogeneous for all $u \in X$;*

$$\varphi^0(u;v) \le L\|v\| \quad \textit{for all } u, v \in X, \tag{2.4}$$
with $L > 0$ being a Lipschitz constant for φ around u.

Lemma 2.3 ([16, Proposition 1.6]). *Let $\varphi, \psi\colon X \to R$ be an l.L. functional. Then*

$$\partial(\lambda\varphi)(u) = \lambda\partial\varphi(u) \qquad \textit{for all } u \in X,\ \lambda \in R; \tag{2.5}$$

$$\partial(\varphi+\psi)(u) \subseteq \partial\varphi(u) + \partial\psi(u) \quad \textit{for all } u \in X. \tag{2.6}$$

Lemma 2.4 ([11, Lemma 6]). *Let $\varphi\colon X \to R$ be a functional with a compact gradient. Then φ is sequentially weakly continuous.*

We say that $u \in X$ is a *critical point* of an l.L. functional φ if $0 \in \partial\varphi(u)$.

In the proof of our main results, we shall use Theorem 2.6. For this, we first present an important definition.

Definition 2.5. An operator $A\colon X \to X^*$ is of type $(\mathrm{S})_+$ if, for any sequence (u_n) in X, $u_n \rightharpoonup u$ and $\limsup_{n\to\infty}\langle A(u_n), u_n - u\rangle \le 0$ imply $u_n \to u$.

The following theorem is a special case of [11, Theorem 14].

Theorem 2.6. *Let $(X, \|\cdot\|)$ be a reflexive Banach space, $I \subseteq R$ be an interval, $\mathcal{N} \in C^1(X)$ be a sequentially weakly l.s.c. functional whose derivative is of type $(\mathrm{S})_+$, $\mathcal{F}\colon X \to R$ be an l.L. functional with a compact gradient, and $\rho \in R$. Assume that*

$$\lim_{\|u\|\to\infty}[\mathcal{N}(u) - \lambda\mathcal{F}(u)] = +\infty \quad \textit{for all } \lambda \in I; \tag{2.7}$$

$$\sup_{\lambda\in I}\inf_{u\in X}[\mathcal{N}(u) + \lambda(\rho - \mathcal{F}(u))] < \inf_{u\in X}\sup_{\lambda\in I}[\mathcal{N}(u) + \lambda(\rho - \mathcal{F}(u))]. \tag{2.8}$$

Then, there exist $\alpha, \beta \in I$ ($\alpha < \beta$) and $r > 0$ with the following property: for any $\lambda \in [\alpha, \beta]$ and any l.L. functional $\mathcal{G}\colon X \to R$ with a compact gradient, there exists $\delta > 0$ such that, for all $\mu \in [0, \delta]$, the functional $\varphi_{\lambda,\mu} = \mathcal{N} - \lambda\mathcal{F} - \mu\mathcal{G}$ admits at least three critical points in X, with norms less than r.

The main hypothesis of Theorem 2.6 above is the minimax inequality (2.8). An easy way to satisfy it is illustrated by the following result due again to Ricceri [18].

Lemma 2.7 ([18, Proposition 3.1]). *Let X be a nonempty set, $\mathcal{N}$, $\mathcal{F}\colon X \to R$ be functions, $\check{u}$, $\widehat{u} \in X$, and $\tau > 0$ be such that*

$$\mathcal{N}(\check{u}) = \mathcal{F}(\check{u}) = 0; \tag{2.9}$$

$$\mathcal{N}(\widehat{u}) > \tau; \tag{2.10}$$

$$\sup_{\mathcal{N}(u)<\tau}\mathcal{F}(u) < \frac{\tau\mathcal{F}(\widehat{u})}{\mathcal{N}(\widehat{u})}. \tag{2.11}$$

Then, there exists $\rho \in R$ such that (2.8) holds.

Here is our main result for problem (1.1). We need to define the constant

$$\overline{\gamma} = 2^{1/q} \times \max\left\{1 \Big/ \left((b-a)^{1/p}\Big(\min_{x\in[a,b]} s(x)\Big)^{1/p}\right), \right.$$
$$\left. (b-a)^{1/q} \Big/ \Big(\min_{x\in[a,b]} \rho(x)\Big)^{1/p}\right\},$$

where $1/p + 1/q = 1$.

THEOREM 2.8. *Assume that* (F_1)–(F_3) *hold. Furthermore, we assume*:

(F_4) *there exist* $k > 0$ *and* $\tau > 0$ *satisfying*

$$\frac{\tau \min \displaystyle\int_0^k F(s)\,ds}{\dfrac{k}{p}\displaystyle\int_a^b s(x)\,dx - \sum_{i=1}^{l}\int_0^k I_i(s)\,ds + \dfrac{\rho(b)\sigma^{p-1}}{p\gamma^{p-1}}k^p + \dfrac{\rho(a)\beta^{p-1}}{p\alpha^{p-1}}k^p}$$
$$> \sup_{|u|<\overline{\gamma}(\tau p)^{1/p}} \min \int_0^u F(s)\,ds,$$

where

$$\frac{k}{p}\int_a^b s(x)\,dx + \frac{\rho(b)\sigma^{p-1}}{p\gamma^{p-1}}k^p + \frac{\rho(a)\beta^{p-1}}{p\alpha^{p-1}}k^p > \tau;$$

(F_5) $\min \displaystyle\int_0^t F(s)\,ds \le \eta(1+|t|^l)$ *for* $t \in R$, $\eta > 0$, $1 < l < p$;

(I_1) $I_i(0) = 0$, $I_i(x)x \le 0$ *for* $x \in R$, $i = 1, \dots, l$.

Then there exist a nondegenerate interval $[\alpha, \beta] \subset (0, +\infty)$ *and* $r > 0$ *with the following property*: *for any* $\lambda \in [\alpha, \beta]$ *and any multifunction* G *satisfying* (G_1)–(G_2), *there exists* $\delta > 0$ *such that, for all* $\mu \in [0, \delta]$, *problem* (1.1) *admits at least three solutions with norms in* X *less than* r.

3. Variational structure and related lemmas

For convenience, we consider the differential inclusion

$$(3.1)\qquad \begin{cases} -(\rho(x)\Phi_p(u'(x)))' + s(x)\Phi_p(u(x)) \in H(x, u(x)) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{in } [a,b] \setminus \{x_1, \dots, x_l\}, \\ -\Delta(\rho(x_k)\Phi_p(u'(x_k))) = I_k(u(x_k)), \qquad k = 1, \dots, l, \\ \alpha u'(a) - \beta u(a) = 0, \quad \gamma u'(b) + \sigma u(b) = 0, \end{cases}$$

where the multifunction H satisfies:

(H_1) $H(x, \cdot)\colon R \to 2^R$ is u.s.c. with compact convex values for almost every $x \in [a,b] \setminus \{x_1, \dots, x_l\}$;

(H_2) $\min H, \max H\colon [a,b] \setminus \{x_1, \dots, x_l\} \times R \to R$ are $\mathcal{L} \times \mathcal{B}$-measurable;

(H_3) $|\xi| \le \delta(1+|s|^{p-1})$ for almost every $x \in [a,b] \setminus \{x_1,\dots,x_l\}$, and all $s \in R$ and $\xi \in H(x,s)$ for some $\delta > 0$.

REMARK 3.1. Notice that condition (H_3) implies that $\xi \in L^q[a,b]$ where $1/p + 1/q = 1$.

We introduce the Banach space $X = W^{1,p}([a,b])$ endowed with the norm

$$\|u\|_X = \bigg(\int_a^b \rho(x)|u'(x)|^p + s(x)|u(x)|^p\,dx\bigg)^{1/p}$$

for all $u \in X$. Clearly, $(X, \|\cdot\|_X)$ is a reflexive Banach space and the norm $\|u\|_X$ is equivalent to the usual one $\big(\int_a^b |u'(x)|^p + |u(x)|^p\,dx\big)^{1/p}$. Clearly, X is compactly embedded into $L^\gamma[a,b]$, endowed with the usual norm $\|\cdot\|_{L^\gamma}$ for all $\gamma \ge 1$.

LEMMA 3.2 ([20, Lemma 2.3]). *For $u \in X$, we have $\|u\|_{C^0} \le \overline{\gamma}\|u\|_X$, where the constant $\overline{\lambda}$ is defined before Theorem* 2.8.

DEFINITION 3.3. A function $u \in X$ is said to be a weak solution of problem (3.1) if there exists $u^* \in L^1([a,b],X)$ such that $u^*(x) \in H(x,u(x))$ for almost every $x \in [a,b]$, and

$$\begin{aligned}
(3.2)\quad &\int_a^b \rho(x)\Phi_p(u'(x))v'(x) + s(x)\Phi_p(u(x))v(x) - u^*(x)v(x)\,dx \\
&- \sum_{i=1}^{l} I_i(u(x_i))v(x_i) + \rho(b)\Phi_p\bigg(\frac{\sigma u(b)}{\gamma}\bigg)v(b) + \rho(a)\Phi_p\bigg(\frac{\beta u(a)}{\alpha}\bigg)v(a) = 0
\end{aligned}$$

for all $v \in X$ and for almost every $x \in [a,b]$.

DEFINITION 3.4. By a solution of the differential inclusion (3.1) we mean a function $u\colon [a,b]\setminus\{x_1,\dots,x_l\} \to R$ that is of class C^1 with $\Phi_p(u')$ being absolutely continuous and satisfying

$$\begin{cases}
-(\rho(x)\Phi_p(u'(x)))' + s(x)\Phi_p(u(x)) \in H(x,u(x)) & \text{in } [a,b]\setminus\{x_1,\dots,x_l\},\\
-\Delta(\rho(x_k)\Phi_p(u'(x_k))) = I_k(u(x_k)), & k = 1,\dots,l,\\
\alpha u'(a) - \beta u(a) = 0, \quad \gamma u'(b) + \sigma u(b) = 0.
\end{cases}$$

LEMMA 3.5. *If a function $u \in X$ is a weak solution of* (3.1), *then u is a* (*classical*) *solution of* (3.1).

PROOF. Let $u \in X$ be a weak solution of (3.1). Then there exists $u^* \in H(x,u(x))$ satisfying (3.2) for all $v \in X$ and $u^* \in H(x,u(x))$ for almost every $x \in [a,b]$. So (3.2) holds for all $v \in C_0^\infty(a,x_1)$ with $v(x) = 0$ for $x > x_1$. Then, (3.2) becomes

$$\int_a^{x_1} \rho(x)\Phi_p(u'(x))v'(x) + s(x)\Phi_p(u(x))v(x) - u^*(x)v(x)\,dx = 0.$$

Hence, $\rho(x)\Phi_p(u'(x))$ has a weak derivative $(\rho(x)\Phi_p(u'(x)))'$ and

$$-(\rho(x)\Phi_p(u'(x)))' + s(x)\Phi_p(u(x)) - u^*(x) = 0 \tag{3.3}$$

for $x \in (0, x_1)$. Similarly,

$$-(\rho(x)\Phi_p(u'(x)))' + s(x)\Phi_p(u(x)) - u^*(x) = 0$$

holds for $x \in [a,b]\backslash\{x_1, \ldots, x_m\}$. Since $u \in X$ and $u^* \in L^1[a,b]$, $\rho(x)\Phi_p(u'(x))$ is continuous in $[a,b]\backslash\{x_1, \ldots, x_m\}$ and absolutely continuous. So u is a solution of the differential inclusion (3.1). We need to show that the boundary and impulsive conditions hold.

Integration of the equality

$$\begin{aligned}
&(\rho(x)\Phi_p(u'(x)))'v(x) \\
&\qquad = \left(\int_a^x (\rho(s)\Phi_p(u'(s)))'\,ds v(x)\right)' - \int_a^x (\rho(s)\Phi_p(u'(s)))'\,ds v'(x)
\end{aligned}$$

from a to b gives

$$\begin{aligned}
&\int_a^b (\rho(x)\Phi_p(u'(x)))'v(x)\,dx \\
&= \int_a^b \left[\left(\int_a^x (\rho(s)\Phi_p(u'(s)))'\,ds v(x)\right)' - \int_a^x (\rho(s)\Phi_p(u'(s)))'\,ds v'(x)\right] dx \\
&= \int_a^b (\rho(x)\Phi_p(u'(x)))'\,dx v(b) \\
&\quad - \int_a^b \Bigg[\rho(x)\Phi_p(u'(x)) - \rho(a)\Phi_p(u'(a)) \\
&\quad - \sum_{a\le x_i<x} \Delta(\rho(x_i)\Phi_p(u'(x_i)))\Bigg] v'(x)\,dx \\
&= \left[\rho(b)\Phi_p(u'(b)) - \rho(a)\Phi_p(u'(a)) - \sum_{i=1}^{l} \Delta(\rho(x_i)\Phi_p(u'(x_i)))\right] v(b) \\
&\quad - \int_a^b \rho(x)\Phi_p(u'(x))v'(x)dx + \rho(a)\Phi_p(u'(a))[v(b) - v(a)] \\
&\quad + \sum_{j=0}^{l} \int_{x_j}^{x_{j+1}} \sum_{a\le x_i<x} \Delta(\rho(x_i)\Phi_p(u'(x_i)))v'(x)\,dx \\
&= \rho(b)\Phi_p(u'(b))v(b) - \rho(a)\Phi_p(u'(a))v(a) \\
&\quad - \sum_{i=1}^{l} \Delta(\rho(x_i)\Phi_p(u'(x_i)))v(b) - \int_a^b \rho(x)\Phi_p(u'(x))v'(x)\,dx \\
&\quad + \sum_{i=1}^{l} \Delta(\rho(x_i)\Phi_p(u'(x_i)))v(b) - \sum_{i=1}^{l} \Delta(\rho(x_i)\Phi_p(u'(x_i)))v(x_i)
\end{aligned} \tag{3.4}$$

$$
\begin{aligned}
&= \rho(b)\Phi_p(u'(b))v(b) - \rho(a)\Phi_p(u'(a))v(a) \\
&\quad - \sum_{i=1}^{l} \Delta(\rho(x_i)\Phi_p(u'(x_i))v(x_i)) - \int_a^b \rho(x)\Phi_p(u'(x))v'(x)\,dx.
\end{aligned}
$$

Substituting (3.4) into (3.2), we have

$$
\begin{aligned}
&\int_a^b [-(\rho(x)\Phi_p(u'(x)))' + s(x)\Phi_p(u(x)) - u^*(x)]v(x)\,dx \\
&+ \rho(b)\bigg[\Phi_p(u'(b)) + \Phi_p\bigg(\frac{\sigma u(b)}{\gamma}\bigg)\bigg]v(b) + \rho(a)\bigg[-\Phi_p(u'(a)) + \Phi_p\bigg(\frac{\beta u(a)}{\alpha}\bigg)\bigg]v(a) \\
&\qquad - \sum_{i=1}^{l}[\Delta(\rho(x_i)\Phi_p(u'(x_i))) + I_i(u(x_i))]v(x_i) = 0.
\end{aligned}
$$

Since u satisfies (3.3), for all $v \in X$, we have

$$
\begin{aligned}
&\rho(b)\bigg[\Phi_p(u'(b)) + \Phi_p\bigg(\frac{\sigma u(b)}{\gamma}\bigg)\bigg]v(b) \\
&\quad + \rho(a)\bigg[-\Phi_p(u'(a)) + \Phi_p\bigg(\frac{\beta u(a)}{\alpha}\bigg)\bigg]v(a) \\
&\quad - \sum_{i=1}^{l}[\Delta(\rho(x_i)\Phi_p(u'(x_i))) + I_i(u(x_i))]v(x_i) = 0.
\end{aligned}
\tag{3.5}
$$

We shall show that u satisfies the impulsive conditions in (3.1). If this is not the case, without loss of generality, we may assume that there exists $i \in \{1, \ldots, l\}$ such that

$$
\Delta(\rho(x_i)\Phi_p(u'(x_i))) + I_i(u(x_i)) \neq 0. \tag{3.6}
$$

Let $v(x) = \prod_{j=0,\, j\neq i}^{l+1}(x - x_j)$. Substituting $v(x)$ into (3.5), we have

$$
\begin{aligned}
&\rho(b)\bigg[\Phi_p(u'(b)) + \Phi_p\bigg(\frac{\sigma u(b)}{\gamma}\bigg)\bigg]v(b) \\
&\quad - \sum_{k=1}^{l}[\Delta(\rho(x_k)\Phi_p(u'(x_k))) + I_k(u(x_k))]v(x_k) \\
&\quad + \rho(a)\bigg[-\Phi_p(u'(a)) + \Phi_p\bigg(\frac{\beta u(a)}{\alpha}\bigg)\bigg]v(a) \\
&= \rho(b)\bigg[\Phi_p(u'(b)) + \Phi_p\bigg(\frac{\sigma u(b)}{\gamma}\bigg)\bigg]\prod_{j=0,\, j\neq i}^{l+1}(x_{l+1} - x_j) \\
&\quad + \rho(a)\bigg[-\Phi_p(u'(a)) + \Phi_p\bigg(\frac{\beta u(a)}{\alpha}\bigg)\bigg]\prod_{j=0,\, j\neq i}^{l+1}(x_0 - x_j)
\end{aligned}
\tag{3.7}
$$

$$
\begin{aligned}
&-\sum_{k=1}^{l}[\Delta(\rho(x_k)\Phi_p(u'(x_k)))+I_k(u(x_k))]\prod_{j=0,\,j\neq i}^{l+1}(x_k-x_j)\\
=&-[\Delta(\rho(x_i)\Phi_p(u'(x_i)))+I_i(u(x_i))]\prod_{j=0,\,j\neq i}^{l+1}(x_i-x_j)\neq 0,
\end{aligned}
$$

which contradicts (3.5). Thus, x satisfies the impulsive conditions in (3.1). Similarly, u satisfies the boundary conditions. Therefore, u is a solution of problem (3.1). □

LEMMA 3.6. *Let $R(u)=\|u'\|_{L^p}^p$ for $u\in X$. Then $R(u)$ is weakly lower semi-continuous.*

PROOF. First we show that $\|u_n'\|_{L^p}^p$ is lower semi-continuous. Let $u_n\in X$ be such that $u_n\to u$ as $n\to\infty$. Hence,

$$\|u_n'-u'\|_{L^p}^p\to 0 \tag{3.8}$$

as $n\to\infty$. Recall that

$$\|x+y\|_{L^p}\le\|x\|_{L^p}+\|y\|_{L^p}\quad\text{for any } x,y\in L^p[0,T]. \tag{3.9}$$

On one hand, by (3.8)–(3.9), we have

$$\|u_n'\|_{L^p}^p-\|u'\|_{L^p}^p\le(\|u_n'-u'\|_{L^p}+\|u'\|_{L^p})^p-\|u'\|_{L^p}^p\to 0 \tag{3.10}$$

as $n\to\infty$. On the other hand, we have

$$
\begin{aligned}
\|u_n'\|_{L^p}^p-\|u'\|_{L^p}^p&=\left\|\frac{u_n'+u'}{2}+\frac{u_n'-u'}{2}\right\|_{L^p}^p-\left\|\frac{u_n'+u'}{2}+\frac{u'-u_n'}{2}\right\|_{L^p}^p \\
&\ge\left(\left\|\frac{u_n'+u'}{2}\right\|_{L^p}-\left\|\frac{u_n'-u'}{2}\right\|_{L^p}\right)^p-\left(\left\|\frac{u_n'+u'}{2}\right\|_{L^p}+\left\|\frac{u'-u_n'}{2}\right\|_{L^p}\right)^p\to 0
\end{aligned} \tag{3.11}
$$

as $n\to\infty$. From (3.10) and (3.11), it follows that $\lim\limits_{n\to\infty}\|u_n'\|_{L^p}^p=\|u'\|_{L^p}^p$ and so $R(u)$ is lower semi-continuous. Also, since $R(u)$ is convex, $R(u)$ is weakly lower semi-continuous (see [22, Proposition 38.7]). □

Let

$$\mathcal{N}(u)=\frac{1}{p}\|u\|_X^p-\sum_{i=1}^{l}\int_0^{u(x_i)}I_i(s)\,ds+\frac{\rho(b)\sigma^{p-1}}{p\gamma^{p-1}}|u(b)|^p+\frac{\rho(a)\beta^{p-1}}{p\alpha^{p-1}}|u(a)|^p. \tag{3.12}$$

Then,

$$
\begin{aligned}
\langle\mathcal{N}'(u),v\rangle=&\int_a^b\rho(x)\Phi_p(u'(x))v'(x)+s(x)\Phi_p(u(x))v(x)\,dx\\
&-\sum_{i=1}^{l}I_i(u(x_i))v(x_i)+\rho(b)\Phi_p\Big(\frac{\sigma u(b)}{\gamma}\Big)v(b)+\rho(a)\Phi_p\Big(\frac{\beta u(a)}{\alpha}\Big)v(a)
\end{aligned}
$$

holds for all $v\in X$.

Lemma 3.7. *$\mathcal{N}'\colon X \to X^*$ is of type* $(\mathrm{S})_+$.

Proof. Let $u_n \rightharpoonup u$ and $\limsup_n \langle \mathcal{N}'(u_n), u_n - u\rangle \le 0$. In view of Definition 2.5, we need to show that $u_n \to u$. Assume, to the contrary, that there exist $\varepsilon > 0$ and a subsequence (u_{n_k}) such that

$$\|u_{n_k} - u\|_X \ge \varepsilon. \tag{3.13}$$

Since $u_n \rightharpoonup u$, we have $\|u_n\|_X \le M_1$, $\|u_n'\|_{L^p}^p \le M_2$, $M_i > 0$, $i = 1, 2$, $u_n \to u$ in $C[a,b]$, and $\|u_{n_k}'\|_{L^p}^p$ has a convergent subsequence, which without loss of generality, we again denote by (u_{n_k}'), i.e.

$$\|u_{n_k}'\|_{L^p}^p \to c_1. \tag{3.14}$$

Since $\limsup_n \langle \mathcal{N}'(u_n), u_n - u\rangle \le 0$, we have

$$\begin{aligned}
&\limsup_n \langle \mathcal{N}'(u_n), u_n - u\rangle \\
&= \limsup_n \bigg[\int_a^b \rho(x)\Phi_p(u_n'(x))(u_n'(x) - u'(x)) + s(x)\Phi_p(u_n(x))(u_n(x) - u(x))\,dx \\
&\quad + \sum_{i=1}^{l} I_i(u_n(x_i))(u_n(x_i) - u(x_i)) \\
&\quad + \rho(b)\Phi_p\bigg(\frac{\sigma u_n(b)}{\gamma}\bigg)[u_n(b) - u(b)] + \rho(a)\Phi_p\bigg(\frac{\beta u_n(a)}{\alpha}\bigg)[u_n(a) - u(a)]\bigg] \le 0.
\end{aligned}$$

Since $u_n \to u$ in $C[a,b]$, it follows from the above inequality that

$$\limsup_n \int_a^b \rho(x)\Phi_p(u_n'(x))(u_n'(x) - u'(x))\,dx \le 0. \tag{3.15}$$

From (3.14)–(3.15) and the convexity of $|u|^p$, we have

$$\begin{aligned}
\int_a^b |u'(x)|^p\,dx \ge \int_a^b |u_{n_k}'(x)|^p\,dx + \int_a^b \Phi_p(u_{n_k}'(x))(u'(x) - u_{n_k}'(x))\,dx \\
\ge \|u_{n_k}'\|_{L^p}^p \to c_1.
\end{aligned} \tag{3.16}$$

By Lemma 3.6, we have

$$\|u'\|_{L^p}^p \le \liminf_{n_k} \|u_{n_k}'\|_{L^p}^p = c_1, \tag{3.17}$$

so from (3.16)–(3.17),

$$\|u'\|_{L^p}^p = c_1. \tag{3.18}$$

From Lemma 3.6, it follows that

$$c_1 = \|u'\|_{L^p}^p \le \liminf_{n_k} \bigg\|\frac{u' + u_{n_k}'}{2}\bigg\|_{L^p}^p. \tag{3.19}$$

From [8, Theorem 2], for any x, $y \in L^p[0,T]$,

$$\left\|\frac{x+y}{2}\right\|_{L^p}^p + \left\|\frac{x-y}{2}\right\|_{L^p}^p \le \frac{1}{2}(\|x\|_{L^p}^p + \|y\|_{L^p}^p), \quad p \ge 2, \tag{3.20}$$

and

$$\left\|\frac{x+y}{2}\right\|_{L^p}^q + \left\|\frac{x-y}{2}\right\|_{L^p}^q \le \left[\frac{1}{2}(\|x\|_{L^p}^p + \|y\|_{L^p}^p)\right]^{q-1}, \quad 1 < p < 2. \tag{3.21}$$

By (3.13), (3.14), (3.18) and (3.20), for $p \ge 2$ we have

$$\begin{aligned} \limsup_{n_k\to\infty} \left\|\frac{u' + u'_{n_k}}{2}\right\|_{L^p}^p & \\ \le \limsup_{n_k\to\infty} &\left[\frac{1}{2}(\|u'\|_{L^p}^p + \|u'_{n_k}\|_{L^p}^p) - \left\|\frac{u' - u'_{n_k}}{2}\right\|_{L^p}^p\right] \le c_1 - \varepsilon_1, \end{aligned} \tag{3.22}$$

for some $\varepsilon_1 > 0$, which contradicts (3.19) and proves $u_n \to u$ in X.

Now by (3.13), (3.14), (3.18) and (3.21), for $1 < p < 2$ we have

$$\begin{aligned} \limsup_{n_k\to\infty} \left\|\frac{u' + u'_{n_k}}{2}\right\|_{L^p}^p & \\ \le \limsup_{n_k\to\infty} &\left\{\left[\frac{1}{2}(\|u'\|_{L^p}^p + \|u'_{n_k}\|_{L^p}^p)\right]^{q-1} - \left\|\frac{u' - u'_{n_k}}{2}\right\|_{L^p}^q\right\}^{p/q} \\ & \le (c_1^{q-1} - \varepsilon_2)^{p/q} < c_1 \end{aligned} \tag{3.23}$$

for some $\varepsilon_2 > 0$. Again, this contradicts (3.19) and so $u_n \to u$ in X. Therefore, $\mathcal{N}'$ is of type $(\mathrm{S})_+$. □

We introduce, for almost every $x \in [a,b]$ and all $s \in R$, the Aumann-type set-valued integral

$$\int_0^s H(x,t)\,dt = \left\{ \int_0^s h(x,t)\,dt \;\middle|\; h\colon [0,T]\times R \to R \text{ is a measurable selection of } H \right\}$$

and set

$$\mathcal{H}(u) = \int_a^b \min \int_0^u H(x,s)\,ds\,dx \quad \text{for all } u \in L^\gamma[a,b].$$

LEMMA 3.8 ([12, Lemma 7]). *Assume that* (H_1)–(H_3) *hold. Then the functional* $\mathcal{H}\colon L^\gamma[a,b] \to R$, $\gamma > 1$, *is well defined and Lipschitz on any bounded subset of* $L^\gamma[a,b]$. *Moreover, for all* $u \in L^\gamma[a,b]$ *and all* $u^* \in \partial\mathcal{H}(u)$, *we have* $u^*(x) \in H(x,u(x))$ *for almost every* $x \in [a,b]$.

We define an energy functional for the problem (3.1) by setting, for all $u \in X$,

$$\varphi(u) = \frac{1}{p}\|u\|_X^p - \mathcal{H}(u) - \sum_{i=1}^m \int_0^{u(x_i)} I_i(s)\,ds + \frac{\rho(b)\sigma^{p-1}}{p\gamma^{p-1}}|u(b)|^p + \frac{\rho(a)\beta^{p-1}}{p\alpha^{p-1}}|u(a)|^p.$$

Lemma 3.9. *The functional $\varphi\colon X \to R$ is l.L. Moreover, for each critical point $u \in X$ of φ, u is a weak solution of* (3.1).

Proof. For any u, v in a bounded domain Ω of X, we shall show that φ is Lipschitz in Ω. Let $\varphi(u) = \varphi_1(u) + \varphi_2(u)$, where

$$\varphi_1(u) = \frac{1}{p}\|u\|_X^p - \sum_{i=1}^m \int_0^{u(x_i)} I_i(s)\,ds + \frac{\rho(b)\sigma^{p-1}}{p\gamma^{p-1}}|u(b)|^p + \frac{\rho(a)\beta^{p-1}}{p\alpha^{p-1}}|u(a)|^p,$$

$\varphi_2(u) = -\mathcal{H}(u)$. Clearly $\varphi_1 \in C^1(X)$. By Lemma 2.1, φ_1 is l.L. on X. By Lemma 3.8, φ_2 is locally Lipschitz on $L^p[a,b]$. Moreover, X is compactly embedded into $L^p[a,b]$. So φ_2 is l.L. on X. Therefore, φ is l.L. on X.

Now we show that each critical point u of φ is a weak solution of (3.1). Let $u \in X$ be a critical point of φ. Then,

$$0 \in \partial\varphi = \{u^* \in X^* : \langle u^*, v\rangle \le \varphi^0(u; v) \text{ for all } v \in X\}. \tag{3.24}$$

Set

$$\begin{aligned}\langle A(u), v\rangle = &\int_a^b \rho(x)\Phi_p(u'(x))v'(x) + s(x)\Phi_p(u(x))v(x)\,dx \\ &- \sum_{i=1}^m I_i(u(x_i))v(x_i) + \rho(b)\Phi_p\bigg(\frac{\sigma u(b)}{\gamma}\bigg)v(b) + \rho(a)\Phi_p\bigg(\frac{\beta u(a)}{\alpha}\bigg)v(a)\end{aligned}$$

for all $u, v \in X$. By (2.2), (2.5), (2.6), and (3.24), $0 \in A(u) - \partial\mathcal{H}(u)$, i.e. there exists $u^* \in \partial\mathcal{H}(u)$ satisfying

$$A(u) = u^* \quad \text{in } X^*. \tag{3.25}$$

We extend u^* to an element of $L^q[a,b]$. Hence, we regard X as a closed subspace of $L^p[a,b]$.

First we observe that u^*, as a linear functional on X, is also continuous with respect to the topology induced by the norm $\|\cdot\|_{L^p}$. By Lemma 3.8, $\mathcal{H}$ admits a Lipschitz constant L around u with respect to $\|\cdot\|_{L^p}$. Then, by (2.4), we obtain

$$\langle u^*, v\rangle \le L\|v\|_{L^p} \quad \text{for all } v \in X. \tag{3.26}$$

Moreover, $\mathcal{H}^0(u;\,\cdot\,)$ is a subadditive, positively homogeneous function on $L^p[a,b]$ and

$$\langle u^*, v\rangle \le \mathcal{H}^0(u; v) \tag{3.27}$$

for all $v \in X$. By the Hahn–Banach theorem, u^* extends to a bounded linear functional defined on $L^p[a,b]$ satisfying (3.27) for all $v \in L^p[a,b]$. This implies two facts. First, by Lemma 3.8, we have $u^*(x) \in H(x, u(x))$ for almost every $x \in [a,b]$. This implies $u^*(x) \in L^q[a,b]$ (see Remark 3.1).

Second, we may rewrite (3.25) as

$$\int_a^b \rho(x)\Phi_p(u'(x))v'(x) + s(x)\Phi_p(u(x))v(x) - u^*(x)v(x)\,dx$$
$$-\sum_{i=1}^{l} I_i(u(x_i))v(x_i) + \rho(b)\Phi_p\left(\frac{\sigma u(b)}{\gamma}\right)v(b) + \rho(a)\Phi_p\left(\frac{\beta u(a)}{\alpha}\right)v(a) = 0.$$

Thus, by Definition 3.3, u is a weak solution of (3.1). □

4. Proof of main results

PROOF OF THEOREM 2.8. We shall apply Theorem 2.6 to prove Theorem 2.8. The operator $\mathcal{N}$ is defined above in (3.12). We define $\mathcal{F}\colon X \to R$ by

$$\mathcal{F}(u) = \int_a^b \min \int_0^{u(x)} F(s)\,ds\,dx$$

for every $u \in X$. Clearly, $\mathcal{N}$ is a sequentially weak l.s.c. functional. By Lemma 3.7, $\mathcal{N}'$ is of type $(\mathrm{S})_+$. By Lemma 3.8, $\mathcal{F}\colon X \to R$ is Lipschitz on bounded subsets of X.

To prove that the gradient $\partial\mathcal{F}\colon X \to 2^{X^*}$ is compact, choose a bounded sequence (u_n) in X with $u_n^* \in \partial\mathcal{F}(u_n)$ for all $n \in N$. Let $L > 0$ be a Lipschitz constant for $\mathcal{F}$, restricted to a bounded set containing the sequence (u_n); then $\|u_n^*\|_{X^*} \le L$ for all $n \in N$. A subsequence of (u_n^*), which we again denote by (u_n^*), weakly converges to some u^* in X^*. We shall show that the convergence is strong.

Assume, to the contrary, that there exists $\varepsilon > 0$ and a subsequence (u_{n_k}) of (u_n) such that $\|u_{n_k}^* - u^*\|_{X^*} > \varepsilon$ for all $k \in N$. Then, for all $k \in N$, we can find $v_k \in X$ with $\|v_k\|_X < 1$ and

$$\langle u_{n_k}^* - u^*, v_k\rangle > \varepsilon. \tag{4.1}$$

Passing if necessary to a subsequence, we can assume that $v_k \rightharpoonup v$ in X, while $v_k \to v$ in $L^1[a,b]$ and $L^p[a,b]$. From (F_3) and the Hölder's inequality,

$$\begin{aligned}\langle u_{n_k}^* - u^*, v_k\rangle &= \langle u_{n_k}^*, v_k - v\rangle + \langle u_{n_k}^* - u^*, v\rangle + \langle u^*, v - v_k\rangle \\ &\le L\|v_k - v\|_{L^p} + \langle u_{n_k}^* - u^*, v\rangle + \langle u^*, v - v_k\rangle \to 0\end{aligned}$$

as $k \to \infty$, which contradicts (4.1).

Next, we verify that condition (2.7) in Theorem 2.6 holds. By (I_1), for all $u \in X$,

$$\int_0^{u(x_i)} I_i(s)\,ds \le 0, \quad i = 1, \dots, l, \tag{4.2}$$

which together with (F_5), (I_1), and Lemma 3.2 implies

$$\begin{aligned}
&\mathcal{N}(u) - \lambda\mathcal{F}(u) \\
&\quad \geq \frac{\|u\|_X^p}{p} - \lambda\int_a^b \eta(1+|u(x)|^l)\,dx + \frac{\rho(b)\sigma^{p-1}}{p\gamma^{p-1}}|u(b)|^p + \frac{\rho(a)\beta^{p-1}}{p\alpha^{p-1}}|u(a)|^p \\
&\qquad\qquad \geq \frac{\|u\|_X^p}{p} - \lambda c_3(1+\|u\|_X^l)
\end{aligned}$$

for some $c_3 > 0$. Since $1 < l < p$, $\lim\limits_{\|u\|_X\to+\infty} \mathcal{N}(u) - \lambda\mathcal{F}(u) = +\infty$.

We shall show that condition (2.8) in Theorem 2.6 holds by using Lemma 2.7. Set $\check{u}(x) = 0$ and $\widehat{u}(x) = k > 0$. Clearly, $\check{u}, \widehat{u} \in X$ and (2.9) in Lemma 2.7 holds. Now

$$\begin{aligned}
\mathcal{N}(\hat{u}) &= \frac{k}{p}\int_a^b s(x)\,dx - \sum_{i=1}^{l}\int_0^k I_i(s)\,ds + \frac{\rho(b)\sigma^{p-1}}{p\gamma^{p-1}}k^p + \frac{\rho(a)\beta^{p-1}}{p\alpha^{p-1}}k^p \\
&> \frac{k}{p}\int_a^b s(x)\,dx + \frac{\rho(b)\sigma^{p-1}}{p\gamma^{p-1}}k^p + \frac{\rho(a)\beta^{p-1}}{p\alpha^{p-1}}k^p.
\end{aligned} \tag{4.3}$$

From (F_4), (2.10) in Lemma 2.7 holds.

For all $u \in X$ with $\mathcal{N}(u) < \tau$, we have

$$\frac{1}{p}\|u\|_X^p - \sum_{i=1}^{l}\int_0^{u(x_i)} I_i(s)\,ds + \frac{\rho(b)\sigma^{p-1}}{p\gamma^{p-1}}|u(b)|^p + \frac{\rho(a)\beta^{p-1}}{p\alpha^{p-1}}|u(a)|^p < \tau,$$

which together with (I_1), gives $\|u\|_X < (\tau p)^{1/p}$. So

$$\{u \in X : \mathcal{N}(u) < \tau\} \subset \{u \in X : \|u\|_{C^0} < \overline{\gamma}(\tau p)^{1/p}\}.$$

Thus,

$$\begin{aligned}
\sup_{\mathcal{N}(u)<\tau} \mathcal{F}(u) &= \sup_{\mathcal{N}(u)<\tau}\int_a^b \min\int_0^{u(x)} F(s)\,ds\,dx \\
&\leq \sup_{|u|<\overline{\gamma}(\tau p)^{1/p}}\int_a^b \min\int_0^{u(x)} F(s)\,ds\,dx \\
&\leq \sup_{|u|<\overline{\gamma}(\tau p)^{1/p}} (b-a)\min\int_0^u F(s)\,ds.
\end{aligned} \tag{4.4}$$

Therefore,

$$\frac{\tau\mathcal{F}(\widehat{u})}{\mathcal{N}(\widehat{u})} = \frac{\tau\displaystyle\int_a^b \min\int_0^k F(s)\,ds\,dx}{\displaystyle\frac{k}{p}\int_a^b s(x)\,dx - \sum_{i=1}^{l}\int_0^k I_i(s)\,ds + \frac{\rho(b)\sigma^{p-1}}{p\gamma^{p-1}}k^p + \frac{\rho(a)\beta^{p-1}}{p\alpha^{p-1}}k^p} \tag{4.5}$$

$$= \frac{\tau(b-a)\min\displaystyle\int_0^k F(s)\,ds}{\dfrac{k}{p}\displaystyle\int_a^b s(x)\,dx - \sum_{i=1}^{l}\int_0^k I_i(s)\,ds + \dfrac{\rho(b)\sigma^{p-1}}{p\gamma^{p-1}}k^p + \dfrac{\rho(a)\beta^{p-1}}{p\alpha^{p-1}}k^p}.$$

From (F_4), we have

$$\frac{\tau(b-a)\min\displaystyle\int_0^k F(s)\,ds}{\dfrac{k}{p}\displaystyle\int_a^b s(x)\,dx - \sum_{i=1}^{l}\int_0^k I_i(s)\,ds + \dfrac{\rho(b)\sigma^{p-1}}{p\gamma^{p-1}}k^p + \dfrac{\rho(a)\beta^{p-1}}{p\alpha^{p-1}}k^p} > \sup_{|u|<\overline{\gamma}(\tau p)^{1/p}} (b-a)\min\int_0^u F(s)\,ds, \tag{4.6}$$

so by (4.4)–(4.6), we see that (2.11) holds. Then, we have that (2.8) holds for some $\rho \in R$.

Let $[\alpha,\beta]$, $0<\alpha<\beta$, and $r>0$ be as in Theorem 2.6. Choose $\lambda \in [\alpha,\beta]$ and a multifunction $\mathcal{G}$ satisfying (G_1)–(G_2). Set

$$\mathcal{G} = \int_a^b \min\int_0^u G(s)\,ds$$

for all $u \in X$. By Lemma 3.8 and an argument analogous to that used for $\mathcal{F}$, it follows that the functional $\mathcal{G}\colon X \to R$ is l.L. and its gradient $\partial\mathcal{G}$ is compact. Then, there is $\delta > 0$ such that, for all $\mu \in [0,\delta]$ the functional

$$\varphi_{\lambda,\mu} = \mathcal{N} - \lambda\mathcal{F} - \mu\mathcal{G}$$

admits at least three critical points $u_0, u_1, u_2 \in X$ with $\|u_i\|_X < r$, $i = 0,1,2$. For all $\lambda > 0$ and $\mu \geq 0$, the multifunction H defined by setting $H(x,s) = \lambda F(s) + \mu G(x,s)$ for all $(x,s) \in [a,b]\times R$ satisfies (H_1) and (H_2). Therefore, by Lemmas 3.5 and 3.9, u_0, u_1, u_2 are three solutions of problem (1.1). $\square$

We conclude this paper with an example.

EXAMPLE 4.1. For all $s \in R$, set $I_1(s) = -s$ and

$$F(s) = \begin{cases} \{0\} & \text{if } |s| < 2^{2/3}, \\ [0,1] & \text{if } |s| = 2^{2/3}, \\ \{(s - 2^{2/3} + 1)^{1.5}\} & \text{if } s > 2^{2/3}, \\ \{|s + 2^{2/3} + 1|^{1.5}\} & \text{if } s < -2^{2/3}. \end{cases}$$

Consider the impulsive differential inclusion

$$\begin{cases} -(\Phi_3(u'(x)))' + \Phi_3(u(x)) \in \lambda F(u(x)) + \mu G(x,u(x)), & x \in [0,1], \\ -\Delta\Phi_3(u'(x_1)) = I_1(u(x_1)), & x_1 = 1/2, \\ u'(0) - u(0) = 0, \quad u'(1) + u(1) = 0. \end{cases} \tag{4.7}$$

For any multifunction G satisfying (G_1)–(G_3), any function I_1 satisfying (I_1), (4.7) admits at least three solutions (uniformly bounded) for λ and μ lying in appropriate intervals.

In (1.1), we have $p=3$, $\rho(x)=1$, $s(x)=1$, $a=0$, $b=1$, and $\alpha=\beta=\gamma=\sigma=1$. Clearly, assumptions (F_1)–(F_3), (F_5), and (I_1) hold. To show (F_4) holds, let $k=2$ and $\tau=1/3$; then

$$\frac{k}{p}\int_a^b s(x)\,dx+\frac{\rho(b)\sigma^{p-1}}{p\gamma^{p-1}}k^p+\frac{\rho(a)\beta^{p-1}}{p\alpha^{p-1}}k^p=\frac{4}{3}>\tau=\frac{1}{3}.$$

Also,

$$\frac{\tau\min\displaystyle\int_0^k F(s)\,ds}{\dfrac{k}{p}\displaystyle\int_a^b s(x)\,dx-\sum_{i=1}^{l}\int_0^k I_1(s)\,ds+\dfrac{\rho(b)\sigma^{p-1}}{p\gamma^{p-1}}k^p+\dfrac{\rho(a)\beta^{p-1}}{p\alpha^{p-1}}k^p}=\frac{1}{10}\min\int_0^1 F(s)\,ds>0$$

and

$$\sup_{|u|<\overline{\gamma}(\tau p)^{1/p}}\min\int_0^u F(s)\,ds=\sup_{|u|<2^{2/3}}\min\int_0^u F(s)\,ds=0,$$

so (F_4) is satisfied. By Theorem 2.8, (4.7) has at least three solutions.

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