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SUBSHIFTS, ROTATIONS AND THE SPECIFICATION PROPERTY

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Abstract. Let $X = \Sigma_2$ and let $F\colon X \times \mathbb{S}^1 \to X \times \mathbb{S}^1$ be a map given by

$$F(x,t) = (\sigma(x), R_{x_0}(t)),$$

where (Σ_2,σ) denotes the full shift over the alphabet $\{0,1\}$ while R_0, R_1 are the rotations of the unit circle $\mathbb{S}^1$ by the angles r_0 and r_1, respectivelly. It was recently proved by X. Wu and G. Chen that if r_0 and r_1 are irrational, then the system $(X\times\mathbb{S}^1, F)$ has an uncountable distributionally δ-scrambled set S_δ for every positive $\delta \le \operatorname{diam} X \times \mathbb{S}^1 = 1$. Moreover, each point in S_δ is recurrent but not weakly almost periodic (this answeres a question from [Wang et al., Ann. Polon. Math. **82** (2003), 265–272]).
We generalize the above result by proving that if $r_0 - r_1 \in \mathbb{R}\setminus\mathbb{Q}$ and $X \subset \Sigma_2$ is a nontrivial subshift with the specification property, then the system $(X \times \mathbb{S}^1, F)$ also has the specification property. As a consequence, there exist a constant $\delta > 0$ and a dense Mycielski distributionally δ-scrambled set for $(X \times \mathbb{S}^1, F)$, in which each point is recurrent but not weakly almost periodic.

1. Introduction

In the field of the qualitative theory of dynamical systems, the objects of a special interest are dynamical systems that exhibit any kind of chaotic behavior. Probably the most famous definition of chaos comes from Devaney [6] and involves such notions as transitivity or sensitivity. In this paper we investigate

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another concept that is known as a distributional chaos and was introduced by Schweizer and Smítal in [18] (under the name "strong chaos"). Roughly speaking, it relates to the complexity of dynamics with a distribution function of average distances between trajectories of two points (see the next section for a definition).

The specification property was first introduced by Bowen [4], and it is one of the strongest mixing properties. On the other hand, rotations of the circle are examples of the simplest dynamical systems – they are, in fact, isometries and their dynamics is, in some sense, rigid. However, it is known that skew product systems involving irrational rotations can lead to surprising examples. Among the most interesting results in this direction are the constructions from [9], where irrational rotations on tori are used as a tool to construct a weakly mixing, minimal and uniformly rigid skew product system. This shows that simple rotations, when properly controlled (or sufficiently randomized, depending on point of view), may lead to chaotic systems with mixing properties. In our work we are going to show that a kind of complicated dynamics, involving the specification property, can arise as a consequence of a proper control (in the sense of a skew product) of two (appropriately chosen) rotations. This result seems to be somewhat surprising, since on each fiber (i.e., the unit circle) such dynamics is, in fact, rigid, in the sense that all points never change their relative positions.

The origins of our model can be found in a paper by Falcó [8], who in fact was inspired by much earlier paper by Afraimovich and Shilnikov, published in 1974 [1]. In [8], Falcó was interested in describing the set of periods for the skew product system in the form of (3.1), defined by N rotations and a full two-sided shift over N symbols. The present paper is motivated by the works of Wang et al. [19] and Wu and Chen [21]. In [19] there is a question concerning the existence of an uncountable distributionally scrambled set S (a dynamical system with a distributional chaos) such that each element of S is a recurrent point. Wang et. al. formulated a sufficient condition for the existence of such a set S (moreover, the chaotic set S they obtained by this condition did not contain almost periodic points). They also presented an example of a class of "shift directed alternating systems" of irrational rotations of the unit circle (see Section 3 of the present paper for precise statement), for which they were not able to verify the introduced condition and hence failed to prove in this case the presence of a distributional chaos. The discussion was then continued in [21], where it was shown that the systems considered in [19] actually contain an uncountable distributionally scrambled set, in which each point is recurrent but not weakly almost periodic.

In our paper we use the concept of the specification property in order to formulate another global condition, guaranteeing the presence of a distributional chaos on a recurrent but not weakly periodic part of the phase space (see Theorem 3.4). Then we show that a large class of skew product systems (containing these introduced in [19]) exhibits the specification property and, as a by-product, we provide a more detailed characterization of chaos (and chaotic sets) for the systems considered in [21] (see Theorem 3.8 and Corollaries 3.11 and 3.12).

2. Preliminaries

In this section we establish a relevant background for further considerations that involve topological and symbolic dynamics. Specifically, we recall some notions related to the concepts of specification properties and distributional chaos. From now on the sets of all nonnegative integers, integers, rational numbers and real numbers are denoted by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$, respectively, and the unit circle of the complex plane is denoted by $\mathbb{S}^1$.

2.1. Topological dynamics. We say that a pair (X, f) is a *dynamical system*, if (X, d) is a compact metric space and $f\colon X \to X$ is a continuous map. A pair of points $(x, y) \in X \times X$ is called *distal* if

$$\liminf_{n\to\infty} d(f^n(x), f^n(y)) > 0.$$

A point $x \in X$ is said to be *recurrent* if

$$\liminf_{n\to\infty} d(f^n(x), x) = 0,$$

and *weakly almost periodic* if for any $\varepsilon > 0$ there exists $N_\varepsilon > 0$ such that

$$\#\{j : d(x, f^j(x)) \le \varepsilon,\ 0 \le j < nN_\varepsilon\} \ge n$$

for every positive integer n. By $\mathrm{Rec}(f)$ and $\mathrm{W}(f)$ we denote the sets of all recurrent points and all weakly almost periodic points, respectively. Clearly $\mathrm{W}(f) \subset \mathrm{Rec}(f)$.

For a positive integer n, points $x, y \in X$ and a constant $t \in \mathbb{R}$ we put

$$\Phi_{xy}^{(n)}(t) = \frac{1}{n}\#\{0 \le i < n : d(f^i(x), f^i(y)) < t\},$$

and then we define the following nondecreasing functions of the variable t:

$$\Phi_{xy}(t) = \liminf_{n\to\infty} \Phi_{xy}^{(n)}(t), \quad \Phi_{xy}^*(t) = \limsup_{n\to\infty} \Phi_{xy}^{(n)}(t).$$

Note that $\Phi_{xy}(t) = \Phi_{xy}^*(t) = 0$ for $t \le 0$ and $\Phi_{xy}(t) = \Phi_{xy}^*(t) = 1$ for $t > \operatorname{diam} X$. For any constant $\delta > 0$ we consider the following symmetric relation on X:

$$\mathbf{DC}_\delta = \{(x, y) \in X \times X : \Phi_{xy}^*(t) = 1 \text{ for } t > 0,\ \Phi_{xy}(\delta) = 0\},$$

and we define

$$\mathbf{DC} = \bigcup_{\delta>0} \mathbf{DC}_\delta\,.$$

Note that **DC** is also a symmetric relation on X. Each member of **DC** is called a DC1 *pair*. A set $S \subset X$ is said to be *distributionally δ-scrambled* (*distributionally scrambled*), if $S \times S \setminus \Delta \subset \mathbf{DC}_\delta$ (resp. $S \times S \setminus \Delta \subset \mathbf{DC}$), where Δ denotes the diagonal in $X \times X$, i.e. $\Delta = \{(x,x) : x \in X\}$.

A dynamical system (X,f) is called *transitive*, if for any nonempty open sets $U,V \subset X$ we can find an integer $n > 0$ such that $f^n(U) \cap V \neq \emptyset$. We say that (X,f) is *weakly mixing*, if $(X \times X, f \times f)$ is a transitive system, and (X,f) is *totally transitive*, if the system (X,f^n) is transitive for every positive integer n.

The concept of specification was introduced by Bowen in [4] (see also [7] and [2]). Note that the property which is called "specification" in [7] is the same as the "strong specification" property considered in [2]. Since we want to use a weaker version of this condition (which does not assume periodicity of a tracing point), we decided to follow the terminology of [2].

We say that a dynamical system (X,f) has the *strong specification property*, if for any $\delta > 0$ there is a positive integer N_δ such that for any integer $s \geq 2$, any set $\{y_1,\dots,y_s\}$ of s points in X, and any sequence $0 = j_1 \leq k_1 < j_2 \leq k_2 < \dots < j_s \leq k_s$ of $2s$ integers satisfying $j_{m+1} - k_m \geq N_\delta$ for $m = 1,\dots,s-1$, we can find a point $x \in X$ such that for each positive integer $m \leq s$ and all integers i satisfying $j_m \leq i \leq k_m$, the following conditions hold:

$$d(f^i(x), f^i(y_m)) < \delta, \tag{2.1}$$

$$f^n(x) = x, \text{ where } n = N_\delta + k_s. \tag{2.2}$$

If (2.1) holds (but not necessarily (2.2)), then we say that (X,f) has the *specification property*. Note that if f is surjective, then we can replace (2.1) by the following condition:

$$d(f^i(x), f^{i-j_m}(y_m)) < \delta. \tag{2.3}$$

In particular, we see that for any dynamical system generated by a surjective map, the specification property implies the weak mixing.

2.2. Symbolic dynamics. Endow the set $\Sigma_2 = \{0,1\}^{\mathbb{N}}$ with the metric d given by: $d(x,y) = 2^{-k}$ if $x \neq y$ and $k = \min\{i \in \mathbb{N} : x_i \neq y_i\}$, and $d(x,y) = 0$ if $x = y$. Then (Σ_2, d) is a compact metric space and the *shift* map $\sigma\colon \Sigma_2 \to \Sigma_2$, defined by $\sigma(x)_i = x_{i+1}$, is continuous. The system (Σ_2,σ) is called the *full shift* (on 2 symbols) and every closed invariant subset of Σ_2, i.e. any subsystem of (Σ_2,σ), is called a *shift* or a *subshift*. When X is a subshift of (Σ_2,σ), we write σ for $\sigma|_X$.

By a *word* we mean any finite sequence $w \in \bigcup_{n=1}^{\infty} \{0,1\}^n$, and by $|w|$ we denote its *length*, i.e. the number of symbols in w. Note that in our convention, a word has at least one symbol, i.e. it cannot be empty. Sometimes it is important to know the number of occurrences of each symbol in a word $w = w_1 \dots w_n$, hence it is convenient to put $|w|_0 = \#\{i : w_i = 0,\ 1 \leq i \leq n\}$ and $|w|_1 = |w| - |w|_0 = \#\{i : w_i = 1,\ 1 \leq i \leq n\}$. By $L(X)$ we denote the language of the subshift X, i.e. $L(X) = \{x_{[0,k)} : x \in X,\ k > 0\}$, where $x_{[i,j)} = x_i x_{i+1} \dots x_{j-1}$ for $0 \leq i < j < \infty$. For any $n \geq 1$ we write $L_n(X) = \{x_{[0,n)} : x \in X\}$, and for any word $w \in L(X)$ we define its *cylinder set* in X, as the set $C_X[w] = \{x \in X : x_{[0,n)} = w\}$, where $n = |w|$.

It is not hard to verify that a subshift X has the *specification property* if and only if there is an integer $k > 0$ such that for any two words $u, w \in L(X)$ we can find a word v with $|v| = k$, such that $uvw \in L(X)$. A subshift X is called *transitive*, if for every two words $u, v \in L(X)$ there is a word s such that $usv \in L(X)$, and X is called *synchronized*, if it is a transitive subshift and there exists a word $u \in L(X)$ (a *synchronizing word*) satisfying the following condition: if a and b are such words that $au, ub \in L(X)$, then also $aub \in L(X)$. It was proved in [3] that any subshift with the specification property is synchronized. Having this result at hand it is easy to see that the specification property is equivalent to the strong specification property for subshifts.

3. Results

Let (X, f) be a dynamical system on a compact metric space (X, d). The following lemma was proved in [16] (see also [15]).

Lemma 3.1. *If (X, f) has the specification property and a distal pair then there is $\delta > 0$ such that the set $\mathbf{DC}_\delta(f)$ is residual in $X \times X$.*

We recall that a subset of a topological space Y is called *residual*, if it contains a countable intersection of open and dense subsets of Y.

We need the following auxiliary lemma.

Lemma 3.2. *Let X be a compact metric space containing at least two points and let (X, f) be a surjective dynamical system with the specification property. Then (X, f) has a distal pair and the set $X \setminus \mathrm{W}(f)$ is residual in X.*

Proof. Since X has at least two points there are nonempty open sets U and V such that $\overline{U} \cap \overline{V} = \emptyset$. Then using the specification property of f and compactness of X it is not hard to prove that there is a closed set $\Lambda \subset \overline{U} \cup \overline{V}$, an integer $k > 0$ and a continuous surjection $\pi \colon \Lambda \to \Sigma_2$ such that $f^k(\Lambda) = \Lambda$ and $\pi \circ f^k = \sigma \circ \pi$. It is known that the full shift contains uncountably many minimal subsets (see, e.g. [11]), therefore we easily obtain that there are uncountably many minimal sets for (Λ, f^k) and, consequently, at least two different minimal sets M_1, M_2 for (X, f). Hence we obtain a distal pair (q_1, q_2), where q_i is any point in M_i $(i = 1, 2)$.

Put $\varepsilon = \mathrm{dist}(M_1, M_2)/10$ and define $A = \bigcap_{m=1}^{\infty} A_m$, where A_m is the set of such points $x \in X$, for which there exists a positive integer n satisfying

$$\#\{j : d(x, f^j(x)) \leq \varepsilon,\ 0 \leq j < nm\} < n.$$

Clearly, for every $\xi > 0$ there is $\gamma > 0$ such that the following implication holds:

$$d(x, y) < \gamma \Rightarrow d(f^j(x), f^j(y)) < \xi \text{ for each } j = 0, 1, \ldots, nm.$$

In particular, for all points $y \in X$ in a sufficiently small neighborhood of x we have

$$\#\{j : d(x, f^j(x)) > \varepsilon,\ 0 \leq j < nm\} \leq \#\{j : d(y, f^j(y)) > \varepsilon,\ 0 \leq j < nm\},$$

which proves that each set A_m is open.

Now establish any integer $m > 0$. Fix an open set $U \subset X$, a point $x \in U$ and a positive constant $\delta < \varepsilon/2$, such that $D = \overline{B}(x, \delta) \subset U$. Then there exists $i \in \{1, 2\}$ such that $\mathrm{dist}(D, M_i) > 3\varepsilon$. Fix any point $q \in M_i$ and let $N > 0$ be provided to δ by the specification property. Then, for $j_1 = k_1 = 0$ and $j_2 = mN$, $k_2 = (m^2 + m)N$ there exists a point $z \in X$ such that $d(z, x) < \delta$ and $d(f^j(z), f^j(q)) < \delta$ for $j_2 \leq j \leq k_2$. But then we have $z \in D$ and furthermore

$$\begin{aligned}
&\#\{j : d(z, f^j(z)) \leq \varepsilon,\ 0 \leq j < N(m+1)m\} \\
&\qquad \leq \#\{j : \mathrm{dist}(f^j(z), M_i) \geq \varepsilon,\ 0 \leq j < N(m+1)m\} \leq mN < (m+1)N.
\end{aligned}$$

This shows that $z \in A_m$, proving that A_m is dense, and, as a result, the set A is residual. But $A \subset X \setminus \mathrm{W}(f)$ completing the proof. □

The next result can be derived from the works of Kuratowski and Mycielski (see, e.g. [13]). Let us recall that a set M is a *Mycielski set*, if it is a countable union of Cantor sets.

Theorem 3.3 (Mycielski). *Let Y be a compact metric space without isolated points and assume that Q is a residual subset of $Y \times Y$. Then there exists a Mycielski set M dense in Y, such that for any distinct $x, y \in M$ we have $(x, y) \in Q$.*

The following theorem, which is one of the main results of the paper, shows that the specification property implies distributional chaos.

Theorem 3.4. *Assume that the system (X, f) has the specification property, the map f is surjective and X has at least two points. Then there exist a constant $\delta > 0$ and a dense Mycielski set M, which is distributionally δ-scrambled, and, additionally, $M \subset \mathrm{Rec}(f) \setminus \mathrm{W}(f)$.*

PROOF. By Lemma 3.2 (X, f) has a distal pair. As a consequence, by Lemma 3.1 there is $\delta > 0$ such that the relation $\mathbf{DC}_\delta(f)$ is residual in $X \times X$. Each surjective map with the specification property is transitive, hence the set of recurrent points $\mathrm{Rec}(f)$ is residual in X. By Lemma 3.2 the set $X \times X \setminus \mathrm{W}(f) \times \mathrm{W}(f)$ is residual in $X \times X$. It is easy to check that any set containing at least two points and admitting a weakly mixing map must be perfect. Now it is enough to apply Theorem 3.3 to the set

$$R = (\mathbf{DC}_\delta(f) \cap \mathrm{Rec}(f) \times \mathrm{Rec}(f)) \setminus \mathrm{W}(f) \times \mathrm{W}(f). \qquad \square$$

Standing assumption. *For the reminder of this paper $X \subset \Sigma_2$ will be a subshift with the specification property such that $\#X \geq 2$, and r_0 and r_1 will be arbitrary real numbers (in each result we will specify their additional properties, if necessary).*

For any $\alpha \in \mathbb{R}$ denote by ρ_α the rotation by the angle α on $\mathbb{S}^1$, i.e.

$$\rho_\alpha(z) = e^{2\pi i \alpha} z \quad \text{for every } z \in \mathbb{S}^1.$$

For a compact metric space (X, d) we will also denote by d the product metric on $X \times S^1$, induced by the metric d on X and the Euclidean metric $|\cdot|$ on S^1 (we hope it will not lead to any confusion). Define a map $F\colon X \times \mathbb{S}^1 \to X \times \mathbb{S}^1$ by

$$F(x, t) = (\sigma(x), R_{x_0}(t)), \tag{3.1}$$

where $R_j(z) = \rho_{r_j}(z)$ for $j = 0, 1$ and every $z \in \mathbb{S}^1$. For simplicity of the notation, we will write $R_v = R_{v_n} \circ \ldots \circ R_{v_0}$ for any word $v = v_0 \ldots v_n \in L(X)$ ($v_i \in \{0, 1\}$).

At first we show the following lemma that supplies some simple consequence of the specification property in a subshift case.

LEMMA 3.5. *There exist words $u, v, w \in L(X)$ such that u is synchronizing, $|v| = |w|$, $|v|_0 > |w|_0$, and $uvu, uwu \in L(X)$.*

PROOF. By [3] the shift X is synchronized. Let u be a synchronizing word. First we claim that there are words p, q such that $|q| = |p| + 1$, $|p| > |p|_0 > 0$ and $upu, uqu \in L(X)$. By the specification property there exists a positive integer k, such that any two words can be joined by the word of the length k. There are words $a, b, c, d \in L(X)$ with $|a| = |b| = |c| = |d| = k$, such that $ua0b1c1du \in L(X)$ (note that $0, 1 \in L(X)$, since $\#X \geq 2$). Hence $ua0b1c \in L(X)$ and so we have a word $e \in L(X)$ with $|e| = k$, such that $ua0b1ceu \in L(X)$. Now it is enough to take $p = a0b1ce$ and $q = a0b1c1d$.

Denote $n = |p| + |u| = |up|$ and take the words $v = (pu)^n p$ and $w = (qu)^{n-1} q$. Then $|v| = |w|$ and, moreover, $uvu, uwu \in L(X)$ are also synchronizing words.

If $|up|_0 \geq |uq|_0$, then

$$|v|_0 = (n+1)|up|_0 - |u|_0 > n|up|_0 - |u|_0 \geq n|uq|_0 - |u|_0 = |w|_0.$$

In the other case we have

$$\begin{aligned}|w|_0 &= n|uq|_0 - |u|_0 = n|up|_0 + n(|uq|_0 - |up|_0) - |u|_0\\ &\geq n|up|_0 + n - |u|_0 > n|up|_0 + |up|_0 - |u|_0 = |v|_0.\end{aligned}$$

The proof is finished. □

The proof of Theorem 3.8 is based on the auxiliary lemmas presented below.

Lemma 3.6. *Let v, w be words such that $|w| = |v| \geq |v|_0 > |w|_0$ and assume that $r_0 - r_1 \in \mathbb{R} \setminus \mathbb{Q}$. Then for every $\delta > 0$ we can find $m > 0$ satisfying the following condition: for every $s, t \in \mathbb{S}^1$ there is $j \in \{0, \ldots, m\}$ such that*

$$|R_{v^j w^{m-j}}(t) - s| < \delta.$$

Proof. Put $k = |v|_0 - |w|_0 > 0$ and fix any $\delta > 0$. Let $U_1, \ldots, U_n$ be a finite cover of $\mathbb{S}^1$ by nonempty open sets with diameters less than $\delta/2$. Since $\rho_{k(r_0-r_1)}$ is topologically transitive, for each $i_1, i_2 \in \{1, \ldots, n\}$ there is $m(i_1, i_2)$ such that

$$\rho_{k(r_0-r_1)}^{m(i_1,i_2)}(U_{i_1}) \cap U_{i_2} \neq \emptyset.$$

Put $m = \max_{i_1,i_2 \in \{1,\ldots,n\}} m(i_1, i_2)$. Then for every $s, u \in \mathbb{S}^1$ there is $j \in \{1, \ldots, m\}$ such that $|\rho^j_{k(r_0-r_1)}(u) - s| < \delta$. But a simple calculation shows that for any $t \in \mathbb{S}^1$, if we put $u = \rho_{mr_1|v|+m(r_0-r_1)|w|_0}(t)$, then for any $0 \leq j \leq m$ we have

$$R_{v^j w^{m-j}}(t) = \rho^j_{k(r_0-r_1)}(u) = \rho_{jk(r_0-r_1)}(u),$$

which shows that j is as desired. □

Lemma 3.7. *Assume that $r_0 - r_1 \in \mathbb{R} \setminus \mathbb{Q}$. Then for every $\delta > 0$ there is a constant $N > 0$ such that for any $x \in X$, $p \in L(X)$ and $t, s \in \mathbb{S}^1$, there is a word $q \in L_N(X)$ such that $pqx \in X$ and $d(F^N(qx, t), (x, s)) < \delta$.*

Proof. Fix any $\delta > 0$. Let u, v, w be words provided by Lemma 3.5. Let m be an integer provided by Lemma 3.6 for δ and the words uv, uw. Let k be an integer provided by the specification property, so that any two words in $L(X)$ can be joined together by a word of length k. Put $N = 2k + m|uw| + |u|$, and fix any $x \in X$, $p \in L(X)$ and $t, s \in \mathbb{S}^1$. We claim that there are words $a, b \in L_k(X)$ such that $pau \in L(X)$ and $ubx \in X$. To see it, note that there exists a sequence of words $(b_n)_{n=1}^{\infty} \subset L_k(X)$ such that $ub_n x_{[0,n)} \in L(X)$. There are finitely many different words of length k, hence there is $b \in L_k(X)$ such that $b_n = b$ for infinitely many n, and so $ubx \in X$. Let $j \in \{0, \ldots, m\}$ be an integer provided by Lemma 3.6 for the points $R_a(t), R_{ub}^{-1}(s) \in \mathbb{S}^1$. The word u is synchronizing, hence if we put $q = \alpha(uv)^j(uw)^{m-j}ub$, then $q \in L_N(X)$, $pqx \in X$. Moreover,

$$\begin{aligned} d(F^N(qx,t),(x,s)) &= d((x,R_q(t)),(x,s)) \\ &= |R_q(t)-s| = |R_{ub}^{-1}(R_q(t)) - R_{ub}^{-1}(s)| \\ &= \left|R_{(uv)^j(uw)^{m-j}}(R_a(t)) - R_{ub}^{-1}(s)\right| < \delta. \end{aligned}$$

The proof is completed. □

Now we are ready to prove the main result of this paper.

THEOREM 3.8. *If $r_0 - r_1 \in \mathbb{R} \setminus \mathbb{Q}$ then the system $(X \times \mathbb{S}^1, F)$ has the specification property.*

PROOF. Fix any $\delta > 0$ and let $N > 0$ be a constant provided by Lemma 3.7. Let $\widehat{N} = N + k$, where k is the smallest positive integer, for which the following condition holds:

$$x_{[0,k)} = y_{[0,k)} \Rightarrow d(x,y) < \delta \text{ for every } x, y \in X.$$

Fix any integer $n \geq 2$, points $(x_1,t_1),\dots,(x_n,t_n) \in X \times \mathbb{S}^1$ and integers $0 = a_1 \leq b_1 < \dots < a_n \leq b_n$ with $a_{j+1} - b_j \geq \widehat{N}$ for all $j \in \{1,\dots,n-1\}$. We are going to apply Lemma 3.7 inductively. At first, we put $p_1 = (x_1)_{[0,a_2-N)}$ and apply Lemma 3.7 with $x = \sigma^{a_2}(x_2) = (x_2)_{[a_2,\infty)}$, $p = p_1$, $t = R_{p_1}(t_1)$ and $s = R_{(x_2)_{[0,a_2)}}(t_2)$, obtaining a word $q_1 \in L_N(X)$ such that $z_1 = p_1 q_1 \sigma^{a_2}(x_2) \in X$ and

$$d(F^N(q_1\sigma^{a_2}(x_2), R_{p_1}(t_1)), (\sigma^{a_2}(x_2), R_{(x_2)_{[0,a_2)}}(t_2))) < \delta.$$

Let us note that $b_1 + k \leq a_2 - \widehat{N} + k = a_2 - N$ and, consequently, $(z_1)_{[0,b_1+k)} = (x_1)_{[0,b_1+k)}$. Then, for any $j \in \{a_1,\dots,b_1\}$ we have $d(F^j(z_1,t_1), F^j(x_1,t_1)) < \delta$, and

$$\begin{aligned} |R_{p_1q_1}(t_1) - R_{(x_2)_{[0,a_2)}}(t_2)| &\leq d(F^{a_2}(z_1,t_1), F^{a_2}(x_2,t_2)) \\ &= d(F^N(q_1\sigma^{a_2}(x_2), R_{p_1}(t_1)), (\sigma^{a_2}(x_2), R_{(x_2)_{[0,a_2)}}(t_2))) < \delta. \end{aligned}$$

Now we apply Lemma 3.7 with $x = \sigma^{a_3}(x_3)$, $p = p_1q_1p_2$, $t = R_{p_1q_1p_2}(t_1)$ and $s = R_{(x_3)_{[0,a_3)}}(t_3)$, where $p_2 = (x_2)_{[a_2,a_3-N)}$. That way we obtain $q_2 \in L_N(X)$ such that $z_2 = p_1q_1p_2q_2\sigma^{a_3}(x_3) \in X$ and

$$d(F^N(q_2\sigma^{a_3}(x_3), R_{p_1q_1p_2}(t_1)), (\sigma^{a_3}(x_3), R_{(x_3)_{[0,a_3)}}(t_3))) < \delta.$$

Again, since sufficiently long prefix of z_2 coincides with a prefix of z_1 we obtain

$$d(F^j(z_2,t_1), F^j(x_1,t_1)) < \delta$$

for any $j \in \{a_1, \ldots, b_1\}$. Moreover,

$$\begin{aligned} &d(F^j(z_2,t_1),F^j(x_2,t_2)) \\ &\qquad = d(F^{j-a_2}(p_2q_2\sigma^{a_3}(x_3),R_{p_1q_1}(t_1)),F^{j-a_2}(\sigma^{a_2}(x_2),R_{(x_2)_{[0,a_2)}}(t_2))) < \delta \end{aligned}$$

for any $j \in \{a_2, \ldots, b_2\}$, because $\sigma^{j-a_2}(p_2q_2x_3)$ and $\sigma^j(x_2)$ have a common prefix of length at least k and

$$|R_{p_1q_1(x_2)_{[a_2,j)}}(t_1) - R_{(x_2)_{[0,j)}}(t_2)| = |R_{p_1q_1}(t_1) - R_{(x_2)_{[0,a_2)}}(t_2)| < \delta.$$

Additionally,

$$\begin{aligned} |R_{p_1q_1p_2q_2}(t_1) - R_{(x_3)_{[0,a_3)}}(t_3)| &= d(F^{a_3}(z_2,t_1),F^{a_3}(x_3,t_3)) \\ &= d(F^N(q_2\sigma^{a_3}(x_3),R_{p_1q_1p_2}(t_1)),(\sigma^{a_3}(x_3),R_{(x_3)_{[0,a_3)}}(t_3))) < \delta. \end{aligned}$$

Consequently, the point $(z_2,t_1) \in X \times \mathbb{S}^1$ δ-traces for $a_1 \le j \le b_1$ and $a_2 \le j \le b_2$ the points (x_1,t_1), (x_2,t_2). After finite number of applications of Lemma 3.7 we obtain a point $(z_n,t_1) \in X \times \mathbb{S}^1$ satisfying all the conditions required in the definition of the specification property (for the last step we can take the additional point $(x_{n+1},t_{n+1}) = (x_1,t_1)$ together with the integers $a_{n+1} = b_{n+1} = b_n + \widehat{N}$). □

If $r_0 - r_1 \in \mathbb{Q}$, the system $(X \times \mathbb{S}^1, F)$ does not have the specification property, because it is not even weakly mixing.

Proposition 3.9. *If $r_0 - r_1 \in \mathbb{Q}$, then the system $(X \times \mathbb{S}^1, F)$ is not weakly mixing.*

Proof. Without loss of generality we can assume that $0 \le r_0 \le r_1 < 1$ and $q := r_1 - r_0 \in [0, r_1] \cap \mathbb{Q}$. Then there exist an integer $n > 0$ and a collection $\mathcal{V} = \{(p_0,q_0), \ldots, (p_n,q_n)\}$ of open intervals, satisfying the following conditions:

(i) $0 = p_0 < q_0 < p_1 < q_1 < \ldots < p_n < q_n < 1$,

(ii) $q_i - p_i = q_j - p_j$ and $p_{i+1} - q_i = p_{j+1} - q_j$ for each pair of indices $i,j \in \{0, \ldots, n\}$, where for technical reasons we put $p_{n+1} := 1$,

(iii) $R_q(V) = V$, where $V = \bigcup_{i=0}^n \{e^{2\pi i\theta} : p_i < \theta < q_i\} \subset \mathbb{S}^1$,

(iv) there is $\alpha > 0$ such that if we denote $V' = R_\alpha(V)$ then for every $\theta \in \mathbb{R}$ we have $R_\theta(V) \cap V = \emptyset$ or $R_\theta(V) \cap V' = \emptyset$.

Since R_θ is an isometry, to ensure (iv) it is enough to demand in (ii) that $q_0 - p_0$ is sufficiently small.

Now let us note that regardless of an established sequence in X, we have guaranteed that under iterations of the map F the set V will always follow in the same way, i.e.

$$F^k(x,V) = F^k(y,V) \quad \text{for any } k > 0 \text{ and } x,y \in X \text{ satisfying } \sigma^k(x) = \sigma^k(y).$$

Indeed, by the condition (iii) we obtain

$$R_{r_1}(V) = R_{r_0+q}(V) = R_{r_0}(R_q(V)) = R_{r_0}(V),$$

which clearly ensures the above statement. Hence, for every $N > 0$ we have $F^N(X \times V) = X \times R_{nr_0}(V)$ and, consequently, by the choice of V and V', we see that $F^N(X \times V) \cap (X \times V) = \emptyset$ or $F^N(X \times V) \cap (X \times V') = \emptyset$ proving that the system $(X \times \mathbb{S}^1, F)$ is not weakly mixing. □

But even in the case $r_0 - r_1 \in \mathbb{Q}$ our map can still be transitive.

PROPOSITION 3.10. *If $\{r_0, r_1\} \cap \mathbb{R} \setminus \mathbb{Q} \neq \emptyset$ then the system $(X \times \mathbb{S}^1, F)$ is totally transitive.*

PROOF. By Theorem 3.8 we may assume that $r_0 - r_1 \in \mathbb{Q}$. Fix any nonempty open sets $U, V \subset X \times \mathbb{S}^1$ and integer $N > 0$. Without loss of generality we may assume that U and V are taken from the base of the product topology, i.e. $U = U_1 \times U_2$ and $V = V_1 \times V_2$. Then, there are words $u, v \in L(X)$ such that their cylinder sets satisfy $C_X[u] \subset U_1$ and $C_X[v] \subset V_1$. Since X is a subshift with the specification property (and hence synchronized), then taking a synchronizing word $w \in L(X)$ we can find words $a, b, c \in L(X)$ such that $uawbwcv \in L(X)$ and b contains both symbols 0 and 1. Note that by the specification property we can control lengths of the words a, b, and c, in particular we may assume that $|ua| = |wb| = |wc| = sN$ for some integer $s > 0$. Let us also notice that for any $k > 0$ we have $\alpha_k = ua(wb)^k wc \in L(X)$ and $\alpha_k v \in L(X)$.

Now, take any $t \in U_2$. Since R_{wb} is an irrational rotation by the assumption and

$$R_{\alpha_k}(t) = R_{ua} \circ R_{(wb)^k} \circ R_{wc}(t) = R_{wb}^k \circ R_{uawc}(t),$$

we have $R_{\alpha_k}(t) \in V_2$ for some $k > 0$. Hence, taking any $z \in X$ with the prefix $\alpha_k v$ and putting $m = |\alpha_k| = (k+2)|wb| = Ns(k+2)$, we obtain

$$(F^N)^{s(k+2)}(z, t) = F^m(z, t) \in F^m(U_1 \times U_2) \cap (V_1 \times V_2),$$

which completes the proof. □

The following statements complete the paper, supplying almost full description of a distributional chaos for the system $(X \times \mathbb{S}^1, F)$.

COROLLARY 3.11. *Assume that $r_0 - r_1 \in \mathbb{R} \setminus \mathbb{Q}$. Then there is a constant $\delta > 0$ such that the system $(X \times \mathbb{S}^1, F)$ has a dense Mycielski set $M \subset X \times \mathbb{S}^1$, which is distributionally δ-scrambled and, additionally, $M \subset \mathrm{Rec}(F) \setminus \mathrm{W}(f)$.*

PROOF. It is enough to combine Theorem 3.4 with Theorem 3.8. □

The proof of the next theorem uses standard techniques (see [14] and [17]), therefore we omit some details.

Corollary 3.12. *Assume that $r_0 - r_1 \in \mathbb{Q}$. Then there is a Cantor set $S \subset X$ and a constant $\gamma > 0$ such that for any $t \in \mathbb{S}^1$ the set $S \times \{t\}$ is distributionally γ-scrambled for the system $(X \times \mathbb{S}^1, F)$ and $S \times \{t\} \subset \operatorname{Rec}(F) \setminus \operatorname{W}(f)$.*

Proof. Since F acts in the same way on each fibre $X \times \{t\}$, the proof does not depend on $t \in \mathbb{S}^1$. Fix any distal pair $(p,q) \in X \times X$ (which exists by the specification property) and any positive constant $\gamma < \inf_{n \in \mathbb{N}} d(\sigma^n(p), \sigma^n(q))$. Let w be a synchronizing word and put $A = C_X[w] \times \{t\}$. Note that if we fix any two disjoint open sets $U, V \subset C_X[w]$, then there exist arbitrarily long words u and v of the same length satisfying $|u|_0 \neq |v|_0$, such that $wuw, wvw \in L(X)$, $C_X[wu] \subset U$, and $C_X[wv] \subset V$ (constructed as in Lemma 3.5).

Since $r_0 - r_1 \in \mathbb{Q}$, there exists $k > 0$ such that $R_{(wv)^k}(t) = R_{(wu)^k}(t)$. Then, taking $x, y \in U$ and $r, z \in V$ given by $x = (wu)^k (wu)^\infty$, $y = wuap$, $r = (wv)^k (wu)^\infty$ and $z = wvbq$, where the words a and b, satisfying $|a| = |b|$, are obtained by the specification property, we easily conclude that the relation $\mathbf{DC}_\gamma(F)$ is a residual subset of $A \times A$ (note that $\Phi^*_{(x,t)(r,t)}(s) = 1$ for any $s > 0$ and $\Phi_{(y,t)(z,t)}(\gamma) = 0$, and we can use the specification property to construct a pair that follows close to, alternately, $((x,t),(r,t))$ and $((y,t),(z,t))$, e.g. see [17] for more details).

Now, if we put $\alpha = (wu)^k$ and $\beta = (wv)^k$ then α^∞ and β^∞ belong to distinct periodic orbits. In particular, there are $\varepsilon > 0$ and an integer $m > 0$, such that if $x \in C_X[\alpha^m]$ and $y \in C_X[\beta^m]$ then for any $i, j \in \{0, \ldots, |\alpha| - 1\}$ we have $d(\sigma^i(x), \sigma^j(y)) > \varepsilon$. Hence, we can find a sequence of positive integers n_i such that

$$z = \alpha^m \beta^{n_1} \alpha^{n_2} \beta^{n_3} \alpha^{n_4} \beta^{n_5} \ldots \in \operatorname{Rec}(\sigma) \setminus \operatorname{W}(\sigma).$$

But for any $s \in \mathbb{S}^1$ we have $R_\alpha(s) = R_\beta(s)$ and, since R_α is a rotation (rational or irrational), return times (to any fixed neighbourhood) of points recurrent under R_α synchronize with return times of any recurrent point (and its neighborhood) in any dynamical system; it is, so-called, product recurrence (see, e.g. [10] for more details). In particular, the point (z,t) is recurrent for $\sigma^{|\alpha|} \times R_\alpha$ and, as a consequence, $(z,t) \in \operatorname{Rec}(F) \setminus \operatorname{W}(f)$ (note that $F^{n|\alpha|}(z,t) = (\sigma^{|\alpha|} \times R_\alpha)^n(z,t)$). This proves that the set $\operatorname{Rec}(F) \setminus \operatorname{W}(f) \cap A$ is dense in A. Now, repeating the argument from the proof of Lemma 3.2, we can verify that $\operatorname{Rec}(F) \setminus \operatorname{W}(f)$ is a G_δ set in A, thus it is residual in A. Then it is easily seen that also the set

$$\mathbf{DC}_\gamma(F) \cap (\operatorname{Rec}(F) \setminus \operatorname{W}(f)) \times (\operatorname{Rec}(F) \setminus \operatorname{W}(f)) \cap A \times A$$

is residual in $A \times A$. The proof is finished by an application of Theorem 3.3. $\square$

A transitive system with a dense set of periodic points is called *Devaney chaotic*, after the definition that appeared in Devaney's book [6]. It may happen, however, that a system with the specification property does not have periodic points (see, e.g. [12, Example 5.6]). This motivates further considerations on the density of the set of periodic points for the system $(X \times \mathbb{S}^1, F)$.

EXAMPLE 3.13. Assume that r_0 and r_1 are rationally independent irrational numbers, i.e. $mr_0 + nr_1 \in \mathbb{R} \setminus \mathbb{Q}$ for every $m, n \in \mathbb{Z}$, where at least one of m, n is not zero. Then the system $(X \times \mathbb{S}^1, F)$ does not have periodic points, since on the second coordinate we always have an irrational rotation of the initial point.

Then a necessary condition for a periodic point to exist, is that there are positive integers m and n such that $mr_0 + nr_1 \in \mathbb{N}$. Note that if we want $(X \times \mathbb{S}^1, F)$ to be additionally transitive, then one of r_0, r_1 (hence both) has to be irrational. But then simple calculations yield that $r_0 - r_1 \in \mathbb{R} \setminus \mathbb{Q}$ and so, by Theorem 3.8, we are in the case when the system $(X \times \mathbb{S}^1, F)$ has the specification property.

For example, if X is the full shift, $r_0 \in (0, 1) \setminus \mathbb{Q}$ and $r_1 = (1 - r_0)/2$, then for any word $w \in X$ we can find a word $v \in X$ such that for any $t \in \mathbb{S}^1$ the point $((wv)^\infty, t)$ is periodic (simply, v must be such word that wv contains twice that many the symbols 1 as the symbols 0). However, if we keep the same r_0 and r_1, but as X we take a subshift with the specification property such that every occurrence of 1 is separated by at least two occurrences of 0, then the above assignment for r_0 and r_1 will not lead to a periodic point for $(X \times \mathbb{S}^1, F)$.

The above example inspires the following question.

QUESTION 3.14. When does the system $(X \times \mathbb{S}^1, F)$ defined by (3.1) possess a dense set of periodic points?

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