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DYNAMICS OF NON-AUTONOMOUS REACTION-DIFFUSION EQUATIONS IN LOCALLY UNIFORM SPACES

Gaocheng Yue — Chengkui Zhong

Abstract. In this paper, we first prove the well-posedness for the non-autonomous reaction-diffusion equations on the entire space $\mathbb{R}^N$ in the setting of locally uniform spaces with singular initial data. Then we study the asymptotic behavior of solutions of such equation and show the existence of $(H_U^{1,q}(\mathbb{R}^N), H_\phi^{1,q}(\mathbb{R}^N))$-uniform(w.r.t. $g \in \mathcal{H}_{L_U^q(\mathbb{R}^N)}(g_0)$) attractor $\mathcal{A}_{\mathcal{H}_{L_U^q(\mathbb{R}^N)}(g_0)}$ with locally uniform external forces being translation uniform bounded but not translation compact in $L_b^p(\mathbb{R}; L_U^q(\mathbb{R}^N))$. We also obtain the uniform attracting property in the stronger topology.

1. Introduction

In this paper, we consider the long-time dynamical behavior of solutions for the following non-autonomous reaction-diffusion equations on $\mathbb{R}^N$:

$$u_t - \Delta u + f(x,u) = g(x,t), \quad \text{in } \mathbb{R}^N \times [\tau, \infty), \tag{1.1}$$

with the initial condition

$$u(x,\tau) = u_\tau(x), \quad x \in \mathbb{R}^N, \tag{1.2}$$

where $f\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a suitable continuous function and g is given the external force.

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Our goal is to prove that under suitable conditions there exist the uniform attractor of reaction-diffusion equation with singular initial data and forcing term $g(x,t)$ being translation uniform bounded but not translation compact in an appropriate sense.

The nonlinear term f is assumed to satisfy the following assumptions:

ASSUMPTION I. Assume that f satisfies $f(x,0)=0$ and there exist constants C and ρ with $C>0, \rho>1$ such that

$$|f(x,u)-f(x,v)| \le C|u-v|\left(1+|u|^{\rho-1}+|v|^{\rho-1}\right), \quad \text{for all } u,v\in\mathbb{R}. \tag{1.3}$$

ASSUMPTION II. There exist two positive constants μ_0, μ_2 such that

$$-\mu_0 s^2-\mu_2|s| \le sf(x,s), \quad \text{for all } s\in\mathbb{R},\ x\in\mathbb{R}^N.$$

We furthermore assume that the external force $g(\,\cdot\,,t)\in \dot{L}^q_U(\mathbb{R}^N)$ with $1<q<N$ for almost every $t\in\mathbb{R}$ and g has finite norm in the space $L^p_b(\mathbb{R},\dot{L}^q_U(\mathbb{R}^N))$ with $p>2$, i.e.

$$\|g\|^p_{L^p_b(\mathbb{R},\dot{L}^q_U(\mathbb{R}^N))}=\sup_{t\in\mathbb{R}}\int_t^{t+1}\|g(\,\cdot\,,s)\|^p_{\dot{L}^q_U(\mathbb{R}^N)}\,ds<+\infty.$$

The main contribution of the present paper is to extend the external force to the case where $g(x,t)$ is considered in much larger space $L^p_b(\mathbb{R},\dot{L}^q_U(\mathbb{R}^N))$ and with weaker assumptions.

The long-time behavior of the solutions of (1.1) is of great current interest. It is well-known that this behavior of solutions of such equations arise from mathematical physics can be described as the existence of the so-called attractors of the corresponding semigroups (or process). In particular, when the domain is bounded, the global attractor of such problems have been extensively studied by using many different methods in the literature (see, e.g. [6], [13], [19], [23], [29], [32], [35], [37] and the references therein).

In contrast to this, the case of unbounded domain becomes much more difficulty and some of methods for bounded domains are on longer valid. The main difficulty lies in the absence of the standard Sobolev compact embedding.

On the other hand, the disadvantage of the standard Lebesgue spaces $L^{p_1}(\mathbb{R}^N)$ $(1\le p_1<\infty)$ is that it does not consider the behavior of the solutions for large spatial values and the family of such spaces are not nested. In order to overcome these difficulties, one should introduce an appropriate functional spaces to study these partial differential equations. With respect to the use of weighted Sobolev spaces, one of the pioneer works is in [7], where Babin and Vishik for the first time introduced weighted Sobolev space as the phase space and showed, under appropriate assumptions, the existence of global attractors for parabolic type evolutionary equations on unbounded domain, however, which requires the initial data and forcing term belonging to the corresponding weighted spaces (see also [1]). Later on, in [5], the authors also proved the compactness of the nonlinear semigroup and the existence of global attractor in weighted Sobolev spaces. It is worth noting that the usual Sobolev spaces $L^{p_1}(\mathbb{R}^N)$ $(1 \le p_1 < \infty)$ do not include the constant solutions and traveling wave connections between two equilibria. For reaction-diffusion equations $u_t = \Delta u + u - u^3$, the solutions $u = \pm 1$ and the traveling wave connections between $u = 0$ and $u = \pm 1$ are no longer included the Sobolev spaces like $L^{p_1}(\mathbb{R}^N)$ $(1 \le p_1 < \infty)$, for example, see [21]. Hence, in [3], [4], [8], [18], [33], the authors introduced locally uniform spaces as the phase space to include these special solutions in the global attractors. The idea of the locally uniform spaces can be traced back to Kato [22]. In [3], J. Arrieta et al. systematically study some properties and embedding relationship of locally uniform spaces. These spaces enjoys suitable nesting properties (e.g. if $p_1 \le q_1$, then $L^{q_1}_U(\mathbb{R}^N) \hookrightarrow L^{p_1}_U(\mathbb{R}^N)$) and have locally compact embeddings.

We should mention that the methods introduced by Arrieta, Moya and Rodríguez–Bernal in [5] will play a crucial role in our work. In [5], by using weighted Sobolev spaces, the authors systematically study the properties of these spaces, including Sobolev type embeddings and the weighted $L^p - L^q$ estimates for heat kernel. Then the authors prove the global existence and regularity of the following reaction-diffusion equations in the space $L^q_\rho(\mathbb{R}^N)$ with $1 < q < \infty$,

$$\begin{cases} \partial_t u - \Delta u = f(x,u), \quad (x,t) \in \mathbb{R}^N \times (0,+\infty), \\ u(0) = u_0. \end{cases}$$

and construct, under quite general assumptions with respect to f the existence of global attractor. In particular, the authors in [15] established a global attractor $\mathcal{A}$ and showed that $\mathcal{A}$ is contained in an ordered interval $[\phi_m, \phi_M]$, where $\phi_m, \phi_M \in \mathcal{A}$ is a pair of stationary solutions. We also refer to [16]. In [24], the authors studied asymptotic behavior of solutions of non-autonomous parabolic equations with singular initial data in the bounded domain of $\mathbb{R}^N$.

The motivation of this paper is to give a detailed study of the uniform attractor of reaction-diffusion equation with singular initial data and forcing term $g(x,t)$ being translation uniform bounded but not translation compact in $L^p_b(\mathbb{R}; L^q_U(\mathbb{R}^N))$.

The rest of this paper is organized as follows. In Section 2 we include some preliminaries concerning locally uniform spaces and uniform Bessel spaces, their embeddings, basic notations of non-autonomous dynamical systems. In Section 3 we consider the well-posedness of the linear non-autonomous reaction-diffusion equations, i.e. $f(x,u) = -\mu_0 u - \mu_2$, and prove a comparison principle in locally uniform spaces which plays an important role in obtaining the existence of uniform attractor. In Section 4 we prove the existence and uniqueness of solutions for problem (1.1)–(1.2). In Section 5, we consider the global well-posedness of problem (1.1)–(1.2) in $H_U^{1,q}(\mathbb{R}^N)$. Motivated by ideas of [14], [18], [33] (see also, for instance, [36], [34]), we obtain continuous property of a family of process in $H_\phi^{1,q}(\mathbb{R}^N)$ by extension solutions obtained in Section 4. In Section 6, by introducing the uniform normal forcing we show the asymptotic compactness properties of the family of processes and obtain the existence and structure of $(H_U^{1,q}(\mathbb{R}^N), H_\phi^{1,q}(\mathbb{R}^N))$-uniform attractor that attracts bounded subset of $H_U^{1,q}(\mathbb{R}^N)$ in the topology of $H_\phi^{1,q}(\mathbb{R}^N)$, in fact, we obtain the attracting property in the topology of $H_\phi^{2\alpha,q}(\mathbb{R}^N)$, $1/2 \le \alpha < 1$. This paper is complemented by some properties of uniform normal functions, stated in Appendix A.

Throughout this paper we will use C to denote positive constants that are independent of the parameter α. They may change from line to line or/and depend on the weight function. In particular, when a constant C depends on some particular parameters, say a, b, c, we shall denote it by $C_{a,b,c}$.

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2. Preliminaries

For completeness, we will recall some functional spaces which will be used throughout of the paper. For more details, see, e.g. [3], [4], [8], [18], [33].

Definition 2.1. A function $\phi\colon \mathbb{R}^N \to (0,+\infty)$ is said to be strictly positive integrable weighted function of class $\mathcal{C}^2(\mathbb{R}^N)$ if for two positive constants ϕ_0, ϕ_1, and $x \in \mathbb{R}^N$, $j,k = 1,\ldots,N$, the following three conditions are satisfied:

(a) $\left|\dfrac{\partial\phi}{\partial x_j}(x)\right| \le \phi_0\phi(x)$,

(b) $\left|\dfrac{\partial^2\phi}{\partial x_j\partial x_k}(x)\right| \le \phi_1\phi(x)$, and

(c) $\displaystyle\int_{\mathbb{R}^N} \phi(x)\,dx < \infty$.

In particular, according to the above definition, we can choose

$$\phi(x) = (1+\varepsilon_0|x|^2)^{-\gamma}, \quad \gamma > \frac{N}{2},\ \varepsilon_0 > 0, \tag{2.1}$$

as a strictly positive integrable weighted function of class $\mathcal{C}^2(\mathbb{R}^N)$ satisfying the estimates $|\nabla\phi| \le C\sqrt{\varepsilon_0}\phi$ and $|\Delta\phi| \le C\varepsilon_0\phi$, and the integral property holds obviously. This weight function has the following properties:

Proposition 2.2. *There exist positive constants C_1, C_2 such that*

$$C_1\phi(x) \le \inf_{y\in B(x,1)} \phi(y) \le \sup_{y\in B(x,1)} \phi(y) \le C_2\phi(x) \quad \text{for all } x \in \mathbb{R}^N,$$

where $B(x,1) = \{y \in \mathbb{R}^N \mid |y-x| \le 1\}$.

Thus, similar to Proposition 1.2 of [34], it follows

PROPOSITION 2.3. *There exist positive constants C_1, C_2 such that for all $u \in L^p_\phi(\mathbb{R}^N)$ with $1 \le p < \infty$,*

$$C_1 \int_{\mathbb{R}^N} |u(x)|^p \phi(x)\,dx \le \int_{\mathbb{R}^N} \phi(y) \int_{B(y,1)} |u(x)|^p\,dx\,dy \le C_2 \int_{\mathbb{R}^N} |u(x)|^p \phi(x)\,dx.$$

For a weight function $\phi(x)$ defined as like (2.1), we define the weighted Sobolev spaces as follows, see also [33].

DEFINITION 2.4.

(a) For $1 \le p < \infty$, define the weighted Sobolev space with weight $\phi(x)$ as

$$L^p_\phi(\mathbb{R}^N) := \left\{ u \in L^p_{\mathrm{loc}}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u(x)|^p \phi(x)\,dx < \infty \right\},$$

with the norm

$$\|u\|_{L^p_\phi(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} |u(x)|^p \phi(x)\,dx \right)^{1/p}.$$

(b) For $p = \infty$, define the weighted Sobolev space with weight $\phi(x)$ by

$$L^\infty_\phi(\mathbb{R}^N) := \left\{ u \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N) : \sup_{x \in \mathbb{R}^N} |u(x)|\phi(x) < \infty \right\},$$

with the norm

$$\|u\|_{L^\infty_\phi(\mathbb{R}^N)} = \sup_{x \in \mathbb{R}^N} |u(x)|\phi(x).$$

Analogously, the weighted Sobolev space $W^{k,p}_\phi(\mathbb{R}^N)$, $k \in N$ is defined as the space of distribution whose derivatives up to the order k inclusively belong to $L^p_\phi(\mathbb{R}^N)$, with the norm

$$\|u\|_{W^{k,p}_\phi(\mathbb{R}^N)} = \sum_{|\alpha| \le k} \|D^\alpha u\|_{L^p_\phi(\mathbb{R}^N)} < \infty.$$

We define also, for $1 \le p < \infty$, the locally uniform space $L^p_U(\mathbb{R}^N)$ as the set of functions $u \in L^p_{\mathrm{loc}}(\mathbb{R}^N)$ such that

$$\sup_{x \in \mathbb{R}^N} \int_{B(x,1)} |u(y)|^p\,dy < \infty,$$

where $B(x,1)$ denote the unit ball centered at x, with norm

$$\|u\|_{L^p_U(\mathbb{R}^N)} = \sup_{x \in \mathbb{R}^N} \|u\|_{L^p(B(x,1))}.$$

In a similar way, for $p = \infty$, we have $L^\infty_U(\mathbb{R}^N) = L^\infty(\mathbb{R}^N)$ endowed with

$$\|u\|_{L^\infty_U(\mathbb{R}^N)} = \sup_{x \in \mathbb{R}^N} \|u\|_{L^\infty(B(x,1))} = \|u\|_{L^\infty(\mathbb{R}^N)}.$$

From the definition we can easily know that $L^p_U(\mathbb{R}^N)$ contains $L^\infty(\mathbb{R}^N)$, $L^r(\mathbb{R}^N)$ and $L^r_U(\mathbb{R}^N)$ for any $r \geq p$.

In order to consider the non-autonomous reaction diffusion equations in much larger spaces with initial data in the uniform space $L^q_U(\mathbb{R}^N)$ for $1 \leq q \leq \infty$, we denote by $\dot{L}^q_U(\mathbb{R}^N)$ the closed subspace of $L^q_U(\mathbb{R}^N)$ consisting of all elements which are translation continuous with respect to $\|\cdot\|_{L^q_U(\mathbb{R}^N)}$ under the action of the group of translations $\{\tau_y, y \in \mathbb{R}^N\}$ by $\tau_y\phi(\,\cdot\,) := \phi(\,\cdot - y)$, that is

$$\|\phi_y(x) - \phi(x)\|_{L^q_U(\mathbb{R}^N)} := \|\tau_y\phi(x) - \phi(x)\|_{L^q_U(\mathbb{R}^N)} \to 0 \quad \text{as } |y| \to 0,$$

and the uniform spaces $W^{m,p}_U(\mathbb{R}^N)$ and $\dot{W}^{m,p}_U(\mathbb{R}^N)$ can be defined, respectively, by $L^p_U(\mathbb{R}^N)$ and $\dot{L}^p_U(\mathbb{R}^N)$ in a similar way as the definition of $W^{k,p}_\phi(\mathbb{R}^N)$.

We recall the locally uniform spaces. For $1 \leq p < \infty$, define the weighted Sobolev space with weight $\phi(x)$ as

$$L^p_{lu}(\mathbb{R}^N) := \Big\{u \in L^p_{\rm loc}(\mathbb{R}^N) : \|u\|_{L^p_{lu}(\mathbb{R}^N)} = \sup_{y\in\mathbb{R}^N} \|u\|_{L^p_{\phi_y}(\mathbb{R}^N)} < \infty\Big\},$$

and its the closed subspace

$$\dot{L}^p_{lu}(\mathbb{R}^N) := \{u \in L^p_{lu}(\mathbb{R}^N) : \|\tau_y u - u\|_{L^p_{lu}(\mathbb{R}^N)} \to 0 \text{ as } |y| \to 0\},$$

and the definition of $W^{k,p}_{lu}(\mathbb{R}^N)$ and $\dot{W}^{k,p}_{lu}(\mathbb{R}^N)$ can be carry out by using the standard way.

If the weight function ϕ satisfies (2.1), then we know from [3] there exist positive constants C_1, C_2 such that the following norms are equivalent:

$$\|u\|^p_{L^p_{lu}(\mathbb{R}^N)} \leq C_1\|u\|^p_{L^p_U(\mathbb{R}^N)} \leq C_2\|u\|^p_{L^p_{lu}(\mathbb{R}^N)}, \quad u \in L^p_{lu}(\mathbb{R}^N). \tag{2.2}$$

This equivalent norms imply that for $k \in \mathbb{N} \cup \{0\}$, the spaces $W^{k,p}_U(\mathbb{R}^N)$ and $W^{k,p}_{lu}(\mathbb{R}^N)$ coincide algebraically and topologically when the weight function ϕ satisfies (2.1), and we recall some well-known embeddings of these spaces:

Lemma 2.5.

(a) *If $s_1 \geq s_2 \geq 0$, $1 < p_1 \leq p_2 < \infty$ and $s_1 - N/p_1 \geq s_2 - N/p_2$, then*

$$W^{s_1,p_1}_U(\mathbb{R}^N) \hookrightarrow W^{s_2,p_2}_U(\mathbb{R}^N)$$

is continuous. Moreover, assume that $1 < p, p_1, p_2 < \infty$, $1/p \leq \theta/p_1 + (1-\theta)/p_2$, $0 \leq \theta \leq 1$ and

$$s - \frac{N}{p} \leq \theta\bigg(s_1 - \frac{N}{p_1}\bigg) + (1-\theta)\bigg(s_2 - \frac{N}{p_2}\bigg),$$

then there is a $C > 0$ such that one has

$$\|u\|_{W^{s,p}_U(\mathbb{R}^N)} \leq C\|u\|^\theta_{W^{s_1,p_1}_U(\mathbb{R}^N)}\|u\|^{1-\theta}_{W^{s_2,p_2}_U(\mathbb{R}^N)}.$$

(b) *If the weight function* ϕ *satisfies* (2.1), *then the inclusion* $W_U^{s_1,p_1}(\mathbb{R}^N) \hookrightarrow W_\phi^{s_2,p_2}(\mathbb{R}^N)$ *is compact provided that* $s_2 \in \mathbb{N}$, $s_1 > s_2$, $1 < p_1 \le p_2 < \infty$ *and* $s_1 - N/p_1 > s_2 - N/p_2$.

For the fractional power operators $(-\Delta + I)^\alpha$, one has from [3]

LEMMA 2.6. *Let* $0 \le \alpha \le 1$. *Then there exist positive constants* $C_{1,\alpha}$ *and* $C_{2,\alpha}$ *such that*

$$(2.3) \quad C_{1,\alpha}\|u\|^2_{W_U^{2\alpha,2}(\mathbb{R}^N)} \le \|u\|^2_{L^2_U(\mathbb{R}^N)} + \|(-\Delta)^\alpha\|^2_{L^2_U(\mathbb{R}^N)} \le C_{2,\alpha}\|u\|^2_{W_U^{2\alpha,2}(\mathbb{R}^N)},$$

for all $u \in W_U^{2\alpha,2}(\mathbb{R}^N)$.

Choosing $\dot{L}^q_U(\mathbb{R}^N)$ with $1 < q < \infty$ as a base space, the unbounded linear operator $\Delta\colon D(\Delta) \subset \dot{L}^q_U(\mathbb{R}^N) \to \dot{L}^q_U(\mathbb{R}^N)$ with the domain $D(\Delta) = \dot{H}^{2,q}_U(\mathbb{R}^N)$ generates an analytic semigroup $e^{\Delta t}$ in $\dot{L}^q_U(\mathbb{R}^N)$, see [2], [30]. The uniform Bessel–Sobolev space $H^{k,p}_U(\mathbb{R}^N)$ is defined as the set of functions $\phi \in H^{k,p}_{\mathrm{loc}}(\mathbb{R}^N)$ such that

$$\|\phi\|_{H^{k,p}_U(\mathbb{R}^N)} = \sup_{x\in\mathbb{R}^N} \|\phi\|_{H^{k,p}(B(x,1)} < \infty,$$

for $k \in N$ and then denote by $\dot{H}^{k,p}_U(\mathbb{R}^N)$ a subspace of $H^{k,p}_U(\mathbb{R}^N)$ consisting of all elements which are translation continuous w.r.t. $\|\cdot\|_{H^{k,p}_U(\mathbb{R}^N)}$, that is,

$$\|\tau_y\phi - \phi\|_{H^{k,p}_U(\mathbb{R}^N)} \to 0 \quad \text{as } |y| \to 0.$$

We denote by $\dot{H}^{s,q}_U(\mathbb{R}^N)$ the Bessel potentials spaces which coincide with the uniform spaces $\dot{W}^{s,q}_U(\mathbb{R}^N)$ for integer s if $1 < q < \infty$ or for all s if $q = 2$. By using the complex interpolation-extrapolation procedure, one can construct the fractional power uniform spaces associated to the operator Δ, which will be denoted $X^\alpha := \dot{H}^{2\alpha,q}_U(\mathbb{R}^N)$ for $0 \le \alpha \le 1$, which is the closed subspaces of $H^{s,q}_U(\mathbb{R}^N)$, where

$$\dot{H}^{2\alpha,q}_U(\mathbb{R}^N) = [\dot{H}^{2,q}_U(\mathbb{R}^N), \dot{L}^q_U(\mathbb{R}^N)]_\alpha.$$

Note that there is no results on the negative part of the scale of uniform Bessel spaces when $\alpha < 0$. Moreover, the fractional power spaces are given by X^α which satisfy the following sharp embedding of Bessel spaces:

$$(2.4) \quad \dot{H}^{s,q}_U(\mathbb{R}^N) \subset \begin{cases} \dot{L}^r_U(\mathbb{R}^N),\ s - \dfrac{N}{q} \ge -\dfrac{N}{r}, 1 \le r < \infty, & \text{if } s - \dfrac{N}{q} < 0, \\ \dot{L}^r_U(\mathbb{R}^N),\ 1 \le r < \infty, & \text{if } s - \dfrac{N}{q} = 0, \\ C^\eta_b(\mathbb{R}^N), & \text{if } s - \dfrac{N}{q} > \eta > 0. \end{cases}$$

Thus, for some constant λ, the operator $\Delta - \lambda I$ generates an analytic semigroup $e^{(\Delta-\lambda I)t}$ in each uniform spaces $\dot{H}^{2\alpha,q}(\mathbb{R}^N)$ with $0 \le \alpha < \infty$ and satisfies the smooth estimates for some $\mu \in \mathbb{R}$

$$\|e^{(\Delta-\lambda I)t}\|_{\mathcal{L}(X^\beta,X^\alpha)} \leq \frac{M_{\alpha,\beta}e^{\mu t}}{t^{\alpha-\beta}}, \quad 0 \leq \beta \leq \alpha < 1. \tag{2.5}$$

We now state the singular Gronwall lemma used below (cf. Lemma 7.1.1 in [21]).

LEMMA 2.7 (Gronwall–Henry inequality). *Let $v(t)$ be a nonnegative function in $L^\infty_{\mathrm{loc}}[0,\tau;\mathbb{R})$ satisfy the inequalities*:

$$v(t) \leq at^{c-1} + M\int_0^t (t-s)^{r-1}v(s)\,ds, \quad t \in (0,\tau),$$

where a and c are positive constants, $M \geq 0$, $0 < \tau \leq \infty$ and $r > 0$. Then there exists a positive constant $C_{r,c}$ such that

$$v(t) \leq C_{r,c}at^{c-1}E_{r,c}(\mu t), \quad t \in (0,\tau),$$

where $E_{r,c}(z) = \sum_{n=0}^{\infty} \frac{\Gamma(c)}{\Gamma(nr+c)}z^{nr}$, $\mu = (M\Gamma(r))^{1/r}$ and $\Gamma(\,\cdot\,)$ is the Gamma function.

Here, for our purpose, we only recall a special case of the Gronwall-Henry inequality, a general case and detailed proof of this inequality can be found in Henry [21], Lemma 7.1.1.

To proceed with our investigation we introduce the concept of attractor of our interest. The definitions we state below are taken from [9], [11], [12].

We now consider a family of equations of the following abstract form with symbols $\sigma(t)$ from the hull $\mathcal{H}(\sigma_0)$ of the symbol $\sigma_0(t)$:

$$\begin{cases} \partial_t u = A_{\sigma(t)}(u), & \sigma \in \mathcal{H}(\sigma_0), \\ u|_{t=\tau} = u_\tau. \end{cases} \tag{2.6}$$

For simplicity, we assume that the set $\mathcal{H}(\sigma_0)$ is a complete metric space. We suppose that, for every symbol $\sigma \in \mathcal{H}(\sigma_0)$, the Cauchy problem (2.6) has a unique solution for any $\tau \in \mathbb{R}$ and for every initial condition $u_\tau \in E$. Thus, we have the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \mathcal{H}(\sigma_0)$ acting in the Banach space E. Then, the solution $u(t) \in E$ of the problem (2.6) can be represented as

$$u(t) = U_\sigma(t,\tau)u_\tau, \quad t \geq \tau,\ \tau \in \mathbb{R},\ \sigma \in \mathcal{H}(\sigma_0).$$

By the unique solvability of problem (2.6), the operators $\{U_\sigma(t,\tau)\}$ posses the following multiplicative properties:

(1) $U_\sigma(\tau,\tau) = \mathrm{Id}$ for all $\tau \in \mathbb{R}$, where Id is the identity operator,
(2) $U_\sigma(t,s) \circ U_\sigma(s,\tau) = U_\sigma(t,\tau)$ for all $t \geq s \geq \tau$, $\tau \in \mathbb{R}$.

We note that the following translation identity holds for the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \mathcal{H}(\sigma_0)$ corresponding to (2.6):

$$U_{T(h)\sigma}(t,\tau) = U_\sigma(t+h,\tau+h), \quad \text{for all } h \geq 0,\ t \geq \tau,\ \tau \in \mathbb{R}, \tag{2.7}$$

where $T(h)\sigma(t) = \sigma(t+h)$.

The family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \mathcal{H}(\sigma_0)$ is called $(E \times \mathcal{H}(\sigma_0), E)$-*weakly continuous* if for any t and τ, $t \geq \tau$ the mapping $(u,\sigma) \to U_\sigma(t,\tau)u$ is weakly continuous from $E \times \mathcal{H}(\sigma_0)$ to E. By $\mathcal{B}(E)$ we denote the family of all bounded (in the norm of E) sets in E. The Kuratowski measure of noncompactness of a bounded set $B \in \mathcal{B}(E)$ is defined as

$$\kappa(B) = \inf\{\delta : B \text{ is covered by finitely many sets of diameter less than } \delta\}.$$

Further straightforward properties can be find in [19]. The family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \mathcal{H}(\sigma_0)$ is said to be *uniformly* (w.r.t. $\sigma \in \mathcal{H}(\sigma_0)$) ω*-limit compact* if for any $\tau \in \mathbb{R}$ and $B \in \mathcal{B}(E)$, $B_t := \bigcup_{\sigma \in \mathcal{H}(\sigma_0)} \bigcup_{s \geq t} U_\sigma(s,\tau)B$ is bounded for every t and $\lim_{t\to\infty} \kappa(B_t) = 0$. A set $B_0 \subset E$ is said to be *uniformly* (w.r.t. $\sigma \in \mathcal{H}(\sigma_0)$) *absorbing* for the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \mathcal{H}(\sigma_0)$ if for any $\tau \in \mathbb{R}$ and $B \in \mathcal{B}(E)$, there exists $t_0 = t_0(\tau, B) \geq \tau$ such that for all $t \geq t_0$,

$$\bigcup_{\sigma \in \mathcal{H}(\sigma_0)} U_\sigma(t,\tau)B \subseteq B_0.$$

A set $P \subset E$ is said to be *uniformly* (w.r.t. $\sigma \in \mathcal{H}(\sigma_0)$) *attracting* for the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \mathcal{H}(\sigma_0)$ if for every set $B \in \mathcal{B}(E)$ and an arbitrary fixed $\tau \in \mathbb{R}$,

$$\lim_{t\to\infty}\left(\sup_{\sigma \in \mathcal{H}(\sigma_0)} \operatorname{dist}_E(U_\sigma(t,\tau)B, P)\right) = 0,$$

where $\operatorname{dist}_E(X,Y) = \sup_{x \in X} \inf_{y \in Y} \|y - x\|_E$, $X, Y \subseteq E$ is called the Hausdorff (non-symmetric) distance from the set X to the set Y.

We now recall the notion of the uniform attractor $\mathcal{A}_{\mathcal{H}(\sigma_0)}$.

Definition 2.8. A closed set $\mathcal{A}_{\mathcal{H}(\sigma_0)} \subset E$ is called the uniform (w.r.t. $\sigma \in \mathcal{H}(\sigma_0)$) attractor of the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \mathcal{H}(\sigma_0)$, if

(a) it is uniformly (w.r.t. $\sigma \in \mathcal{H}(\sigma_0)$) attracting (*attracting property*);
(b) it is contained in any closed uniformly (w.r.t. $\sigma \in \mathcal{H}(\sigma_0)$) attracting set $\mathcal{A}'$, that is $\mathcal{A}_{\mathcal{H}(\sigma_0)} \subseteq \mathcal{A}'$ (*minimality property*).

To describe the general structure of the uniform global attractor of the family of processes, we need the notion of the kernel of the process that generalizes the notion of a kernel of a semigroup. A function $u(s)$, $s \in \mathbb{R}$ with values in E is said to be a complete trajectory of the process $\{U_\sigma(t,\tau)\}$ if

$$U_\sigma(t,\tau)u(\tau) = u(t), \quad \text{for all } t \geq \tau,\ \tau \in \mathbb{R}. \tag{2.8}$$

A complete trajectory $u(s)$ is called *bounded* if the set $\{u(s); s \in \mathbb{R}\}$ is bounded in E.

Definition 2.9. The kernel $\mathcal{K}_\sigma$ of the process $\{U_\sigma(t,\tau)\}$ is the family of all bounded complete trajectories of this process:

$$\mathcal{K}_\sigma = \{u(\,\cdot\,)|u \text{ satisfies (2.8) and } \|u(s)\|_E \le C_u \text{ for all } s \in \mathbb{R}\}$$

The set $\mathcal{K}_\sigma(t) = \{u(t)|u(\,\cdot\,) \in \mathcal{K}_\sigma\} \subset E$, $t \in \mathbb{R}$ is called the *kernel section* at time t.

3. Linear problem

This section is devoted to study the following linear non-autonomous reaction diffusion equations of the type (1.1)–(1.2),

$$\begin{cases} u_t - \Delta u - \mu_0 u - \mu_2 = g(x,t), & \text{in } \mathbb{R}^N \times [\tau, T], \\ u(x,\tau) = u_\tau(x), & x \in \mathbb{R}^N, \end{cases} \tag{3.1}$$

where $\mu_0 > 0$ is a fixed constant and $g \in L^p([\tau,T], \dot{L}^q_U(\mathbb{R}^N))$.

Our main result in this section is the following well-posedness theorem concerning the linear problem in the locally uniform spaces.

Theorem 3.1. *Assume that $u_\tau \in \dot{H}^{1,q}_U(\mathbb{R}^N)$ and $g \in L^p([\tau,T], \dot{L}^q_U(\mathbb{R}^N))$ with $p > 2$, $1 < q < \infty$. Then for any interval $[\tau, T]$, problem (3.1) with the initial data u_τ is well defined and there exists a unique mild solution $u(t)$, which is given by the variations of constants formula*

$$u(t) = e^{(\Delta+\mu_0 I)(t-\widetilde{\tau})}u_{\widetilde{\tau}} + \int_{\widetilde{\tau}}^t e^{(\Delta+\mu_0 I)(t-s)}(g(x,s)+\mu_2)\,ds, \tag{3.2}$$

where $u_{\widetilde{\tau}}$ converges to u_τ in the norm of $\dot{H}^{1,q}_U(\mathbb{R}^N)$ as $\widetilde{\tau} \to \tau$, and $\tau < \widetilde{\tau} \le t \le \widetilde{t}$, and satisfies

$$u \in C([\tau,T], \dot{H}^{1,q}_U(\mathbb{R}^N)), \tag{3.3}$$

and for every $\tau < t < T$, there exist positive constant c_1 and c_2 such that

$$\|u(t)\|_{\dot{H}^{1,q}_U(\mathbb{R}^N)} \le c_1\|u_\tau\|_{\dot{H}^{1,q}_U(\mathbb{R}^N)} + c_2\|g\|_{L^p([\tau,T],\dot{L}^q_U(\mathbb{R}^N))} + C_{\mu_2}, \tag{3.4}$$

where the constants c_1 depends only on T and c_2 depends on T, τ and p.

Proof. The existence of solutions can be obtained in a standard way (Theorem 3.3.3 in [21, p. 54]). We deduce only a priori estimate (3.4) for the solutions of the problem (3.1). By using (2.5) and Hölder inequality we obtain from (3.2)

$$\begin{aligned} \|u(t)\|_{\dot{H}^{1,q}_U(\mathbb{R}^N)} &\le \|e^{(\Delta+\mu_0 I)(t-\tau)}u_\tau\|_{\dot{H}^{1,q}_U(\mathbb{R}^N)} \\ &\quad + \int_{\widetilde{\tau}}^t \|e^{(\Delta+\mu_0 I)(t-s)}(g(x,s)+\mu_2)\|_{\dot{H}^{1,q}_U(\mathbb{R}^N)}\,ds \end{aligned} \tag{3.5}$$

$$\begin{aligned}
&\le Me^{\mu t}\|u_\tau\|_{\dot{H}^{1,q}_U(\mathbb{R}^N)} + M\int_\tau^t \frac{e^{\mu(t-s)}}{(t-s)^{1/2}}\|g(s)\|_{\dot{L}^q_U(\mathbb{R}^N)}\,ds + C_{\mu_2}\\
&\le Me^{\mu t}\|u_\tau\|_{\dot{H}^{1,q}_U(\mathbb{R}^N)}\\
&\quad + M\|g\|_{L^p([\tau,T],\dot{L}^q_U(\mathbb{R}^N))}\bigg(\int_\tau^t \frac{e^{p\mu(t-s)/(p-1)}}{(t-s)^{p/(2(p-1))}}\,ds\bigg)^{(p-1)/p} + C_{\mu_2}\\
&\le M_T\|u_\tau\|_{\dot{H}^{1,q}_U(\mathbb{R}^N)} + M_{T,\tau,p}\|g\|_{L^p([\tau,T],\dot{L}^q_U(\mathbb{R}^N))} + C_{\mu_2},
\end{aligned}$$

which gives the estimate (3.4). □

Based on this result, we will prove a comparison principle for the non-autonomous reaction diffusion equations in the locally uniform spaces that allows us to obtain the global existence of solutions to (1.1)–(1.2).

Lemma 3.2. *Let* $u_i \in C([\tau,T],\dot{H}^{1,q}_U(\mathbb{R}^N)) \cap C((\tau,T],\dot{H}^{2,q}_U(\mathbb{R}^N))$ *and let the following equality be hold almost everywhere in* $\mathbb{R}^N \times [\tau,T]$,

$$\begin{cases} u_{i_t} - \Delta u_i = f_i(x,u_i) + g(x,t), & \\ u_i(x,\tau) = u_{\tau_i}(x), & i = 1,2. \end{cases} \tag{3.6}$$

Assume that the nonlinear terms $f_i(x,u_i)$, $i=1,2$, *satisfy Assumptions* I *and* II *and let* $f_1(x,u_1) \le f_2(x,u_2)$. *Then for any two initial values* $u_{\tau_1}, u_{\tau_2} \in \dot{H}^{1,q}_U(\mathbb{R}^N)$ *having* $u_{\tau_1} \le u_{\tau_2}$, *we have* $u_1(t,\tau;u_{\tau_1}) \le u_2(t,\tau;u_{\tau_2})$ *almost everywhere in* $\mathbb{R}^N \times [\tau,T]$.

Proof. The existence and uniqueness of solutions of problem (3.8) can be obtained from the standard Theorem 3.3.3 in [21, p. 54], in the base space $X = \dot{L}^q_U(\mathbb{R}^N)$, for the detail of proof we refer to the proof of Theorem 4.2 in the next section. The corresponding solution can be given by the variation of constants formula

$$u_i(t,\tau;u_{\tau_i}) = e^{\Delta(t-\tau)}u_{\tau_i} + \int_\tau^t e^{\Delta(t-s)}(f_i(x,u_i(s)) + g(x,s))\,ds. \tag{3.7}$$

Since the operator $e^{\Delta t}$ is order preserving in $\dot{L}^q_U(\mathbb{R}^N)$, which follows from Proposition 5.3 in [3], then the right hand side of (3.7) preserve the ordering. Hence we can obtain $u_1(t,\tau;u_{\tau_1}) \le u_2(t,\tau;u_{\tau_2})$, $t \in [\tau,T]$. □

Similar to the arguments in [5], from the above lemma we know that

$$u(t,\tau;-|u_\tau|) \le u(t,\tau;u_\tau) \le u(t,\tau;|u_\tau|). \tag{3.8}$$

If $0 \le g \in L^p([\tau,T],\dot{L}^q_U(\mathbb{R}^N))$, then we have

$$U(t,\tau;|u_\tau|) = e^{(\Delta+\mu_0 I)(t-\tau)}|u_\tau| + \int_\tau^t e^{(\Delta+\mu_0 I)(t-s)}(g(x,s)+\mu_2)\,ds \ge 0,$$

which is a solution of the following equations

$$\begin{cases} U_t - \Delta U = \mu_2 + g(x,t), \\ U(x,\tau) = |u_\tau|. \end{cases}$$

Note that $-\mu_0 s^2 - \mu_2|s| + g(x,t)s \le sf(x,s) + g(x,t)s$, we have

$$\begin{aligned} f(x,s) + g(x,t) &\ge -\mu_0 s - \mu_2 + g(x,t), \quad s \ge 0, \\ f(x,s) + g(x,t) &\le -\mu_0 s + \mu_2 + g(x,t), \quad s \le 0, \end{aligned}$$

it follows from the property of order preserving of the semigroup $e^{(\Delta+\mu_0 I)t}$ in locally uniform spaces $\dot{L}^q_U(\mathbb{R}^N)$,

$$\begin{aligned} U(t,\tau;|u_\tau|) &\ge e^{(\Delta+\mu_0 I)(t-\tau)}|u_\tau| \\ &\quad + \int_\tau^t e^{(\Delta+\mu_0 I)(t-s)}(-f(x,u(s,\tau;|u_\tau|)) + g(x,s) - \mu_0 u(s,\tau;|u_\tau|))\,ds \\ &= u(t,\tau;|u_\tau|), \end{aligned}$$

which implies $U(t,\tau;|u_\tau|)$ is a supersolution and it follows from (3.8)

$$u(t,\tau;u_\tau) \le U(t,\tau;|u_\tau|).$$

A similar argument can be applied to $-U(t,\tau;|u_\tau|)$, which shows that

$$-U(t,\tau;|u_\tau|) \le u(t,\tau;u_\tau).$$

Hence it follows immediately that

$$|u(t,\tau;u_\tau)| \le U(t,\tau;|u_\tau|). \tag{3.9}$$

4. Well-posedness

In order to initiate the discussion let us recall the definition of a mild solution.

Definition 4.1. We say that $u\colon [\tau,T] \to X^\alpha$ is a mild solution to the problem (1.1) with the initial data $u_\tau \in X^\alpha$ if $u \in C([\tau,T],X^\alpha) \cap C^1((\tau,T],X) \cap C((\tau,T),D(A))$ and $u(t)$ satisfies

$$u(t) = e^{\Delta(t-\widetilde{\tau})}u_{\widetilde{\tau}} + \int_{\widetilde{\tau}}^t e^{\Delta(t-s)}(g(x,s) - f(x,u(s)))\,ds$$

where $u_{\widetilde{\tau}}$ converges to u_τ in the norm of X^α as $\widetilde{\tau} \to \tau$, and $\tau < \widetilde{\tau} \le t \le T$.

It is well known from [3] that the operator $-\Delta$ generates a strongly continuous analytic semigroup in $X^\alpha = \dot{H}^{2\alpha,q}_U(\mathbb{R}^N)$ and $e^{\Delta t}$ is uniformly bounded, that is,

$$t^{\alpha-\beta}\|e^{\Delta t}u_\tau\|_{\dot{H}^{2\alpha,q}_U(\mathbb{R}^N)} \le M_{\alpha,\beta}e^{\xi_0 t}\|u_\tau\|_{\dot{H}^{2\beta,q}_U(\mathbb{R}^N)}, \tag{4.1}$$

for $\tau \in \mathbb{R}$, $u_\tau \in \dot{H}^{2\beta,q}_U(\mathbb{R}^N)$, $0 \le \beta \le \alpha < 1$ and $\xi_0 > 0$ arbitrary.

Our main result in this section is Theorem 4.2 on well-posedness of problem (1.1)–(1.2). This theorem also contains some auxiliary properties of mild solution which are needed to prove the existence of the uniform attractor. Before we prove Theorem 4.2 we will need some lemmas.

Theorem 4.2. *Let Assumptions* I *and* II *be in force and the external force* $g_0 \in L^p_b(\mathbb{R}, \dot{L}^q_U(\mathbb{R}^N))$ *with* $1 < q < N$ *and* $\rho \leq N/(N-q)$. *Then, for each* $u_\tau \in \dot{H}^{1,q}_U(\mathbb{R}^N)$, *there exists a unique mild solution of* (1.1)–(1.2) *starting at* u_τ. *This solution satisfies,*

$$u \in C\big([\tau, T], \dot{H}^{1,q}_U(\mathbb{R}^N)\big).$$

Moreover, if $u^1_\tau, u^2_\tau \in \dot{H}^{1,q}_U(\mathbb{R}^N)$ *and* $g^1, g^2 \in L^p_b(\mathbb{R}, \dot{L}^q_U(\mathbb{R}^N))$, *then*

$$\begin{aligned}\|u(t,u^1_\tau) - u(t,u^2_\tau)\|_{\dot{H}^{1,q}_U(\mathbb{R}^N)} \\ \leq C_{M,C_{T-\tau},T,\theta_2,p}(\|w(\tau)\|_{\dot{H}^{1,q}_U(\mathbb{R}^N)} + \|g^1 - g^2\|_{L^p_b(\mathbb{R};\dot{L}^q_U(\mathbb{R}^N))}),\end{aligned} \tag{4.2}$$

for $\tau < t \leq T$.

The main properties of the Nemitskiĭ operator associated with f are included in the following lemma, which will be used to obtain the well-posedness of solutions.

Lemma 4.3. *Assume that* $f(x,u)$ *satisfies Assumption* I. *Then there exists a positive constant* C *such that the following estimates hold*:

$$\|f(x,u)\|_{\dot{L}^q_U(\mathbb{R}^N)} \leq C\Big(1 + \|u\|^{(\rho-1)q}_{\dot{H}^{1,q}_U(\mathbb{R}^N)}\Big), \tag{4.3}$$

$$\begin{aligned}\|f(x,u) - f(x,v)\|_{\dot{L}^q_U(\mathbb{R}^N)} \\ \leq C\|u - v\|_{\dot{H}^{1,q}_U(\mathbb{R}^N)}\Big(1 + \|u\|^{(\rho-1)q}_{\dot{H}^{1,q}_U(\mathbb{R}^N)} + \|u\|^{(\rho-1)q}_{\dot{H}^{1,q}_U(\mathbb{R}^N)}\Big),\end{aligned} \tag{4.4}$$

for all $u, v \in \dot{H}^{1,q}_U(\mathbb{R}^N)$.

Proof. For any $u_1, u_2 \in \dot{H}^{1,q}_U(\mathbb{R}^N)$ and $y \in \mathbb{R}^N$, we have

$$\begin{aligned}&\int_{\mathbb{R}^N} \phi_y(x)|f(x,u_1) - f(x,u_2)|^q\,dx \\ &\leq C\int_{\mathbb{R}^N} \phi_y(x)|u_1 - u_2|^q(1 + |u_1|^{\rho-1} + |u_2|^{\rho-1})^q\,dx \\ &\leq C\int_{\mathbb{R}^N} \phi_y(z)\int_{B(z,1)} |u_1 - u_2|^q(1 + |u_1|^{\rho-1} + |u_2|^{\rho-1})^q\,dx\,dz.\end{aligned} \tag{4.5}$$

In the ball $B(z,1)$, by using the sharp embeddings of Bessel spaces: if $1 < q < N$ and $1/q - 1/r \leq 1/N$ with $1 \leq r < \infty$, then $H^{1,q}(B(z,1)) \subset L^r(B(z,1))$, we have

$$\begin{aligned}
&\int_{B(z,1)} |u_1-u_2|^q(1+|u_1|^{\rho-1}+|u_2|^{\rho-1})^q\,dx \\
&\le C\|u_1-u_2\|^q_{L^{Nq/(N-q)}(B(z,1))}\Big(1+\|u_1\|^{(\rho-1)q}_{L^{(\rho-1)N}(B(z,1))}+\|u_2\|^{(\rho-1)q}_{L^{(\rho-1)N}(B(z,1))}\Big) \\
&\le C\|u_1-u_2\|^q_{H^{1,q}(B(z,1))}\Big(1+\|u_1\|^{(\rho-1)q}_{H^{1,q}(B(z,1))}+\|u_2\|^{(\rho-1)q}_{H^{1,q}(B(z,1))}\Big).
\end{aligned}$$

Inserting this estimate into (4.5), it follows from the definition of the locally uniform spaces that the conclusions of this lemma holds true. □

Proof of Theorem 4.2. We will use Theorem 3.3.3 in [21, p. 54], in the base space $X=\dot L^q_U(\mathbb{R}^N)$. It is well known from [3] that the operator $-\Delta$ with domain $\dot H^{2,q}_U(\mathbb{R}^N)$ is a sectorial operator in $\dot L^q_U(\mathbb{R}^N)$ for $1<q<\infty$. Since $g_0\in L^p_b(\mathbb{R},\dot L^q_U(\mathbb{R}^N))$ with $1<q<\infty$, it is enough to prove that $f\colon X^{1/2}=\dot H^{1,q}_U(\mathbb{R}^N)\to X=\dot L^q_U(\mathbb{R}^N)$ is Lipschitz on bounded sets. From Lemma 4.3 we easily get

$$\|f(x,u_1)-f(x,u_2)\|_{\dot L^q_U(\mathbb{R}^N)}\le C\|u_1-u_2\|_{\dot H^{1,q}_U(\mathbb{R}^N)}.$$

Thus, we obtain the local existence of solutions to problem (1.1)–(1.2), that is, there exists a time $\widetilde t>0$ such that

$$u\in C([\tau,\widetilde t),\dot H^{1,q}_U(\mathbb{R}^N))\cap C^1((t,\widetilde t),\dot H^{2\gamma,q}_U(\mathbb{R}^N))\cap C((\tau,\widetilde t),\dot H^{2,q}_U(\mathbb{R}^N))$$

and satisfies

$$u(t)=e^{(\Delta+\mu_0 I)(t-\widetilde\tau)}u_{\widetilde\tau}+\int_{\widetilde\tau}^t e^{(\Delta+\mu_0 I)(t-s)}(g(x,s)-f(x,u(s))-\mu_0u(s))\,ds,$$

where $u_{\widetilde\tau}$ converges to u_τ in the norm of $\dot H^{1,q}_U(\mathbb{R}^N)$ as $\widetilde\tau\to\tau$, and $\tau<\widetilde\tau\le t\le\widetilde t$.

To prove the global existence of solutions of (1.1)–(1.2), we need only to show that $\dot H^{1,q}_U(\mathbb{R}^N)$-norm of such a local solution do not blow up in a finite time. In fact, using the embedding of Bessel spaces in (2.4), if $q<N$, we have

$$\dot H^{1,q}_U(\mathbb{R}^N)\subset\dot L^r_U(\mathbb{R}^N),\quad\text{with } \frac1q-\frac1r\le\frac1N,\ 1\le r<\infty.$$

Hence we can obtain from (3.9) and Theorem 3.1,

$$\begin{aligned}
\|u(t,\tau;u_\tau)\|_{\dot L^r_U(\mathbb{R}^N)}&\le\|U(t,\tau;|u_\tau|)\|_{\dot L^r_U(\mathbb{R}^N)}\le\|U(t,\tau;|u_\tau|)\|_{\dot H^{1,q}_U(\mathbb{R}^N)}\\
&\le c_1\|u_\tau\|_{\dot H^{1,q}_U(\mathbb{R}^N)}+c_2\|g\|_{L^p([\tau,T],\dot L^q_U(\mathbb{R}^N))}+C_{\mu_2},
\end{aligned}\tag{4.6}$$

where the constants c_1 depends only on T and c_2 depends on T, τ and p.

Now, it follows from the uniform boundedness of $e^{(\Delta+\mu_0)t}$ in (4.1) and the formula (4.3) of Lemma 4.3 that, for some $\xi_0>0$,

$$\begin{aligned}
\|u(t)\|_{\dot H^{1,q}_U(\mathbb{R}^N)}&=\|e^{(\Delta+\mu_0 I)(t-\tau)}u_\tau\|_{\dot H^{1,q}_U(\mathbb{R}^N)}\\
&\quad+\int_\tau^t\|e^{(\Delta+\mu_0 I)(t-s)}(g(x,s)-f(x,u(s))-\mu_0u(s))\|_{\dot H^{1,q}_U(\mathbb{R}^N)}\,ds
\end{aligned}\tag{4.7}$$

$$
\begin{aligned}
&\le Me^{\xi_0(t-\tau)}\|u_\tau\|_{\dot H_U^{1,q}(\mathbb{R}^N)} \\
&\quad + M\int_\tau^t \frac{e^{\xi_0(t-s)}}{(t-s)^{1/2}}\|(g(x,s)-f(x,u(s))-\mu_0 u(s))\|_{\dot L_U^q(\mathbb{R}^N)}\,ds \\
&\le Me^{\xi_0(t-\tau)}\|u_\tau\|_{\dot H_U^{1,q}(\mathbb{R}^N)} \\
&\quad + C_T\sup_{s\in[\tau,T]}\|f(x,u(s))+\mu_0 u(s)\|_{\dot L_U^q(\mathbb{R}^N)}\,ds + C_T\|g\|_{L_b^p(\mathbb{R},\dot L_U^q(\mathbb{R}^N))}.
\end{aligned}
$$

Note that $\rho \le N/(N-q)$ and Bessel embedding

$$H^{1,q}(B(z,1)) \subset L^{Nq/(N-q)}(B(z,1)),$$

we can especially choose $r = q$ and $r = \rho q$ in (4.6), and obtain

$$
\begin{aligned}
&\|f(x,u(s))+\mu_0 u(s)\|^q_{\dot L_U^q(\mathbb{R}^N)} \\
&\quad \le C\int_{\mathbb{R}^N}\phi_y(x)(|u(s)|^q+|u(s)|^{\rho q})\,dx + \mu_0\|u(s)\|^q_{\dot L_U^q(\mathbb{R}^N)} \\
&\quad \le C\int_{\mathbb{R}^N}\phi_y(x)(|U(s)|^q+|U(s)|^{\rho q})\,dx + \mu_0\|U(s)\|^q_{\dot L_U^q(\mathbb{R}^N)} \\
&\quad \le C\int_{\mathbb{R}^N}\phi_y(z)\int_{B(z,1)}(|U(s)|^q+|U(s)|^{\rho q})\,dx\,dz + \mu_0\|U(s)\|^q_{\dot L_U^q(\mathbb{R}^N)} \\
&\quad \le (C+\mu_0)\|U(s)\|^q_{\dot H_U^{1,q}(\mathbb{R}^N)} + C\|U(s)\|^{\rho q}_{\dot H_U^{1,q}(\mathbb{R}^N)}.
\end{aligned}
\tag{4.8}
$$

Thus, according to Theorem 3.1 and plugging (4.8) into (4.7) it yields, for $\tau \le t \le T < \infty$,

$$\|u(t)\|_{\dot H_U^{1,q}(\mathbb{R}^N)} \le C_{\mu_0,\mu_2,T,\|u_\tau\|_{\dot H_U^{1,q}(\mathbb{R}^N)},\|g\|_{L^p([\tau,T],\dot L_U^q(\mathbb{R}^N))}}. \tag{4.9}$$

Hence, the solution is global in $\dot H_U^{1,q}(\mathbb{R}^N)$.

Last, the proof of (4.2), that is, continuous dependence of solutions with respect to the initial data and external force, is completely similar to the proof of (5.5) in Theorem 5.4. So, we here omit details. $\square$

5. Global weak solution: existence, uniqueness and dissipativity

In this section, we are concerned with global weak solution to problem (1.1) subject to the initial data (1.2), based on smooth global solutions to (1.1)–(1.2) defined as in Theorem 4.2 and some continuities obtained in Theorem 5.4 below and motivated by the definition of weak solution of [14], we can extend smooth solution of this kind to global weak solution in the whole of $H_U^{1,q}(\mathbb{R}^N)$.

For the symbol of the original equation (1.1) $g_0(t) := g_0(x,t) \in L_b^p(\mathbb{R}, L_U^q(\mathbb{R}^N))$, consider the translation group $\{T(h), h \in \mathbb{R}\}$ acting by the formula $T(h)g_0(t) = g_0(t+h)$, $h \in \mathbb{R}$ and denote by

$$\sigma_0 = \{T(h)g_0(t) \mid h \in \mathbb{R}\} = \{g_0(t+h) \mid h \in \mathbb{R}\}.$$

The resulting family of symbols σ_0 forms the hull

$$\mathcal{H}_{L^q_U(\mathbb{R}^N)}(g_0) := [\sigma_0]_{L^p_{\mathrm{loc}}(\mathbb{R};L^q_U(\mathbb{R}^N))}$$

of the original symbol $g_0(t)$ that is the closure of σ_0 in the space $L^{p,w}_{\mathrm{loc}}(\mathbb{R};L^q_U(\mathbb{R}^N))$, which is the subspace of $L^p_{\mathrm{loc}}(\mathbb{R};L^q_U(\mathbb{R}^N))$ equipped with the local weak convergence topology. Similar to Proposition 2.3 of [11], for any $g(t)$ belongs to the hull $\mathcal{H}_{L^q_U(\mathbb{R}^N)}(g_0)$, we have

$$\|g(t)\|_{L^p_b(\mathbb{R};L^q_U(\mathbb{R}^N))} \leq \|g_0(t)\|_{L^p_b(\mathbb{R};L^q_U(\mathbb{R}^N))}. \tag{5.1}$$

Now, we give the following definition of a weak $H^{1,q}_U(\mathbb{R}^N)$ global solution to problem (1.1)–(1.2).

Definition 5.1. The function $u(t) \in C([\tau,\infty), H^{1,q}_U(\mathbb{R}^N))$ is called a global weak solution to problem (1.1) with the initial data $u_\tau \in H^{1,q}_U(\mathbb{R}^N)$ and the external force $g \in L^p_b(\mathbb{R};L^q_U(\mathbb{R}^N))$ if and only if there exist a bounded (in $H^{1,q}_U(\mathbb{R}^N)$) sequence $\{u^n_\tau\}_{n=1}^\infty \subset \dot{H}^{1,q}_U(\mathbb{R}^N)$ and a bounded sequence $\{g_n\}_{n=1}^\infty \subset L^p_b(\mathbb{R};L^q_U(\mathbb{R}^N))$ satisfying $u^n_\tau \to u_\tau$ in $H^{1,q}_\phi(\mathbb{R}^N)$ and $g_n \to g$ in $L^p_b(\mathbb{R};L^q_\phi(\mathbb{R}^N))$ such that a sequence $\{u_n(t)\}$ of smooth global solutions to (1.1)–(1.2) converges to $u(t)$ in $C([\tau,T], H^{1,q}_\phi(\mathbb{R}^N))$ on each compact interval $[\tau,T]$.

In the sequel we will require the following known results, which can be found in [27], [3] and [36], respectively.

Lemma 5.2. *The closure in $H^{1,q}_\phi$ of a bounded subset of $H^{1,q}_U$ consists of elements in a bounded subset of $H^{1,q}_U$.*

Lemma 5.3. *Let $m,q \in \mathbb{N}\cup\{0\}$ and $\phi(\,\cdot\,)$ be given by Definition 2.1. Then $\dot{H}^{m,q}_U(\mathbb{R}^N)$ is a dense set of $H^{m,q}_U(\mathbb{R}^N)$ w.r.t. the $H^{m,q}_\phi(\mathbb{R}^N)$. Moreover, for each $w \in H^{m,q}_U(\mathbb{R}^N)$, there exists a bounded(in $w \in H^{m,q}_U(\mathbb{R}^N)$) sequence $\{w_n\}_{n=1}^\infty \subset \dot{H}^{m,q}_U(\mathbb{R}^N)$ such that $w_n \to w$ w.r.t. the $H^{m,q}_\phi(\mathbb{R}^N)$.*

In the following theorem, we will prove the existence, uniqueness of global weak solutions and their continuous dependence with respect to the initial data in the $H^{1,q}_\phi(\mathbb{R}^N)$-norm.

Theorem 5.4. *Suppose that the nonlinearity f satisfies Assumptions* I *and* II *and the external force $g_0 \in L^p_b(\mathbb{R}, L^q_U(\mathbb{R}^N))$ with $1 < q < N$ and $\rho \leq N/(N-q)$. Then, for each $u_\tau \in H^{1,q}_U(\mathbb{R}^N)$, there exists a unique global weak solution of problem* (1.1)–(1.2) *that generate the process $\{U_g(t,\tau) \mid t \geq \tau,\ \tau \in \mathbb{R}\}$, $g \in \mathcal{H}_{L^q_U(\mathbb{R}^N)}(g_0)$ acting in $H^{1,q}_U(\mathbb{R}^N)$ by the formula*

$$U_g(t,\tau)u_\tau = u(t), \quad U_g(t,\tau)\colon H^{1,q}_U(\mathbb{R}^N) \to H^{1,q}_U(\mathbb{R}^N), \quad t \geq \tau,\ \tau \in \mathbb{R},$$

where $u(t)$ is a solution of (1.1)–(1.2)*. Moreover, $\{U_g(t,\tau)\}$, $g \in \mathcal{H}_{L^q_U(\mathbb{R}^N)}(g_0)$ is $(H^{1,q}_U(\mathbb{R}^N) \times \mathcal{H}_{L^q_U(\mathbb{R}^N)}(g_0), H^{1,q}_U(\mathbb{R}^N))$-continuous in the $H^{1,q}_\phi(\mathbb{R}^N)$-norm, i.e. for* *any* $t \geq \tau$, $\tau \in \mathbb{R}$, *the mapping* $(u, g) \to U_g(t,\tau)u$ *is continuous from* $H_U^{1,q}(\mathbb{R}^N) \times \mathcal{H}_{L_U^q(\mathbb{R}^N)}(g_0)$ *to* $H_U^{1,q}(\mathbb{R}^N)$ *with respect to* $H_\phi^{1,q}(\mathbb{R}^N)$*-norm.*

PROOF. We will split the proof into two steps.

Step 1. (Continuity) We begin with continuous dependence on initial conditions in the space $H_\phi^{1,q}(\mathbb{R}^N)$ for the solution obtained by Theorem 4.2.

To this end, let $u^i(t)$, $i = 1,2$ be the solution of problem (1.1) with the initial data u_τ^i belongings to $\dot{H}_U^{1,q}(\mathbb{R}^N)$ and associated with the external force $g^i \in L_b^p(\mathbb{R}, \dot{L}_U^q(\mathbb{R}^N))$, and for convenience, set $w(t) = u^1(t) - u^2(t)$ and $f^i(s) = f(x, u^i(s))$, exploiting the variation of constant formula we get

$$\begin{aligned} w(t) = {} & e^{(\Delta+\mu_0 I)(t-\tau)} w_\tau \\ & + \int_\tau^t e^{(\Delta+\mu_0 I)(t-s)} \big[(g^1(s) - g^2(s)) - (f^1(s) - f^2(s)) - \mu_0 w(s)\big]\, ds. \end{aligned}$$

Thus, in a similar way as in the proof of Theorem 4.2, it follows for some $\theta_2 > 0$ and $t \in [\tau, T]$,

$$\begin{aligned} \|w(t)\|_{H^{1,q}_{\phi_y}(\mathbb{R}^N)} \leq {} & \|e^{(\Delta+\mu_0 I)(t-\tau)} w_\tau\|_{H^{1,q}_{\phi_y}(\mathbb{R}^N)} \\ & + Q \int_\tau^t \|e^{(\Delta+\mu_0 I)(t-s)} [(g^1(s) - g^2(s)) \\ & - (f^1(s) - f^2(s)) - \mu_0 w(s)]\|_{H^{1,q}_{\phi_y}(\mathbb{R}^N)}\, ds \\ \leq {} & M e^{\theta_2 (t-\tau)} \|w(\tau)\|_{H^{1,q}_{\phi_y}(\mathbb{R}^N)} \\ & + M \int_\tau^t \frac{e^{\theta_2(t-s)}}{(t-s)^{1/2}} \|(g^1 - g^2) - (f^1 - f^2) - \mu_0 w(s))\|_{L^q_{\phi_y}(\mathbb{R}^N)}\, ds \\ \leq {} & M C_{T-\tau} \|w(\tau)\|_{H^{1,q}_{\phi_y}(\mathbb{R}^N)} + M_{T-\tau,\theta_2,p} \|g^1 - g^2\|_{L^p_b(\mathbb{R}; L^q_{\phi_y}(\mathbb{R}^N))} \\ & + M C_{T-\tau} \int_\tau^t (t-s)^{-1/2} \|(f^1 - f^2) + \mu_0 w(s))\|_{L^q_{\phi_y}(\mathbb{R}^N)}\, ds. \end{aligned} \tag{5.2}$$

Now, we estimate the difference of the nonlinearity $f^i(s)$, $i = 1,2$, in the weighted space $L_\phi^q(\mathbb{R}^N)$. Similar to (4.5), it fields

$$\begin{aligned} & \|(f^1 - f^2) + \mu_0 w(s))\|_{L^q_{\phi_y}(\mathbb{R}^N)} \\ & \leq \bigg(\int_{\mathbb{R}^N} \phi_y |f^1 - f^2|^q\, dx\bigg)^{1/q} + \mu_0 \|w(s)\|_{L^q_{\phi_y}(\mathbb{R}^N)} \\ & \leq C\bigg(\int_{\mathbb{R}^N} \phi_y(x) |u_1 - u_2|^q (1 + |u_1|^{\rho-1} + |u_2|^{\rho-1})^q\, dx\bigg)^{1/q} \\ & \quad + \mu_0 \|w(s)\|_{L^q_{\phi_y}(\mathbb{R}^N)} \end{aligned} \tag{5.3}$$

$$
\begin{aligned}
&\leq C_{\|u^1\|_{\dot{H}^{1,q}_U(\mathbb{R}^N)},\|u^2\|_{\dot{H}^{1,q}_U(\mathbb{R}^N)}}\bigg(\int_{\mathbb{R}^N}\phi_y(z)\|w\|^q_{H^{1,q}(B(z,1))}\,dz\bigg)^{1/q}\\
&\quad+\mu_0\|w(s)\|_{L^q_{\phi_y}(\mathbb{R}^N)}\\
&\leq C_{\|u^1\|_{\dot{H}^{1,q}_U(\mathbb{R}^N)},\|u^2\|_{\dot{H}^{1,q}_U(\mathbb{R}^N)},\mu_0}\|w(s)\|_{H^{1,q}_{\phi_y}(\mathbb{R}^N)},
\end{aligned}
$$

where we used Sobolev type inclusion $H^{1,q}_\phi(\mathbb{R}^N)\hookrightarrow L^q_\phi(\mathbb{R}^N)$ for $q<N$ and a weight function given as in Definition 2.1. Then, by substituting (5.3) into the right hand side of inequality (5.2), one obtains from Lemma 2.7 that there exists a positive constant $\widetilde{C}=C_{M,C_{T-\tau},T,\|u^1\|_{\dot{H}^{1,q}_U(\mathbb{R}^N)},\|u^2\|_{\dot{H}^{1,q}_U(\mathbb{R}^N)},\mu_0}$ such that

$$
\|w(t)\|_{H^{1,q}_{\phi_y}(\mathbb{R}^N)}\leq\widetilde{C}\big(MC_{T-\tau}\|w(\tau)\|_{H^{1,q}_{\phi_y}(\mathbb{R}^N)}+M_{T-\tau,\theta_2,p}\|g^1-g^2\|_{L^p_b(\mathbb{R};L^q_{\phi_y}(\mathbb{R}^N))}\big).
$$

Consequently, we obtain

$$
\begin{aligned}
&\sup_{t\in[\tau,T]}\|w(t)\|_{H^{1,q}_\phi(\mathbb{R}^N)}\\
&\qquad\leq C_{\theta_2,p,\widetilde{C}}\big(\|w(\tau)\|_{H^{1,q}_\phi(\mathbb{R}^N)}+\|g^1-g^2\|_{L^p_b(\mathbb{R};L^q_\phi(\mathbb{R}^N))}\big)
\end{aligned}
\tag{5.4}
$$

and

$$
\begin{aligned}
&\sup_{t\in[\tau,T]}\|w(t)\|_{H^{1,q}_U(\mathbb{R}^N)}\\
&\qquad\leq C_{\theta_2,p,\widetilde{C}}\big(\|w(\tau)\|_{H^{1,q}_U(\mathbb{R}^N)}+\|g^1-g^2\|_{L^p_b(\mathbb{R};L^q_U(\mathbb{R}^N))}\big).
\end{aligned}
\tag{5.5}
$$

The above two estimates indicate that if the initial conditions u_τ belongs to a bounded set of $\dot{H}^{1,q}_U(\mathbb{R}^N)$ and the external force g belongs to a bounded set of $L^p_b(\mathbb{R},\dot{L}^q_U(\mathbb{R}^N))$, then the solution $u(t)$ of problem (1.1) is continuous dependence on initial data in the topology of weighted(*resp. uniform*) spaces $H^{1,q}_\phi(\mathbb{R}^N)$ (resp. $H^{1,q}_U(\mathbb{R}^N)$) with the corresponding external force belonging to $L^p_b(\mathbb{R},L^q_\phi(\mathbb{R}^N))$ (resp. $L^p_b(\mathbb{R},L^q_U(\mathbb{R}^N))$), uniformly with respect to t on any bounded subintervals of $[0,\infty)$. Consequently, the estimate (5.4) gives the uniqueness.

Step 2. (Existence) As a consequence of Theorem 4.2, one can represent smooth global solutions given by this theorem as $\widetilde{U}_{g_n}(t,\tau)u^n_\tau=u_n(t)$, $t\geq\tau$, $\tau\in\mathbb{R}$, where the initial data $u^n_\tau\in\dot{H}^{1,q}_U(\mathbb{R}^N)$ and the external force $g_n\in\mathcal{H}_{\dot{L}^q_U(\mathbb{R}^N)}(g_0)$, $n=1,2,\ldots$ In addition, from (4.2) we know that $\{\widetilde{U}_{g_n}(t,\tau)\}$, $g_n\in\mathcal{H}_{\dot{L}^q_U(\mathbb{R}^N)}(g_0)$ is $(\dot{H}^{1,q}_U(\mathbb{R}^N)\times\mathcal{H}_{\dot{L}^q_U(\mathbb{R}^N)}(g_0),\dot{H}^{1,q}_U(\mathbb{R}^N))$ continuous in the $\dot{H}^{1,q}_U(\mathbb{R}^N)$-norm.

On the other hand, from Lemma 5.3 we have that for each $u_\tau\in H^{1,q}_U(\mathbb{R}^N)$ and $g\in\mathcal{H}_{L^q_U(\mathbb{R}^N)}(g_0)$, there exists $\{u^n_\tau\}\subset\dot{H}^{1,q}_U(\mathbb{R}^N)$ that is bounded in $H^{1,q}_U(\mathbb{R}^N)$ and $\{g_n\}\subset\mathcal{H}_{\dot{L}^q_U(\mathbb{R}^N)}(g_0)$ that is bounded in $\mathcal{H}_{L^q_U(\mathbb{R}^N)}(g_0)$ such that $u^n_\tau\to u_\tau$ in the $H^{1,q}_\phi(\mathbb{R}^N)$-norm and $g_n\to g$ in the $L^p_b(\mathbb{R};L^q_\phi(\mathbb{R}^N))$-norm.

Thus, for each $u_\tau \in H_U^{1,q}(\mathbb{R}^N)$ and $g \in \mathcal{H}_{L_U^q(\mathbb{R}^N)}(g_0)$, we can define the following limit of the process $\{\widetilde{U}_{g_n}(t,\tau)\}$:

$$U_g(t,\tau)u_\tau := \lim_{n\to\infty} \widetilde{U}_{g_n}(t,\tau)u_\tau^n \quad \text{w.r.t. } H_\phi^{1,q}(\mathbb{R}^N)\text{-norm}, \tag{5.6}$$

for all $t \geq \tau$, $\tau \in \mathbb{R}$.

It follows from Theorem 4.2 and Lemma 5.2 that the process $U_g(t,\tau)u_\tau$ belongs to $H_U^{1,q}(\mathbb{R}^N)$. Since the estimates of continuity (5.4) and (5.5), $U_g(t,\tau)u_\tau$ does not depend on the choice of $\{u_\tau^n\}_{n=1}^\infty$ and $\{g_n\}_{n=1}^\infty$, and is the limit of $\widetilde{U}_{g_n}(t,\tau)u_\tau^n$ in the space $C([\tau,T],H_\phi^{1,q}(\mathbb{R}^N))$, where $T>\tau$, $\tau\in\mathbb{R}$. $\{U_g(t,\tau)\}$, $g \in \mathcal{H}_{L_U^q(\mathbb{R}^N)}(g_0)$ forms the family of the process of the space $H_U^{1,q}(\mathbb{R}^N)$ and is $(H_U^{1,q}(\mathbb{R}^N)\times\mathcal{H}_{L_U^q(\mathbb{R}^N)}(g_0), H_U^{1,q}(\mathbb{R}^N))$ continuous in the $H_\phi^{1,q}(\mathbb{R}^N)$-norm. Finally, it is obvious that if $g_n \in \mathcal{H}_{\dot{L}_U^q(\mathbb{R}^N)}(g_0)$, then $U_g(t,\tau) = \widetilde{U}_g(t,\tau)$ on the dense subspace $\dot{H}_U^{1,q}(\mathbb{R}^N)$ of $H_U^{1,q}(\mathbb{R}^N)$.

Therefore, $U_g(t,\tau)u_\tau = u(t)$ is a unique global weak solution of (1.1)–(1.2) associated with the initial data $u_\tau \in H_U^{1,q}(\mathbb{R}^N)$ and the corresponding external force $g \in \mathcal{H}_{L_U^q(\mathbb{R}^N)}(g_0)$. □

We are now able to state our result on dissipativity of the non-autonomous reaction-diffusion equations in the locally uniform space $H_U^{2\alpha,q}(\mathbb{R}^N)$.

THEOREM 5.5. *Let $g \in L_b^p(\mathbb{R};L_U^q(\mathbb{R}^N))$ with $p>2$. Suppose that the analytic semigroup $e^{(\Delta+\mu_0 I)t}$ generated by the operator $\Delta+\mu_0 I$ with domain $H_U^{2,q}(\mathbb{R}^N)$ is exponential decay. In addition, let $v \in H_U^{2,q}(\mathbb{R}^N)$ be the unique solution of the corresponding elliptic equations*

$$-\Delta\widetilde{v} - \mu_0\widetilde{v} = \mu_2, \quad x \in \mathbb{R}^N. \tag{5.7}$$

Under the assumptions of Theorem 4.2, there is a positive constant $C_{\|\widetilde{v}\|_{H_U^{2,q}(\mathbb{R}^N)}}$ such that for any bounded (in $H_U^{1,q}(\mathbb{R}^N)$) subset $B \subset H_U^{1,q}(\mathbb{R}^N)$, there exists a positive constant $T_0 = T+1$, $T = T(B,\tau)$ which depends on the $H_U^{1,q}(\mathbb{R}^N)$-norm of B and τ, such that for each $0 \leq \alpha < 1$ and $t \geq T_0$

$$\|u(t)\|_{H_U^{2\alpha,q}(\mathbb{R}^N)} \leq \widetilde{C}_{g,\widetilde{v}}, \tag{5.8}$$

where $\widetilde{C}_{g,\widetilde{v}}$ depends on $\|\widetilde{v}(x)\|_{L_U^q(\mathbb{R}^N)}$, $\|g\|_{L_b^p(\mathbb{R};L_U^q(\mathbb{R}^N))}$ and the positive constants θ_0, θ_1, α, p, λ_0.

PROOF. For the solution $U(x,t;|u_\tau|)$ of the linear problem (3.1) associated with the initial $|u_\tau|$, we now decompose $U(x,t;|u_\tau|)$ as follows:

$$U(x,t;|u_\tau|) = \widetilde{v}(x) + v(x,t),$$

where $\widetilde{v}(x)$ is the solution of (5.7) and $v(x,t)$ satisfies the following non-autonomous linear equation:

$$\begin{cases} v_t - \Delta v - \mu_0 v = g(x,t), & \text{in } \mathbb{R}^N \times [\tau,\infty), \\ v(x,\tau) = u_\tau - \widetilde{v}(x), & x \in \mathbb{R}^N. \end{cases} \tag{5.9}$$

By using the variation of constants formula, we have

$$v(x,t) = e^{(\Delta+\mu_0 I)(t-\tau)}(u_\tau - \widetilde{v}(x)) + \int_\tau^t e^{(\Delta+\mu_0 I)(t-s)} g(x,s)\,ds,$$

and then, for some $\theta_0 > 0$, it follows from (4.1)

$$\begin{aligned} \|v(x,t)\|_{L^q_U(\mathbb{R}^N)} \le{}& Me^{-\theta_0(t-\tau)}(\|u_\tau\|_{L^q_U(\mathbb{R}^N)} \\ &+ \|\widetilde{v}(x)\|_{L^q_U(\mathbb{R}^N)}) + M\int_\tau^t e^{-\theta_0(t-s)}\|g\|_{L^q_U(\mathbb{R}^N)}\,ds. \end{aligned} \tag{5.10}$$

Note that

$$\begin{aligned} \int_\tau^t e^{-\theta_0(t-s)}\|g\|_{L^q_U(\mathbb{R}^N)}\,ds \le{}& \int_{t-1}^t \|g\|_{L^q_U(\mathbb{R}^N)}\,ds \\ &+ e^{-\theta_0}\int_{t-1}^t \|g\|_{L^q_U(\mathbb{R}^N)}\,ds + e^{-2\theta_0}\int_{t-1}^t \|g\|_{L^q_U(\mathbb{R}^N)}\,ds + \ldots \\ &\le (1-e^{-\theta_0})^{-1}\|g\|_{L^p_b(\mathbb{R};L^q_U(\mathbb{R}^N))}. \end{aligned}$$

Thus, from the inequality $(1-e^{-\theta_0})^{-1} < 1 + 1/\theta_0$ and (5.10) we obtain

$$\begin{aligned} \|v(x,t)\|_{L^q_U(\mathbb{R}^N)} \le{}& Me^{-\theta_0(t-\tau)}(\|u_\tau\|_{L^q_U(\mathbb{R}^N)} \\ &+ \|\widetilde{v}(x)\|_{L^q_U(\mathbb{R}^N)}) + \left(1+\frac{1}{\theta_0}\right)\|g\|_{L^p_b(\mathbb{R};L^q_U(\mathbb{R}^N))}. \end{aligned} \tag{5.11}$$

Let $\widetilde{M} = 2(1+1/\theta_0)\|g\|_{L^p_b(\mathbb{R};L^q_U(\mathbb{R}^N))}$. Since $U(x,t;|u_\tau|)$ is a supersolution, we conclude from (5.1) that

$$\|u(x,t;u_\tau)\|_{L^q_U(\mathbb{R}^N)} \le \|U(x,t;|u_\tau|)\|_{L^q_U(\mathbb{R}^N)} \le \widetilde{M} + \|\widetilde{v}(x)\|_{L^q_U(\mathbb{R}^N)} := \widetilde{M}_0 \tag{5.12}$$

and $B_0 = \{u \in L^q_U(\mathbb{R}^N) \mid \|u\|_{L^q_U(\mathbb{R}^N)} \le \widetilde{M}_0\}$ is a bounded uniformly (w.r.t. $g \in \mathcal{H}_{L^q_U(\mathbb{R}^N)}(g_0)$) absorbing set of the family of processes $\{U_g(t,\tau)\}$, $g \in \mathcal{H}_{L^q_U(\mathbb{R}^N)}(g_0)$, i.e. for any $\tau \in \mathbb{R}$ and bounded set $B \subset L^q_U(\mathbb{R}^N)$, there exists $T = T(B,\tau) > \tau$ such that

$$\bigcup_{g\in\mathcal{H}_{L^q_U(\mathbb{R}^N)}(g_0)} U_g(t,\tau)B \subset B_0, \quad \text{for all } t \ge T. \tag{5.13}$$

On the other hand, we can choose λ_0 large enough such that the semigroup $e^{\Delta-\lambda_0 I}$ generated by $\Delta - \lambda_0 I$ decays exponentially in $L^q_U(\mathbb{R}^N)$. Thus, for equation (1.1) we use the variation of constants formula from t to $t+1$ and have

$$(5.14)\quad u(t+1,u_\tau) = e^{\Delta-\lambda_0 I}u(t) + \int_t^{t+1} e^{(\Delta-\lambda_0 I)(t+1-s)}(g(x,t) - f(x,u) + \lambda_0 u)\, ds$$

and then, for some $\theta_1 > 0$,

$$(5.15)\quad \|u(t+1,u_\tau)\|_{H^{2\alpha,q}_U(\mathbb{R}^N)} \le Me^{-\theta_1}\|u(t)\|_{L^q_U(\mathbb{R}^N)} + M\int_t^{t+1} \frac{e^{-\theta_1(t+1-s)}}{(t+1-s)^\alpha}\|g(x,t) - f(x,u) + \lambda_0 u\|_{L^q_U(\mathbb{R}^N)}\, ds.$$

For any $\tau \in \mathbb{R}$ and bounded set $B \subset H^{1,q}_U(\mathbb{R}^N)$, there exists $T = T(B,\tau) > \tau$ such that for all $t \ge T$, similar to (4.8), we get

$$\begin{aligned}\|-f(x,u(t)) + \lambda_0 u(t)\|_{L^q_U(\mathbb{R}^N)} &\le C_{\lambda_0}\|U(t)\|_{H^{1,q}_U(\mathbb{R}^N)} + C\|U(t)\|^\rho_{H^{1,q}_U(\mathbb{R}^N)} \\ &\le C_{\lambda_0}\widetilde{M}_0 + C\widetilde{M}_0^\rho,\end{aligned}$$

and

$$\begin{aligned}&\int_t^{t+1} \frac{e^{-\theta_1(t+1-s)}}{(t+1-s)^\alpha}\|g(x,t)\|_{L^q_U(\mathbb{R}^N)}\, ds \\ &\le \|g(x,t)\|_{L^p_b(\mathbb{R};L^q_U(\mathbb{R}^N))}\left(\int_t^{t+1} \frac{e^{-\theta_1 p(t+1-s)/(p-1)}}{(t+1-s)^{p\alpha/(p-1)}}\, ds\right)^{(p-1)/p} \\ &\le C_{\alpha,\theta_1,p}\|g(x,t)\|_{L^p_b(\mathbb{R};L^q_U(\mathbb{R}^N))}.\end{aligned}$$

Now, substituting the above two estimates in (5.15) it follows for all $t \ge T$.

$$\begin{aligned}\|u(t+1,u_\tau)\|_{H^{2\alpha,q}_U(\mathbb{R}^N)} \le{}& Me^{-\theta_1}\widetilde{M}_0 + MC_{\alpha,\theta_1}(C_{\lambda_0}\widetilde{M}_0 \\ &+ C\widetilde{M}_0^\rho) + C_{\alpha,\theta_1,p}\|g(x,t)\|_{L^p_b(\mathbb{R};L^q_U(\mathbb{R}^N))} := \widetilde{C}_{g,\widetilde{v}},\end{aligned}$$

which implies that for all $t \ge T+1$, the estimate (5.8) holds true. □

It follows from (5.8) and (5.1) that the family of processes $\{U_g(t,\tau)\}$, $g \in \mathcal{H}_{L^q_U(\mathbb{R}^N)}(g_0)$ has a bounded uniformly (w.r.t. $g \in \mathcal{H}_{L^q_U(\mathbb{R}^N)}(g_0)$) absorbing set B_α:

$$(5.16)\qquad B_\alpha = \left\{u \in H^{2\alpha,q}_U(\mathbb{R}^N) \mid \|u\|_{H^{2\alpha,q}_U(\mathbb{R}^N)} \le \widetilde{C}_{g,\widetilde{v}}\right\}$$

for all $1/2 \le \alpha < 1$, that is, for any $\tau \in \mathbb{R}$ and bounded $B \subset H^{1,q}_U(\mathbb{R}^N)$, there exists $T_0 = T(B,\tau) + 1 \ge \tau$ such that

$$\bigcup_{g\in\mathcal{H}_{L^q_U(\mathbb{R}^N)}(g_0)} U_g(t,\tau)B \subset B_\alpha, \quad \text{for all } t \ge T_0.$$

6. Uniform asymptotic compactness and attractor

Now we are in a position to formulate the main result of our paper.

Theorem 6.1. *Assume that $f(x,u)$ satisfies Assumptions* I *and* II *and the external force $g \in L^p_b(\mathbb{R}; L^q_U(\mathbb{R}^N))$ is the uniform normal with $p > 2$, $1 < q < N$ and $\rho \le N/(N-q)$. Then, the family of processes $\{U_g(t,\tau)\}$, $g \in \mathcal{H}_{L^q_U(\mathbb{R}^N)}(g_0)$, generated by solutions of problem* (1.1)-(1.2) *has a compact uniform (w.r.t. $g \in \mathcal{H}_{L^q_U(\mathbb{R}^N)}(g_0)$) attractor $\mathcal{A}_{\mathcal{H}_{L^q_U(\mathbb{R}^N)}(g_0)}$ in $H^{1,q}_\phi(\mathbb{R}^N)$, which satisfies*

(a) *$\mathcal{A}_{\mathcal{H}_{L^q_U(\mathbb{R}^N)}(g_0)}$ is compact in $H^{1,q}_\phi(\mathbb{R}^N)$ and attracts every bounded subset of $H^{1,q}_U(\mathbb{R}^N)$ with respect to the $H^{1,q}_\phi(\mathbb{R}^N)$-norm;*

(b) *$\mathcal{A}_{\mathcal{H}_{L^q_U(\mathbb{R}^N)}(g_0)}$ is contained in any closed uniformly (w.r.t. $g\in\mathcal{H}_{L^q_U(\mathbb{R}^N)}(g_0)$) attracting set $\mathcal{A}'$, $\mathcal{A}' \subset H^{1,q}_U(\mathbb{R}^N)$;*

(c) *$\mathcal{A}_{\mathcal{H}_{L^q_U(\mathbb{R}^N)}(g_0)}$ is closed and bounded in $H^{1,q}_U(\mathbb{R}^N)$;*

(d) *the following formula holds:*

$$\begin{aligned}\mathcal{A}_{\mathcal{H}_{L^q_U(\mathbb{R}^N)}(g_0)} = \omega_{0,\mathcal{H}_{L^q_U(\mathbb{R}^N)}(g_0)}(B_{1/2}) &= \omega_{\tau,\mathcal{H}_{L^q_U(\mathbb{R}^N)}(g_0)}(B_{1/2}) \\ &= \bigcup_{B\in\mathcal{B}(H^{1,q}_U(\mathbb{R}^N))} \omega_{\tau,\mathcal{H}_{L^q_U(\mathbb{R}^N)}(g_0)}(B),\end{aligned}$$

for all $\tau \in \mathbb{R}$ and $B_{1/2}$ is the bounded uniformly (w.r.t. $g \in \mathcal{H}_{L^q_U(\mathbb{R}^N)}(g_0)$) absorbing set defined as in (5.16).

Furthermore, $\mathcal{A}_{\mathcal{H}_{L^q_U(\mathbb{R}^N)}(g_0)} \subset H^{2\alpha,q}_\phi(\mathbb{R}^N)$, $1/2 \le \alpha < 1$, that attracts uniformly (w.r.t. $\sigma \in \mathcal{H}(\sigma_0)$) bounded set of $H^{1,q}_U(\mathbb{R}^N)$ in the norm of $H^{2\alpha,q}_\phi(\mathbb{R}^N)$.

In order to prove the family of $\{U_\sigma(t,\tau)\}$ possesses some compactness in the sense of the process, we will recall the concept of uniform asymptotic compactness that is proposed by Moise et.al. in [28] and is different from the one given by Haraux [20] and Chepyzhov and Vishik [12]. From Example 2.2 of [27] we know that there is no compactness of the semigroup in the norm of the locally uniform spaces, but in the corresponding weighted spaces. Hence, we will prove the asymptotically compact of the family of processes $\{U_g(t,\tau), g \in \mathcal{H}_{L^q_U(\mathbb{R}^N)}(g_0)\}$ in the weighted norm. First of all, we recall the following definition.

Definition 6.2. Let E and $\mathcal{H}(\sigma_0)$ be two complete metric spaces. A family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, on E with the symbol $\sigma \in \mathcal{H}(\sigma_0)$ is said to be uniformly (w.r.t. $\sigma \in \mathcal{H}(\sigma_0)$) asymptotically compact, if and only if for any fixed $\tau \in \mathbb{R}$, a bounded sequence $\{u_n\}_{n=1}^\infty \subset E$, $\{\sigma_n\}_{n=1}^\infty \subset \mathcal{H}(\sigma_0)$, and any $\{t_n\}_{n=1}^\infty \subset [\tau,\infty)$ with $t_n \to \infty$ as $n \to \infty$, the sequence $\{U_{\sigma_n}(t_n,\tau)u_n\}_{n=1}^\infty$ is precompact in E.

In view of our scopes, the main tool is the following abstract result from [25]. For convenience, we denote by $\mathcal{B}(E)$ all of the bounded set of E.

THEOREM 6.3. *Let E be a complete metric space and the translation semigroup $\{T(t)\}$ be continuous invariant, that is, $T(t)\mathcal{H}(\sigma_0) = \mathcal{H}(\sigma_0)$ satisfying the translation identity* (2.7). *The family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \mathcal{H}(\sigma_0)$ possesses a compact uniform* (*w.r.t. $\sigma \in \mathcal{H}(\sigma_0)$*) *attractor $\mathcal{A}_{\mathcal{H}(\sigma_0)}$ in E satisfying*:

$$\mathcal{A}_{\mathcal{H}(\sigma_0)} = \omega_{0,\mathcal{H}(\sigma_0)}(B_0) = \omega_{\tau,\mathcal{H}(\sigma_0)}(B_0) = \bigcup_{B\in\mathcal{B}(E)} \omega_{\tau,\mathcal{H}(\sigma_0)}(B)$$

for all $\tau \in \mathbb{R}$, where

$$\omega_{\tau,\mathcal{H}(\sigma_0)}(B) := \bigcap_{t\geq\tau} \overline{\bigcup_{\sigma\in\mathcal{H}(\sigma_0)} \bigcup_{s\geq t} U_\sigma(s,\tau)B},$$

if and only if it

(a) *has a bounded uniformly* (*w.r.t. $\sigma \in \mathcal{H}(\sigma_0)$*) *absorbing set B_0, and*
(b) *is uniformly* (*w.r.t. $\sigma \in \mathcal{H}(\sigma_0)$*) *asymptotically compact.*

Next, we will derive some priori estimates about the difference of two solutions which will be used to obtain required uniform (w.r.t. $\sigma \in \Sigma$) asymptotic compactness. To this end, let $u^i(t)$, $i = 1,2$ be the solution of problem (1.1) with the initial data u^i_τ belongings to $B_{1/2} \subset H^{1,q}_U(\mathbb{R}^N)$ and associated with the external force $g^i \in L^p_b(\mathbb{R}, L^q_U(\mathbb{R}^N))$, and for convenience, set $w(t) = u^1(t) - u^2(t)$ and $f^i(s) = f(x, u^i(s))$. Thanks to the translation identity (2.7) and the invariant property of $T(t)$, it follows that for any fixed $\tau \in \mathbb{R}$ and $g \in \mathcal{H}_{L^q_U(\mathbb{R}^N)}(g_0)$, there is $\widetilde{g} \in \mathcal{H}_{L^q_U(\mathbb{R}^N)}(g_0)$ such that

$$U_{\widetilde{g}}(t+\tau,\tau)u = U_g(t,0)u \quad \text{for } t \geq 0,\ u \text{ is a solution of (1.1)–(1.2).}$$

Thus, without loss of generality, we start with $\tau = 0$ and $w(t)$ solves the following integral equations:

$$\begin{aligned} w(t) &= e^{(\Delta+\mu_0 I)t} w_\tau \\ &\quad + \int_0^t e^{(\Delta+\mu_0 I)(t-s)} \left[(g^1(s) - g^2(s)) - (f^1(s) - f^2(s)) - \mu_0 w(s) \right] ds. \end{aligned}$$

Since $w_0(x) \in H^{1,q}_U(\mathbb{R}^N)$, by taking the norm $H^{2\alpha,q}_\phi(\mathbb{R}^N)$ with $1/2 \leq \alpha < 1$ on the two sides of the above equation, it follows for some $\theta_3 > 0$ and $t \in [0,T]$,

$$\begin{aligned} \|w(t)\|_{H^{2\alpha,q}_\phi(\mathbb{R}^N)} &\leq \|e^{(\Delta+\mu_0 I)t} w_0\|_{H^{2\alpha,q}_\phi(\mathbb{R}^N)} \\ &\quad + \int_0^t \|e^{(\Delta+\mu_0 I)(t-s)} \\ &\quad \cdot \left[(g^1(s) - g^2(s)) - (f^1(s) - f^2(s)) - \mu_0 w(s) \right] \|_{H^{2\alpha,q}_\phi(\mathbb{R}^N)} ds \\ &\leq M \frac{e^{\theta_3 t}}{t^\alpha} \|w(0)\|_{L^q_\phi(\mathbb{R}^N)} \\ &\quad + M \int_0^t \frac{e^{\theta_3(t-s)}}{(t-s)^\alpha} \|(g^1 - g^2) - (f^1 - f^2) - \mu_0 w(s))\|_{L^q_\phi(\mathbb{R}^N)} ds \end{aligned} \tag{6.1}$$

$$\le M\frac{e^{\theta_3 t}}{t^\alpha}\|w(0)\|_{L^q_\phi(\mathbb{R}^N)} + M_{T,\theta_3,p}\|g^1-g^2\|_{L^p_b(\mathbb{R};L^q_\phi(\mathbb{R}^N))} + MC_T\int_0^t (t-s)^{-\alpha}\|(f^1-f^2)+\mu_0 w(s))\|_{L^q_\phi(\mathbb{R}^N)}\,ds.$$

Now, we estimate the difference of the nonlinearity $f^i(s)$, $i=1,2$, in the weighted space $L^q_\phi(\mathbb{R}^N)$. Similar to (4.5), it fields

$$\begin{aligned}
\text{(6.2)}\qquad &\|(f^1-f^2)+\mu_0 w(s))\|_{L^q_\phi(\mathbb{R}^N)}\\
&\le\bigg(\int_{\mathbb{R}^N}\phi|f^1-f^2|^q\,dx\bigg)^{1/q}+\mu_0\|w(s)\|_{L^q_\phi(\mathbb{R}^N)}\\
&\le C\bigg(\int_{\mathbb{R}^N}\phi(x)|u_1-u_2|^q(1+|u_1|^{\rho-1}+|u_2|^{\rho-1})^q\,dx\bigg)^{1/q}\\
&\quad+\mu_0\|w(s)\|_{L^q_\phi(\mathbb{R}^N)}\\
&\le C_{\|u^1\|_{H^{1,q}_U(\mathbb{R}^N)},\|u^2\|_{H^{1,q}_U(\mathbb{R}^N)}}\bigg(\int_{\mathbb{R}^N}\phi(z)\|w\|^q_{H^{1,q}(B(z,1))}\,dz\bigg)^{1/q}\\
&\quad+\mu_0\|w(s)\|_{L^q_\phi(\mathbb{R}^N)}\\
&\le C_{\|u^1\|_{H^{1,q}_U(\mathbb{R}^N)},\|u^2\|_{H^{1,q}_U(\mathbb{R}^N)},\mu_0}\|w(s)\|_{H^{2\alpha,q}_\phi(\mathbb{R}^N)},
\end{aligned}$$

where we used Sobolev type inclusion $H^{2\alpha,q}_\phi(\mathbb{R}^N)\hookrightarrow L^q_\phi(\mathbb{R}^N)$ for $q<N$, $1/2\le\alpha<1$ and a weight function given as in Definition 2.1. Then, we insert (6.2) into the right hand side of inequality (6.1) and use Lemma 2.7, it follows that there exists a positive constant $\widehat{C}=C_{M,C_T,\|u^1\|_{H^{1,q}_U(\mathbb{R}^N)},\|u^2\|_{H^{1,q}_U(\mathbb{R}^N)},\mu_0}$ such that

$$\text{(6.3)}\quad \|w(t)\|_{H^{2\alpha,q}_\phi(\mathbb{R}^N)}\le\frac{C_{\theta_3,p,\widehat{C}}}{t^\alpha}\|w(0)\|_{L^q_\phi(\mathbb{R}^N)}+C_{\theta_3,p,\widehat{C}}\|g^1-g^2\|_{L^p_b(\mathbb{R};L^q_\phi(\mathbb{R}^N))},$$

for $t\in(0,T]$.

Before starting with the existence of compact uniform attractor, we first introduce a new class of external forces which is an extension of the Definition 3.1 of [25] in the setting of locally uniform spaces. More properties will be given in Appendix A.

Definition 6.4. A function $g\in L^p_{\rm loc}(\mathbb{R};L^q_U(\mathbb{R}^N))$ with $p>2$ is said to be uniform normal if for any $\varepsilon>0$, there exists $\eta>0$ such that

$$\sup_{t\in\mathbb{R}}\int_t^{t+\eta}\|g(s)\|^p_{L^q_U(\mathbb{R}^N)}\,ds\le\varepsilon.$$

The set of all the uniform normal functions is denoted by $L^p_{un}(\mathbb{R};L^q_U(\mathbb{R}^N))$. It is obvious that $L^p_{un}(\mathbb{R};L^q_U(\mathbb{R}^N))\subset L^p_b(\mathbb{R};L^q_U(\mathbb{R}^N))$.

Theorem 6.5. *Let $g\in L^p_b(\mathbb{R};L^q_U(\mathbb{R}^N))$ with $p>2$ is the uniform normal. Under the assumptions in Theorem 5.5, the family of processes $\{U_g(t,\tau)\}$, $g\in$*

$\mathcal{H}_{L^q_U(\mathbb{R}^N)}(g_0)$, *generated by solutions of problem* (1.1)–(1.2) *is uniformly* (*w.r.t.* $g \in \mathcal{H}_{L^q_U(\mathbb{R}^N)}(g_0)$) *asymptotically compact in* $H^{2\alpha,q}_{\phi}(\mathbb{R}^N)$.

Proof. Let u_n be the solutions corresponding to initial data $u_0^n \in B_{1/2} \subset H^{1,q}_U(\mathbb{R}^N)$ w.r.t. the symbol $g_n \in \mathcal{H}_{L^q_U(\mathbb{R}^N)}(g_0))$, $n = 1, 2, \ldots$, and let $\tau^* > 0$ be fixed. Then, from the term (b) of Lemma 2.5 we know that the embedding $H^{1,q}_U(\mathbb{R}^N) \hookrightarrow L^q_\phi(\mathbb{R}^N)$ is compact, we extract a subsequence $\{u^n\}_{n=1}^\infty$, where $u^n = U_{g_n}(t_n - \tau^*, 0)u_0^n$ satisfying $t_n \gg \tau^*$, without loss of generality we still denote by $\{u^n\}_{n=1}^\infty$, which is a Cauchy sequence in $L^q_\phi(\mathbb{R}^N)$.

Since the symbol space $\mathcal{H}_{L^q_U(\mathbb{R}^N)}(g_0)$ is invariant w.r.t. the translation semigroup $\{T(s)\}$, $s \geq 0$, then it follows from (5.1)

$$\begin{aligned}
\sup_{t\in\mathbb{R}} &\int_t^{t+1} \|\widetilde{g}(\tau^* + s_n) - \widetilde{g}(\tau^* + s_m)\|^p_{L^q_\phi(\mathbb{R}^N)}\, d\tau^* \\
&= \sup_{t\in\mathbb{R}} \int_t^{t+1} \|T(s_n)\widetilde{g}(\tau^*) - T(s_m)\widetilde{g}(\tau^*)\|^p_{L^q_\phi(\mathbb{R}^N)}\, d\tau^* \\
&\leq \sup_{t\in\mathbb{R}} \int_t^{t+1} [\|T(s_n)\widetilde{g}(\tau^*)\|^p_{L^q_\phi(\mathbb{R}^N)} + \|T(s_m)\widetilde{g}(\tau^*)\|^p_{L^q_\phi(\mathbb{R}^N)}]\, d\tau^* \\
&= \sup_{t\in\mathbb{R}} \int_t^{t+1} [\|\widetilde{g}(\tau^* + s_n)\|^p_{L^q_\phi(\mathbb{R}^N)} + \|\widetilde{g}(\tau^* + s_m)\|^p_{L^q_\phi(\mathbb{R}^N)}]\, d\tau^* \\
&\leq \sup_{t\in\mathbb{R}} \int_t^{t+1} [\|\widetilde{g}(\tau^* + s_n)\|^p_{L^q_U(\mathbb{R}^N)} + \|\widetilde{g}(\tau^* + s_m)\|^p_{L^q_U(\mathbb{R}^N)}]\, d\tau^* \\
&\leq \sup_{t\in\mathbb{R}} \int_t^{t+1} \|g_0(\tau^* + s_n)\|^p_{L^q_U(\mathbb{R}^N)}\, d\tau^* \\
&\quad + \sup_{t\in\mathbb{R}} \int_t^{t+1} \|g_0(\tau^* + s_m)\|^p_{L^q_U(\mathbb{R}^N)}\, d\tau^* := I_{g_n} + I_{g_m}.
\end{aligned}$$

Note that g_0 is the uniform normal, then according to the definition 6.4, for any $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that

$$\sup_{t\in\mathbb{R}} \int_{t+(k-1)\eta}^{t+k\eta} \|g_0(\tau^* + s_n)\|^p_{L^q_U(\mathbb{R}^N)}\, d\tau^* \leq \frac{\varepsilon}{1 + [1/\eta]}, \quad k \in \mathbb{Z}^+,$$

which implies for $1/\eta < k \leq 1 + [1/\eta]$, where $[r]$ stands for the integer part of r,

$$\begin{aligned}
I_{g_n} &\leq \sup_{t\in\mathbb{R}} \int_t^{t+k\eta} \|g_0(\tau^* + s_n)\|^p_{L^q_U(\mathbb{R}^N)}\, d\tau^* \\
&\leq \sup_{t\in\mathbb{R}} \int_t^{t+\eta} \|g_0\|^p_{L^q_U(\mathbb{R}^N)}\, d\tau^* + \sup_{t\in\mathbb{R}} \int_{t+\eta}^{t+2\eta} \|g_0\|^p_{L^q_U(\mathbb{R}^N)}\, d\tau^* \\
&\quad + \ldots + \sup_{t\in\mathbb{R}} \int_{t+(k-1)\eta}^{t+k\eta} \|g_0\|^p_{L^q_U(\mathbb{R}^N)}\, d\tau^* \leq k\frac{\varepsilon}{1 + [1/\eta]} \leq \varepsilon,
\end{aligned}$$

and $I_{g_m} \le \varepsilon$. Thus, from the translation identity (2.7), we have

$$
\begin{aligned}
(6.4)\qquad &\|\widetilde{u}^n(t_n) - \widetilde{u}^m(t_m)\|_{H_\phi^{2\alpha,q}(\mathbb{R}^N)} \\
&= \|U_{g_n}(t_n,0)u_0^n - U_{g_m}(t_n,0)u_0^m\|_{H_\phi^{2\alpha,q}(\mathbb{R}^N)} \\
&= \|U_{g_n}(t_n,t_n-\tau^*)U_{g_n}(t_n-\tau^*,0)u_0^n \\
&\quad - U_{g_m}(t_m,t_m-\tau^*)U_{g_m}(t_m-\tau^*,0)u_0^m\|_{H_\phi^{2\alpha,q}(\mathbb{R}^N)} \\
&= \|U_{g_n}(t_n,t_n-\tau^*)u^n - U_{g_m}(t_m,t_m-\tau^*)u^m\|_{H_\phi^{2\alpha,q}(\mathbb{R}^N)} \\
&= \|U_{\widetilde{g}_n}(\tau^*,0)u^n - U_{\widetilde{g}_m}(\tau^*,0)u^m\|_{H_\phi^{2\alpha,q}(\mathbb{R}^N)} \\
&\le \frac{C_{\theta_3,p,\widehat{C}}}{(\tau^*)^\alpha}\|u^n - u^m\|_{L_\phi^q(\mathbb{R}^N)} + C_{\theta_3,p,\widehat{C}}\|\widetilde{g}_n(\tau^*) - \widetilde{g}_m(\tau^*)\|_{L_b^p(\mathbb{R};L_\phi^q(\mathbb{R}^N))} \\
&\le C_{\theta_3,p,\widehat{C}}\bigg(\frac{1}{(\tau^*)^\alpha} + 2\bigg)\varepsilon.
\end{aligned}
$$

Thus the proof of Theorem 6.5 is complete. □

In particular, if the symbol $g_0(t)$ belonging to the hull $\mathcal{H}_{L_U^q(\mathbb{R}^N)}(g_0)$ is a tr.c. function, then according to Theorems 5.4 and 6.5, we have from Corollary 5.2 of [11],

$$
\mathcal{A}_{\mathcal{H}_{L_U^q(\mathbb{R}^N)}(g_0)} = \omega_{0,\mathcal{H}_{L_U^q(\mathbb{R}^N)}(g_0)}(B_{1/2}) = \bigcup_{g\in\mathcal{H}_{L_U^q(\mathbb{R}^N)}(g_0)} \mathcal{K}_g(0),
$$

where $\mathcal{K}_g(0)$ is the kernel section of the family of the processes $\{U_g(t,\tau)\}$, $g \in \mathcal{H}_{L_U^q(\mathbb{R}^N)}(g_0)$ at time 0 generated by the solutions of problem (1.1)–(1.2).

Proof of Theorem 6.1. The first part can be immediately obtained by applying the results of Theorems 5.5, 6.5 to the abstract Theorem 6.3 of existence.

Now, we will prove the second part of this theorem by contradiction. Assume that there exists $\varepsilon_0 \ge 0$, the sequence $\{t_n\}_{n=1}^\infty \subset [\tau,\infty)$ with $t_n \to \infty$ as $n \to \infty$ and $g_n \in \mathcal{H}_{L_U^q(\mathbb{R}^N)}(g_0)$ such that for each $x_n \in B$ and $y_n \in \mathcal{A}_{\mathcal{H}_{L_U^q(\mathbb{R}^N)}(g_0)}$,

$$
\operatorname{dist}_{H_\phi^{2\alpha,q}(\mathbb{R}^N)}(U_{g_n}(t_n,\tau)x_n, y_n) \ge \varepsilon_0. \tag{6.5}
$$

On the other hand, from the attracting property of $\mathcal{A}_{\mathcal{H}_{L_U^q(\mathbb{R}^N)}(g_0)}$ in the norm of $H_\phi^{1,q}(\mathbb{R}^N)$ we know that for any $\varepsilon > 0$, $t_n \to \infty$ as $n \to \infty$ and $g_n \in \mathcal{H}_{L_U^q(\mathbb{R}^N)}(g_0)$ and any $x_n \in B$, there exists $y_n \in \mathcal{A}_{\mathcal{H}_{L_U^q(\mathbb{R}^N)}(g_0)}$ such that

$$
\operatorname{dist}_{H_\phi^{1,q}(\mathbb{R}^N)}(U_{g_n}(t_n,\tau)x_n, y_n) < \varepsilon. \tag{6.6}
$$

Thanks to any ε-neighbourhood of $\mathcal{A}_{\mathcal{H}_{L_U^q(\mathbb{R}^N)}(g_0)}$ is a uniformly (w.r.t. $g_n \in \mathcal{H}_{L_U^q(\mathbb{R}^N)}(g_0)$) absorbing set, it follows from minimality property of $\mathcal{A}_{\mathcal{H}_{L_U^q(\mathbb{R}^N)}(g_0)}$,

$$
\operatorname{dist}_{H_\phi^{2\alpha,q}(\mathbb{R}^N)}(U_{g_n}(t_n,t_n-t^*)y_n, y_n) < \varepsilon. \tag{6.7}
$$

Let $t^* > 0$ be fixed, we can obtain from the translation identity (2.7), a priori estimate (6.3) and (6.6),

$$
\begin{aligned}
(6.8)\quad &\|U_{g_n}(t_n,\tau)x_n - U_{g_n}(t_n,t_n-t^*)y_n\|_{H_\phi^{2\alpha,q}(\mathbb{R}^N)}\\
&= \|U_{g_n}(t_n,t_n-t^*)U_{g_n}(t_n-t^*,\tau)x_n - U_{g_n}(t_n,t_n-t^*)y_n\|_{H_\phi^{2\alpha,q}(\mathbb{R}^N)}\\
&= \|U_{\widetilde{g}_n}(t^*,0)U_{g_n}(t_n-t^*,\tau)x_n - U_{\widetilde{g}_n}(t^*,0)y_n\|_{H_\phi^{2\alpha,q}(\mathbb{R}^N)}\\
&\le \frac{C_{\theta_3,p,\widehat{C}}}{(t^*)^\alpha}\|U_{g_n}(t_n-t^*,\tau)x_n - y_n\|_{L_\phi^q(\mathbb{R}^N)} + C_{\theta_3,p,\widehat{C}}k\varepsilon\\
&\le C_{\theta_3,p,\widehat{C}}\left(\frac{1}{(t^*)^\alpha}+2\right)\varepsilon.
\end{aligned}
$$

Therefore, combining with (6.7) and (6.8), one has

$$
\operatorname{dist}_{H_\phi^{2\alpha,q}(\mathbb{R}^N)}(U_{g_n}(t_n,\tau)x_n, y_n) \to 0, \quad \text{as } n \to \infty,
$$

which contradicts (6.5). $\square$

Appendix A. Some properties of uniform normal external forces

We present some result concerning the locally uniform spaces $L^p_{\mathrm{loc}}(\mathbb{R}; L^q_U(\mathbb{R}^N))$ with $1 < p$ and $1 < q < N$. Recall that a function $\sigma(s) \in L^p_{\mathrm{loc}}(\mathbb{R}; L^q_U(\mathbb{R}^N))$ is said to be translation compact (tr.c.) if the closure of $\{\sigma(s+h) \mid h \in \mathbb{R}\}$ is compact in $L^p_{\mathrm{loc}}(\mathbb{R}; L^q_U(\mathbb{R}^N))$. The set of all tr.c. functions in $L^p_{\mathrm{loc}}(\mathbb{R}; L^q_U(\mathbb{R}^N))$ is denoted by $L^p_c(\mathbb{R}; L^q_U(\mathbb{R}^N))$.

We first recall two propositions (see [10], [11]) that give the compactness criterion in $L^p(t_1,t_2; L^q_U(\mathbb{R}^N))$.

PROPOSITION A.1. *A set $\Sigma \subset L^p_{\mathrm{loc}}(\mathbb{R}; L^q_U(\mathbb{R}^N))$ is precompact in $L^p_{\mathrm{loc}}(\mathbb{R}; L^q_U(\mathbb{R}^N))$ if and only if the set $\Sigma|_\Omega$ is precompact in $L^p_{\mathrm{loc}}(\mathbb{R}; L^q(\Omega))$ for every bounded subset $\Omega \subset \mathbb{R}^N$. Here $\Sigma|_\Omega$ denotes the restriction of the set Σ to the subset Ω.*

According to the definition of translation compact function in $L^p_{\mathrm{loc}}(\mathbb{R}; L^q_U(\mathbb{R}^N))$ and this proposition, a function $g \in L^p_{\mathrm{loc}}(\mathbb{R}; L^q_U(\mathbb{R}^N))$ is tr.c. if and only if

$$
\{T(h)g_0(t) \mid h \in \mathbb{R},\ t \in [0,1]\} \quad \text{is precompact in } L^p([0,1]; L^q(\Omega)).
$$

The following proposition gives a compactness criterion in $L^p_{\mathrm{loc}}(\mathbb{R}; L^q_U(\mathbb{R}^N))$, whose proof is essentially given in Proposition 3.2 of [11].

PROPOSITION A.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. Then a function $\psi(s)$ is tr.c. in $L^p_{\mathrm{loc}}(\mathbb{R}; L^q(\Omega))$ if and only if*

(a) *for any fixed $h \in \mathbb{R}$ the set $\left\{\int_t^{t+h}\psi(s)\,ds \mid t \in \mathbb{R}\right\}$ is precompact in $L^q(\Omega)$;*

(b) *there exists a positive function* $\xi(s)$ *with* $\xi(s) \to 0^+$ *as* $s \to 0^+$ *such that*

$$\int_t^{t+1} \|\psi(s) - \psi(s+l)\|^p_{L^q(\Omega)}\, ds \le \xi(|l|) \quad \text{for all } t \in \mathbb{R}. \tag{A.1}$$

The following relationship between tr.c. function space $L^p_c(\mathbb{R}; L^q_U(\mathbb{R}^N))$, uniform normal function space $L^p_{un}(\mathbb{R}; L^q_U(\mathbb{R}^N))$ and $L^p_b(\mathbb{R}; L^q_U(\mathbb{R}^N))$ has been essentially established for the bounded domain of $\mathbb{R}^N$ in [25]. According to Propositions A.1, A.2 and the definition of topology of these spaces, we can prove the following conclusion by using the method of [25]. Hence, we omit details here.

Theorem A.3. *Let* $p, q \ge 1$*, then*

(a) $L^p_c(\mathbb{R}; L^q_U(\mathbb{R}^N))$ *is a closed subspace of* $L^p_b(\mathbb{R}; L^q_U(\mathbb{R}^N))$;
(b) $L^p_c(\mathbb{R}; L^q_U(\mathbb{R}^N)) \subset L^p_{un}(\mathbb{R}; L^q_U(\mathbb{R}^N))$;
(c) $L^p_{un}(\mathbb{R}; L^q_U(\mathbb{R}^N))$ *is a closed subspace of* $L^p_b(\mathbb{R}; L^q_U(\mathbb{R}^N))$.

In particular, if the external force $g(x,t)$ is given by $g(x,t) = m(t)n(x)$, $n(x) \in L^q_U(\mathbb{R}^N)$, and let $m(t) \in L^p_{loc}(\mathbb{R})$ be the uniform normal, i.e.

$$\sup_{t\in\mathbb{R}} \int_t^{t+\eta} \|m(s)\|^p\, ds \le \varepsilon,$$

then

$$\begin{aligned}\sup_{t\in\mathbb{R}} \int_t^{t+\eta} \|g(s)\|^p_{L^q_U(\mathbb{R}^N)}\, ds &= \sup_{t\in\mathbb{R}} \int_t^{t+\eta} \|m(t)n(x)\|^p_{L^q_U(\mathbb{R}^N)}\, ds \\ &\le \|n(x)\|^p_{L^q_U(\mathbb{R}^N)} \sup_{t\in\mathbb{R}} \int_t^{t+\eta} |m(s)|^p\, ds \le \varepsilon \|n(x)\|^p_{L^q_U(\mathbb{R}^N)}.\end{aligned}$$

Now, we will give some examples to explain the inclusion relationship of these spaces given by Theorem A.3.

Example A.4. For $k_1, k_2 = 1, 2, \ldots$, we take

$$n(x) = \begin{cases} k_1, & |x| \in [k_1, k_1 + 1/k_1^q], \\ 0, & \text{otherwise}, \end{cases} \quad \text{and} \quad m(s) = \begin{cases} k_2^{1/p}, & s \in [k_2, k_2 + 1/k_2], \\ 0, & \text{otherwise}, \end{cases}$$

it follows that

$$\begin{aligned}\|g(x,s)\|^p_{L^p_b(\mathbb{R}; L^q_U(\mathbb{R}^N))} &= \|m(s)n(x)\|^p_{L^p_b(\mathbb{R}; L^q_U(\mathbb{R}^N))} \\ &= \sup_{t\in\mathbb{R}} \int_t^{t+1} \|m(s)n(x)\|^p_{L^q_U(\mathbb{R}^N))} ds \le C.\end{aligned}$$

On the other hand, for any $\eta > 0$ there exist k_2 such that $1/k_2 \le \eta$, set $t = k_2$, then

$$\sup_{t\in\mathbb{R}} \int_t^{t+\eta} \|m(s)n(x)\|^p_{L^q_U(\mathbb{R}^N))}\, ds \ge \int_{k_2}^{k_2+1/k_2} \|m(s)\|^p\, ds = 1,$$

which implies $g(x,t)$ belonging to $L^p_b(\mathbb{R}; L^q_U(\mathbb{R}^N))$ but not $L^p_{un}(\mathbb{R}; L^q_U(\mathbb{R}^N))$.

EXAMPLE A.5. For $k_1, k_2 = 1, 2, \ldots$, we take

$$n(x) = \begin{cases} k_1, & |x| \in [k_1, k_1 + 1/k_1^q], \\ 0, & \text{otherwise}, \end{cases}$$

and

$$m(s) = \begin{cases} k_2^{1/p}, & s \in [k_2 + i/k_2, k_2 + i/k_2 + 1/k_2^p], \ i = 0, 1, \ldots, k_2^{p-1} - 1, \\ 0, & \text{otherwise}, \end{cases}$$

it yields

$$\begin{aligned} &\int_0^\eta \|m(s)n(x)\|^p_{L^q_U(\mathbb{R}^N))}\, ds \\ &\qquad \leq \begin{cases} (i+1)\dfrac{1}{k_2^p}(k_2^{1/p})^p \leq \dfrac{2i}{k_2^{p-1}} \leq 2\eta^{p-1} & \text{if } \dfrac{i}{k_2} \leq \eta < \dfrac{i+1}{k_2}, \ i \geq 1, \\ \dfrac{1}{k_2^p}(k_2^{1/p})^p \leq \dfrac{1}{k_2^{p-1}} \leq \eta^{(p-1)/p} & \text{if } \dfrac{1}{k_2^p} \leq \eta < \dfrac{1}{k_2}, \\ \eta(k_2^{1/p})^p \leq \eta\left(\dfrac{1}{\eta}\right)^{1/p} = \eta^{(p-1)/p} & \text{if } 0 \leq \eta < \dfrac{1}{k_2^p}. \end{cases} \end{aligned}$$

Thus, for any $0 < \varepsilon < 1/2^{1/(p-1)}$, let $\eta = \varepsilon^{p/(p-1)}$, and we have

$$\begin{aligned} \sup_{t\in\mathbb{R}} \int_t^{t+\eta} \|m(s)n(x)\|^p_{L^q_U(\mathbb{R}^N))}\, ds &= \sup_{t\in\mathbb{R}} \int_0^\eta \|m(t+s)n(x)\|^p_{L^q_U(\mathbb{R}^N))}\, ds \\ &= \sup_{k_2\in N} \int_0^\eta \|m(s+k_2)n(x)\|^p_{L^q_U(\mathbb{R}^N))}\, ds \leq \varepsilon. \end{aligned}$$

However, it is easy to see that for any $1/k_2^p \leq l < 1/k_2 - 1/k_2^p$ satisfying $k_2 > 2^{1/(p-1)}$,

$$\int_0^1 \|m(s+k_2)n(x) - m(s+k_2-l)n(x)\|^p_{L^q_U(\mathbb{R}^N))}\, ds = \int_0^1 k_2 \frac{1}{k_2^p}(2k_2^{p-1})2\, ds = 4,$$

which implies from Proposition A.2 that $g(x,t) = m(t)n(x)$ does not belong to $L^p_c(\mathbb{R}; L^q_U(\mathbb{R}^N))$ but in $L^p_{un}(\mathbb{R}; L^q_U(\mathbb{R}^N))$.

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Gaocheng Yue
Department of Mathematics
Nanjing University of Aeronautics and Astronautics
Nanjing, 211106, P.R. CHINA
E-mail address: yuegch@nuaa.edu.cn

Chengkui Zhong
Department of Mathematics
Nanjing University
Nanjing, 210093, P.R. CHINA