Topological **M**ethods in **N**onlinear **A**nalysis
Volume 46, No. 2, 2015, 665–693
DOI: 10.12775/TMNA.2015.063

EXISTENCE OF SOLUTIONS IN THE SENSE OF DISTRIBUTIONS OF ANISOTROPIC NONLINEAR ELLIPTIC EQUATIONS WITH VARIABLE EXPONENT

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Abstract. The aim of this paper is to study the existence of solutions in the sense of distributions for a strongly nonlinear elliptic problem where the second term of the equation f is in $W^{-1,\overrightarrow{p}'(\cdot)}(\Omega)$ which is the dual space of the anisotropic Sobolev $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ and later f will be in $L^1(\Omega)$.

1. Introduction

Let Ω be a bounded domain of $\mathbb{R}^N$ $(N \geq 2)$ with smooth boundary $\partial\Omega$. For the variable vectorial exponent $\overrightarrow{p}(\cdot) = (p_0(\cdot), \ldots, p_N(\cdot))$, we assume that for $i = 0, \ldots, N$, the functions $p_i(x) \in \mathcal{C}_+(\overline{\Omega})$ (defined in Section 2), where

$$p_0(x) \geq \max\{p_i(x),\ i = 1, \ldots, N\}, \quad \text{for any } x \in \overline{\Omega}. \tag{1.1}$$

2010 *Mathematics Subject Classification.* Primary: 35J66, 46E35; Secondary: 35D30.
Key words and phrases. Anisotropic Sobolev spaces, variable exponent, strongly nonlinear elliptic equations, boundary value problems, solution in the sense of distributions.

Our aim is to prove the existence of solutions in the sense of distributions to the anisotropic nonlinear elliptic problem:

$$
\begin{cases} -\sum_{i=1}^{N} \partial_{x_i} a_i(x,u,\nabla u) + g(x,u,\nabla u) + d(x)|u|^{p_0(x)-2}u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}
$$

where the right-hand side f is in $W^{-1,\overrightarrow{p}'(\,\cdot\,)}(\Omega)$ which is the dual space of the anisotropic Sobolev space $W_0^{1,\overrightarrow{p}(\,\cdot\,)}(\Omega)$ and later f will be in $L^1(\Omega)$. The positive function $d(x)$ belong to $L^\infty(\Omega)$, and there exists a constant $d_0 > 0$ such that $d(x) \geq d_0$ almost everywhere in Ω.

We assume that for $i = 1,\dots,N$ the function $a_i\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \mapsto \mathbb{R}$ is Carathéodory function (i.e. measurable with respect to x in Ω for every (s,ξ) in $\mathbb{R}\times\mathbb{R}^N$ and continuous with respect to (s,ξ) in $\mathbb{R}\times\mathbb{R}^N$ for almost every x in Ω) which satisfies the following conditions:

$$
|a_i(x,s,\xi)| \leq \beta\,(K_i(x) + |s|^{p_i(x)-1} + |\xi_i|^{p_i(x)-1}) \quad \text{for } i = 1,\dots,N, \tag{1.3}
$$

$$
a_i(x,s,\xi)\xi_i \geq \alpha|\xi_i|^{p_i(x)} \quad \text{for } i = 1,\dots,N, \tag{1.4}
$$

$a_i(\,\cdot\,,\,\cdot\,,\,\cdot\,)$ is strictly monotone, i.e. for all $\xi = (\xi_1,\dots,\xi_N)$ and $\xi' = (\xi'_1,\dots,\xi'_N)$ in $\mathbb{R}^N$, we have

$$
(a_i(x,s,\xi) - a_i(x,s,\xi'))(\xi_i - \xi'_i) > 0, \quad \text{for } \xi_i \neq \xi'_i, \tag{1.5}
$$

for almost every $x \in \Omega$ and all $(s,\xi) \in \mathbb{R}\times\mathbb{R}^N$, where $K_i(\,\cdot\,)$ is a non-negative function lying in $L^{p'_i(\,\cdot\,)}(\Omega)$ where $1/p_i(x) + 1/p'_i(x) = 1$ and $\alpha, \beta > 0$ are two positive constants.

Note that, Gwiazda et al. in [17] studied a steady and in [18] a dynamic model for non-Newtonian fluids under an additional strict monotonicity assumption on the operator. The authors used Young measure techniques in place of a monotonicity method. Moreover, a version of the Minty–Browder trick adapted to the setting of generalized Orlicz spaces was introduced in [27] (see also [19]) in framework of non-Newtonian fluids.

The nonlinear term $g(x,s,\xi)$ is a Carathéodory function which satisfies

$$
g(x,s,\xi)s \geq 0, \tag{1.6}
$$

$$
|g(x,s,\xi)| \leq b(|s|)(c(x) + \sum_{i=1}^{N} |\xi_i|^{p_i(x)}), \tag{1.7}
$$

where $b(\,\cdot\,)\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous non-decreasing function, and $c(\,\cdot\,)\colon \Omega \to \mathbb{R}^+$ with $c(\,\cdot\,) \in L^1(\Omega)$.

In view of (1.7), the Carathéodory function $g(x,u,\nabla u)$ does not define a mapping from $W_0^{1,\overrightarrow{p}(\,\cdot\,)}(\Omega)$ into its dual, but from $W_0^{1,\overrightarrow{p}(\,\cdot\,)}(\Omega) \cap L^\infty(\Omega)$ into $L^1(\Omega)$ (see also [9]).

In this paper, we prove that $g(x,u,\nabla u) \in L^1(\Omega)$ for $u \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ solution in the sense of distribution, in both cases of $f \in W^{-1,\overrightarrow{p}'(\cdot)}(\Omega)$ and $f \in L^1(\Omega)$.

The study of nonlinear elliptic equation involving p-Laplace operator is based on the theory of standard Sobolev spaces $W^{m,p}(\Omega)$ in order to find weak solutions. These spaces consist of functions that have weak derivatives and satisfy certain integrability conditions. In the case of $p(\cdot)$-Laplace equations, the natural setting for this approach is the use of the variable exponent Lebesgue and Sobolev spaces $L^{p(\cdot)}(\Omega)$ and $W^{m,p(\cdot)}(\Omega)$. These spaces were thoroughly studied in [12], [28]–[30] and the references therein. Partial differential equations and variational problems involving $p(\cdot)$-growth conditions have been extensively studied in the last decades; see e.g. [1]–[4], [10], [11]. The reason is that they can model various phenomena arising from the study of elastic mechanics, electrorheological fluids or image restoration (for more details see [25]).

In this paper, the operator involved in (1.2) is more general than the $p(\cdot)$-Laplace operator. Thus, the variable exponent Sobolev space $W^{m,p(\cdot)}(\Omega)$ is not adequate to study nonlinear problems of this type. This leads us to seek weak solutions for problem (1.2) in a more general variable exponent Sobolev space, the anisotropic variable exponent Sobolev space $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ which was recently introduced by Mihalescu–Pucci–Raduslescu in [25].

Note that, Benboubker et al. studied in [7] the following problem which is quite close to (1.2):

$$\text{(1.8)} \qquad \begin{cases} -\operatorname{div}(a(x,u,\nabla u)+\phi(u)) + g(x,u,\nabla u) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\mu \in W^{-1,p'(\cdot)}(\Omega) + L^1(\Omega)$. In [7] the authors proved among others result, existence of entropy solutions. Here, our operator is anisotropic and more general than the one used in [7]. In the case where $g \equiv 0$, $d \equiv 0$ and $f \in L^\infty(\Omega)$ the authors proved in [23] the existence and uniqueness of a weak energy solution.

We refer the reader to [20] for results on existence of renormalized solutions of elliptic problems of type

$$\text{(1.9)} \qquad \begin{cases} \beta(x,u) - \operatorname{div}(a(x,\nabla u)+\phi(u)) \ni f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with right-hand side $f \in L^1(\Omega)$. The function ϕ is assumed to be locally Lipschitz. The vector field $a(\cdot\,|\,\cdot)$ is monotone in the second variable and satisfies a non-standard growth condition described by an x-dependent convex function that generalizes both $L^{p(x)}$ and classical Orlicz settings. $\beta(x,\cdot)$ is a maximal monotone operator. In the case where $\beta \equiv 0$ and $\phi \equiv 0$, the authors proved in [16] existence results for the elliptic problem in a generalized Orlicz–Musielak spaces. For the basic properties of anisotropic Orlicz–Musielak spaces see [15] and [26]. We mention that in [26] the author considered an anisotropic parabolic problem.

Motivated by the papers [3], [23] and the ideas in [7], the main goal of this paper is to study the existence of solutions in the sense of distributions for problem (1.2). Indeed, using the Galerkin method we can thereby prove existence of a weak solution u_n of some approximate problem. In a second step we show that a subsequence of the approximate solutions u_n converges to a solution of problem (1.2). In this step we combine truncation techniques and basic properties of pseudo-monotone operators. We also use some idea devoted to renormalized solutions for elliptic problem [20].

The remaining part of this paper is organized as follows: Section 2 is devoted to mathematical preliminaries, including among other things, a brief discussion of anisotropic variable exponent Sobolev space. In Section 3, some technical Lemmas are given. The main existence results are stated and proved in Section 4.

2. Preliminaries

As the exponent $p_i(\,\cdot\,)$ appearing in (1.3), (1.4) and (1.7) depends on the variable x, we must work with Lebesgue and Sobolev spaces with variable exponent. For this purpose, in this section, we start by recalling some definitions of Lebesgue, Sobolev and anisotropic Sobolev spaces with variable exponent and give some of their properties. Roughly speaking, anistropic Lebesgue and Sobolev spaces are functional spaces of Lebesgue's and Sobolev's type in which different space directions have different roles.

Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$, we denote

$$\mathcal{C}_+(\overline{\Omega}) = \{\text{continuous function } p(\,\cdot\,)\colon \overline{\Omega} \mapsto \mathbb{R} \text{ such that } 1 < p^- \leq p^+ < \infty\},$$

where $p^+ = \max_{x\in\overline{\Omega}} p(x)$ and $p^- = \min_{x\in\overline{\Omega}} p(x)$.

We define the Lebesgue space with variable exponent $L^{p(\,\cdot\,)}(\Omega)$ as the set of all measurable functions $u\colon \Omega \mapsto \mathbb{R}$ for which the convex modular

$$\rho_{p(\,\cdot\,)}(u) := \int_\Omega |u|^{p(x)}\,dx$$

is finite. If the exponent is bounded, i.e. if $p^+ < +\infty$, then the expression

$$\|u\|_{p(\,\cdot\,)} = \inf\{\lambda > 0 : \rho_{p(\,\cdot\,)}(u/\lambda) \leq 1\}$$

defines a norm in $L^{p(\,\cdot\,)}(\Omega)$, called the Luxemburg norm.

The space $(L^{p(\,\cdot\,)}(\Omega), \|\cdot\|_{p(\,\cdot\,)})$ is a separable Banach space. Moreover, if $1 < p^- \leq p^+ < +\infty$, then $L^{p(\,\cdot\,)}(\Omega)$ is uniformly convex, hence reflexive, and its dual space is isomorphic to $L^{p'(\,\cdot\,)}(\Omega)$, where $1/p(x) + 1/p'(x) = 1$.

Finally, we have the Hölder type inequality:

$$\left| \int_\Omega uv\,dx \right| \le \left(\frac{1}{p^-} + \frac{1}{p'^-} \right) \|u\|_{p(\,\cdot\,)} \|v\|_{p'(\,\cdot\,)} \tag{2.1}$$

for all $u \in L^{p(\,\cdot\,)}(\Omega)$ and $v \in L^{p'(\,\cdot\,)}(\Omega)$.

An important role in manipulating the generalized Lebesgue and Sobolev spaces is played by the modular $\rho_{p(\,\cdot\,)}$ of the space $L^{p(\,\cdot\,)}(\Omega)$. We have the following result:

PROPOSITION 2.1 (see [14], [28]). *If $u_n, u \in L^{p(\,\cdot\,)}(\Omega)$, then the following properties hold true*:

(a) $\|u\|_{p(\,\cdot\,)} < 1$ *(resp.* $= 1, > 1$*) if and only if* $\rho(u) < 1$ *(resp.* $= 1, > 1$*),*

(b) *if* $\|u\|_{p(\,\cdot\,)} > 1$ *then* $\|u\|^{p^-}_{p(\,\cdot\,)} \le \rho(u) \le \|u\|^{p^+}_{p(\,\cdot\,)}$ *, and*
if $\|u\|_{p(\,\cdot\,)} < 1$ *then* $\|u\|^{p^+}_{p(\,\cdot\,)} \le \rho(u) \le \|u\|^{p^-}_{p(\,\cdot\,)}$*,*

(c) $\|u_n\|_{p(\,\cdot\,)} \to 0$ *if and only if* $\rho(u_n) \to 0$*, and*
$\|u_n\|_{p(\,\cdot\,)} \to \infty$ *if and only if* $\rho(u_n) \to \infty$*,*

which implies that the norm convergence and the modular convergence are equivalent.

Now, we define the variable exponent Sobolev space by

$$W^{1,p(\,\cdot\,)}(\Omega) = \{\, u \in L^{p(\,\cdot\,)}(\Omega) \text{ and } |\nabla u| \in L^{p(\,\cdot\,)}(\Omega)\},$$

which is a Banach space equipped with the following norm

$$\|u\|_{1,p(\,\cdot\,)} = \|u\|_{p(\,\cdot\,)} + \|\nabla u\|_{p(\,\cdot\,)} \quad \text{for all } u \in W^{1,p(\,\cdot\,)}(\Omega).$$

The space $(W^{1,p(\,\cdot\,)}(\Omega), \|\cdot\|_{1,p(\,\cdot\,)})$ is a separable and reflexive Banach space.

We denote by $W_0^{1,p(\,\cdot\,)}(\Omega)$ the closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{1,p(\,\cdot\,)}(\Omega)$, and we define the Sobolev exponent by

$$p^*(\,\cdot\,) = \begin{cases} \dfrac{Np(\,\cdot\,)}{N - p(\,\cdot\,)} & \text{for } p(\,\cdot\,) < N, \\ \infty & \text{for } p(\,\cdot\,) \ge N. \end{cases}$$

PROPOSITION 2.2 (see [13]).

(a) *Assuming* $p(\,\cdot\,) \in \mathcal{C}_+(\overline{\Omega})$ *i.e.* $1 < p^- \le p^+ < \infty$*, the spaces* $W^{1,p(\,\cdot\,)}(\Omega)$ *and* $W_0^{1,p(\,\cdot\,)}(\Omega)$ *are separable and reflexive Banach spaces.*

(b) *If* $p(\,\cdot\,)$*,* $q(\,\cdot\,)$ *are in* $\mathcal{C}_+(\overline{\Omega})$ *and* $q(x) < p^*(x)$ *for any* $x \in \overline{\Omega}$*, then the embedding* $W_0^{1,p(\,\cdot\,)}(\Omega) \hookrightarrow\hookrightarrow L^{q(\,\cdot\,)}(\Omega)$ *is continuous and compact.*

REMARK 2.3. Recall that the definition of these spaces requires only the measurability of $p(\,\cdot\,)$, in this work we do not need to use Sobolev and Poincaré inequality. Note that the sharp Sobolev inequality is proved for $p(\,\cdot\,)$ log-Hölder continuous, while the Poincaré inequality requires only the continuity of $p(\,\cdot\,)$, (see [12], [21]).

Now, we present the anisotropic Sobolev space with variable exponent which is used for the study of problem (1.2).

Let $p_0(x),\dots,p_N(x)$ be $N+1$ variable exponents in $\mathcal{C}_+(\overline{\Omega})$. We denote

$$\overrightarrow{p}(\,\cdot\,)=\{p_0(\,\cdot\,),\dots,p_N(\,\cdot\,)\},\quad \partial_{x_0}u=u \quad\text{and}\quad \partial_{x_i}u=\frac{\partial u}{\partial x_i} \quad \text{for } i=1,\dots,N,$$

and we define

$$\underline{p}=\min\{p_0^-,\dots,p_N^-\}\quad \text{then } \underline{p}>1. \tag{2.2}$$

The anisotropic variable exponent Sobolev space $W^{1,\overrightarrow{p}(\,\cdot\,)}(\Omega)$ is defined as follow:

$$W^{1,\overrightarrow{p}(\,\cdot\,)}(\Omega)=\{u\in L^{p_0(\,\cdot\,)}(\Omega),\ \partial_{x_i}u\in L^{p_i(\,\cdot\,)}(\Omega),\ i=1,\dots,N\},$$

endowed with the norm

$$\|u\|_{1,\overrightarrow{p}(\,\cdot\,)}=\sum_{i=0}^{N}\|\partial_{x_i}u\|_{L^{p_i(\,\cdot\,)}(\Omega)}. \tag{2.3}$$

We define also $W_0^{1,\overrightarrow{p}(\,\cdot\,)}(\Omega)$ as the closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{1,\overrightarrow{p}(\,\cdot\,)}(\Omega)$ with respect to the norm (2.3). The space $\big(W_0^{1,\overrightarrow{p}(\,\cdot\,)}(\Omega),\|u\|_{1,\overrightarrow{p}(\,\cdot\,)}\big)$ is a reflexive Banach space (cf. [25]).

Lemma 2.4. *We have the following continuous and compact embeddings*:

(a) *if* $\underline{p}<N$ *then* $W_0^{1,\overrightarrow{p}(\,\cdot\,)}(\Omega)\hookrightarrow\hookrightarrow L^q(\Omega)$, *for* $q\in[\underline{p},\underline{p}^*[$, *where* $\underline{p}^*=N\underline{p}/(N-\underline{p})$,

(b) *if* $\underline{p}=N$ *then* $W_0^{1,\overrightarrow{p}(\,\cdot\,)}(\Omega)\hookrightarrow\hookrightarrow L^q(\Omega)$, *for all* $q\in[\underline{p},+\infty[$,

(c) *if* $\underline{p}>N$ *then* $W_0^{1,\overrightarrow{p}(\,\cdot\,)}(\Omega)\hookrightarrow\hookrightarrow L^\infty(\Omega)\cap\mathcal{C}^0(\overline{\Omega})$.

The proof of this lemma follows from the fact that the embedding $W_0^{1,\overrightarrow{p}(\,\cdot\,)}(\Omega)\hookrightarrow W_0^{1,\underline{p}}(\Omega)$ is continuous, and in view of the compact embedding theorem for Sobolev spaces.

Proposition 2.5. *The dual of* $W_0^{1,\overrightarrow{p}(\,\cdot\,)}(\Omega)$ *is denote by* $W^{-1,\overrightarrow{p}'(\,\cdot\,)}(\Omega)$, *where* $\overrightarrow{p}'(\,\cdot\,)=\{p_0'(\,\cdot\,),\dots,p_N'(\,\cdot\,)\}$, $1/p_i'(\,\cdot\,)+1/p_i(\,\cdot\,)=1$, (*cf.* [8] *for the constant exponent case*), *and for each* $F\in W^{-1,\overrightarrow{p}'(\,\cdot\,)}(\Omega)$ *there exists* $f_i\in L^{p_i'(\,\cdot\,)}(\Omega)$ *for* $i=0,\dots,N$, *such that* $F=f_0-\sum_{i=1}^{N}\partial_{x_i}f_i$. *Moreover, for all* $u\in W_0^{1,\overrightarrow{p}(\,\cdot\,)}(\Omega)$, *we have*

$$\langle F,u\rangle=\sum_{i=0}^{N}\int_\Omega f_i\partial_{x_i}u\,dx.$$

We define a norm on the dual space by

$$\|F\|_{-1,\overrightarrow{p}'(\,\cdot\,)}=\sum_{i=0}^{N}\|f_i\|_{p_i'(\,\cdot\,)}.$$

3. Some technical lemmas

LEMMA 3.1 (see [22, Theorem 13.47]). *Let $(u_n)_n$ be a sequence in $L^1(\Omega)$ and $u \in L^1(\Omega)$ such that $u_n \to u$ almost everywhere, $u_n, u \geq 0$ almost everywhere and $\int_\Omega u_n\,dx \to \int_\Omega u\,dx$, then $u_n \to u$ in $L^1(\Omega)$.*

LEMMA 3.2 (see [6]). *Let $g \in L^{p(\cdot)}(\Omega)$ and $g_n \in L^{p(\cdot)}(\Omega)$ with $\|g_n\|_{p(\cdot)} \leq C$ for $1 < p(x) < \infty$. If $g_n(x) \to g(x)$ almost everywhere in Ω, then $g_n \rightharpoonup g$ in $L^{p(\cdot)}(\Omega)$.*

LEMMA 3.3. *Let $(u_n)_n$ be a bounded sequence in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$. If $u_n \rightharpoonup u$ in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$, then $T_k(u_n) \rightharpoonup T_k(u)$ in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$, where, for any $k > 0$, $T_k(\cdot)$ is the truncation function defined by $T_k(s) := \max\{-k, \min\{k, s\}\}$.*

PROOF. Since $u_n \rightharpoonup u$ in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$, we have $u_n \to u$ in $L^{\underline{p}}(\Omega)$. It follows that $u_n \to u$ almost everywhere in Ω, therefore $T_k(u_n) \to T_k(u)$ almost everywhere in Ω. Consequently,

$$\sum_{i=1}^N \int_\Omega |\partial_{x_i} T_k(u_n)|^{p_i(x)}\,dx = \sum_{i=1}^N \int_{\{|u_n| \leq k\}} |\partial_{x_i} u_n|^{p_i(x)}\,dx \leq \sum_{i=1}^N \int_\Omega |\partial_{x_i} u_n|^{p_i(x)}\,dx < \infty,$$

we deduce that $(T_k(u_n))_n$ is bounded in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$, then $T_k(u_n) \rightharpoonup v_k$ weakly in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$, therefore $v_k = T_k(u)$ and we obtain

$$T_k(u_n) \rightharpoonup T_k(u) \quad \text{in } W_0^{1,\overrightarrow{p}(\cdot)}(\Omega).$$ □

LEMMA 3.4. *Let $u \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ then $T_k(u) \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ for all $k > 0$. Moreover, we have $T_k(u) \to u$ in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ as $k \to \infty$.*

PROOF. We have $u \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$, it is clear that $T_k(u) \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ and

$$\begin{aligned}\sum_{i=0}^N \int_\Omega |\partial_{x_i} T_k(u) - \partial_{x_i} u|^{p_i(x)}\,dx &= \sum_{i=0}^N \int_{\{|u| \leq k\}} |\partial_{x_i} T_k(u) - \partial_{x_i} u|^{p_i(x)}\,dx \\ &\quad + \sum_{i=0}^N \int_{\{|u| > k\}} |\partial_{x_i} T_k(u) - \partial_{x_i} u|^{p_i(x)}\,dx \\ &= \int_{\{|u| > k\}} |T_k(u) - u|^{p_0(x)}\,dx + \sum_{i=1}^N \int_{\{|u| > k\}} |\partial_{x_i} u|^{p_i(x)}\,dx.\end{aligned}$$

Since $T_k(u) \to u$ almost everywhere in Ω as $k \to \infty$ and by using the dominated convergence theorem, we obtain

$$\int_{\{|u| > k\}} |T_k(u) - u|^{p_0(x)}\,dx + \sum_{i=1}^N \int_{\{|u| > k\}} |\partial_{x_i} u|^{p_i(x)}\,dx \to 0 \quad \text{as } k \to \infty.$$

Finally, $\|T_k(u) - u\|_{1,\overrightarrow{p}(\,\cdot\,)} \to 0$ as $k \to \infty$. □

Lemma 3.5. *Assuming that (1.3)–(1.5) hold and let $(u_n)_n$ be a sequence in $W_0^{1,\overrightarrow{p}(\,\cdot\,)}(\Omega)$ such that $u_n \rightharpoonup u$ in $W_0^{1,\overrightarrow{p}(\,\cdot\,)}(\Omega)$ and*

$$\int_\Omega (|u_n|^{p_0(x)-2}u_n - |u|^{p_0(x)-2}u)(u_n - u)\,dx + \sum_{i=1}^N \int_\Omega (a_i(x,u_n,\nabla u_n) - a_i(x,u_n,\nabla u))(\partial_{x_i}u_n - \partial_{x_i}u)\,dx \to 0, \tag{3.1}$$

then $u_n \to u$ in $W_0^{1,\overrightarrow{p}(\,\cdot\,)}(\Omega)$ for a subsequence.

Proof. Let

$$S_n = (|u_n|^{p_0(x)-2}u_n - |u|^{p_0(x)-2}u)(u_n - u) + \sum_{i=1}^N (a_i(x,u_n,\nabla u_n) - a_i(x,u_n,\nabla u))(\partial_{x_i}u_n - \partial_{x_i}u)$$

thanks to (1.5) we have S_n is a positive function, and by (3.1), $S_n \to 0$ in $L^1(\Omega)$ as $n \to \infty$.

Since $u_n \rightharpoonup u$ in $W_0^{1,\overrightarrow{p}(\,\cdot\,)}(\Omega)$, using the compact embedding we obtain $u_n \to u$ in $L^{\underline{p}}(\Omega)$, and since $S_n \to 0$ almost everywhere in Ω, there exists a subset B in Ω of measure zero such that for all $x \in \Omega \setminus B$: $|u(x)| < \infty$, $|\partial_{x_i}u(x)| < \infty$, $K_i(x) < \infty$, $u_n \to u$ and $S_n \to 0$ almost everywhere in Ω.

By using (1.3)–(1.4) and some estimates, we obtain

$$\begin{aligned}
S_n(x) &= \sum_{i=1}^N (a_i(x,u_n,\nabla u_n) - a_i(x,u_n,\nabla u))(\partial_{x_i}u_n - \partial_{x_i}u) \\
&\quad + (|u_n|^{p_0(x)-2}u_n - |u|^{p_0(x)-2}u)(u_n - u) \\
&= \sum_{i=1}^N a_i(x,u_n,\nabla u_n)\partial_{x_i}u_n + \sum_{i=1}^N a_i(x,u_n,\nabla u)\partial_{x_i}u \\
&\quad - \sum_{i=1}^N a_i(x,u_n,\nabla u)\partial_{x_i}u_n - \sum_{i=1}^N a_i(x,u_n,\nabla u_n)\partial_{x_i}u \\
&\quad + |u_n|^{p_0(x)} + |u|^{p_0(x)} - |u_n|^{p_0(x)-2}u_n u - |u|^{p_0(x)-2}u u_n \\
&\geq \underline{\alpha}\sum_{i=0}^N |\partial_{x_i}u_n|^{p_i(x)} + \underline{\alpha}\sum_{i=0}^N |\partial_{x_i}u|^{p_i(x)} - |u_n|^{p_0(x)-1}|u| - |u|^{p_0(x)-1}|u_n| \\
&\quad - \beta\sum_{i=1}^N (K_i(x) + |u_n|^{p_i(x)-1} + |\partial_{x_i}u|^{p_i(x)-1})|\partial_{x_i}u_n| \\
&\quad - \beta\sum_{i=1}^N (K_i(x) + |u_n|^{p_i(x)-1} + |\partial_{x_i}u_n|^{p_i(x)-1})|\partial_{x_i}u|
\end{aligned}$$

$$\geq \underline{\alpha}\sum_{i=0}^{N}|\partial_{x_i}u_n|^{p_i(x)} - C_x\sum_{i=0}^{N}(1+|\partial_{x_i}u_n|^{p_i(x)-1}+|\partial_{x_i}u_n|),$$

with $\underline{\alpha} = \min(\alpha, 1)$ and C_x depending on x, without dependence on n. (Since $u_n(x) \to u(x)$ then $(u_n(x))_n$ is bounded), we obtain

$$S_n(x) \geq \sum_{i=0}^{N}|\partial_{x_i}u_n|^{p_i(x)}\bigg(\underline{\alpha} - \frac{C_x}{|\partial_{x_i}u_n|^{p_i(x)}} - \frac{C_x}{|\partial_{x_i}u_n|} - \frac{C_x}{|\partial_{x_i}u_n|^{p_i(x)-1}}\bigg),$$

by the standard argument $(\partial_{x_i}u_n)_n$ is bounded almost everywhere in Ω for all $i = 0, \dots, N$. (Indeed, if $|\partial_{x_i}u_n| \to \infty$ in a measurable subset $E \subset \Omega$ which has a positive measure, then

$$\lim_{n\to\infty}\int_\Omega S_n(x)\,dx \geq \lim_{n\to\infty}\sum_{i=0}^{N}\int_E|\partial_{x_i}u_n|^{p_i(x)} \times \bigg(\underline{\alpha} - \frac{C_x}{|\partial_{x_i}u_n|^{p_i(x)}} - \frac{C_x}{|\partial_{x_i}u_n|} - \frac{C_x}{|\partial_{x_i}u_n|^{p_i(x)-1}}\bigg)\,dx = \infty,$$

which is absurd since $S_n \to 0$ in $L^1(\Omega)$).

Let ξ_i^* be an accumulation point of $(\partial_{x_i}u_n)_n$ for $i = 1, \dots, N$. We have $|\xi_i^*| < \infty$ and by the continuity of $a(x, \cdot, \cdot)$, we obtain

$$(a_i(x, u, \xi^*) - a_i(x, u, \nabla u))(\xi_i^* - \partial_{x_i}u) = 0 \quad \text{for } i = 1, \dots, N,$$

thanks to (1.5) we have $\xi^* = \nabla u$, the uniqueness of the accumulation point implies that $\nabla u_n \to \nabla u$ almost everywhere in Ω. Since $(a_i(x, u_n, \nabla u_n))_n$ is bounded in $L^{p_i'(\cdot)}(\Omega)$ and $a_i(x, u_n, \nabla u_n) \to a_i(x, u, \nabla u)$ almost everywhere in Ω, by the Lemma 3.2, we can establish that

$$a_i(x, u_n, \nabla u_n) \rightharpoonup a_i(x, u, \nabla u) \quad \text{in } L^{p_i'(\cdot)}(\Omega) \text{ for } i = 1, \dots, N.$$

Using (3.1) and the Lemma 3.1, we deduce that

$$|u_n|^{p_0(x)} \to |u|^{p_0(x)} \quad \text{in } L^1(\Omega), \tag{3.2}$$

$$a_i(x, u_n, \nabla u_n)\partial_{x_i}u_n \to a_i(x, u, \nabla u)\partial_{x_i}u \quad \text{in } L^1(\Omega). \tag{3.3}$$

According to the condition (1.4), we have

$$\alpha|\partial_{x_i}u_n|^{p_i(x)} \leq a_i(x, u_n, \nabla u_n)\partial_{x_i}u_n \quad \text{for } i = 1, \dots, N.$$

Let $y_n^i = (1/\alpha)a_i(x, u_n, \nabla u_n)\partial_{x_i}u_n$ and $y^i = (1/\alpha)a_i(x, u, \nabla u)\partial_{x_i}u$, we have

$$0 \leq y_n^i + y^i - |\partial_{x_i}u_n|^{p_i(x)} - |\partial_{x_i}u|^{p_i(x)} \leq y_n^i + y^i - \frac{1}{2^{p_i^+-1}}|\partial_{x_i}u_n - \partial_{x_i}u|^{p_i(x)},$$

using Fatou's Lemma, we get

$$\int_\Omega 2y^i\,dx \leq \liminf_{n\to\infty}\int_\Omega\bigg(y_n^i + y^i - \frac{1}{2^{p_i^+-1}}|\partial_{x_i}u_n - \partial_{x_i}u|^{p_i(x)}\bigg)\,dx,$$

then

$$0 \leq -\limsup_{n\to\infty} \int_\Omega |\partial_{x_i} u_n - \partial_{x_i} u|^{p_i(x)}\,dx,$$

and since

$$0 \leq \liminf_{n\to\infty} \int_\Omega |\partial_{x_i} u_n - \partial_{x_i} u|^{p_i(x)}\,dx \leq \limsup_{n\to\infty} \int_\Omega |\partial_{x_i} u_n - \partial_{x_i} u|^{p_i(x)}\,dx \leq 0,$$

it follows that

$$\int_\Omega |\partial_{x_i} u_n - \partial_{x_i} u|^{p_i(x)}\,dx \to 0 \quad \text{as } n \to \infty,$$

and finally we obtain $\partial_{x_i} u_n \to \partial_{x_i} u$ in $L^{p_i(\,\cdot\,)}(\Omega)$ for $i = 1, \ldots, N$. In view of (3.2) we deduce that $u_n \to u$ in $W_0^{1,\overrightarrow{p}(\,\cdot\,)}(\Omega)$. □

4. Main results

4.1. The case of $f \in W^{-1,\overrightarrow{p}'(\,\cdot\,)}(\Omega)$.

Definition 4.1. In the case of $f \in W^{-1,\overrightarrow{p}'(\,\cdot\,)}(\Omega)$, a measurable function u is said to be a solution in the sense of distributions to the problem (1.2), if

$$\begin{cases} \displaystyle\sum_{i=1}^N \int_\Omega a_i(x,u,\nabla u)\partial_{x_i} v\,dx + \int_\Omega g(x,u,\nabla u)v\,dx \\ \displaystyle\qquad + \int_\Omega d(x)|u|^{p_0(x)-2}uv\,dx = \int_\Omega fv\,dx, \\ u \in W_0^{1,\overrightarrow{p}(\,\cdot\,)}(\Omega), \quad g(x,u,\nabla u) \in L^1(\Omega), \quad g(x,u,\nabla u)u \in L^1(\Omega) \end{cases} \tag{4.1}$$

for any $v \in W_0^{1,\overrightarrow{p}(\,\cdot\,)}(\Omega) \cap L^\infty(\Omega)$.

Theorem 4.2. *Assuming that* (1.3)–(1.7) *hold and* $f \in W^{-1,\overrightarrow{p}'(\,\cdot\,)}(\Omega)$. *Then the problem* (1.2) *has at least one solution in the sense of distributions.*

Remark 4.3. The assumption (1.1) is essential to ensure that $|a_i(x,u,\nabla u)|$ belongs to $L^{p_i'(\,\cdot\,)}(\Omega)$. In the case of $Au = -\sum_{i=1}^N \partial_{x_i} a_i(x,\nabla u)$ the existence of solution in the sense of distributions is guaranteed, without using this assumption.

Proof of the Theorem 4.2.
Step 1. Approximate problems. We consider the approximate problem:

$$\begin{cases} A_n u_n + g_n(x,u_n,\nabla u_n) + d(x)|u_n|^{p_0(x)-2}u_n = f, \\ u_n \in W_0^{1,\overrightarrow{p}(\,\cdot\,)}(\Omega). \end{cases} \tag{4.2}$$

Let us define the operator A_n from $W_0^{1,\overrightarrow{p}(\,\cdot\,)}(\Omega)$ into its dual $W^{-1,\overrightarrow{p}'(\,\cdot\,)}(\Omega)$, by

$$A_n v = -\sum_{i=1}^N \frac{\partial}{\partial x_i} a_i(x, T_n(v), \nabla v),$$

and

$$g_n(x,s,\xi)=\frac{g(x,s,\xi)}{1+|g(x,s,\xi)|/n}.$$

Note that $g_n(x,s,\xi)s\geq 0$, $|g_n(x,s,\xi)|\leq|g(x,s,\xi)|$ and $|g_n(x,s,\xi)|\leq n$ for all $n\in\mathbb{N}\setminus\{0\}$.

We define $G_n\colon W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)\mapsto W^{-1,\overrightarrow{p}'(\cdot)}(\Omega)$, by

$$\langle G_nu,v\rangle=\int_\Omega g_n(x,u,\nabla u)v\,dx+\int_\Omega d(x)|u|^{p_0(x)-2}u\ v\,dx$$

for all $u,v\in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$. Thanks to the Hölder inequality, we have for all $u,v\in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$

$$\begin{aligned}
(4.3)\qquad &\bigg|\int_\Omega g_n(x,u,\nabla u)v\,dx+\int_\Omega d(x)|u|^{p_0(x)-2}uv\,dx\bigg|\\
&\leq\bigg(\frac{1}{p_0^-}+\frac{1}{(p_0')^-}\bigg)(\|g_n(x,u,\nabla u)\|_{p_0'(x)}\\
&\quad+\|d\|_\infty\||u|^{p_0(x)-1}\|_{p_0'(x)})\|v\|_{p_0(x)}\\
&\leq\bigg(\frac{1}{p_0^-}+\frac{1}{(p_0')^-}\bigg)\bigg(\bigg(\int_\Omega n^{p_0'(x)}\,dx+1\bigg)^{1/(p_0')^-}\\
&\quad+\|d\|_\infty\bigg(\int_\Omega|u|^{p_0(x)}\,dx+1\bigg)^{1/(p_0')^-}\bigg)\|v\|_{1,\overrightarrow{p}(\cdot)}\\
&\leq\bigg(\frac{1}{p_0^-}+\frac{1}{(p_0')^-}\bigg)\bigg((n^{(p_0')^+}.\mathrm{meas}(\Omega)+1)^{1/(p_0')^-}\\
&\quad+\|d\|_\infty\bigg(\int_\Omega|u|^{p_0(x)}\,dx+1)^{1/(p_0')^-}\bigg)\bigg)\|v\|_{1,\overrightarrow{p}(\cdot)}\leq C_0\|v\|_{1,\overrightarrow{p}(\cdot)}.
\end{aligned}$$

LEMMA 4.4. *The operator $B_n=A_n+G_n$ from $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ into $W^{-1,\overrightarrow{p}'(\cdot)}(\Omega)$ is pseudo-monotone. Moreover, B_n is coercive in the following sense*:

$$\frac{\langle B_nv,v\rangle}{\|v\|_{1,\overrightarrow{p}(\cdot)}}\to+\infty\quad\textit{if } \|v\|_{1,\overrightarrow{p}(\cdot)}\to+\infty \textit{ for } v\in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega).$$

PROOF OF LEMMA 4.4. Using the Hölder's inequality and the growth condition (1.3) we can show that the operator A_n is bounded, and by using (4.3) we conclude that B_n is bounded.

For the coercivity, we have for any $u\in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$,

$$\begin{aligned}
\langle B_nu,u\rangle&=\langle A_nu,u\rangle+\langle G_nu,u\rangle\\
&=\sum_{i=1}^N\int_\Omega a_i(x,T_n(u),\nabla u)\partial_{x_i}u\,dx+\int_\Omega g_n(x,u,\nabla u)u\,dx+\int_\Omega d(x)|u|^{p_0(x)}\,dx\\
&\geq\delta\sum_{i=0}^N\int_\Omega|\partial_{x_i}u|^{p_i(x)}\,dx\geq\delta\|u\|^{\underline{p}}_{1,\overrightarrow{p}(\cdot)}-\delta(N+1),
\end{aligned}$$

with $\delta = \min(\alpha, d_0)$, then

$$\frac{\langle B_n u, u\rangle}{\|u\|_{1,\overrightarrow{p}(\,\cdot\,)}} \to +\infty \quad \text{as } \|u\|_{1,\overrightarrow{p}(\,\cdot\,)} \to +\infty.$$

It remains to show that B_n is pseudo-monotone. Let $(u_k)_k$ be a sequence in $W_0^{1,\overrightarrow{p}(\,\cdot\,)}(\Omega)$ such that

$$\begin{cases} u_k \rightharpoonup u & \text{in } W_0^{1,\overrightarrow{p}(\,\cdot\,)}(\Omega), \\ B_n u_k \rightharpoonup \chi_n & \text{in } W^{-1,\overrightarrow{p}'(\,\cdot\,)}(\Omega), \\ \limsup_{k\to\infty} \langle B_n u_k, u_k\rangle \le \langle \chi_n, u\rangle. \end{cases} \tag{4.4}$$

We have to prove that $\chi_n = B_n u$ and $\langle B_n u_k, u_k\rangle \to \langle \chi_n, u\rangle$ as $k \to +\infty$.

Firstly, since $W_0^{1,\overrightarrow{p}(\,\cdot\,)}(\Omega) \hookrightarrow\hookrightarrow L^{\underline{p}}(\Omega)$, then $u_k \to u$ in $L^{\underline{p}}(\Omega)$ for a subsequence still denoted $(u_k)_k$.

The sequence $(u_k)_k$ is bounded in $W_0^{1,\overrightarrow{p}(\,\cdot\,)}(\Omega)$, then by using the growth condition $a_i(x, T_n(u_k), \nabla u_k)$ is bounded in $L^{p_i'(\,\cdot\,)}(\Omega)$, therefore there exists a function $\varphi_i^n \in L^{p_i'(\,\cdot\,)}(\Omega)$ such that

$$a_i(x, T_n(u_k), \nabla u_k) \rightharpoonup \varphi_i^n \quad \text{in } L^{p_i'(\,\cdot\,)}(\Omega) \text{ as } k \to \infty. \tag{4.5}$$

Similarly, since $(g_n(x, u_k, \nabla u_k))_k$ is bounded in $L^{\underline{p}'}(\Omega)$, with $\underline{p}'$ is the conjugate exponent of $\underline{p}$, then there exists a function $\psi_n \in L^{\underline{p}'}(\Omega)$ such that

$$g_n(x, u_k, \nabla u_k) \rightharpoonup \psi_n \quad \text{in } L^{\underline{p}'}(\Omega) \text{ as } k \to \infty. \tag{4.6}$$

For all $v \in W_0^{1,\overrightarrow{p}(\,\cdot\,)}(\Omega)$, we have

$$\begin{aligned} \langle \chi_n, v\rangle &= \lim_{k\to\infty} \langle B_n u_k, v\rangle \\ &= \lim_{k\to\infty} \sum_{i=1}^N \int_\Omega a_i(x, T_n(u_k), \nabla u_k)\partial_{x_i} v\,dx \\ &\quad + \lim_{k\to\infty} \int_\Omega g_n(x, u_k, \nabla u_k) v\,dx + \lim_{k\to\infty} \int_\Omega d(x)|u_k|^{p_0(x)-2} u_k\ v\,dx \\ &= \sum_{i=1}^N \int_\Omega \varphi_i^n \partial_{x_i} v\,dx + \int_\Omega \psi_n v\,dx + \int_\Omega d(x)|u|^{p_0(x)-2} u\ v\,dx. \end{aligned} \tag{4.7}$$

By using (4.4) and (4.7), we obtain

$$\begin{aligned} \limsup_{k\to\infty} \langle B_n u_k, u_k\rangle &= \limsup_{k\to\infty} \bigg\{ \sum_{i=1}^N \int_\Omega a_i(x, T_n(u_k), \nabla u_k)\partial_{x_i} u_k\,dx \\ &\qquad + \int_\Omega g_n(x, u_k, \nabla u_k) u_k\,dx + \int_\Omega d(x)|u_k|^{p_0(x)}\,dx \bigg\} \\ &\le \sum_{i=1}^N \int_\Omega \varphi_i^n \partial_{x_i} u\,dx + \int_\Omega \psi_n u\,dx + \int_\Omega d(x)|u|^{p_0(x)}\,dx, \end{aligned} \tag{4.8}$$

thanks to (4.6), and since $u_k \to u$ in $L^{\underline{p}}(\Omega)$, then

$$\int_\Omega g_n(x, u_k, \nabla u_k) u_k \, dx \to \int_\Omega \psi_n u \, dx, \tag{4.9}$$

therefore

$$\limsup_{k\to\infty} \left\{ \sum_{i=1}^N \int_\Omega a_i(x, T_n(u_k), \nabla u_k) \partial_{x_i} u_k \, dx + \int_\Omega d(x) |u_k|^{p_0(x)} \, dx \right\} \leq \sum_{i=1}^N \int_\Omega \varphi_i^n \partial_{x_i} u \, dx + \int_\Omega d(x) |u|^{p_0(x)} \, dx. \tag{4.10}$$

On the other hand, using (1.5), we have

$$\sum_{i=1}^N \int_\Omega (a_i(x, T_n(u_k), \nabla u_k) - a_i(x, T_n(u_k), \nabla u))(\partial_{x_i} u_k - \partial_{x_i} u) \, dx + \int_\Omega d(x)(|u_k|^{p_0(x)-2} u_k - |u|^{p_0(x)-2} u)(u_k - u) \, dx \geq 0,$$

then

$$\begin{aligned} &\sum_{i=1}^N \int_\Omega a_i(x, T_n(u_k), \nabla u_k) \partial_{x_i} u_k \, dx + \int_\Omega d(x) |u_k|^{p_0(x)} \, dx \\ &\quad \geq \sum_{i=1}^N \int_\Omega a_i(x, T_n(u_k), \nabla u_k) \partial_{x_i} u \, dx + \int_\Omega d(x) |u_k|^{p_0(x)-2} u_k u \, dx \\ &\quad\quad + \sum_{i=1}^N \int_\Omega a_i(x, T_n(u_k), \nabla u)(\partial_{x_i} u_k - \partial_{x_i} u) \, dx + \int_\Omega d(x) |u|^{p_0(x)-2} u (u_k - u) \, dx. \end{aligned}$$

In view of Lebesgue dominated convergence theorem, we have $T_n(u_k) \to T_n(u)$ in $L^{p_i(\,\cdot\,)}(\Omega)$ then $a_i(x, T_n(u_k), \nabla u) \to a_i(x, T_n(u), \nabla u)$ in $L^{p_i'(\,\cdot\,)}(\Omega)$, and by (4.5), we get

$$\liminf_{k\to\infty} \left\{ \sum_{i=1}^N \int_\Omega a_i(x, T_n(u_k), \nabla u_k) \partial_{x_i} u_k \, dx + \int_\Omega d(x) |u_k|^{p_0(x)} \, dx \right\} \geq \sum_{i=1}^N \int_\Omega \varphi_i^n \partial_{x_i} u \, dx + \int_\Omega d(x) |u|^{p_0(x)} \, dx,$$

this implies, thanks to (4.10), that

$$\lim_{k\to\infty} \left\{ \sum_{i=1}^N \int_\Omega a_i(x, T_n(u_k), \nabla u_k) \partial_{x_i} u_k \, dx + \int_\Omega d(x) |u_k|^{p_0(x)} \, dx \right\} = \sum_{i=1}^N \int_\Omega \varphi_i^n \partial_{x_i} u \, dx + \int_\Omega d(x) |u|^{p_0(x)} \, dx. \tag{4.11}$$

By combining (4.8), (4.9) and (4.11), we deduce that $\langle B_n u_k, u_k\rangle \to \langle \chi_n, u\rangle$ as $k \to +\infty$. Now, by (4.11) we obtain

$$\lim_{k\to+\infty}\bigg\{\sum_{i=1}^N\int_\Omega (a_i(x,T_n(u_k),\nabla u_k)-a_i(x,T_n(u_k),\nabla u))(\partial_{x_i}u_k-\partial_{x_i}u)\,dx + d_0\int_\Omega (|u_k|^{p_0(x)-2}u_k-|u|^{p_0(x)-2}u)(u_k-u)\,dx\bigg\}=0.$$

In view of Lemma 3.5, we get $u_k \to u$ in $W_0^{1,\vec{p}(\cdot)}(\Omega)$ and $\partial_{x_i}u_k \to \partial_{x_i}u$ almost everywhere in Ω, then $a_i(x,T_n(u_k),\nabla u_k) \rightharpoonup a_i(x,T_n(u),\nabla u)$ in $L^{p_i'(\cdot)}(\Omega)$for $i=1,\dots,N$ and $g_n(x,u_k,\nabla u_k) \rightharpoonup g_n(x,u,\nabla u)$ in $L^{p_0'(\cdot)}(\Omega)$, we deduce that $\chi_n = B_n u$, which completes the proof of Lemma 4.4.

In view of Lemma 4.4, there exists at least one weak solution $u_n \in W_0^{1,\vec{p}(\cdot)}(\Omega)$ of the problem (4.2), (cf. [24, Theorem 2.7, p. 180]).

Step 2. A priori estimates. Taking u_n as a test function in (4.2), we obtain

$$\sum_{i=1}^N\int_\Omega a_i(x,T_n(u_n),\nabla u_n)\partial_{x_i}u_n\,dx+\int_\Omega g_n(x,u_n,\nabla u_n)u_n\,dx + \int_\Omega d(x)|u_n|^{p_0(x)}\,dx = \langle f,u_n\rangle_{W^{-1,\vec{p}'(\cdot)}(\Omega),W_0^{1,\vec{p}(\cdot)}(\Omega)},$$

since $g_n(x,u_n,\nabla u_n)u_n \geq 0$, using the generalized Hölder inequality and (1.4), we deduce that

$$\begin{aligned}\delta\|u_n\|^{\underline{p}}_{1,\vec{p}(\cdot)}-\alpha N-d_0 &\leq \alpha\sum_{i=1}^N\int_\Omega|\partial_{x_i}u_n|^{p_i(x)}\,dx+d_0\int_\Omega|u_n|^{p_0(x)}\,dx\\ &\leq \sum_{i=0}^N\bigg(\frac{1}{p_i^-}+\frac{1}{(p_i')^-}\bigg)\|f_i\|_{p_i'(\cdot)}\|u_n\|_{p_i(\cdot)} \leq 2\|f\|_{-1,\vec{p}'(\cdot)}\|u_n\|_{1,\vec{p}(\cdot)}\end{aligned}$$

with $\delta=\min(\alpha,d_0)$. Therefore,

$$\|u_n\|_{1,\vec{p}(\cdot)}\leq C_1, \tag{4.12}$$

with C_1 is a constant that does not depend on n. Then there exists a subsequence still denoted $(u_n)_n$ such that

$$\begin{cases} u_n \rightharpoonup u & \text{in } W_0^{1,\vec{p}(\cdot)}(\Omega),\\ u_n \to u & \text{in } L^{\underline{p}}(\Omega).\end{cases} \tag{4.13}$$

Finally, by using Lemma 3.3 and the Lebesgue dominated convergence theorem, we conclude for any $k>0$ that

$$\begin{cases} T_k(u_n) \rightharpoonup T_k(u) & \text{in } W_0^{1,\vec{p}(\cdot)}(\Omega),\\ T_k(u_n) \to T_k(u) & \text{in } L^{p_0(\cdot)}(\Omega).\end{cases} \tag{4.14}$$

Step 3. The strong convergence of $(T_k(u_n))_n$ in $W_0^{1,\vec{p}(\cdot)}(\Omega)$. In the sequel, we denote by $\varepsilon_i(n)$ $i = 1, 2, \dots$ various functions of real numbers which converge to 0 as n tends to infinity.

Let $\varphi_k(s) = s.\exp(\gamma s^2)$ where $\gamma = (b(k)/(2\alpha))^2$, it is obvious that (see [9, Lemma 1])

$$\varphi_k'(s) - \frac{b(k)}{\alpha}|\varphi_k(s)| \geq \frac{1}{2} \quad \text{for all } s \in \mathbb{R}.$$

We set $\omega_n = T_k(u_n) - T_k(u)$. By taking $\varphi_k(\omega_n)$ as a test function in the approximate problem (4.2), we obtain

$$\sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_n), \nabla u_n)\varphi_k'(\omega_n)\partial_{x_i}\omega_n \, dx + \int_\Omega g_n(x, u_n, \nabla u_n)\varphi_k(\omega_n)\, dx$$
$$+ \int_\Omega d(x)|u_n|^{p_0(x)-2}u_n\varphi_k(\omega_n)\, dx = \langle f, \varphi_k(\omega_n)\rangle_{W^{-1,\vec{p}'(\cdot)}(\Omega), W_0^{1,\vec{p}(\cdot)}(\Omega)}.$$

It is easy to see that ω_n have the same sign that u_n on the set $\{|u_n| > k\}$, (indeed if $u_n > k$ then $\omega_n = T_k(u_n) - T_k(u) = k - T_k(u) \geq 0$, and similarly, we prove that $\omega_n \leq 0$ for $u_n < -k$). Using (1.6) we get

$$\begin{aligned}
\text{(4.15)} \qquad & \sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_n), \nabla u_n)\varphi_k'(\omega_n)\partial_{x_i}\omega_n \, dx \\
& + \int_{\{|u_n|\leq k\}} g_n(x, T_k(u_n), \nabla T_k(u_n))\varphi_k(\omega_n)\, dx \\
& + \int_{\{|u_n|\leq k\}} d(x)|T_k(u_n)|^{p_0(x)-2}T_k(u_n)\varphi_k(\omega_n)\, dx \\
& = \langle f, \varphi_k(\omega_n)\rangle_{W^{-1,\vec{p}'(\cdot)}(\Omega), W_0^{1,\vec{p}(\cdot)}(\Omega)}.
\end{aligned}$$

On the one hand, we have

$$\begin{aligned}
\text{(4.16)} \qquad & \int_\Omega a_i(x, T_n(u_n), \nabla u_n)\varphi_k'(\omega_n)\partial_{x_i}\omega_n \, dx \\
& = \int_\Omega a_i(x, T_k(u_n), \nabla T_k(u_n))\varphi_k'(\omega_n)(\partial_{x_i}T_k(u_n) - \partial_{x_i}T_k(u))\, dx \\
& \quad + \int_{\{|u_n|>k\}} a_i(x, T_k(u_n), \nabla T_k(u_n))\varphi_k'(\omega_n)\partial_{x_i}T_k(u)\, dx \\
& \quad - \int_{\{|u_n|>k\}} a_i(x, T_n(u_n), \nabla u_n)\varphi_k'(\omega_n)\partial_{x_i}T_k(u)\, dx \\
& = \int_\Omega (a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u)))\varphi_k'(\omega_n) \\
& \quad \times (\partial_{x_i}T_k(u_n) - \partial_{x_i}T_k(u))\, dx \\
& \quad + \int_\Omega a_i(x, T_k(u_n), \nabla T_k(u))\varphi_k'(\omega_n)(\partial_{x_i}T_k(u_n) - \partial_{x_i}T_k(u))\, dx
\end{aligned}$$

$$+\int_{\{|u_n|>k\}} a_i(x,T_k(u_n),\nabla T_k(u_n))\varphi_k'(\omega_n)\partial_{x_i}T_k(u)\,dx$$
$$-\int_{\{|u_n|>k\}} a_i(x,T_n(u_n),\nabla u_n)\varphi_k'(\omega_n)\partial_{x_i}T_k(u)\,dx.$$

We have $T_k(u_n)\to T_k(u)$ in $L^{p_i(\,\cdot\,)}(\Omega)$, using the Lebesgue dominated convergence theorem and (1.3), we get $a_i(x,T_k(u_n),\nabla T_k(u))\to a_i(x,T_k(u),\nabla T_k(u))$ in $L^{p_i'(\,\cdot\,)}(\Omega)$, and since $\partial_{x_i}T_k(u_n)$ tends weakly to $\partial_{x_i}T_k(u)$ in $L^{p_i(\,\cdot\,)}(\Omega)$, we obtain

$$\begin{aligned}(4.17)\quad \varepsilon_1(n)&=\bigg|\int_\Omega a_i(x,T_k(u_n),\nabla T_k(u))\varphi_k'(\omega_n)(\partial_{x_i}T_k(u_n)-\partial_{x_i}T_k(u))\,dx\bigg|\\&\le\varphi_k'(2k)\int_\Omega|a_i(x,T_k(u_n),\nabla T_k(u))|\cdot|\partial_{x_i}T_k(u_n)-\partial_{x_i}T_k(u)|\,dx\to0\end{aligned}$$

as $n\to\infty$. Concerning the third term on the right-hand side of (4.16), the sequence $(a_i(x,T_k(u_n),\nabla T_k(u_n)))_n$ is bounded in $L^{p_i'(\,\cdot\,)}(\Omega)$, then, there exists $\xi_i\in L^{p_i'(\,\cdot\,)}(\Omega)$ such that $|a_i(x,T_k(u_n),\nabla T_k(u_n))|\rightharpoonup\xi_i$ in $L^{p_i'(\,\cdot\,)}(\Omega)$. Also, we have $|\partial_{x_i}T_k(u)|\chi_{\{|u_n|>k\}}\to|\partial_{x_i}T_k(u)|\chi_{\{|u|>k\}}$ in $L^{p_i(\,\cdot\,)}(\Omega)$, it follows that

$$\begin{aligned}(4.18)\qquad \varepsilon_2(n)&=\bigg|\int_{\{|u_n|>k\}} a_i(x,T_k(u_n),\nabla T_k(u_n))\varphi_k'(\omega_n)\partial_{x_i}T_k(u)\,dx\bigg|\\&\le\varphi_k'(2k)\int_{\{|u_n|>k\}}|a_i(x,T_k(u_n),\nabla T_k(u_n))||\partial_{x_i}T_k(u)|\,dx\\&\to\varphi_k'(2k)\int_{\{|u|>k\}}\xi_i|\partial_{x_i}T_k(u)|\,dx=0.\end{aligned}$$

For the last term on the right-hand side of (4.16), thanks to (4.13) we know that $a_i(x,T_n(u_n),\nabla u_n)$ is bounded in $L^{p_i'(\,\cdot\,)}(\Omega)$. Then, similarly as in (4.18) we can prove that

$$(4.19)\qquad \varepsilon_3(n)=\int_{\{|u_n|>k\}} a_i(x,T_n(u_n),\nabla u_n)\varphi_k'(\omega_n)\partial_{x_i}T_k(u)\,dx\to0$$

as $n\to\infty$. By combining (4.16)–(4.19), we get

$$\begin{aligned}(4.20)\quad &\int_\Omega(a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u)))\\&\qquad\times\varphi_k'(\omega_n)(\partial_{x_i}T_k(u_n)-\partial_{x_i}T_k(u))\,dx\\&\qquad\qquad=\int_\Omega a_i(x,T_n(u_n),\nabla u_n)\varphi_k'(\omega_n)\partial_{x_i}\omega_n\,dx+\varepsilon_4(n).\end{aligned}$$

On the other hand, using (1.4) and (1.7) we have

$$\begin{aligned}&\bigg|\int_{\{|u_n|\le k\}} g_n(x,T_k(u_n),\nabla T_k(u_n))\varphi_k(\omega_n)\,dx\bigg|\\&\qquad\le b(k)\int_{\{|u_n|\le k\}}(c(x)+\sum_{i=1}^N|\partial_{x_i}T_k(u_n)|^{p_i(x)})|\varphi_k(\omega_n)|\,dx\end{aligned}$$

$$
\begin{aligned}
&\le b(k)\int_{\{|u_n|\le k\}} c(x)|\varphi_k(\omega_n)|\,dx\\
&\quad+\frac{b(k)}{\alpha}\sum_{i=1}^{N}\int_\Omega a_i(x,T_k(u_n),\nabla T_k(u_n))\partial_{x_i}T_k(u_n)|\varphi_k(\omega_n)|\,dx\\
&\le b(k)\int_{\{|u_n|\le k\}} c(x)|\varphi_k(\omega_n)|\,dx\\
&\quad+\frac{b(k)}{\alpha}\sum_{i=1}^{N}\int_\Omega (a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u)))\\
&\quad\times(\partial_{x_i}T_k(u_n)-\partial_{x_i}T_k(u))|\varphi_k(\omega_n)|\,dx\\
&\quad+\frac{b(k)}{\alpha}\sum_{i=1}^{N}\int_\Omega a_i(x,T_k(u_n),\nabla T_k(u))(\partial_{x_i}T_k(u_n)-\partial_{x_i}T_k(u))|\varphi_k(\omega_n)|\,dx\\
&\quad+\frac{b(k)}{\alpha}\sum_{i=1}^{N}\int_\Omega a_i(x,T_k(u_n),\nabla T_k(u_n))\partial_{x_i}T_k(u)|\varphi_k(\omega_n)|\,dx,
\end{aligned}
$$

then

$$
\begin{aligned}
&\frac{b(k)}{\alpha}\sum_{i=1}^{N}\int_\Omega (a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u)))\\
&\quad\times(\partial_{x_i}T_k(u_n)-\partial_{x_i}T_k(u))|\varphi_k(\omega_n)|\,dx\\
&\quad\ge\left|\int_{\{|u_n|\le k\}} g_n(x,T_k(u_n),\nabla T_k(u_n))\varphi_k(\omega_n)\,dx\right|\\
&\quad-b(k)\int_{\{|u_n|\le k\}} c(x)|\varphi_k(\omega_n)|\,dx\\
&\quad-\frac{b(k)}{\alpha}\sum_{i=1}^{N}\int_\Omega a_i(x,T_k(u_n),\nabla T_k(u))\\
&\quad\times(\partial_{x_i}T_k(u_n)-\partial_{x_i}T_k(u))|\varphi_k(\omega_n)|\,dx\\
&\quad-\frac{b(k)}{\alpha}\sum_{i=1}^{N}\int_\Omega a_i(x,T_k(u_n),\nabla T_k(u_n))\partial_{x_i}T_k(u)|\varphi_k(\omega_n)|\,dx.
\end{aligned}
\tag{4.21}
$$

Firstly, since $\varphi_k(\omega_n)\rightharpoonup 0$ weak-$\star$ in $L^\infty(\Omega)$, then

$$
\int_{\{|u_n|\le k\}} c(x)|\varphi_k(\omega_n)|\,dx\to 0. \tag{4.22}
$$

Concerning the third term on the right-hand side of (4.21), thanks to (4.17), we have

$$
\left|\sum_{i=1}^{N}\int_\Omega a_i(x,T_k(u_n),\nabla T_k(u))(\partial_{x_i}T_k(u_n)-\partial_{x_i}T_k(u))|\varphi_k(\omega_n)|\,dx\right| \tag{4.23}
$$

$$\leq \varphi_k(2k)\sum_{i=1}^{N}\int_\Omega |a_i(x,T_k(u_n),\nabla T_k(u))|\,|\partial_{x_i}T_k(u_n)-\partial_{x_i}T_k(u)|\,dx\to 0$$

as $n\to\infty$. For the last term of the right-hand side of (4.21), the sequence $(a_i(x,T_k(u_n),\nabla T_k(u_n)))_n$ is bounded in $L^{p_i'(\,\cdot\,)}(\Omega)$, and since $\partial_{x_i}T_k(u)|\varphi_k(\omega_n)|\to 0$ in $L^{p_i(\,\cdot\,)}(\Omega)$, it follows that

$$\int_\Omega a_i(x,T_k(u_n),\nabla T_k(u_n))\partial_{x_i}T_k(u)|\varphi_k(\omega_n)|\,dx\to 0. \tag{4.24}$$

By combining (4.21)–(4.23) and (4.24), we get

$$\begin{aligned}\frac{b(k)}{\alpha}\sum_{i=1}^{N}\int_\Omega &(a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u)))\\ &\times(\partial_{x_i}T_k(u_n)-\partial_{x_i}T_k(u))|\varphi_k(\omega_n)|\,dx\\ &\geq\bigg|\int_{\{|u_n|\leq k\}}g_n(x,T_k(u_n),\nabla T_k(u_n))\varphi_k(\omega_n)\,dx\bigg|+\varepsilon_5(n).\end{aligned} \tag{4.25}$$

Concerning the third term on the left-hand side of (4.15), we have

$$\begin{aligned}\int_{\{|u_n|\leq k\}}&d(x)|T_k(u_n)|^{p_0(x)-2}T_k(u_n)\varphi_k(\omega_n)\,dx\\ &=\int_\Omega d(x)(|T_k(u_n)|^{p_0(x)-2}T_k(u_n)\\ &\quad-|T_k(u)|^{p_0(x)-2}T_k(u))(T_k(u_n)-T_k(u))\exp(\gamma\omega_n^2)\,dx\\ &\quad+\int_\Omega d(x)|T_k(u)|^{p_0(x)-2}T_k(u)(T_k(u_n)-T_k(u))\exp(\gamma\omega_n^2)\,dx\\ &\quad-\int_{\{|u_n|>k\}}d(x)|T_k(u_n)|^{p_0(x)-2}T_k(u_n)(T_k(u_n)-T_k(u))\exp(\gamma\omega_n^2)\,dx\\ &\geq d_0\int_\Omega(|T_k(u_n)|^{p_0(x)-2}T_k(u_n)-|T_k(u)|^{p_0(x)-2}T_k(u))(T_k(u_n)-T_k(u))\,dx\\ &\quad-\exp(\gamma(2k)^2)\|d\|_\infty\int_\Omega|T_k(u)|^{p_0(x)-1}\,|T_k(u_n)-T_k(u)|\,dx\\ &\quad-\exp(\gamma(2k)^2)\|d\|_\infty\int_{\{|u_n|>k\}}k^{p_0(x)-1}\,|T_k(u_n)-T_k(u)|\,dx.\end{aligned}$$

In view of the Lebesgue dominated convergence theorem, we have $T_k(u_n)\to T_k(u)$ in $L^{p_0(\,\cdot\,)}(\Omega)$, then the second and the last term on the right-hand side of the previous inequality tend to zero as n tends to ∞. Therefore, we get

$$\begin{aligned}d_0\int_\Omega&(|T_k(u_n)|^{p_0(x)-2}T_k(u_n)-|T_k(u)|^{p_0(x)-2}T_k(u))(T_k(u_n)-T_k(u))\,dx\\ &\leq\int_{\{|u_n|\leq k\}}d(x)|T_k(u_n)|^{p_0(x)-2}T_k(u_n)\varphi_k(\omega_n)\,dx+\varepsilon_6(n).\end{aligned} \tag{4.26}$$

Thanks to (4.20) and (4.25)–(4.26), we obtain

$$
\begin{aligned}
(4.27)\qquad & \frac{1}{2}\sum_{i=1}^{N}\int_\Omega (a_i(x,T_k(u_n),\nabla T_k(u_n)) - a_i(x,T_k(u_n),\nabla T_k(u))) \\
&\quad \times (\partial_{x_i}T_k(u_n) - \partial_{x_i}T_k(u))\,dx \\
&\quad + d_0\int_\Omega (|T_k(u_n)|^{p_0(x)-2}T_k(u_n) \\
&\quad - |T_k(u)|^{p_0(x)-2}T_k(u))(T_k(u_n)-T_k(u))\,dx \\
&\le \sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_n),\nabla u_n)\varphi_k'(\omega_n)\partial_{x_i}\omega_n\,dx \\
&\quad - \left|\int_{\{|u_n|\le k\}} g_n(x,T_k(u_n),\nabla T_k(u_n))\varphi_k(\omega_n)\,dx\right| \\
&\quad + \int_{\{|u_n|\le k\}} d(x)|T_k(u_n)|^{p_0(x)-2}T_k(u_n)\varphi_k(\omega_n)\,dx + \varepsilon_7(n), \\
&\le \langle f,\varphi_k(\omega_n)\rangle_{W^{-1,\vec{p}\,'(\cdot)}(\Omega),W_0^{1,\vec{p}(\cdot)}(\Omega)} + \varepsilon_7(n).
\end{aligned}
$$

Since $\varphi_k(\omega_n) = \varphi_k(T_k(u_n) - T_k(u)) \rightharpoonup 0$ in $W_0^{1,\vec{p}(\cdot)}(\Omega)$. Then, by letting n tends to infinity in (4.27), it follows that

$$
\begin{aligned}
(4.28)\qquad & \sum_{i=1}^{N}\int_\Omega (a_i(x,T_k(u_n),\nabla T_k(u_n)) - a_i(x,T_k(u_n),\nabla T_k(u))) \\
&\qquad \times (\partial_{x_i}T_k(u_n) - \partial_{x_i}T_k(u))\,dx \\
&+ \int_\Omega (|T_k(u_n)|^{p_0(x)-2}T_k(u_n) - |T_k(u)|^{p_0(x)-2}T_k(u))(T_k(u_n)-T_k(u))\,dx \to 0
\end{aligned}
$$

as $n\to\infty$. Now using Lemma 3.5, we deduce that

$$T_k(u_n) \to T_k(u) \quad \text{in } W_0^{1,\vec{p}(\cdot)}(\Omega).$$

In view of (4.12) and by letting k tends to infinity, we get

$$
(4.29)\qquad \begin{cases} u_n \to u & \text{in } W_0^{1,\vec{p}(\cdot)}(\Omega), \\ \nabla u_n \to \nabla u & \text{a.e. in } \Omega. \end{cases}
$$

Step 4. Equi-integrability and passage to the limit. Thanks to (4.29), we have

$$
\begin{aligned}
a_i(x,T_n(u_n),\nabla u_n) &\to a_i(x,u,\nabla u) && \text{a.e. in } \Omega, \\
g_n(x,u_n,\nabla u_n) &\to g(x,u,\nabla u) && \text{a.e. in } \Omega, \\
g_n(x,u_n,\nabla u_n)u_n &\to g(x,u,\nabla u)u && \text{a.e. in } \Omega.
\end{aligned}
$$

On the other hand, thanks to (4.12) we have

$$\|u_n\|_{1,\vec{p}(\cdot)} \le C_1,$$

then $(a_i(x,u_n,\nabla u_n))_n$ is bounded in $L^{p_i'(\,\cdot\,)}(\Omega)$, therefore, by Lemma 3.2, we obtain

$$a_i(x,T_n(u_n),\nabla u_n) \rightharpoonup a_i(x,u,\nabla u) \quad \text{in } L^{p_i'(\,\cdot\,)}(\Omega).$$

Now, we have to prove that

$$g_n(x,u_n,\nabla u_n) \to g(x,u,\nabla u) \quad \text{in } L^1(\Omega).$$

Using Vitali's theorem, it is sufficient to prove that $g_n(x,u_n,\nabla u_n)$ is uniformly equi-integrable. Indeed, taking $T_1(u_n - T_h(u_n))$ as a test function in (4.2), we obtain

$$\begin{aligned}\sum_{i=1}^{N}\int_{\{h\le|u_n|\le h+1\}} & a_i(x,T_n(u_n),\nabla u_n)\partial_{x_i}u_n\,dx \\ &+\int_{\{h\le|u_n|\}} g_n(x,u_n,\nabla u_n)T_1(u_n-T_h(u_n))\,dx \\ &+\int_{\{h\le|u_n|\}} d(x)|u_n|^{p_0(x)-2}u_nT_1(u_n-T_h(u_n))\,dx \\ &=\langle f, T_1(u_n-T_h(u_n))\rangle_{W^{-1,\vec{p}\,'(\,\cdot\,)}(\Omega),W_0^{1,\vec{p}(\,\cdot\,)}(\Omega)},\end{aligned} \tag{4.30}$$

since $T_1(u_n - T_h(u_n))$ and u_n have the same sign, then

$$\begin{aligned}\int_{\{h+1\le|u_n|\}}|g_n(x,u_n,\nabla u_n)|\,dx &\le \int_{\{h\le|u_n|\}} g_n(x,u_n,\nabla u_n)T_1(u_n-T_h(u_n))\,dx \\ &\le \langle f, T_1(u_n-T_h(u_n))\rangle_{W^{-1,\vec{p}\,'(\,\cdot\,)}(\Omega),W_0^{1,\vec{p}(\,\cdot\,)}(\Omega)} \\ &\le C\|f\|_{-1,\vec{p}\,'(\,\cdot\,)}\|T_1(u_n-T_h(u_n))\|_{1,\vec{p}(\,\cdot\,)}\to 0\end{aligned}$$

as $h\to\infty$, thus, for all $\eta>0$, there exists $h(\eta)>1$ such that

$$\int_{\{h(\eta)\le|u_n|\}}|g_n(x,u_n,\nabla u_n)|\,dx \le \frac{\eta}{2}. \tag{4.31}$$

On the other hand, for any measurable subset $E\subset\Omega$, we have

$$\begin{aligned}\int_E|g_n(x,u_n,\nabla u_n)|\,dx \le b(h(\eta))\int_E\Big(c(x)+\sum_{i=1}^N|\partial_{x_i}u_n|^{p_i(x)}\Big)\,dx \\ +\int_{\{|u_n|\ge h(\eta)\}}|g_n(x,u_n,\nabla u_n)|\,dx.\end{aligned} \tag{4.32}$$

In view of (4.29), there exists $\ell(\eta)>0$ such that

$$b(h(\eta))\int_E\bigg(c(x)+\sum_{i=1}^N|\partial_{x_i}u_n|^{p_i(x)}\bigg)\,dx \le \frac{\eta}{2} \tag{4.33}$$

for all E such that $\operatorname{meas}(E)\le\ell(\eta)$. Finally, combining (4.31)–(4.33), we obtain

$$\int_E|g_n(x,u_n,\nabla u_n)|\,dx\le\eta \tag{4.34}$$

for all E such that $\text{meas}(E) \leq \ell(\eta)$, then we obtain the equi-integrability of $g_n(x, u_n, \nabla u_n)$.

Taking $v \in W_0^{1,\vec{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$ as a test function in (4.2), we have

$$\sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_n), \nabla u_n) \partial_{x_i} v \, dx + \int_\Omega g_n(x, u_n, \nabla u_n) v \, dx + \int_\Omega d(x)|u_n|^{p_0(x)-2} u_n v \, dx = \langle f, v \rangle_{W^{-1,\vec{p}'(\cdot)}(\Omega), W_0^{1,\vec{p}(\cdot)}(\Omega)}.$$

By letting n to tend to infinity, we get

$$\sum_{i=1}^{N} \int_\Omega a_i(x, u, \nabla u) \partial_{x_i} v \, dx + \int_\Omega g(x, u, \nabla u) \, v \, dx + \int_\Omega d(x)|u|^{p_0(x)-2} uv \, dx = \langle f, v \rangle_{W^{-1,\vec{p}'(\cdot)}(\Omega), W_0^{1,\vec{p}(\cdot)}(\Omega)}.$$

Moreover, by taking $v = u_n$ in our approximate problem, we obtain

$$\sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_n), \nabla u_n) \partial_{x_i} u_n \, dx + \int_\Omega g_n(x, u_n, \nabla u_n) u_n \, dx + \int_\Omega d(x)|u_n|^{p_0(x)} \, dx = \int_\Omega f u_n \, dx.$$

Thanks to (4.12), we get

$$\int_\Omega g_n(x, u_n, \nabla u_n) u_n \, dx \leq C\|f\|_{-1,\vec{p}'(\cdot)} \|u_n\|_{1,\vec{p}(\cdot)} \leq C \; C_1 \|f\|_{-1,p'(x)} = C_2.$$

Since $g_n(x, u_n, \nabla u_n) u_n \geq 0$, by using Fatou's Lemma, we deduce that

$$0 \leq \int_\Omega g(x, u, \nabla u) u \, dx \leq C_2,$$

then $g(x, u, \nabla u) u \in L^1(\Omega)$, which completes the proof of Theorem 4.2. □

4.2. The case of $f \in L^1(\Omega)$. In this section, we consider the following assumptions:

$$\begin{cases} f \in L^1(\Omega), \\ \exists \rho_1, \; \rho_2 > 0, \text{ such that} \\ \qquad \text{if } |s| > \rho_1 \text{ then } |g(x, s, \xi)| \geq \rho_2 \Big(|s|^{p_0(x)} + \sum_{i=1}^{N} |\xi_i|^{p_i(x)} \Big). \end{cases} \tag{4.35}$$

DEFINITION 4.5. In the case of $f \in L^1(\Omega)$, a measurable function u is said to be solution in the sense of distributions to the problem (1.2), if

$$\begin{cases} \sum_{i=1}^{N} \int_\Omega a_i(x,u,\nabla u)\partial_{x_i} v\,dx + \int_\Omega g(x,u,\nabla u)v\,dx \\ \qquad + \int_\Omega d(x)|u|^{p_0(x)-2}uv\,dx = \int_\Omega fv\,dx, \\ u \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega), \quad g(x,u,\nabla u) \in L^1(\Omega), \end{cases} \tag{4.36}$$

for any $v \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$.

THEOREM 4.6. *Let* $f \in L^1(\Omega)$. *assuming that* (1.3)–(1.7) *and* (4.35) *hold, then the problem* (1.2) *has at least one solution in the sense of distributions.*

PROOF OF THE THEOREM 4.6.

Step 1. The approximate problems. Let $(f_n)_n$ be a sequence in $W^{-1,\overrightarrow{p}'(\cdot)}(\Omega) \cap L^1(\Omega)$ such that $f_n \to f$ in $L^1(\Omega)$ with $|f_n| \le |f|$ and we consider the approximate problem

$$\begin{cases} Au_n + g(x,u_n,\nabla u_n) + d(x)|u_n|^{p_0(x)-2}u_n = f_n \quad \text{in } \Omega, \\ u_n \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega). \end{cases} \tag{4.37}$$

Thanks to Theorem 4.2, there exists at least one solution in the sense of distributions for the approximate problem (4.37).

Step 2. A priori estimates. Taking $T_k(u_n)$ as a test function in (4.37), then

$$\sum_{i=1}^{N} \int_\Omega a_i(x,u_n,\nabla u_n)\partial_{x_i} T_k(u_n)\,dx + \int_\Omega g(x,u_n,\nabla u_n)T_k(u_n)\,dx \\ + \int_\Omega d(x)|u_n|^{p_0(x)-2}u_n T_k(u_n)\,dx = \int_\Omega f_n T_k(u_n)\,dx.$$

Thanks to (1.4) and (1.6), we obtain

$$\delta \sum_{i=0}^{N} \int_\Omega |\partial_{x_i} T_k(u_n)|^{p_i(x)}\,dx \le \sum_{i=1}^{N} \int_\Omega a_i(x,u_n,\nabla u_n)\partial_{x_i} T_k(u_n)\,dx \\ + \int_\Omega d(x)|T_k(u_n)|^{p_0(x)}\,dx \le k\int_\Omega |f|\,dx \le k\,C_1,$$

with $\delta = \min(\alpha, d_0)$, then

$$\sum_{i=0}^{N} \int_\Omega |\partial_{x_i} T_k(u_n)|^{p_i(x)}\,dx \le \frac{k}{\delta}\,C_1, \tag{4.38}$$

also, we have

$$k\int_{\{|u_n|>k\}} |g(x,u_n,\nabla u_n)|\,dx \le k\,C_1. \tag{4.39}$$

By combining (4.35), (4.38) and (4.39), for $k \geq \rho_1$, we deduce that

$$\begin{aligned}\sum_{i=0}^{N} \int_\Omega |\partial_{x_i} u_n|^{p_i(x)}\,dx &= \sum_{i=0}^{N} \int_\Omega |\partial_{x_i} T_k(u_n)|^{p_i(x)}\,dx + \sum_{i=0}^{N} \int_{\{|u_n|>k\}} |\partial_{x_i} u_n|^{p_i(x)}\,dx \\ &\leq \frac{k}{\alpha}\,C_1 + \frac{1}{\rho_2}\int_{\{|u_n|>k\}} |g(x,u_n,\nabla u_n)|\,dx \\ &\leq \frac{k}{\alpha}\,C_1 + \frac{C_1}{\rho_2} = C_2.\end{aligned}$$

It follows that

$$\sum_{i=0}^{N} \|\partial_{x_i} u_n\|_{p_i(\,\cdot\,)}^{\underline{p}} - N - 1 \leq \sum_{i=0}^{N} \int_\Omega |\partial_{x_i} u_n|^{p_i(x)}\,dx \leq C_2,$$

then $\|u_n\|_{1,\overrightarrow{p}(\,\cdot\,)} \leq C$. We conclude that

$$\begin{cases} u_n \rightharpoonup u & \text{in } W_0^{1,\overrightarrow{p}(\,\cdot\,)}(\Omega), \\ u_n \to u & \text{in } L^{\underline{p}}(\Omega). \end{cases} \tag{4.40}$$

Step 3. The strong convergence of $(T_k(u_n))_n$ in $W_0^{1,\overrightarrow{p}(\,\cdot\,)}(\Omega)$. Let $h \geq k > 0$ and $\omega_n = T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u))$, we set $\varphi_k(s) = s.\exp(\gamma s^2)$ where $\gamma = (b(k)/(2\alpha))^2$.

By taking $\varphi_k(\omega_n)$ as a test function in the approximate problem (4.37), we obtain

$$\begin{aligned}\sum_{i=1}^{N} \int_\Omega a_i(x,u_n,\nabla u_n)\partial_{x_i}\varphi_k(\omega_n)\,dx + \int_\Omega g(x,u_n,\nabla u_n)\varphi_k(\omega_n)\,dx \\ + \int_\Omega d(x)|u_n|^{p_0(x)-2}u_n\varphi_k(\omega_n)\,dx = \int_\Omega f_n\varphi_k(\omega_n)\,dx.\end{aligned}$$

Let $M = 4k + h$. Since $\partial_{x_i}\omega_n = 0$ on $\{|u_n| > M\}$ and $\varphi_k(\omega_n)$ has the same sign as u_n on the set $\{|u_n| > k\}$ (indeed, if $u_n > k$ then $u_n - T_h(u_n) \geq 0$ and $T_k(u_n) - T_k(u) \geq 0$, it follows that $\omega_n \geq 0$.

Similarly, we show that $\omega_n \leq 0$ on the set $\{u_n < -k\}$) we get

$$\begin{aligned}\sum_{i=1}^{N} \int_\Omega a_i(x,T_M(u_n),\nabla T_M(u_n))\varphi_k'(\omega_n)\partial_{x_i}\omega_n\,dx \\ + \int_{\{|u_n|\leq k\}} g(x,u_n,\nabla u_n)\varphi_k(\omega_n)\,dx \\ + \int_{\{|u_n|\leq k\}} d(x)|u_n|^{p_0(x)-2}u_n\varphi_k(\omega_n)\,dx \leq \int_\Omega f_n\varphi_k(\omega_n)\,dx.\end{aligned} \tag{4.41}$$

Taking $z_n = u_n - T_h(u_n) + T_k(u_n) - T_k(u)$, we have

$$
\begin{aligned}
(4.42)\quad &\sum_{i=1}^{N}\int_\Omega a_i(x,T_M(u_n),\nabla T_M(u_n))\varphi_k'(\omega_n)\partial_{x_i}\omega_n\,dx\\
&= \sum_{i=1}^{N}\int_{\{|u_n|\le k\}} a_i(x,T_k(u_n),\nabla T_k(u_n))\varphi_k'(\omega_n)(\partial_{x_i}T_k(u_n)-\partial_{x_i}T_k(u))\,dx\\
&\quad+\sum_{i=1}^{N}\int_{\{|u_n|>k\}\cap\{|z_n|\le 2k\}} a_i(x,T_M(u_n),\nabla T_M(u_n))\\
&\quad\times\varphi_k'(\omega_n)\partial_{x_i}(u_n-T_h(u_n)+T_k(u_n)-T_k(u))\,dx\\
&\ge \sum_{i=1}^{N}\int_\Omega a_i(x,T_k(u_n),\nabla T_k(u_n))\varphi_k'(\omega_n)(\partial_{x_i}T_k(u_n)-\partial_{x_i}T_k(u))\,dx\\
&\quad+\sum_{i=1}^{N}\int_{\{|u_n|>k\}} a_i(x,T_k(u_n),\nabla T_k(u_n))\varphi_k'(\omega_n)\partial_{x_i}T_k(u)\,dx\\
&\quad-\varphi_k'(2k)\sum_{i=1}^{N}\int_{\{|u_n|>k\}} |a_i(x,T_M(u_n),\nabla T_M(u_n))|\,|\partial_{x_i}T_k(u)|\,dx,
\end{aligned}
$$

that is equivalent to

$$
\begin{aligned}
&\sum_{i=1}^{N}\int_\Omega (a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u)))\\
&\qquad\times(\partial_{x_i}T_k(u_n)-\partial_{x_i}T_k(u))\,\varphi_k'(\omega_n)\,dx\\
&\le \sum_{i=1}^{N}\int_\Omega a_i(x,T_M(u_n),\nabla T_M(u_n))\varphi_k'(\omega_n)\partial_{x_i}\omega_n\,dx\\
&\quad-\sum_{i=1}^{N}\int_{\{|u_n|>k\}} a_i(x,T_k(u_n),\nabla T_k(u_n))\varphi_k'(\omega_n)\partial_{x_i}T_k(u)\,dx\\
&\quad+\ \varphi_k'(2k)\sum_{i=1}^{N}\int_{\{|u_n|>k\}} |a_i(x,T_M(u_n),\nabla T_M(u_n))|\,|\partial_{x_i}T_k(u)|\,dx\\
&\quad-\sum_{i=1}^{N}\int_\Omega a_i(x,T_k(u_n),\nabla T_k(u))(\partial_{x_i}T_k(u_n)-\partial_{x_i}T_k(u))\,\varphi_k'(\omega_n)\,dx.
\end{aligned}
$$

Similarly to (4.17) and (4.18), we can prove that the last three terms tends to 0 as n goes to infinity, then

$$
\begin{aligned}
(4.43)\quad &\sum_{i=1}^{N}\int_\Omega (a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u)))\\
&\qquad\times(\partial_{x_i}T_k(u_n)-\partial_{x_i}T_k(u))\,\varphi_k'(\omega_n)\,dx
\end{aligned}
$$

$$\le \sum_{i=1}^{N} \int_{\Omega} a_i(x, T_M(u_n), \nabla T_M(u_n)) \varphi_k'(\omega_n) \partial_{x_i} \omega_n \, dx + \varepsilon_8(n).$$

Also, we can easily prove, as (4.25) and (4.26), that

$$(4.44)\quad \frac{b(k)}{\alpha} \sum_{i=1}^{N} \int_{\Omega} (a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u))) \times (\partial_{x_i} T_k(u_n) - \partial_{x_i} T_k(u)) |\varphi_k(\omega_n)| \, dx \ge \left| \int_{\{|u_n| \le k\}} g(x, u_n, \nabla u_n) \varphi_k(\omega_n) \, dx \right| + \varepsilon_9(n)$$

and

$$(4.45)\quad d_0 \int_{\Omega} (|T_k(u_n)|^{p_0(x)-2} T_k(u_n) - |T_k(u)|^{p_0(x)-2} T_k(u))(T_k(u_n) - T_k(u)) \, dx \le \int_{\{|u_n| \le k\}} d(x) |u_n|^{p_0(x)-2} u_n \varphi_k(\omega_n) \, dx + \varepsilon_{10}(n).$$

Thanks to (4.41) and (4.43)–(4.45), we obtain

$$(4.46)\quad \frac{1}{2} \sum_{i=1}^{N} \int_{\Omega} (a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u))) \times (\partial_{x_i} T_k(u_n) - \partial_{x_i} T_k(u)) \, dx + d_0 \int_{\Omega} (|T_k(u_n)|^{p_0(x)-2} T_k(u_n) - |T_k(u)|^{p_0(x)-2} T_k(u)) \times (T_k(u_n) - T_k(u)) \, dx \le \int_{\Omega} f_n \varphi_k(\omega_n) \, dx + \varepsilon_{11}(n) \le \int_{\Omega} f \varphi_k(T_{2k}(u - T_h(u))) \, dx + \varepsilon_{12}(n),$$

since $f_n \to f$ in $L^1(\Omega)$ and $\varphi_k(\omega_n) \rightharpoonup \varphi_k(T_{2k}(u - T_h(u)))$ weak-$*$ in $L^\infty(\Omega)$. Then by letting h tends to infinity in the previous inequality, we get

$$(4.47)\quad \frac{1}{2} \sum_{i=1}^{N} \int_{\Omega} (a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u))) \times (\partial_{x_i} T_k(u_n) - \partial_{x_i} T_k(u)) \, dx + d_0 \int_{\Omega} (|T_k(u_n)|^{p_0(x)-2} T_k(u_n) - |T_k(u)|^{p_0(x)-2} T_k(u))(T_k(u_n) - T_k(u)) \, dx \to 0$$

as $n \to \infty$, using Lemma 3.5, we deduce that $T_k(u_n) \to T_k(u)$ in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$, in view of (4.42) we have $(u_n)_n$ and u belong to $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$, and thanks to Lemma 3.4 we obtain

$$\|u_n - u\|_{1,\overrightarrow{p}(\cdot)} \le \|u_n - T_k(u_n)\|_{1,\overrightarrow{p}(\cdot)} + \|T_k(u_n) - T_k(u)\|_{1,\overrightarrow{p}(\cdot)} + \|T_k(u) - u\|_{1,\overrightarrow{p}(\cdot)} \to 0,$$

as k, n tends to ∞, it follows that

$$\begin{cases} u_n \to u & \text{in } W_0^{1,\vec{p}(\,\cdot\,)}(\Omega), \\ \nabla u_n \to \nabla u & \text{a.e. in } \Omega. \end{cases} \tag{4.48}$$

Step 4. The equi-integrability of $g(x, u_n, \nabla u_n)$ and passage to the limit. Thanks to (4.48), we have

$$\begin{aligned} a_i(x, u_n, \nabla u_n) &\to a_i(x, u, \nabla u) \quad \text{a.e. in } \Omega, \\ g(x, u_n, \nabla u_n) &\to g(x, u, \nabla u) \quad \text{a.e. in } \Omega, \end{aligned}$$

since $(a_i(x, u_n, \nabla u_n))_n$ is bounded in $L^{p_i'(\,\cdot\,)}(\Omega)$, and using the Lemma 3.2, we obtain

$$a_i(x, u_n, \nabla u_n) \rightharpoonup a_i(x, u, \nabla u) \quad \text{in } L^{p_i'(\,\cdot\,)}(\Omega).$$

Now, let E be a measurable subset of Ω, for all $m > 0$ we have

$$\begin{aligned} &\int_E |g(x, u_n, \nabla u_n)|\, dx \\ &= \int_{E\cap\{|u_n|\le m\}} |g(x, u_n, \nabla u_n)|\, dx + \int_{E\cap\{|u_n|>m\}} |g(x, u_n, \nabla u_n)|\, dx \\ &\le b(m) \int_E \Big(c(x) + \sum_{i=1}^N |\partial_{x_i} T_m(u_n)|^{p_i(x)}\Big)\, dx + \int_{\{|u_n|>m\}} |g(x, u_n, \nabla u_n)|\, dx, \end{aligned}$$

since $(\partial_{x_i} T_m(u_n))_n$ converges strongly in $L^{p_i(\,\cdot\,)}(\Omega)$, then for all $\sigma > 0$, there exists $\delta > 0$ such that $\operatorname{meas}(E) < \delta$ and

$$b(m) \sum_{i=1}^N \int_E |\partial_{x_i} T_m(u_n)|^{p_i(x)}\, dx < \frac{\sigma}{3} \quad \text{and} \quad b(m) \int_E c(x)\, dx < \frac{\sigma}{3}. \tag{4.49}$$

On the other hand, using $T_1(u_n - T_{m-1}(u_n))$ as a test function in (4.37) for $m > 1$, we obtain

$$\begin{aligned} \sum_{i=1}^N \int_\Omega a_i(x, u_n, \nabla u_n) \partial_{x_i} T_1(u_n - T_{m-1}(u_n))\, dx \\ + \int_\Omega g(x, u_n, \nabla u_n) T_1(u_n - T_{m-1}(u_n))\, dx \\ + \int_\Omega d(x) |u_n|^{p_0(x)-2} u_n T_1(u_n - T_{m-1}(u_n))\, dx = \int_\Omega f_n T_1(u_n - T_{m-1}(u_n))\, dx, \end{aligned}$$

then

$$\int_{\{|u_n|>m\}} |g(x, u_n, \nabla u_n)|\, dx \le \int_{\{|u_n|>m-1\}} |f|\, dx,$$

there exists $m_0 > 0$ such that

$$\int_{\{|u_n|>m\}} |g(x, u_n, \nabla u_n)|\, dx < \frac{\sigma}{3} \quad \text{for all } m > m_0. \tag{4.50}$$

Using (4.49) and (4.50), we deduce the equi-integrability of $g(x, u_n, \nabla u_n)$. In view of Vitali's theorem, we obtain $g(x, u_n, \nabla u_n) \to g(x, u, \nabla u)$ in $L^1(\Omega)$. By taking $v \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$ as a test function in (4.37), it is easy to pass to the limit in

$$\sum_{i=1}^N \int_\Omega a_i(x, u_n, \nabla u_n)\partial_{x_i} v\, dx + \int_\Omega g(x, u_n, \nabla u_n) v\, dx \\ + \int_\Omega d(x)|u_n|^{p_0(x)-2} u_n v\, dx = \int_\Omega f_n v\, dx$$

to obtain

$$\sum_{i=1}^N \int_\Omega a_i(x, u, \nabla u)\partial_{x_i} v\, dx + \int_\Omega g(x, u, \nabla u) v\, dx \\ + \int_\Omega d(x)|u|^{p_0(x)-2} u v\, dx = \int_\Omega f v\, dx, \tag{4.51}$$

which completes our proof. □

EXAMPLE 4.7. A prototype example that is covered by the assumptions (1.3)–(1.5) and (1.6)–(1.7) respectively is the following anisotropic $(p_0(\,\cdot\,), \dots, p_N(\,\cdot\,))$-problem: set

$$d(x) \equiv 1, \quad a_i(x, u, \nabla u) = |\partial_{x_i} u|^{p_i(x)-2} \partial_{x_i} u \quad \text{for } i = 1, \dots, N$$

and

$$g(x, u, \nabla u) = u \sum_{i=0}^N |\partial_{x_i} u|^{p_i(x)}.$$

Then we get the following problem:

$$\begin{cases} -\displaystyle\sum_{i=1}^N \frac{\partial}{\partial x_i}(|\partial_{x_i} u|^{p_i(x)-2} \partial_{x_i} u) \\ \qquad + u \displaystyle\sum_{i=0}^N |\partial_{x_i} u|^{p_i(x)} + |u|^{p_0(x)-2} u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.52}$$

which has at least one solution in the sense of distributions for all $f \in W^{-1,\vec{p}\,'(\cdot)}(\Omega)$, with

$$u \sum_{i=0}^N |\partial_{x_i} u|^{p_i(x)} \in L^1(\Omega) \quad \text{and} \quad |u|^2 \sum_{i=0}^N |\partial_{x_i} u|^{p_i(x)} \in L^1(\Omega).$$

Moreover, let f be in $L^1(\Omega)$. The function $g(x, u, \nabla u)$ verifies the condition (4.35), (we assume $\rho_1 = \rho_2 = 1$). Applying theorem 4.6 allows to conclude that problem (4.52) has at least one solution in the sense of distributions, with $u \sum_{i=0}^N |\partial_{x_i} u|^{p_i(x)} \in L^1(\Omega)$.

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Manuscript received February 20, 2014
accepted March 13, 2015

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