PARTIALLY SYMMETRIC SOLUTIONS
OF THE GENERALIZED HÉNON EQUATION
IN SYMMETRIC DOMAINS

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ABSTRACT. We study the generalized Hénon equation in a symmetric domain Ω. Let H and G be closed subgroups of the orthogonal group such that $H \subsetneq G$ and Ω is G invariant. Then we show the existence of a positive solution which is H invariant but G non-invariant under suitable assumptions of H, G and the coefficient function of the equation.

1. Introduction

We study the existence of partially symmetric positive solutions of the generalized Hénon equation

$$(1.1) \qquad -\Delta u = f(x)u^p, \quad u > 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

Here Ω is a bounded domain in $\mathbb{R}^N$ with piecewise smooth boundary $\partial\Omega$, $1 < p < \infty$ when $N = 2$, $1 < p < (N+2)/(N-2)$ when $N \geq 3$, $f \in L^\infty(\Omega)$ and $f(x)$ may be positive or may change its sign. Let G be a closed subgroup of the orthogonal group $O(N)$. We call Ω a G *invariant domain* if $g(\Omega) = \Omega$ for all $g \in G$. We call $f(x)$ a G *invariant function* if $f(gx) = f(x)$ for all $g \in G$ and $x \in \Omega$. In the same manner, we define a G *invariant solution* of (1.1). Throughout the paper, we

assume that the set of $x \in \Omega$ satisfying $f(x) > 0$ has positive Lebesgue measure. Otherwise, $f(x) \leq 0$ almost everywhere in Ω and no positive solution exists by the maximum principle. If Ω and f are G invariant, (1.1) has a G invariant positive solution, which will be proved in this section. However, we are looking for a positive solution without G invariance. In particular, we investigate the existence of an H invariant G non-invariant solution for closed subgroups H and G such that $H \subsetneq G \subset O(N)$. This solution has a partial symmetry, that is, it is invariant under the H action, but not invariant under the G action.

To get such a solution, we define a least energy solution in the following. First, we introduce the *Rayleigh quotient*

$$R(u) := \left(\int_\Omega |\nabla u|^2 \, dx \right) \left(\int_\Omega f(x)|u|^{p+1} \, dx \right)^{-2/(p+1)},$$

with the definition domain

$$D(R) := \left\{ u \in H_0^1(\Omega) : \int_\Omega f(x)|u|^{p+1} \, dx > 0 \right\}.$$

Here $H_0^1(\Omega)$ denotes the Sobolev space. Second, we define the *Nehari manifold* by

$$\mathcal{N} := \left\{ u \in H_0^1(\Omega) \setminus \{0\} : \int_\Omega (|\nabla u|^2 - f(x)|u|^{p+1}) \, dx = 0 \right\}.$$

Last, we define the *least energy* R_0 by

$$R_0 := \inf\{R(u) : u \in D(R)\} = \inf\{R(u) : u \in \mathcal{N}\}.$$

The equality of the two infimums above can be proved by using two facts that $R(\lambda u) = R(u)$ for any $\lambda > 0$ and $u \in D(R)$; for any $u \in D(R)$ there exists a $\lambda > 0$ such that $\lambda u \in \mathcal{N}$. Because of the Sobolev embedding theorem, $R(u)$ has a positive lower bound. Hence R_0 is well defined and positive. We call R_0 a *least energy* and u a *least energy solution* if $u \in \mathcal{N}$ and $R(u) = R_0$. Such a minimizer exists and becomes a solution of (1.1). Note that $R(u) = R(|u|)$. Thus, if u is a least energy solution, so is $|u|$. Then $|u|$ also satisfies (1.1). By the strong maximum principle, u is positive or negative in Ω. Throughout the paper, we choose a positive solution as a least energy solution. A least energy solution is not necessarily unique.

To explain our problem and motivation, we introduce the original Hénon equation

(1.2) $-\Delta u = |x|^\lambda u^p, \quad u > 0 \quad \text{in } B, \qquad u = 0, \quad \text{on } \partial B,$

where B is a unit ball in $\mathbb{R}^N$. Smets, Willem and Su [49] have proved that if $\lambda > 0$ is large enough, a least energy solution of (1.2) is non-radial. Therefore the equation has at least two positive solutions: a non-radial solution and a radial solution. There are many contributions which have studied the Hénon equation ([4], [7], [10], [12]–[15], [18], [25], [27], [46], [48]).

The result [49] by Smets, Willem and Su means that if λ is large enough, then a least energy solution breaks its radial symmetry. In this case, the weight function $|x|^\lambda$ has a thicker density near the boundary $|x| = 1$ than the value in the interior. We study the problem above with B replaced by Ω and $|x|^\lambda$ by $f(x)$, where $f(x)$ is a G invariant function and Ω is a G invariant domain with a closed subgroup G of $O(N)$. Moreover, we look for an H invariant G non-invariant solution. To find such a solution, we define a G invariant least energy solution as below. When Ω is G invariant, we set

$$H_0^1(\Omega, G) := \{u \in H_0^1(\Omega) : u \text{ is } G \text{ invariant}\},$$

(1.3) $$D(R, G) := D(R) \cap H_0^1(\Omega, G), \quad \mathcal{N}(G) := \mathcal{N} \cap H_0^1(\Omega, G),$$

$$R_G := \inf\{R(u) : u \in D(R, G)\} = \inf\{R(u) : u \in \mathcal{N}(G)\}.$$

We call R_G a *G invariant least energy* and u a *G invariant least energy solution* if $u \in \mathcal{N}(G)$ and $R(u) = R_G$. There exists a G invariant least energy solution and it solves (1.1). To avoid confusion, a usual least energy solution is called a *global least energy solution*.

When Ω is a hollow thin symmetric domain, we proved in [32] the existence of an H invariant G non-invariant solution. To state this result, we begin with the annulus

$$A_N(a, b) := \{x \in \mathbb{R}^N : a < |x| < b\}.$$

Coffman [19], Li [34] and Byeon [9] considered the Emden-Fowler equation

(1.4) $$-\Delta u = u^p, \quad u > 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

with $\Omega = A_N(a, b)$ annulus. They proved that the number of non-equivalent positive solutions diverges to infinity as $(b - a)/a \to 0$. Moreover, a least energy solution is not radially symmetric when $(b - a)/a$ is small enough. Here solutions u and v are said to be *equivalent* if $u(gx) = v(x)$ with some $g \in O(N)$. For the related results to the annulus, we refer the readers to [2], [16], [17], [21]–[23], [35]–[40], [45], [50], [51]. Even if Ω is not an exact annulus but it is like an annular domain without $O(N)$ invariance, the existence of multiple positive solutions was proved by Byeon and Tanaka [11]. For the related results, we refer the readers to [1], [5], [24].

More general equations than (1.4) were studied by papers [6], [33], [43]. Indeed, the existence of a sequence of sign-changing solutions for the critical polyharmonic equation was proved by Bartch, Schneider and Weth [6]. For the p-Laplace equation in $\mathbb{R}^N$, Kristály and Marzantowicz [33] proved the existence of multiple sequences of sign-changing solutions. Musso, Pacard and Wei [43] proved the existence of a sequence of nonradial sign-changing solutions for a semilinear elliptic equation.

We shall state our earlier result. Let Ω be a hollow regular polygon, which is defined as below. Let P be an interior of a regular n polygon with center origin. We define

$$(1 + \varepsilon)P := \{(1 + \varepsilon)x : x \in P\}, \quad \text{with } \varepsilon > 0,$$

which is a regular polygon larger than P. We remove $\overline{P}$ from $(1+\varepsilon)P$ and define $\Omega := (1 + \varepsilon)P \setminus \overline{P}$. Then Ω is a hollow polygon. We define

$$r(\theta) := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

(1.5) $$G_n := \{r(2j\pi/n) : j = 0, 1, \ldots, n-1\}.$$

Then Ω is G_n invariant. Let $1 = n_1 < \ldots < n_d = n$ be all divisors of n and let u_i be a G_{n_i} invariant least energy solution of (1.4). Then we proved in [32] that for $\varepsilon > 0$ small enough, u_i is not equivalent to u_j if $i \neq j$. This is caused by the assumption that Ω is a hollow thin domain.

Replace (1.4) by (1.1). In this case, we do not need Ω to have a hole. Instead of a hole, we assume that the ratio of the density of $f(x)$ near $\partial\Omega$ to that of the interior of Ω is large enough. Then we shall prove the same result as above in Example 3.3.

For a closed subgroup G of $O(N)$ and $x \in \mathbb{R}^N$, we define the *orbit* of G through x by

(1.6) $$G(x) := \{gx : g \in G\}.$$

Let H and G be closed subgroups of $O(N)$ satisfying $H \subsetneq G$. The purpose of this paper is to prove that no H invariant least energy solution is G invariant if $H(x) \subsetneq G(x)$ for all $x \in \Gamma$. Here $H(x)$ and $G(x)$ are the orbits of H and G, respectively, and Γ is the set of points farthest in $\overline{\Omega}$ from the fixed point set of G. By choosing several pairs of H and G, we obtain multiple positive solutions of (1.1).

This paper is organized into six sections. In Section 2, we state the main results. In Section 3, we give several examples of symmetric domains Ω and groups H, G and explain how to apply our theorem to the problem. In Section 4, we give a few lemmas to prove the main results. In Section 5, we give an a priori $L^\infty(\Omega)$ estimate of a G invariant least energy solution and investigate the $L^2(\Omega)$ norm. In Section 6, we prove the main theorems.

2. Main results

In this section, we state the main results. We define the *fixed point set* of G by $F = \text{Fix}(G) := \{x \in \mathbb{R}^N : gx = x \text{ for all } g \in G\}$. Then F is a linear subspace of $\mathbb{R}^N$. Let $F^\perp$ denote the orthogonal complement of F in $\mathbb{R}^N$. Hereafter, the

notation $x = x' + x''$ stands for the orthogonal decomposition of $x \in \mathbb{R}^N$ with $x' \in F$ and $x'' \in F^\perp$. We define a notation

$$\Omega(s < |x''| < t) := \{x' + x'' \in \Omega : s < |x''| < t\}.$$

Similarly, we define $\Omega(|x''| < t)$. We put

$$\operatorname{dist}(x, F) := \inf\{|x - y| : y \in F\},$$

(2.1) $$A := \max_{x \in \overline{\Omega}} \operatorname{dist}(x, F) = \max_{x \in \partial\Omega} \operatorname{dist}(x, F).$$

We denote the set of points farthest in $\overline{\Omega}$ from F by Γ, i.e.

(2.2) $$\Gamma := \{x \in \partial\Omega : \operatorname{dist}(x, F) = A\},$$

where A is given by (2.1).

If $f \geq 0$, $\not\equiv 0$ in Ω, we define, for $N \geq 3$ and $A_0 \in (0, A)$,

$$\mu(f, A_0) := \|f\|_{L^\infty(\Omega(|x''|<A_0))}^{N+2-(N-2)p} \|f\|_{L^\infty(\Omega)}^{(N-2)(p-1)} \left(\int_\Omega f(x)\operatorname{dist}(x, \partial\Omega)^{p+1} dx \right)^{-4},$$

and for $N = 2$ with $\theta \in (0, 1)$,

$$\mu(f, A_0, \theta) := \|f\|_{L^\infty(\Omega(|x''|<A_0))} \|f\|_{L^\infty(\Omega)}^{(1-\theta)/\theta} \left(\int_\Omega f(x)\operatorname{dist}(x, \partial\Omega)^{p+1} dx \right)^{-1/\theta}.$$

Here $\|\cdot\|_{L^q(\Omega)}$ denotes the $L^q(\Omega)$ norm and $\operatorname{dist}(x, \partial\Omega)$ is the distance function from x to $\partial\Omega$.

Let G and H be closed subgroups of $O(N)$ such that Ω is G invariant and $H \subsetneqq G$. We introduce an assumption

(2.3) $$H(x) \subsetneqq G(x) \quad \text{for all } x \in \Gamma.$$

Here $H(x)$ and $G(x)$ denote the orbits defined by (1.6) and Γ is defined by (2.2). Define $f_+(x) := \max(f(x), 0)$. We consider the following two conditions separately.

(A) $f(x) \leq 0$ in $\Omega(|x''| < A_0)$ and $f_+(x) \not\equiv 0$ in $\Omega(A_0 < |x''| < A)$.

(B) $f(x) \geq 0$, $\not\equiv 0$ in Ω and $\mu(f, A_0)$ is small enough when $N \geq 3$ and $\mu(f, A_0, \theta)$ with a certain $\theta \in (0, 1)$ is small enough when $N = 2$.

Recall that R_G is defined by (1.3). We state the first main result.

THEOREM 2.1. *Let G and H be closed subgroups of $O(N)$ such that Ω is G invariant, $H \subsetneqq G$ and they satisfy (2.3). Then there exists an $\varepsilon \in (0, A)$ independent of $f(x)$ such that if either (A) or (B) holds at some $A_0 \in (A - \varepsilon, A)$, then $R_H < R_G$. Therefore if u is an H invariant least energy solution, it is not G invariant. Thus (1.1) has at least two positive solutions: one is an H invariant G non-invariant positive solution and another is a G invariant positive solution.*

Denote the set of G invariant continuous functions on $\overline{\Omega}$ by $C(\overline{\Omega}, G)$. Instead of assumption (B), we consider the next one.

(B)' $f(x) := f_0(x)^\lambda$ with $\lambda > 0$, where $f_0 \in C(\overline{\Omega}, G)$ satisfies

$$(2.4) \qquad 0 \le f_0(x) < \max_{y \in \Gamma} f_0(y) \quad \text{for } x \in \overline{\Omega} \setminus \Gamma.$$

Then we have the next corollary.

COROLLARY 2.2. *Let G, H and Ω be as in Theorem 2.1. Let $f \in C(\overline{\Omega}, G)$ satisfy either assumption (A) with A_0 sufficiently close to A or assumption (B)' with $\lambda > 0$ large enough. Then no H invariant least energy solution of (1.1) can be G invariant.*

Now, we consider the case where $\text{Fix}(G) = \{0\}$. This condition is satisfied if, for example, $G = \{I, -I\}$ with the unit matrix I, $G = G_n$ defined by (1.5) or $G = O(N)$, etc. Then A defined by (2.1) is the radius of the circumscribed sphere of Ω and Γ is the intersection of this sphere and $\partial\Omega$. Let $f(r)$ with $r = |x|$ be radially symmetric. Then assumption (A) is rewritten as

$$(2.5) \qquad f(r) \le 0 \quad \text{in } (0, A_0), \qquad f_+(r) \not\equiv 0 \quad \text{in } (A_0, A).$$

Condition (2.4) is reduced to

$$(2.6) \qquad 0 \le f_0(r) < f_0(A) \quad \text{for } 0 \le r < A,$$

where $f(r) = f_0(r)^\lambda$ and $f_0 \in C[0, A]$. Then Corollary 2.2 leads to the next one.

COROLLARY 2.3. *Let G, H and Ω be as in Theorem 2.1. Moreover, assume that $\text{Fix}(G) = \{0\}$. Let $f(r)$ with $r = |x|$ be a radial continuous function satisfying either (2.5) with $A_0 \in (0, A)$ sufficiently close to A or (2.6) with $f(r) := f_0(r)^\lambda$ and λ large enough. Then no H invariant least energy solution of (1.1) is G invariant.*

There are various functions $f(|x|)$ satisfying (2.6), e.g. $f(|x|) = |x|^\lambda$ (the original Hénon equation), $f(|x|) = e^{\lambda|x|}$, $(|x|/(1 + |x|))^\lambda$ and $(\sin(\pi|x|/2A))^\lambda$, etc. The simplest example satisfying (2.5) is $f(|x|) = |x| - A_0$.

REMARK 2.4. We mention that (2.3) is related to the necessary condition for the existence of an H invariant G non-invariant solution. Suppose that

$$H(x) = G(x) \quad \text{for all } x \in \Omega.$$

Under this condition, any H invariant function coincides with a G invariant function. Therefore the existence of an H invariant G non-invariant solution needs the condition,

$$H(x) \subsetneq G(x) \quad \text{at some } x \in \Omega.$$

This is a necessary condition and (2.3) is stronger than this condition.

Since G is an isometric transformation group on $\mathbb{R}^N$, it becomes that on the unit sphere S^{N-1},

$$S^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\}.$$

If $G(x) = S^{N-1}$ for $x \in S^{N-1}$, G is said to be *transitive* on S^{N-1}. If G is transitive, then a G invariant solution becomes radially symmetric. If G is not transitive, then $G(x) \subsetneq S^{N-1} = O(N)(x)$ for $x \in S^{N-1}$, which implies that $G(x) \subsetneq O(N)(x)$ for all $x \neq 0$. This condition means that the pair $(G, O(N))$ satisfies (2.3). Then Corollary 2.3 implies the next result.

COROLLARY 2.5. *Let Ω be an annulus or a ball, i.e.*

$$(2.7) \qquad \Omega := \{x \in \mathbb{R}^N : a < |x| < A\} \quad or \quad \Omega := \{x \in \mathbb{R}^N : |x| < A\}.$$

Let $f(|x|)$ be a radial continuous function satisfying either (2.5) *with A_0 close to A or* (2.6) *with $\lambda > 0$ large enough. Let G be a closed subgroup of $O(N)$. Then the following two assertions are equivalent.*

(a) *G is not transitive on S^{N-1}.*
(b) *No G invariant least energy solution can be radially symmetric.*

REMARK 2.6. In our paper [28], we studied sign-changing solutions of (1.1) in a ball and an annulus, where we took $f(x) \equiv 1$ and replaced u^p by $|u|^{p-1}u$. Then we proved that (a) in Corollary 2.5 is equivalent to the existence of a sequence u_k of sign-changing solutions which are G invariant but non-radial and the $H_0^1(\Omega)$ norm of u_k diverges to infinity as $k \to \infty$.

All transitive Lie groups have been classified by Montgomery and Samelson [41] and Borel [8] as below (see also [26, p. 186, Theorem 2.6] and [44, p. 267, Theorem 3]).

THEOREM 2.7 (Montgomery and Samelson [41], Borel [8]). *Let $N \geq 2$ and G be a connected closed subgroup of $SO(N)$. Here $SO(N)$ is the special orthogonal group (rotation group). Then G is transitive on S^{N-1} if and only if G is locally isomorphic to one of the following groups: $SO(N)$; $SU(m)$, $U(m)$ if $N = 2m$; $Sp(m)$, $Sp(m)Sp(1)$, $Sp(m)U(1)$ if $N = 4m$; $\mathrm{Spin}(9)$ if $N = 16$; $\mathrm{Spin}(7)$ if $N = 8$; G_2 if $N = 7$.*

When G is not necessarily connected, we denote by G_0 the connected component of G which has the unit matrix. Then G_0 becomes a closed subgroup of G. Moreover, G is transitive if and only if so is G_0. Therefore a closed subgroup G of $O(N)$ is transitive if and only if G_0 is $O(N)$ locally isomorphic to one of the groups listed in the theorem above. Since a finite subgroup of $O(N)$ is not transitive, we have the following result.

Corollary 2.8. *Let Ω and f be as in Corollary 2.5 and let G be a finite subgroup of $O(N)$. Then no G invariant least energy solution can be radially symmetric.*

Example 2.9. Let Ω and f be as in the corollary above. Then an even least energy solution is not radially symmetric. Indeed, we choose $G = \{I, -I\}$ with the unit matrix I. Then a G invariant function is even. Therefore Corollary 2.8 shows the conclusion.

3. Examples

In this section, we give several examples of G, H and Ω and demonstrate how to apply our theorem to the problem. Then we obtain multiple positive solutions.

Example 3.1. Let Ω be a bounded and point symmetric domain, i.e., $x \in \Omega$ implies $-x \in \Omega$. Put $A := \max_{x \in \overline{\Omega}} |x|$. Let $f(x)$ be a continuous even function on $\overline{\Omega}$ satisfying either (A) or (B) below.

(A) $f(x) \le 0$ in $\Omega(0 < |x| < A_0)$ and $f_+(x) \not\equiv 0$ in $\Omega(A_0 < |x| < A)$.
(B) $f(x) = f_0(x)^\lambda$, where $f_0(x)$ is a continuous even function satisfying $0 \le f_0(x) < \max_{|y|=A} f_0(y)$ for $0 \le |x| < A$.

If (A) holds with A_0 sufficiently close to A or if (B) holds with $\lambda > 0$ large enough, then a global least energy solution is not even. Therefore (1.1) has both an even positive solution and a non-even positive solution. We shall show this claim. Put $H := \{I\}$ and $G := \{I, -I\}$ with the unit matrix I. Then $\mathrm{Fix}(G) = \{0\}$ and (2.3) holds. In this case, an H invariant least energy solution coincides with a global least energy solution. Corollary 2.2 shows that a global least energy solution is not even.

Example 3.2. Let $N = 2$ and Ω be a square defined by

$$\Omega := \{(x_1, x_2) \in \mathbb{R}^2 : |x_1| < 1, \ |x_2| < 1\}.$$

Let $f(r)$ with $r = |x|$ be a radial continuous function defined for $r \in [0, \sqrt{2}]$ and hence it is defined on the whole Ω. Assume that $f(r)$ satisfies either (2.5) with A_0 slightly less than $\sqrt{2}$ or (2.6) with $\lambda > 0$ large enough. Let $H := \{I, -I\}$ with the unit matrix I and define

$$G := \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}.$$

Then G is a reflection group with respect to the x-axis and the y-axis. H is a point reflection with respect to the origin. Clearly, H is a subgroup of G. It holds that $\mathrm{Fix}(G) = \{0\}$ and Γ consists of four vertices of Ω, i.e.

(3.1) $$\Gamma = \{(1, 1), \ (1, -1), \ (-1, 1), \ (-1, -1)\},$$

The constant A defined by (2.1) is equal to $\sqrt{2}$, the radius of the circumscribed circle of Ω. From the definition of the orbit, it follows that

$$H(x) = \{x, -x\}, \quad G(x) = \Gamma \quad \text{for } x \in \Gamma,$$

which means that $H(x) \subsetneqq G(x)$ for all $x \in \Gamma$. Thus (H, G, Ω) satisfies (2.3). By Corollary 2.2, a point symmetric (i.e. even) least energy solution has no reflective symmetry with respect to either the x-axis or the y-axis. Observe that a combination of a point symmetry and the x-axis reflective symmetry yields the y-axis reflective symmetry. This claim is still valid by exchanging x with y. Therefore our theorem says that a point symmetric least energy solution has neither the x-axis reflective symmetry nor the y-axis reflective symmetry.

In the following, we consider various groups G. Note that Γ defined by (2.2) depends only on Ω and G. Since Ω is fixed, we denote Γ by $\Gamma(G)$. Let G_4 and H be as in (1.5) and as above, respectively. Then H is a subgroup of G_4. It is easy to compute that $\text{Fix}(G_4) = \{0\}$ and $\Gamma(G_4)$ is equal to Γ in (3.1). Hence (H, G_4, Ω) satisfies (2.3). Accordingly, a point symmetric least energy solution is not invariant under the rotation by angle $\pi/4$.

Define

$$K := \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}.$$

Then K is the reflection with respect to the lines $y = x$ and $y = -x$. H is a subgroup of K and $\Gamma(K)$ is equal to Γ in (3.1). In this case, (H, K, Ω) does not satisfy (2.3). Indeed, we have

$$H(x) = K(x) = \{x, -x\} \quad \text{for } x \in \Gamma.$$

The square Ω has four lines of symmetry: x-axis, y-axis, $y = x$ and $y = -x$. Our theorem ensures that a point symmetric least energy solution does not have any symmetries of the x-axis reflection, the y-axis reflection and the rotation by $\pi/4$. However our theorem says nothing about whether a point symmetric least energy solution has no reflective symmetry with respect to the line $y = x$ or $y = -x$.

We consider a global least energy solution. Let $H = \{I\}$ and $G = \{I, -I\}$. Then an H invariant least energy solution is a global least energy solution. For this G, $\Gamma(G)$ coincides with Γ in (3.1). Since $H(x) \subsetneqq G(x)$ for $x \in \Gamma$, a global least energy solution is not even. We shall show that a global least energy solution is not symmetric with respect to the x-axis reflection. Let $H := \{I\}$ and define

$$G := \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}.$$

Then G is the reflection group of the x-axis. Then $\text{Fix}(G)$ is the x-axis and hence

$$\Gamma(G) = \{(x_1, x_2) : -1 \leq x_1 \leq 1, \ x_2 = \pm 1\}.$$

We compute

$$H(x) = \{x\}, \quad G(x) = \{(x_1, x_2), \ (x_1, -x_2)\} \quad \text{for } x = (x_1, x_2) \in \Gamma(G).$$

Therefore (H, G) satisfies (2.3). Since $\text{Fix}(G)$ is the x-axis, we compute

$$\Omega(|x''| < A_0) = \{(x_1, x_2) \in \Omega : |x_2| < A_0\}.$$

At the beginning of the example, we defined a radial continuous function $f(r)$. For G above, $f(r)$ still satisfies (A) or (B)', which can be proved by using the expression of $\Omega(|x''| < A_0)$ above. Hence a global least energy solution is not symmetric with respect to the x-axis reflection. In the same way, we see that a global least energy solution does not have any symmetries of the x-axis reflection, the y-axis reflection and the rotation by $\pi/4$.

We have already stated in Introduction that when Ω is a hollow thin polygon, (1.4) has an H invariant G non-invariant solution. We consider the usual polygon without a hole in the next example.

EXAMPLE 3.3. Let Ω be a regular n polygon with center origin. Let $1 = n_1 < \ldots < n_d = n$ be all the divisors of n and u_i be a G_{n_i} invariant least energy solution. Here G_n is defined by (1.5). Let A be the distance between the center and a vertex of Ω, i.e., the radius of the circumscribed circle of Ω. Let $f(x)$ be a G_n invariant function satisfying either (A) in Example 3.1 with A_0 sufficiently close to A or (B) with $\lambda > 0$ large enough. Then u_i is not equivalent to u_j for $i \neq j$. For example, consider the regular hexagon Ω. Then it is G_6 invariant. All subgroups of G_6 are G_1, G_2, G_3 and G_6. Let u_i be a G_i invariant least energy solution. Then u_1, u_2, u_3 and u_6 are not equivalent to each other.

The above example and the next one will be proved in the last section.

EXAMPLE 3.4. Let Ω be a cylinder in $\mathbb{R}^3$, which is defined by

$$\Omega := \{(x_1, x_2, x_3) : x_1^2 + x_2^2 < \alpha^2, \ |x_3| < \beta\},$$

with $\alpha, \beta > 0$. Let $f(x)$ be a continuous function which is rotationally symmetric around the x_3 axis and satisfies either (A) or (B) below.

(A) $f(x) \leq 0$ in $\Omega(|(x_1, x_2)| < a)$ and $f_+(x) \not\equiv 0$ in $\Omega(a < |(x_1, x_2)| < \alpha)$.
(B) $f(x_1, x_2, x_3) = f_0(|(x_1, x_2)|)^\lambda$, where $f_0(|(x_1, x_2)|)$ is a radial function satisfying $0 \leq f_0(t) < f_0(\alpha)$ for $0 \leq t = |(x_1, x_2)| < \alpha$.

Then the number of non-equivalent positive solutions diverges to infinity as $a \to \alpha$ in Case (A) or $\lambda \to \infty$ in Case (B).

EXAMPLE 3.5. Let Ω be an interior of a regular tetrahedron centered at the coordinate origin O in $\mathbb{R}^3$. Let A be the radius of the circumscribed sphere of Ω. Let $f(r)$ with $r = |x|$ be a radial continuous function defined on $r \in [0, A]$. Hence it is well defined on the whole Ω. Suppose that $f(r)$ satisfies either (2.5) with A_0 sufficiently close to A or (2.6) with $\lambda > 0$ large enough. Then (1.1) has at least 3 non-equivalent positive solutions. We shall show this claim. We denote the symmetry group of Ω by $O(\Omega)$ and the rotational symmetry group by $SO(\Omega)$, which are defined by

$$O(\Omega) := \{g \in O(3) : g(\Omega) = \Omega)\}, \qquad SO(\Omega) := \{g \in SO(3) : g(\Omega) = \Omega)\}.$$

Then it is known that $SO(\Omega)$ is isomorphic to the alternating group A_4, and $O(\Omega)$ to the symmetric group S_4 (for the proof, see [20, p. 49] or [3, pp. 37–38]). Denote the unit element in S_4 by e. Then all subgroups of S_4 which are not conjugate to each other are listed below.

$$
\begin{aligned}
G_1 &:= \{e\}, \\
G_2 &:= \{e, (1\,2)\}, \\
G_3 &:= \{e, (1\,2)(3\,4)\}, \\
G_4 &:= \{e, (1\,2\,3), (1\,3\,2)\}, \\
G_5 &:= \{e, (1\,3\,2\,4), (1\,2)(3\,4), (1\,4\,2\,3)\}, \\
G_6 &:= \{e, (1\,2), (3\,4), (1\,2)(3\,4)\}, \\
G_7 &:= \{e, (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)\}, \\
G_8 &:= S_3 = \{e, (1\,2), (1\,3), (2\,3), (1\,2\,3), (1\,3\,2)\}, \\
G_9 &:= \{e, (1\,2), (3\,4), (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3), (1\,3\,2\,4), (1\,4\,2\,3)\}, \\
G_{10} &:= A_4, \\
G_{11} &:= O(\Omega) = S_4.
\end{aligned}
$$

Let O be the center of the tetrahedron Ω and V_i with $1 \le i \le 4$ be vertices of Ω. Let V_{ij} be the middle point of the edge connecting V_i with V_j. Let L_{12} denote the reflection matrix with respect to the plane through three points V_{12}, V_3 and V_4. This reflection leaves the tetrahedron Ω invariant and interchanges V_1 with V_2, but leaves V_3 and V_4 invariant. Thus L_{12} corresponds to the transposition $(1\,2)$. For simplicity, we write $L_{12} = (1\,2)$. Similarly, we can define $L_{ij} = (i\,j)$. For each vertex V_i, let $R(V_i, 2\pi/3)$ denote the rotation matrix around the line through V_i and O by angle $2\pi/3$. Then the action $R(V_i, 2\pi/3)$ conserves Ω invariant. Observe that $R(V_1, 2\pi/3)$ leaves V_1 invariant and maps V_2 to V_3, V_3 to V_4 and V_4 to V_2. Thus we write $R(V_1, 2\pi/3) = (2\,3\,4) \in S_4$. Let $R(V_{ij}, \pi)$ be the rotation matrix around the line through V_{ij} and O by angle π. Note that

the line through V_{12} and O goes through V_{34} also. Then we write

$$R(V_{12}, \pi) = R(V_{34}, \pi) = (1\,2)(3\,4) \in S_4.$$

All subgroups G_1 through G_{11} have orthogonal representations by using L_{ij}, $R(V_{ij}, \pi)$ and $R(V_i, 2j\pi/3)$.

Recall that two solutions u and v are said to be equivalent if $u(x) = v(gx)$ with some $g \in O(N)$. If H and K are subgroups of $O(\Omega)$ and they are conjugate, then any H invariant solution is equivalent to a certain K invariant solution. Indeed, let $K = g^{-1}Hg$ with some $g \in O(\Omega)$ and let $u(x)$ be an H invariant solution. Putting $v(x) := u(gx)$, we have

$$v(g^{-1}hgx) = u(hgx) = u(gx) = v(x) \quad \text{for } h \in H,$$

which shows that $v(x)$ is K invariant. Accordingly, the H invariant solution u is equivalent to the $g^{-1}Hg$ invariant solution v. To get non-equivalent solutions, it is enough to consider G_1 through G_{11} only and we ignore conjugate subgroups.

Let u_i be a G_i invariant least energy solution. We shall show that u_1, u_6 and u_{11} have different energy. Observe that $G_1 \subset G_6 \subset G_{11}$. Since Γ defined by (2.2) depends on Ω and G_i, we write it as $\Gamma(G_i)$. Recall that V_{ij} denotes the middle point of the edge connecting V_i with V_j. It is easy to verify that $\mathrm{Fix}(G_6)$ is the line through the points V_{12} and V_{34}. The farthest points in $\overline{\Omega}$ from this line are all vertices V_1, V_2, V_3 and V_4, i.e.

$$(3.2) \qquad\qquad \Gamma(G_6) = \{V_1, V_2, V_3, V_4\}.$$

Then we see that $G_1(x) \subsetneq G_6(x)$ for $x \in \Gamma(G_6)$. Therefore if $\lambda > 0$ is large enough or if A_0 is sufficiently close to A, then it holds that $R(u_1) < R(u_6)$. Since $\mathrm{Fix}(G_{11})$ consists only of the center O of the tetrahedron, $\Gamma(G_{11})$ is the set of all vertices, i.e.

$$(3.3) \qquad\qquad \Gamma(G_{11}) = \{V_1, V_2, V_3, V_4\}.$$

We easily compute that $G_6(x) \subsetneq G_{11}(x)$ for $x \in \Gamma(G_{11})$. Thus $R(u_6) < R(u_{11})$. Therefore u_1, u_6 and u_{11} have different energies and hence they are not equivalent.

As long as we use Theorem 2.1 only, we cannot find more solutions. We shall show it. Recall that $\Gamma(G_6)$ is given by (3.2). Since $G_2 \subset G_6$ but $G_2(V_1) = G_6(V_1) = \{V_1, V_2\}$, our theorem does not ensure that $R(u_2) < R(u_6)$, i.e. we cannot distinguish u_2 from u_6. Observe the inclusions

$$G_3 \subset G_6, \quad G_5 \subset G_{11}, \quad G_7 \subset G_{11}, \quad G_9 \subset G_{11}, \quad G_{10} \subset G_{11}.$$

For all these pairs $G_i \subset G_j$, by using (3.2) and (3.3), one can prove that $G_i(x) = G_j(x)$ at some $x \in \Gamma(G_j)$. Therefore for these pairs i, j, it is unclear whether

u_i is not equivalent to u_j. The groups G_4 and G_8 remain. We easily see that $\mathrm{Fix}(G_4) = \mathrm{Fix}(G_8)$ and it is the line through two points V_4 and O. Hence

$$\Gamma(G_4) = \Gamma(G_8) = \{V_1, V_2, V_3\}.$$

Therefore $f(r)$ satisfy neither (A) nor (B)' in Corollary 2.2 because in the present example, $f(r)$ is a radial function and so it takes the same value at all vertices. Consequently, we have at least 3 non-equivalent positive solutions.

4. Group invariant least energy

In this section, we shall show some lemmas which will be needed for the proof of the main results. Let $L^q(\Omega, G)$ denote the set of G invariant functions in $L^q(\Omega)$. Define the $L^2(\Omega)$ inner product and the $H_0^1(\Omega)$ inner product by

$$(u, v)_{L^2} := \int_\Omega uv\, dx, \qquad (u, v)_{H_0^1} := \int_\Omega \nabla u \nabla v\, dx.$$

We define the orthogonal complements of $L^2(\Omega, G)$ and $H_0^1(\Omega, G)$ by

$$L^2(\Omega, G)^\perp := \{u \in L^2(\Omega) : (u, v)_{L^2} = 0 \text{ for all } v \in L^2(\Omega, G)\},$$

$$H_0^1(\Omega, G)^\perp := \{u \in H_0^1(\Omega) : (u, v)_{H_0^1} = 0 \text{ for all } v \in H_0^1(\Omega, G)\}.$$

The next two lemmas have been proved in our paper [31].

LEMMA 4.1 ([31]). *The following assertions hold.*
(a) $H_0^1(\Omega, G)^\perp \subset L^2(\Omega, G)^\perp$.
(b) *Let* $1 \le p, q \le \infty$ *with* $1/p + 1/q = 1$. *For* $u \in L^p(\Omega) \cap L^2(\Omega, G)^\perp$ *and* $v \in L^q(\Omega, G)$, *it holds that*

$$\int_\Omega uv\, dx = 0.$$

The Rayleigh quotient $R(u)$ belongs to $C^2(H_0^1(\Omega))$ in the sense of the Fréchet derivative. The second derivative $R''(u)vw$ is a continuous bilinear form of v and w. We give the expression of $R''(u)w^2$ in the next lemma.

LEMMA 4.2 ([30]). *Let* u *be a positive solution of* (1.1). *For* $w \in H_0^1(\Omega)$, *it holds that*

$$(4.1) \quad R''(u)w^2 = 2 \int_\Omega |\nabla w|^2\, dx \left(\int_\Omega |\nabla u|^2\, dx \right)^{-2/(p+1)}$$

$$+ 2(p-1) \left(\int_\Omega \nabla u \nabla w\, dx \right)^2 \left(\int_\Omega |\nabla u|^2\, dx \right)^{-(p+3)/(p+1)}$$

$$- 2p \int_\Omega f u^{p-1} w^2\, dx \left(\int_\Omega |\nabla u|^2\, dx \right)^{-2/(p+1)}.$$

The proofs of the main theorems are based on the next proposition, which ensures that $R_H < R_G$. Here R_H and R_G have been defined by (1.3).

PROPOSITION 4.3. *Let H and G be closed subgroups of $O(N)$ such that $H \subsetneq G$. Let Ω and Ω_1 be G invariant bounded domains in $\mathbb{R}^N$ such that $\Omega \subset \Omega_1$. Let u be a G invariant least energy solution of* (1.1) *and ϕ be a function satisfying*

$$(4.2) \qquad \phi \in W^{1,\infty}(\Omega_1) \cap H_0^1(\Omega_1, G)^{\perp} \cap H_0^1(\Omega_1, H),$$

$$(4.3) \qquad \int_{\Omega} |\nabla \phi|^2 u^2 dx < \frac{p-1}{2(2p-1)} \int_{\Omega} |\nabla u|^2 \phi^2 \, dx.$$

Put $v := (1 + \varepsilon\phi)u$. Then $R(v) < R(u)$ for $\varepsilon > 0$ small enough. Therefore $R_H < R_G$, and no H invariant least energy solution can be G invariant.

PROOF. This proposition has already been proved in our paper [30], however we give a proof for the reader's convenience. Since $\phi \in W^{1,\infty}(\Omega_1)$ and $u \in H_0^1(\Omega)$, the functions ϕu and v belong to $H_0^1(\Omega)$. Since u is a solution of (1.1), $R'(u)$ vanishes. By the Taylor theorem, we have

$$(4.4) \qquad R(v) = R(u) + \frac{\varepsilon^2}{2} R''(u)(\phi u)^2 + o(\varepsilon^2),$$

where $o(\varepsilon^2)/\varepsilon^2 \to 0$ as $\varepsilon \to 0$. Let us show that $R''(u)(\phi u)^2 < 0$.

We use the Schwarz inequality to get

$$2|u\phi\nabla u\nabla\phi| \leq 2|\nabla\phi|^2 u^2 + \frac{1}{2}|\nabla u|^2\phi^2,$$

which with (4.3) leads to

$$\int_{\Omega} |\nabla\phi|^2 u^2 dx - 2(p-1)\int_{\Omega} u\phi\nabla u\nabla\phi dx < (p-1)\int_{\Omega} |\nabla u|^2\phi^2 dx.$$

This inequality is rewritten as

$$\int_{\Omega} (|\nabla u|^2\phi^2 + 2u\phi\nabla u\nabla\phi + |\nabla\phi|^2 u^2)\, dx < p\int_{\Omega}(|\nabla u|^2\phi^2 + 2u\phi\nabla u\nabla\phi)\, dx.$$

On the other hand, multiplying (1.1) by $\phi^2 u$ and integrating it over Ω, we have

$$\int_{\Omega}\left(|\nabla u|^2\phi^2 + 2u\phi\nabla u\nabla\phi\right) dx = \int_{\Omega} fu^{p+1}\phi^2 \, dx.$$

Combining two expressions above, we obtain

$$\int_{\Omega}(|\nabla u|^2\phi^2 + 2u\phi\nabla u\nabla\phi + |\nabla\phi|^2 u^2)\, dx < p\int_{\Omega} fu^{p+1}\phi^2 \, dx,$$

or equivalently,

$$(4.5) \qquad \int_{\Omega} |\nabla(\phi u)|^2 \, dx < p\int_{\Omega} fu^{p+1}\phi^2 \, dx.$$

Extend u by zero outside Ω. Since $|\nabla u|^2 \in L^1(\Omega_1, G)$ and ϕ belongs to $H_0^1(\Omega_1, G)^{\perp} \cap L^{\infty}(\Omega_1)$, Lemma 4.1 shows that

$$\int_{\Omega} |\nabla u|^2\phi \, dx = \int_{\Omega_1} |\nabla u|^2\phi \, dx = 0.$$

By the elliptic regularity theorem, u belongs to $L^\infty(\Omega)$. Since $\nabla(u^2) = 2u\nabla u \in L^2(\Omega)$, i.e. $u^2 \in H_0^1(\Omega, G)$, we have

$$\int_\Omega u\nabla u\nabla\phi \, dx = \frac{1}{2}\int_{\Omega_1} \nabla(u^2)\nabla\phi \, dx = 0.$$

From the two identities above, it follows that

$$\int_\Omega \nabla u\nabla(\phi u) \, dx = 0.$$

Substituting $w = \phi u$ into (4.1) and using the identity above, we obtain

$$R''(u)(\phi u)^2 = 2\int_\Omega |\nabla(\phi u)|^2 \, dx \left(\int_\Omega |\nabla u|^2 \, dx\right)^{-2/(p+1)}$$
$$- 2p\int_\Omega fu^{p+1}\phi^2 \, dx \left(\int_\Omega |\nabla u|^2 \, dx\right)^{-2/(p+1)}.$$

By (4.5), the right hand side is negative and hence $R''(u)(\phi u)^2 < 0$. From (4.4), it follows that $R(v) < R(u)$ for $\varepsilon > 0$ small enough. Since $v \in H_0^1(\Omega, H)$, it holds that $R_H \leq R(v) < R(u) = R_G$. Consequently, no H invariant least energy solution can be G invariant. $\qquad\square$

5. Estimate of G invariant solutions

In this section, we give an a priori $L^\infty(\Omega)$ estimate of a G invariant least energy solution. Furthermore, we investigate the $L^2(\Omega)$ norm of solutions in detail. If Ω is a G invariant domain, then $\mathrm{dist}(x, \partial\Omega)$ is a G invariant function. Using this function with the same method as in our paper [29], we have the next result.

LEMMA 5.1 ([29, Lemma 4.2]). *Suppose that $f(x) \geq 0$, $\not\equiv 0$ in Ω. Let u be a G invariant least energy solution. Then for $N \geq 3$, there exists a constant $C > 0$ independent of u and f such that*

$$\|u\|_\infty \leq C\|f\|_\infty^\alpha \left(\int_\Omega f(x)\mathrm{dist}(x, \partial\Omega)^{p+1} dx\right)^\beta,$$

where $\|\cdot\|_\infty$ denotes the $L^\infty(\Omega)$ norm and α and β are given by

$$\alpha := \frac{N-2}{N+2-(N-2)p}, \qquad \beta := \frac{-4}{(p-1)(N+2-(N-2)p)}.$$

When $N = 2$, for any $\theta \in (0,1)$ there exists a $C_\theta > 0$ independent of u and f such that

$$\|u\|_\infty \leq C_\theta\|f\|_\infty^{(1-\theta)/\theta(p-1)} \left(\int_\Omega f(x)\mathrm{dist}(x, \partial\Omega)^{p+1} dx\right)^{-1/\theta(p-1)}.$$

Hereafter we denote $\mathrm{Fix}(G)$ by F. When $n := \dim F \geq 1$, we can assume without loss of generality that

(5.1) $$F = \{(x_1, \ldots, x_n, 0, \ldots, 0) : x_i \in \mathbb{R}\}.$$

Indeed, choose an orthogonal matrix h satisfying

$$h(F) = \{(x_1, \ldots, x_n, 0, \ldots, 0) : x_i \in \mathbb{R}\},$$

and put $\widetilde{u}(x) := u(h^{-1}x)$, $\widetilde{f}(x) := f(h^{-1}x)$ and $\widetilde{\Omega} := h(\Omega)$. Then (1.1) is rewritten as

$$-\Delta \widetilde{u} = \widetilde{f}(x)\widetilde{u}^p, \quad \widetilde{u} > 0 \quad \text{in } \widetilde{\Omega}, \qquad \widetilde{u} = 0 \quad \text{on } \partial\widetilde{\Omega}.$$

Put $\widetilde{G} := hGh^{-1}$. Then u is a G invariant solution of (1.1) if and only if $\widetilde{u}$ is a $\widetilde{G}$ invariant solution of the equation above. Furthermore, it holds that $\mathrm{Fix}(\widetilde{G}) = h(\mathrm{Fix}(G))$. Therefore we can assume either (a) or (b) below.

(a) $F = \{(x_1, \ldots, x_n, 0, \ldots, 0) : x_i \in \mathbb{R}\}$ when $\dim F = n \geq 1$.

(b) $F = \{0\}$ when $\dim F = 0$.

Some lemmas below have already been proved in our paper [31], however it has not been published yet. Therefore we shall prove them for the reader's convenience.

LEMMA 5.2 ([31]). *If* $\dim F = N - 1$, *then* G *takes the form*

(5.2) $$G = \left\{ \begin{pmatrix} I' & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} I' & 0 \\ 0 & -1 \end{pmatrix} \right\},$$

where I' *denotes the* $(N-1) \times (N-1)$ *unit matrix. Therefore any* G *invariant function* u *is even with respect to* x_N, *i.e.*

(5.3) $$u(x_1, \ldots, x_{N-1}, x_N) = u(x_1, \ldots, x_{N-1}, -x_N).$$

PROOF. Since $\dim F = N - 1$, it holds that

$$F = \mathrm{Fix}(G) = \{(x_1, \ldots, x_{N-1}, 0) : x_i \in \mathbb{R}\},$$

which implies that $gx = x$ for all $x = (x_1, \ldots, x_{N-1}, 0)$ and $g \in G$. Hence each $g \in G$ has the form

$$g = \begin{pmatrix} I' & * \\ 0 & * \end{pmatrix}.$$

Since g is orthogonal, it is equal to

$$\begin{pmatrix} I' & 0 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} I' & 0 \\ 0 & -1 \end{pmatrix}.$$

Since $\dim F = N - 1$, it holds that $\{I\} \subsetneq G$. Therefore G takes the form of (5.2). $\square$

By (a) before Lemma 5.2, the orthogonal complement $F^\perp$ of F is written as

$$F^\perp := \{(0, \ldots, 0, x_{n+1}, \ldots, x_N) : x_i \in \mathbb{R} \ (1 \le i \le n)\}.$$

Any point $x \in \mathbb{R}^N$ is decomposed into $x = x' + x''$ with $x' \in F$ and $x'' \in F^\perp$. Moreover, in the space $F^\perp$, we introduce polar coordinates

$$x'' = r\sigma, \quad \text{where } r = |x''|, \quad \sigma = x''/|x''| \in S^{N-n-1}.$$

Here S^{N-n-1} denotes the unit sphere in $F^\perp$. Then any function $u(x)$ is represented as $u(x) = u(x', x'') = u(x', \sigma, r)$. For any $u \in H_0^1(\Omega)$, we extend u by 0 outside Ω. Then $u \in H^1(\mathbb{R}^N)$. For $r > 0$, we define

$$F_r := \{x \in \mathbb{R}^N : \text{dist}(x, F) = r\} = \{x' + x'' : |x''| = r\}.$$

DEFINITION 5.3. We put $n := \dim F$. For a G invariant positive solution u, we define

$$(5.4) \qquad U(r) := \frac{1}{r^{N-n-1}} \int_{F_r} u(x)^2 ds_r,$$

where ds_r is the standard measure on F_r, i.e.,

$$ds_r = r^{N-n-1} \, dx' \, d\sigma = r^{N-n-1} \, dx_1 \ldots dx_n \, d\sigma.$$

Here $d\sigma$ denotes the usual measure on the unit sphere S^{N-n-1} in $F^\perp$.

We rewrite U in the following form.

(i) If $1 \le n \le N - 2$, then

$$(5.5) \qquad U(r) = \int_{S^{N-n-1}} \left(\int_{\mathbb{R}^n} u(x', \sigma, r)^2 \, dx' \right) d\sigma.$$

(ii) If $n = 0$, then $F = \{0\}$, $F^\perp = \mathbb{R}^N$ and we have

$$(5.6) \qquad U(r) = \int_{S^{N-1}} u(\sigma, r)^2 \, d\sigma.$$

(iii) If $n = N - 1$, F_r consists of two hyperplanes parallel to F. By (5.3), we get

$$(5.7) \qquad U(r) = \int_{\mathbb{R}^{N-1}} u(x', r)^2 \, dx' + \int_{\mathbb{R}^{N-1}} u(x', -r)^2 \, dx'$$

$$= 2 \int_{\mathbb{R}^{N-1}} u(x', r)^2 \, dx'.$$

For $a > 0$, we define the weighted Lebesgue and Sobolev spaces:

$$L^1(0, \infty; r^a) := \left\{ v : \int_0^\infty |v(r)| r^a \, dr < \infty \right\},$$

$$W^{1,1}(0, \infty; r^a) := \left\{ v : \int_0^\infty (|v(r)| + |v_r(r)|) r^a \, dr < \infty \right\},$$

where $v_r(r)$ denotes the derivative of $v(r)$.

Lemma 5.4 ([31]). *Let u be a G invariant positive solution. Then the following assertions hold.*

(a) *U belongs to* $L^\infty(0,\infty) \cap W^{1,1}(0,\infty; r^{N-n-1}) \cap C[0,\infty) \cap C^1(0,\infty)$.

(b) *When* $1 \le n \le N-2$,

$$(5.8) \qquad U_r(r) = 2 \int_{S^{N-n-1}} \left(\int_{\mathbb{R}^n} uu_r \, dx' \right) d\sigma.$$

(c) *When* $n = 0$,

$$(5.9) \qquad U_r(r) = 2 \int_{S^{N-1}} uu_r \, d\sigma.$$

(d) *When* $n = N-1$,

$$(5.10) \qquad U_r(r) = 4 \int_{\mathbb{R}^{N-1}} u(x',r)u_{x_N}(x',r) \, dx'.$$

(e) *For any* $0 \le n \le N-1$ *and* $0 < s < t < A$, *we have*

$$(5.11) \qquad t^{N-n-1}U_r(t) - s^{N-n-1}U_r(s) = 2 \int_{\Omega(s<|x''|<t)} \left(|\nabla u|^2 - fu^{p+1} \right) dx.$$

Proof. It is easy to verify (a)–(d). We shall show (e). Let $0 < s < t < A$ and put $D := \Omega(s < |x''| < t) = \{x' + x'' \in \Omega : s < |x''| < t\}$. Multiplying (1.1) by u and integrating it over D, we have

$$(5.12) \qquad \int_D \left(|\nabla u|^2 - fu^{p+1} \right) dx = \int_{\partial D} \frac{\partial u}{\partial n} u \, d\mu,$$

where $d\mu$ is the standard measure on ∂D and $\partial u/\partial n$ denotes the outward normal derivative. We divide ∂D into $\partial D = S \cup \Omega(|x''| = s) \cup \Omega(|x''| = t)$, where $S := \partial\Omega \cap \{x' + x'' : s < |x''| < t\}$, $\Omega(|x''| = r) := \{x' + x'' \in \Omega : |x''| = r\}$ with $r = s, t$.

Let $n \le N-2$. Note that $u = 0$ on S. Since $\partial/\partial n$ is the outward normal derivative on ∂D, it is computed as

$$\frac{\partial u}{\partial n} = \frac{\partial u}{\partial r} \quad \text{on } \Omega(|x''| = t), \qquad \frac{\partial u}{\partial n} = -\frac{\partial u}{\partial r} \quad \text{on } \Omega(|x''| = s).$$

Note that $d\mu = r^{N-n-1} \, dx' \, d\sigma$ on $\Omega(|x''| = r)$, where $d\sigma$ is a standard measure on S^{N-n-1}. Using (5.8) or (5.9), we obtain

$$(5.13) \qquad \int_{\partial D} \frac{\partial u}{\partial n} u \, d\mu = \int_{\Omega(|x''|=t)} \frac{\partial u}{\partial r} u \, d\mu - \int_{\Omega(|x''|=s)} \frac{\partial u}{\partial r} u \, d\mu$$

$$= t^{N-n-1} \int_{\Omega(|x''|=t)} \frac{\partial u}{\partial r} u \, dx' \, d\sigma$$

$$- s^{N-n-1} \int_{\Omega(|x''|=s)} \frac{\partial u}{\partial r} u \, dx' \, d\sigma$$

$$= t^{N-n-1} U_r(t)/2 - s^{N-n-1} U_r(s)/2.$$

Let $n = N - 1$. Then $\Omega(|x''| = r) = \{(x', x_N) \in \Omega : x_N = r, -r\}$. Since $u(x', x_N)$ is even in x_N by Lemma 5.2, we use (5.10) to get

$$\int_{\Omega(|x''|=t)} \frac{\partial u}{\partial r} u \, dx' = 2 \int_{\mathbb{R}^{N-1}} \frac{\partial u}{\partial x_N}(x', t) u(x', t) \, dx' = U_r(t)/2.$$

Therefore (5.13) is valid for $n = N - 1$ also. Substituting (5.13) into (5.12), we obtain (5.11). The proof is complete. $\qquad\square$

We study the relation between the L^∞ norm of $U(r)$ and the L^2 norm of ∇u.

LEMMA 5.5. ([31]) *Let $u(x)$ and $U(r)$ be as in Lemma 5.4. Put $m := N - n - 1$. Then for any $a \in (0, A)$, it holds that*

$$(5.14) \qquad \max_{a \leq r \leq A} U(r) \leq 4(A - a)a^{-m} \int_{\Omega(a<|x''|<A)} |\nabla u|^2 \, dx.$$

PROOF. Let b be a maximum point of $U(r)$ on $[a, A]$. Let $1 \leq n \leq N - 2$ and set $D := S^m \times \mathbb{R}^n$. Then (5.5) and (5.8) are rewritten as

$$U(r) = \int_D u^2 \, dx' d\sigma, \qquad U_r(r) = 2 \int_D u u_r \, dx' d\sigma.$$

Applying the Schwarz inequality to the second equation above, we have

$$(5.15) \qquad |U_r(r)| \leq 2 \left(\int_D u^2 \, dx' \, d\sigma \right)^{1/2} \left(\int_D u_r^2 \, dx' \, d\sigma \right)^{1/2}$$

$$\leq 2U(b)^{1/2} \left(\int_D u_r^2 \, dx' \, d\sigma \right)^{1/2},$$

for $r \in [a, A]$. Using the inequality above and $U(A) = 0$, we get

$$(5.16) \qquad 0 \leq U(b) \leq \int_b^A |U_r(r)| \, dr$$

$$\leq \left(\int_b^A |U_r(r)|^2 r^m \, dr \right)^{1/2} \left(\int_b^A r^{-m} \, dr \right)^{1/2}$$

$$\leq 2((A - b)b^{-m})^{1/2} U(b)^{1/2} \left(\int_b^A \int_D u_r^2 r^m \, dx' \, d\sigma \, dr \right)^{1/2}$$

$$\leq 2((A - b)b^{-m})^{1/2} U(b)^{1/2} \|\nabla u\|_{L^2(\Omega(b \leq |x''| \leq A))}.$$

Since $a \leq b$, we have

$$U(b) \leq 4(A - a)a^{-m} \|\nabla u\|^2_{L^2(\Omega(a \leq |x''| \leq A))}.$$

This proves (5.14). Let $n = N - 1$. Then $m = 0$ and

$$U(r) = 2 \int_{\mathbb{R}^{N-1}} u^2 \, dx', \qquad U_r(r) = 4 \int_{\mathbb{R}^{N-1}} u u_{x_N} \, dx'.$$

Instead of (5.15), we have

$$|U_r(r)| \leq (4/\sqrt{2}) U(b)^{1/2} \left(\int_{\mathbb{R}^{N-1}} u_{x_N}^2 \, dx' \right)^{1/2}.$$

In the same way as in (5.16) with the inequality above, we obtain

$$0 \leq U(b) \leq (4/\sqrt{2})(A-b)^{1/2}U(b)^{1/2}\left(\int_b^A \int_{\mathbb{R}^{N-1}} u_{x_N}^2 \, dx' \, dr\right)^{1/2}.$$

Note that

$$\int_b^A \int_{\mathbb{R}^{N-1}} u_{x_N}^2 \, dx' \, dr = \|\nabla u\|_{L^2(\Omega(b<r<A))}^2 = (1/2)\|\nabla u\|_{L^2(\Omega(b<|x_N|<A))}^2.$$

Therefore we obtain (5.16) for $n = N - 1$. The method above is still valid for $n = 0$ also. $\qquad\square$

LEMMA 5.6 ([31]). *If $f(x) \leq 0$ in $\Omega(|x''| < a)$, then $U(r)$ is nondecreasing in $(0, a)$.*

PROOF. Multiplying (1.1) by u and integrating it over Ω, we have

$$(5.17) \qquad \int_\Omega (|\nabla u|^2 - fu^{p+1}) \, dx = 0.$$

We set $m := N - n - 1$. Then (5.11) is reduced to

$$(5.18) \qquad t^m U_r(t) - s^m U_r(s) = 2\int_{\Omega(s<|x''|<t)} (|\nabla u|^2 - fu^{p+1}) \, dx.$$

Since u is extended by 0 outside Ω, it belongs to $H_0^1(\mathbb{R}^N)$. Since $U(t) = 0$ for $t \geq A$, we substitute $t = A$ to obtain

$$-s^m U_r(s) = 2\int_{\Omega(s<|x''|<A)} (|\nabla u|^2 - fu^{p+1}) \, dx.$$

As $s \to 0$, the right hand side converges to

$$2\int_\Omega (|\nabla u|^2 - fu^{p+1}) \, dx = 0,$$

because of (5.17). Thus $s^m U_r(s)$ converges to zero. Letting $s \to 0$ in (5.18), we find

$$t^m U_r(t) = 2\int_{\Omega(|x''|<t)} (|\nabla u|^2 - fu^{p+1}) \, dx \geq 0,$$

for $0 < t < a$. Therefore $U(t)$ is nondecreasing in $(0, a)$. $\qquad\square$

6. Proof of the main results

In this section, we prove the main theorems. Our idea is based on the methods used in our papers [30], [31] and [32]. Since a closed subgroup G of $O(N)$ is a compact Lie group, it has a unique Haar measure dg. It is a positive Lebesgue measure which satisfies

$$(6.1) \qquad \int_G f(hg) \, dg = \int_G f(gh) \, dg = \int_G f(g^{-1}) \, dg = \int_G f(g) \, dg,$$

$$\int_G f(g) \, dg > 0 \quad \text{if } f \geq 0, \not\equiv 0,$$

(6.2) $$\int_G 1 \, dg = 1,$$

for any $h \in G$ and any real valued integrable function f on G (see [47] or [42, Chapter 2] for more details of the Haar measure).

Let $M(N)$ denote a linear space consisting of all $N \times N$ real matrices, which is equipped with the norm

$$\|g\| := \max_{|x| \le 1} |gx| \quad \text{for } g \in M(N).$$

Here $|x|$ is the usual Euclidean norm in $\mathbb{R}^N$. For $g_0 \in G$ and $r > 0$, we define a ball $B(g_0, r; G)$ in G by

(6.3) $$B(g_0, r; G) := \{g \in G : \|g - g_0\| < r\}.$$

Then the volume of $B(g_0, r; G)$ is defined by

$$|B(g_0, r; G)| := \int_{B(g_0, r; G)} 1 \, dg.$$

The invariance of the Haar measure implies the next lemma.

LEMMA 6.1 ([32]). *Let G be a closed subgroup of $O(N)$. Then the volume $|B(g_0, r; G)|$ is independent of $g_0 \in G$ but depends only on r.*

PROOF. Put $\chi(g) := 1$ if $g \in B(g_0, r; G)$ and $\chi(g) := 0$ if $g \notin B(g_0, r; G)$. Then

$$|B(g_0, r; G)| = \int_G \chi(g) \, dg.$$

Let $g, g_1 \in G$. Since g, g_1, g_0 are isometric in $\mathbb{R}^N$, it holds that

$$\|g_0 g_1^{-1} g - g_0\| = \max_{|x| \le 1} |g_0 g_1^{-1} gx - g_0 x| = \max_{|x| \le 1} |gx - g_1 x| = \|g - g_1\|.$$

Therefore $\chi(g_0 g_1^{-1} g) = 1$ if $\|g - g_1\| < r$ and $\chi(g_0 g_1^{-1} g) = 0$ if $\|g - g_1\| \ge r$. By the invariance of the Haar measure, we get

$$\int_G \chi(g) \, dg = \int_G \chi(g_0 g_1^{-1} g) \, dg = \int_{B(g_1, r; G)} 1 \, dg. \qquad \square$$

For closed subgroups G and H of $O(N)$, we define

(6.4) $$Q(x, g) := \min_{h \in H} |gx - hx|, \qquad P(x) := \max_{g \in G} Q(x, g).$$

LEMMA 6.2. ([32]) *It holds that $|P(x) - P(y)| \le 2|x - y|$ for $x, y \in \mathbb{R}^N$.*

PROOF. Let $x, y \in \mathbb{R}^N$, $g \in G$ and $h \in H$. Since g and h are orthogonal matrices, we have

$$Q(x, g) \le |gx - hx| \le |gx - gy| + |gy - hy| + |hy - hx| = 2|x - y| + |gy - hy|.$$

Taking the minimum on $h \in H$, we obtain $Q(x, g) \leq 2|x - y| + Q(y, g)$. By the definition of P, $Q(x, g) \leq 2|x - y| + P(y)$. Taking the maximum on $g \in G$, we get $P(x) \leq 2|x - y| + P(y)$. Exchanging x with y implies the conclusion. $\quad\square$

By Lemma 6.2, $P(x)$ is continuous. Assumption (2.3) implies that $P(x) > 0$ for $x \in \Gamma$, and so the minimum of $P(x)$ on Γ is positive. We define

$$(6.5) \qquad\qquad \delta := (1/8) \min_{\Gamma} P(x) > 0.$$

We choose $A_1 \in (0, A)$ slightly less than A such that

$$(6.6) \qquad\qquad P(x) > 4\delta \quad \text{for } x \in \Omega(A_1 \leq |x''| \leq A).$$

Define

$$(6.7) \qquad\qquad \Omega_0 := \Omega(A_1 < |x''| < A).$$

By (6.4) and (6.6), for any $x \in \overline{\Omega}_0$ there is a $g \in G$ such that

$$(6.8) \qquad\qquad |gx - hx| > 4\delta > 0 \quad \text{for all } h \in H.$$

Let $\Phi \in C_0^\infty(\mathbb{R}^N)$ be a radial function such that $0 \leq \Phi(|x|) \leq 1$ in $\mathbb{R}^N$ and $\Phi(|x|) = 1$ when $|x| \leq \delta$ and $\Phi(|x|) = 0$ when $|x| \geq 2\delta$. We denote the Haar measures on H and G by dh and dg, respectively. We define

$$(6.9) \qquad\qquad \phi(x) := \int_G \Phi(|x - gx_0|) \, dg - \int_H \Phi(|x - hx_0|) \, dh,$$

where $x_0 \in \overline{\Omega}$ will be determined later on. Moreover, we define

$$\text{dist}(x, \Omega) := \inf\{|x - y| : \ y \in \Omega\}, \qquad \Omega_1 := \{x \in \mathbb{R}^N : \text{dist}(x, \Omega) < 2\delta\}.$$

Clearly $\Omega \subset \Omega_1$. The function ϕ defined by (6.9) satisfies

LEMMA 6.3. $\phi \in C_0^\infty(\Omega_1) \cap H_0^1(\Omega_1, G)^\perp \cap H_0^1(\Omega_1, H)$.

PROOF. Since $x_0 \in \overline{\Omega}$ and $\Phi(|x|) = 0$ for $|x| \geq 2\delta$, the support of ϕ is in Ω_1. Give $\psi \in H_0^1(\Omega_1, G)$ arbitrarily. Let $g \in G$. Since ψ is G invariant, the change of variables $x = gy$ yields

$$\int_{|x - gx_0| < 2\delta} \nabla\Phi(|x - gx_0|)\nabla\psi \, dx = \int_{|gy - gx_0| < 2\delta} \nabla\Phi(|gy - gx_0|)\nabla\psi(gy) \, dy$$
$$= \int_{B(x_0, 2\delta)} \nabla\Phi(|y - x_0|)\nabla\psi(y) \, dy.$$

Since the support of $\Phi(|x - gx_0|)$ is in the ball $B(gx_0, 2\delta)$, the identity above is rewritten as

$$\int_{\Omega_1} \nabla\Phi(|x - gx_0|)\nabla\psi \, dx = \int_{\Omega_1} \nabla\Phi(|x - x_0|)\nabla\psi(x) \, dx,$$

for any $g \in G$. Using this relation with the Fubini theorem and employing (6.2), we obtain

$$\int_{\Omega_1} \nabla \left(\int_G \Phi(|x - gx_0|) \, dg \right) \nabla \psi \, dx = \int_G \left(\int_{\Omega_1} \nabla \Phi(|x - gx_0|) \nabla \psi \, dx \right) dg$$
$$= \int_{\Omega_1} \nabla \Phi(|x - x_0|) \nabla \psi \, dx.$$

Similarly, we get

$$\int_{\Omega_1} \nabla \left(\int_H \Phi(|x - hx_0|) \, dh \right) \nabla \psi \, dx = \int_{\Omega_1} \nabla \Phi(|x - x_0|) \nabla \psi \, dx.$$

Subtracting the latter equation from the former one, we obtain

$$\int_{\Omega_1} \nabla \phi \nabla \psi \, dx = 0,$$

which means that $\phi \in H_0^1(\Omega_1, G)^\perp$.

We shall show that $\phi \in H_0^1(\Omega_1, H)$. Give $k \in H$ arbitrarily. By the invariance of the Haar measure, we have

$$\phi(kx) = \int_G \Phi(|kx - gx_0|) \, dg - \int_H \Phi(|kx - hx_0|) \, dh$$
$$= \int_G \Phi(|x - k^{-1} gx_0|) \, dg - \int_H \Phi(|x - k^{-1}hx_0|) \, dh$$
$$= \int_G \Phi(|x - gx_0|) \, dg - \int_H \Phi(|x - hx_0|) \, dh = \phi(x).$$

Thus $\phi \in H_0^1(\Omega_1, H)$ and the proof is complete. $\square$

Hereafter $B(x, r)$ denotes the ball in $\mathbb{R}^N$ which is centered at x with radius r and $|B(x, r)|$ stands for its Lebesgue measure. Recall that $B(g, r; G)$ defined by (6.3) denotes a ball in G and its volume $|B(g, r; G)|$ depends only on r. Since Ω is bounded, we choose a constant $M > 0$ such that $|x| \leq M$ for $x \in \Omega$. Put $\nu := \delta/(2M)$ and define $c_\nu := |B(g, \nu; G)|$. Let Ω_0 be defined by (6.7). Then we have

LEMMA 6.4. *For any $x_0 \in \overline{\Omega}_0$, there exists a $g_0 \in G$ such that*

(6.10) $\phi(x) \geq c_\nu > 0 \quad for \ x \in B(g_0 x_0, \delta/2).$

In particular, $\phi \not\equiv 0$ in Ω.

PROOF. Let $x_0 \in \overline{\Omega}_0$. By (6.8), we choose a $g_0 \in G$ such that

$$|g_0 x_0 - hx_0| > 4\delta \quad \text{for all } h \in H.$$

Then $B(g_0 x_0, 2\delta) \cap B(hx_0, 2\delta) = \emptyset$ for $h \in H$, which shows that

$$\Phi(|x - hx_0|) = 0 \quad \text{for } x \in B(g_0 x_0, 2\delta), \ h \in H.$$

The definition of ϕ implies

(6.11) $$\phi(x) = \int_G \Phi(|x - gx_0|)dg \quad \text{for } x \in B(g_0 x_0, 2\delta).$$

For $x \in B(g_0 x_0, \delta/2)$ and $g \in B(g_0, \nu; G)$, it holds that

$$|x - gx_0| \leq |x - g_0 x_0| + |g_0 x_0 - gx_0| < \delta/2 + \nu M = \delta.$$

Hence we have

$$\Phi(|x - gx_0|) = 1 \quad \text{for } x \in B(g_0 x_0, \delta/2), \quad g \in B(g_0, \nu; G).$$

Consequently, (6.11) implies

$$\phi(x) \geq \int_{B(g_0, \nu; G)} \Phi(|x - gx_0|)dg = |B(g_0, \nu; G)| = c_\nu > 0,$$

for $x \in B(g_0 x_0, \delta/2)$. This completes the proof.

For Ω_0 defined by (6.7), we determine $k \in \mathbb{N}$ by the smallest positive integer such that

(6.12) $$\overline{\Omega}_0 \subset \bigcup_{i=1}^k B(y_i, \delta/2),$$

with some $y_1, \ldots, y_k \in \overline{\Omega}_0$.

LEMMA 6.5. *Let u be a G invariant least energy solution and k be as above. Then there exist $x_0 \in \overline{\Omega}_0$ and $g_0 \in G$ such that*

$$\int_{\Omega_0} |\nabla u|^2 \, dx \leq k c_\nu^{-2} \int_{B(g_0 x_0, \delta/2)} |\nabla u|^2 \phi^2 \, dx.$$

PROOF. Choose $x_0 \in \overline{\Omega}_0$ which satisfies

$$\max_{y \in \overline{\Omega}_0} \int_{B(y, \delta/2)} |\nabla u|^2 \, dx = \int_{B(x_0, \delta/2)} |\nabla u|^2 \, dx.$$

By (6.12), we have

$$\int_{\Omega_0} |\nabla u|^2 \, dx \leq k \int_{B(x_0, \delta/2)} |\nabla u|^2 \, dx.$$

By Lemma 6.4, we choose $g_0 \in G$ satisfying (6.10). Since u is G invariant, we have

$$\int_{\Omega_0} |\nabla u|^2 \, dx \leq k \int_{B(g_0 x_0, \delta/2)} |\nabla u|^2 \, dx \leq k c_\nu^{-2} \int_{B(g_0 x_0, \delta/2)} |\nabla u|^2 \phi^2 \, dx. \quad \square$$

We give the definition of $\phi(x)$ strictly. First, we define $\delta > 0$ by (6.5). Second, we define A_1 by (6.6) and then Ω_0 by (6.7). Third, we fix x_0 by Lemma 6.5. Last, $\phi(x)$ is well defined by (6.9).

LEMMA 6.6. *Let u be a G invariant least energy solution. Let g_0, x_0, k, c_ν be as in Lemma 6.5 and put $m := N - n - 1$. Then for any $a \in [A_1, A]$, we have*

$$(6.13) \qquad \max_{a \le r \le A} U(r) \le 4k(A - a)a^{-m}c_\nu^{-2} \int_{B(g_0 x_0, \delta/2)} |\nabla u|^2 \phi^2 \, dx.$$

PROOF. Since $\Omega_0 = \Omega(A_1 < |x''| < A)$, we use Lemma 6.5 to get

$$\int_{\Omega(a < |x''| < A)} |\nabla u|^2 \, dx \le \int_{\Omega(A_1 < |x''| < A)} |\nabla u|^2 \, dx \le kc_\nu^{-2} \int_{B(g_0 x_0, \delta/2)} |\nabla u|^2 \phi^2 \, dx.$$

Combining the inequality above with (5.14), we get the conclusion. $\qquad\square$

We note that δ, Ω_0, c_ν, k and m depend only on G, H, Ω and not on $f(x)$. Choose $A_0 \in (A_1, A)$ sufficiently close to A such that

$$32k(m + 1)^{-1} A^{m+1}(A - A_0)A_0^{-m}c_\nu^{-2}\|\nabla\Phi\|_\infty^2 < \frac{p - 1}{2(2p - 1)}.$$

Observe that Φ depends only on δ, which is determined by G, H and Ω. Consequently, A_0 depends only on G, H, Ω and does not on $u(x)$ and $f(x)$. By (6.9), we have

$$\|\nabla\phi\|_{L^\infty(\mathbb{R}^N)} \le 2\|\nabla\Phi\|_{L^\infty(\mathbb{R}^N)}.$$

Combining two inequalities above, we obtain

$$(6.14) \qquad 8k(m + 1)^{-1} A^{m+1}(A - A_0)A_0^{-m}c_\nu^{-2}\|\nabla\phi\|_\infty^2 < \frac{p - 1}{2(2p - 1)}.$$

To prove Theorem 2.1, it is enough to show that ϕ satisfies (4.2) and (4.3). Lemma 6.3 proves (4.2). Let us verify (4.3).

LEMMA 6.7. *Let u be a G invariant least energy solution. If the maximum of $U(r)$ on $[0, A]$ is achieved at a point in $[A_0, A]$, then (4.3) holds.*

PROOF. Denote the maximum point of $U(r)$ on $[A_0, A]$ by a. By assumption,

$$\max_{0 \le r \le A} U(r) = \max_{A_0 \le r \le A} U(r) = U(a).$$

By Lemma 6.6, we have

$$(6.15) \qquad U(a) = \max_{A_0 \le r \le A} U(r) \le 4k(A - A_0)A_0^{-m}c_\nu^{-2} \int_{B(g_0 x_0, \delta/2)} |\nabla u|^2 \phi^2 \, dx.$$

Put $D := S^m \times \mathbb{R}^N$. Then it holds clearly that

$$\int_\Omega u^2 dx = \int_0^A \left(\int_D u^2 r^m \, dx' \, d\sigma \right) dr \le (m + 1)^{-1} A^{m+1} U(a),$$

which implies

$$\int_\Omega |\nabla\phi|^2 u^2 \, dx \le \|\nabla\phi\|_\infty^2 \int_\Omega u^2 \, dx \le \|\nabla\phi\|_\infty^2 (m + 1)^{-1} A^{m+1} U(a).$$

Substituting (6.15) into the right hand side of the inequality above and using (6.14), we obtain (4.3). $\qquad\square$

The next lemma shows that if assumption (A) before Theorem 2.1 is valid, then (4.3) holds.

LEMMA 6.8. *Let u be a G invariant least energy solution. If $f(x) \leq 0$ in $\Omega(|x''| < A_0))$, then (4.3) holds.*

PROOF. Under the assumption of the lemma, $U(r)$ is nondecreasing in $(0, A_0)$ by Lemma 5.6. Thus the maximum of $U(r)$ on $[0, A]$ is achieved at a point in $[A_0, A]$. Therefore Lemma 6.7 ensures the conclusion.   □

By Lemma 6.7, inequality (4.3) holds if the maximum of $U(r)$ on $[0, A]$ is achieved at a point in $[A_0, A]$. In particular, if $f \leq 0$ in $\Omega(|x''| < A_0)$, i.e. assumption (A) is valid, then (4.3) holds. We consider the remained case where assumption (B) holds and the maximum of $U(r)$ on $[0, A]$ is achieved at a point in $[0, A_0)$.

LEMMA 6.9. *Assume that $f \geq 0$, $\not\equiv 0$ in Ω. Let u be a G invariant least energy solution. Suppose that the maximum of $U(r)$ on $[0, A]$ is achieved at a point $a \in [0, A_0)$. If $\mu(f, A_0)$ is small enough with $N = 3$ or $\mu(f, A_0, \theta)$ is small enough with $N = 2$, then (4.3) holds.*

PROOF. Since $U(r)$ has a maximum at a, $U_r(a)$ vanishes when $a > 0$. We substitute $s = a$ in (5.18) to get

$$t^m U_r(t) = 2 \int_{\Omega(a < |x''| < t)} (|\nabla u|^2 - f u^{p+1}) \, dx.$$

As proved in Lemma 5.6, $s^m U_r(s)$ converges to 0 as $s \to 0$. Hence, even if $a = 0$, the identity above holds. We estimate it as

$$-t^m U_r(t) \leq 2 \int_{\Omega(a < |x''| < t)} f u^{p+1} \, dx = 2 \int_a^t \left(\int_D f u^{p+1} r^m dx' d\sigma \right) dr,$$

with $D := S^m \times \mathbb{R}^n$. For $t \in [a, A_0]$, we compute

$$\int_a^t \left(\int_D f u^{p+1} r^m \, dx' d\sigma \right) dr$$

$$\leq \|f\|_{L^\infty(\Omega(|x''| < A_0))} \|u\|_{L^\infty(\Omega)}^{p-1} \int_a^t \left(\int_D u^2 r^m dx' d\sigma \right) dr$$

$$\leq t^m (t - a) \|f\|_{L^\infty(\Omega(|x''| < A_0))} \|u\|_{L^\infty(\Omega)}^{p-1} U(a).$$

Therefore we have

$$-U_r(t) \leq 2(t - a) \|f\|_{L^\infty(\Omega(|x''| < A_0))} \|u\|_{L^\infty(\Omega)}^{p-1} U(a).$$

Integrating both sides over (a, A_0), we obtain

(6.16) $0 \leq U(a) - U(A_0) \leq A_0^2 \|f\|_{L^\infty(\Omega(|x''| < A_0))} \|u\|_{L^\infty(\Omega)}^{p-1} U(a).$

We shall show that $A_0^2 \|f\|_{L^\infty(\Omega(|x''| < A_0))} \|u\|_{L^\infty(\Omega)}^{p-1} \leq 1/2$.

Let $N \geq 3$. Then Lemma 5.1 implies that

$$A_0^2 \|f\|_{L^\infty(\Omega(|x''|<A_0))} \|u\|_{L^\infty(\Omega)}^{p-1}$$

$$\leq A_0^2 C^{p-1} \|f\|_{L^\infty(\Omega(|x''|<A_0))} \|f\|_{L^\infty(\Omega)}^{\alpha(p-1)} \left(\int_\Omega f(x)\mathrm{dist}(x,\partial\Omega)^{p+1}\,dx \right)^{\beta(p-1)}$$

$$= A_0^2 C^{p-1} \mu(f,A_0)^{1/(N+2-(N-2)p)} \leq 1/2,$$

provided that $\mu(f,A_0)$ is small enough. Here α and β have been defined by Lemma 5.1 and $C > 0$ is independent of a, $f(x)$ and $u(x)$.

Let $N = 2$. By Lemma 5.1, we have

$$A_0^2 \|f\|_{L^\infty(\Omega(|x''|<A_0))} \|u\|_{L^\infty(\Omega)}^{p-1}$$

$$\leq A_0^2 \|f\|_{L^\infty(\Omega(|x''|<A_0))} C_\theta^{p-1} \|f\|_{L^\infty(\Omega)}^{(1-\theta)/\theta} \left(\int_\Omega f(x)\mathrm{dist}(x,\partial\Omega)^{p+1}\,dx \right)^{-1/\theta}$$

$$= A_0^2 C_\theta^{p-1} \mu(f,A_0,\theta) \leq 1/2,$$

provided that $\mu(f,A_0,\theta)$ is small enough. Consequently, if $\mu(f,A_0)$ and $\mu(f,A_0,\theta)$ are small enough, we have

$$A_0^2 \|f\|_{L^\infty(\Omega(|x''|<A_0))} \|u\|_{L^\infty(\Omega)}^{p-1} \leq 1/2.$$

This inequality with (6.16) implies that $U(a) \leq 2U(A_0)$. By Lemma 6.6 and (6.14), we obtain

$$\int_\Omega |\nabla\phi|^2 u(x)^2\,dx \leq \|\nabla\phi\|_\infty^2 \int_0^A \left(\int_D u^2 r^m\,dx'\,d\sigma \right) dr$$

$$\leq \|\nabla\phi\|_\infty^2 (m+1)^{-1} A^{m+1} U(a) \leq 2\|\nabla\phi\|_\infty^2 (m+1)^{-1} A^{m+1} U(A_0)$$

$$\leq 8k(A-A_0) A_0^{-m} c_\nu^{-2} \|\nabla\phi\|_\infty^2 (m+1)^{-1} A^{m+1} \int_{B(g_0 x_0, \delta/2)} |\nabla u|^2 \phi^2\,dx$$

$$< \frac{p-1}{2(2p-1)} \int_\Omega |\nabla u|^2 \phi^2\,dx. \qquad \square$$

We are now in a position to prove Theorem 2.1. Because of Proposition 4.3, it is enough to show (4.3).

PROOF OF THEOREM 2.1. If (A) holds, then Lemma 6.8 ensures (4.3). If (B) holds, then Lemmas 6.7 and 6.9 imply (4.3). $\qquad \square$

PROOF OF COROLLARY 2.2. We shall show that (B)' implies (B). We deal with $N \geq 3$ only because the method below is valid for $N = 2$ also. We fix A_0 so close to A that Theorem 2.1 is valid under assumption (B) with A_0. Define

$$M := \max_{x \in \Gamma} f_0(x), \qquad m := \|f_0\|_{L^\infty(\Omega(|x''|\leq A_0))}.$$

Since $m < M$ by assumption, we can choose $M_1 \in (0, M)$ sufficiently close to M such that

$$m^{N+2-(N-2)p} M^{(N-2)(p-1)} M_1^{-4} < 1.$$

Choose $x_0 \in \Gamma$ satisfying $f_0(x_0) = M$. Take an $\varepsilon > 0$ such that $f_0(x) > M_1$ for $x \in \Omega \cap B(x_0, \varepsilon)$. Then we have

$$\int_\Omega f(x) \operatorname{dist}(x, \partial\Omega)^{p+1} \, dx \geq \int_{\Omega \cap B(x_0, \varepsilon)} f(x) \operatorname{dist}(x, \partial\Omega)^{p+1} \, dx \geq C M_1^\lambda,$$

where $C > 0$ is independent of λ. Therefore we obtain

$$\mu(f, A_0) \leq C(m^{N+2-(N-2)p} M^{(N-2)(p-1)} M_1^{-4})^\lambda \to 0 \quad \text{as } \lambda \to \infty.$$

Hence (B) holds for $\lambda > 0$ large enough and the proof is complete.   □

We shall show Examples 3.3 and 3.4

PROOF OF EXAMPLE 3.3. Let G_n be as in (1.5). Then it is easy to see that

(6.17) $$g^{-1} G_n g = G_n \quad \text{for } g \in O(2).$$

For a subset A of $O(2)$, let $\langle A \rangle$ denote the group generated by A. Let $1 < n < m < k$ be integers such that k is the least common multiple of n and m. Then it holds that

(6.18) $$\langle G_n \cup G_m \rangle = G_k.$$

Indeed, using $r(\theta)$ before (1.5), we get

$$\langle G_n \cup G_m \rangle = \{r(2i\pi/n + 2j\pi/m) : i, j \in \mathbb{Z}\}.$$

Since k is the least common multiple, there exist $i, j \in \mathbb{Z}$ such that $i/n + j/m = 1/k$. Hence (6.18) holds.

Let G_{n_i} and u_i be as in Example 3.3. We shall show that u_i is not equivalent to u_j if $i \neq j$. Let $i < j$. If n_j is a multiple of n_i, G_{n_i} is a subgroup of G_{n_j} and $G_{n_i}(x) \subsetneqq G_{n_j}(x)$ for $x \in \Gamma(G_{n_j})$. Therefore $R(u_i) < R(u_j)$.

Let n_j be not a multiple of n_i. The least common multiple of them is equal to an n_k with a certain $k \leq d$. Then $R(u_i), R(u_j) < R(u_k)$. In contradiction to our claim, we assume that u_i is equivalent to u_j, i.e. $u_i(gx) = u_j(x)$ with some $g \in O(2)$. Then u_j is $g^{-1} G_{n_i} g$ invariant because

$$u_j(g^{-1} h g x) = u_i(h g x) = u_i(g x) = u_j(x) \quad \text{for } h \in G_{n_i}.$$

Thus u_j is invariant under both G_{n_j} and $g^{-1} G_{n_i} g$. From (6.17) and (6.18), it follows that

$$\langle G_{n_j} \cup g^{-1} G_{n_i} g \rangle = \langle G_{n_j} \cup G_{n_i} \rangle = G_{n_k}.$$

Hence u_j is G_{n_k} invariant. However this contradicts $R(u_j) < R(u_k)$. Therefore u_i is not equivalent to u_j.   □

PROOF OF EXAMPLE 3.4. Let G_n be as in (1.5). We define a subgroup H_n of $O(3)$ by

$$H_n := \left\{ \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} : g \in G_n \right\}.$$

Then H_n is a subgroup of H_{2n} and $H_n(x) \subsetneq H_{2n}(x)$ for $x \in \Gamma(H_{2n})$. Let u_n be an H_n invariant least energy solution. Fix n arbitrary. Assume either (A) with a sufficiently close to α or (B) with $\lambda > 0$ large enough. Then

$$R(u_1) < R(u_2) < R(u_4) < \ldots < R(u_{2^n}).$$

Therefore the assertion of Example 3.4 holds. $\qquad\square$

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