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CURVED SQUEEZING OF UNBOUNDED DOMAINS AND TAIL ESTIMATES

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This paper is dedicated to Professor Jean Mawhin on the occasion on his 70th birthday

Abstract. Using a resolvent convergence result from [7] we prove Conley index and index braid continuation results for reaction-diffusion equations on singularly perturbed unbounded curved squeezed domains

1. Introduction

Let ω be an arbitrary domain in $\mathbb{R}^\ell$, bounded or not, with Lipschitz boundary. We consider the following semilinear parabolic Neumann boundary problem

$$\begin{aligned} &\widetilde{u}_t = \Delta_\omega \widetilde{u} + \widetilde{G}_\omega(\widetilde{x},\widetilde{u}), && t>0,\ \widetilde{x}\in\omega, \\ &\partial_{\nu_\omega} u = 0, && t>0,\ \widetilde{x}\in\partial\omega, \end{aligned} \tag{1.1}$$

on ω. Here, ν_ω is the outer normal vector field to $\partial\omega$ and $\widetilde{G}_\omega\colon \omega\times\mathbb{R}\to\mathbb{R}$ is a suitable nonlinearity. Define the bilinear forms $\widetilde{a}_\omega$ and $\widetilde{b}_\omega$ by

$$\begin{aligned} &\widetilde{a}_\omega\colon H^1(\omega)\times H^1(\omega)\to\mathbb{R}, && (\widetilde{u},\widetilde{v})\mapsto \int_\omega \nabla\widetilde{u}(x)\cdot\nabla\widetilde{v}(x)\,dx; \\ &\widetilde{b}_\omega\colon L^2(\omega)\times L^2(\omega)\to\mathbb{R}, && (\widetilde{u},\widetilde{v})\mapsto \int_\omega \widetilde{u}(x)\widetilde{v}(x)\,dx. \end{aligned}$$

Then the pair $(\widetilde{a}_\omega,\widetilde{b}_\omega)$ generates a densely defined selfadjoint operator B_ω on $L^2(\omega)$, which is commonly interpreted as the Laplace operator $-\Delta_\omega$ on ω with Neumann boundary condition.

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We are interested in the case $\omega = \Omega_\varepsilon$, where Ω_ε, for $\varepsilon > 0$ small, is 'thin' of order ε. As $\varepsilon \to 0^+$, the domain Ω_ε 'degenerates' to some limit set, which may no longer be a domain in $\mathbb{R}^\ell$.

More specifically, let $\mathcal{M} \subset \mathbb{R}^\ell$ be a smooth $\mathbf{k}$-dimensional submanifold of $\mathbb{R}^\ell$ and $\mathcal{U} \supset \mathcal{M}$ be a normal (tubular) neighbourhood of $\mathcal{M}$ with normal projection ϕ. For $\varepsilon \in [0,1]$ define the *squeezing operator* $\Gamma_\varepsilon \colon \mathcal{U} \to \mathcal{U}$ by $x \mapsto \varepsilon x + (1-\varepsilon)\phi(x)$. For any domain Ω in $\mathbb{R}^\ell$ with Lipschitz boundary and $\mathrm{Cl}\,\Omega \subset \mathcal{U}$ we set $\Omega_\varepsilon = \Gamma_\varepsilon(\Omega)$ and $\mathbf{B}_\varepsilon = B_{\Omega_\varepsilon}$ for $\varepsilon \in \,]0,1]$. A particular case is the *flat squeezing* case in which, writing $\mathbb{R}^\ell = \mathbb{R}^{\mathbf{k}} \times \mathbb{R}^{\ell-\mathbf{k}}$, $x = (x_1, x_2)$, we set $\mathcal{M} = \mathbb{R}^{\mathbf{k}} \times \{0\}$, $\mathcal{U} = \mathbb{R}^\ell$ and $\phi(x) = (x_1, 0)$.

We also consider a family $G_\varepsilon \colon \mathcal{U} \times \mathbb{R} \to \mathbb{R}$ of suitable nonlinearities, $\varepsilon \in [0,1]$ and, for $\varepsilon \in \,]0,1]$, set $\widetilde{G}_{\Omega_\varepsilon} := G_\varepsilon | \Omega_\varepsilon \times \mathbb{R}$.

Equation (1.1) can be written abstractly as

$$\dot{\widetilde{u}} = -\mathbf{B}_\varepsilon \widetilde{u} + \widetilde{f}_\varepsilon(\widetilde{u}) \tag{1.2}$$

on $H^1(\Omega_\varepsilon)$ where $\widetilde{f}_\varepsilon$ is the Nemitski operator from $H^1(\Omega_\varepsilon)$ to $L^2(\Omega_\varepsilon)$ defined by $\widetilde{G}_{\Omega_\varepsilon} \colon \Omega_\varepsilon \times \mathbb{R} \to \mathbb{R}$.

Now using the change of variables defined by Γ_ε we may pull $\mathbf{B}_\varepsilon$ back to $L^2(\Omega)$ and thus obtain the densely defined selfadjoint operator $\mathbf{A}_\varepsilon$ in $L^2(\Omega)$ given by:

(a) $\widetilde{u} \in D(\mathbf{B}_\varepsilon)$ if and only if $u = \widetilde{u} \circ \Gamma_\varepsilon \in D(\mathbf{A}_\varepsilon)$;
(b) $\mathbf{A}_\varepsilon(u) = (\mathbf{B}_\varepsilon \widetilde{u}) \circ \Gamma_\varepsilon$ for $\widetilde{u} \in D(\mathbf{B}_\varepsilon)$.

Equation (1.2) can then be pulled back to yield the equation

$$\dot{u} = -\mathbf{A}_\varepsilon u + f_\varepsilon(u) \tag{1.3}$$

on $H^1(\Omega)$. Here, $f_\varepsilon \colon H^1(\Omega) \to L^2(\Omega)$ is defined by

$$f_\varepsilon(\widetilde{u} \circ \Gamma_\varepsilon) = \widetilde{f}_\varepsilon(\widetilde{u}) \circ \Gamma_\varepsilon, \quad \widetilde{u} \in H^1(\Omega_\varepsilon).$$

More explicitly,

$$f_\varepsilon(u)(x) = G_\varepsilon(\Gamma_\varepsilon(x), u(x))$$

for $u \in H^1(\Omega)$ and $x \in \Omega$.

If Ω is bounded, then there is a closed linear subspace $L^2_s(\Omega)$ of $L^2(\Omega)$ and a densely defined selfadjoint operator $\mathbf{A}_0$ on $L^2_s(\Omega)$ such that, as $\varepsilon \to 0^+$, the eigenvalues and eigenfunctions of $\mathbf{A}_\varepsilon$ converge, in some sense, to the eigenvalues and eigenfunctions of $\mathbf{A}_0$, cf. [10], [13].

This spectral convergence theorem implies various Trotter-Kato-like linear convergence theorems of the C^0-semigroups $e^{-t\mathbf{A}_\varepsilon}$ to $e^{-t\mathbf{A}_0}$, cf. [10], [2], [13], which are used to prove attractor semicontinuity and Conley index continuation results for reaction-diffusion equations with nonlinearities satisfying certain growth assumptions, cf. [10], [13], [2], [3], [16].

If Ω is unbounded, then, in general, the operators $\mathbf{A}_\varepsilon$ do not have compact resolvents and so spectral convergence results in the above form are not expected to hold. However, as shown in [1] in the flat squeezing case, there is again a closed linear subspace $L^2_s(\Omega)$ of $L^2(\Omega)$ and a densely defined selfadjoint operator $\mathbf{A}_0$ on $L^2_s(\Omega)$ such that the resolvents of $\mathbf{A}_\varepsilon$ converge in some sense to the resolvents of $\mathbf{A}_0$. It turns out that this is sufficient for the validity of a corresponding linear convergence result.

In the recent paper [7] the above results from [1] were extended to the curved squeezing case provided the manifold $\mathcal{M}$ has *bounded normal curvature*. This condition is trivially satisfied in the flat squeezing case (the normal curvature of $\mathcal{M} = \mathbb{R}^{\mathbf{k}} \times \{0\}$ being zero) and for compact manifolds $\mathcal{M}$. A simple example of a noncompact manifold with bounded normal curvature is provided by the graph of the exponential function $\exp\colon \mathbb{R} \to \mathbb{R}$, while the graph of the function $g\colon]0,\infty[\to \mathbb{R}$, $x \mapsto \sin(1/x)$, is a manifold with unbounded normal curvature.

Using the results from [7] we will obtain in this paper Conley index continuation results for problem (1.3) as $\varepsilon \to 0$.

This paper is organized as follows.

In Section 2 we introduce an abstract linear convergence concept (Lin) and relate it to resolvent convergence concept from [7]. We then define a nonlinear convergence concept (Conv) and establish some abstract nonlinear singular convergence results. Finally, we define the concept of singular H^ε-admissibility and show that this seemingly weaker concept actually implies the usual singular admissibility as defined in [4]. This implies various Conley index continuation results.

In Section 3 we will apply these abstract results to reaction-diffusion equations on singularly perturbed unbounded domains.

Some results on attractors on singularly perturbed unbounded domains are contained in [17].

2. An abstract singular convergence result

Definition 2.1. We say that the family $(H^\varepsilon, \langle\cdot\,,\cdot\rangle_{H^\varepsilon}, A_\varepsilon)_{\varepsilon\in[0,\overline{\varepsilon}]}$ *satisfies condition* (Lin) if the following properties are satisfied:

(a) $\overline{\varepsilon} \in]0,\infty[$ and for every $\varepsilon \in [0,\overline{\varepsilon}]$, $(H^\varepsilon, \langle\cdot\,,\cdot\rangle_{H^\varepsilon})$ is a Hilbert space and $A_\varepsilon\colon D(A_\varepsilon) \subset H^\varepsilon \to H^\varepsilon$ is a densely defined nonnegative self-adjoint operator on $(H^\varepsilon, \langle\cdot\,,\cdot\rangle_{H^\varepsilon})$. For $\alpha \in [0,\infty[$ write $H^\varepsilon_\alpha := D((A_\varepsilon + I_\varepsilon)^{\alpha/2})$, where $I_\varepsilon = \mathrm{Id}_{H^\varepsilon}$, and $\langle\cdot\,,\cdot\rangle_{H^\varepsilon_\alpha} := \langle\cdot\,,\cdot\rangle_{(A_\varepsilon+I_\varepsilon)^{\alpha/2}}$ with the corresponding norm $|\cdot|_{H^\varepsilon_\alpha}$. In particular, $H^\varepsilon_0 = H^\varepsilon$;

(b) for each $\varepsilon \in]0,\overline{\varepsilon}]$, H^0 is a linear subspace of H^ε and H^0_1 is a linear subspace of H^ε_1;

(c) there exists a constant $C \in \,]1,\infty[$ such that, for $\varepsilon \in \,]0,\overline{\varepsilon}]$,

$$|u|_{H_1^\varepsilon} \le C|u|_{H_1^0} \quad \text{and} \quad |u|_{H_1^0} \le C|u|_{H_1^\varepsilon}, \quad \text{whenever } u \in H_1^0;$$

(d) If $u_0 \in H^0$ and $(u_n)_n$ is a sequence such that $u_n \in H^{\varepsilon_n}$ for each n and $|u_n - u_0|_{H^{\varepsilon_n}} \to 0$ as $n \to \infty$, and if $(\varepsilon_n)_n$ is a sequence in $]0,\overline{\varepsilon}]$ with $\varepsilon_n \to 0$ then

$$|e^{-tA_{\varepsilon_n}}u_n - e^{-tA_0}u_0|_{H_1^{\varepsilon_n}} \to 0 \quad \text{as } n \to \infty,$$

uniformly on compact intervals in $]0,\infty[$.

(e) If $u_0 \in H_1^0$ and $(u_n)_n$ is a sequence such that $u_n \in H_1^{\varepsilon_n}$ for each n and $|u_n - u_0|_{H_1^{\varepsilon_n}} \to 0$ as $n \to \infty$, and if $(\varepsilon_n)_n$ is a sequence in $]0,\overline{\varepsilon}]$ with $\varepsilon_n \to 0$ then

$$|e^{-tA_{\varepsilon_n}}u_n - e^{-tA_0}u_0|_{H_1^{\varepsilon_n}} \to 0 \quad \text{as } n \to \infty,$$

uniformly on compact intervals in $[0,\infty[$.

Proposition 2.2. *If* $(H^\varepsilon, \langle\,\cdot\,,\,\cdot\,\rangle_{H^\varepsilon}, A_\varepsilon)_{\varepsilon\in[0,\overline{\varepsilon}]}$ *satisfy condition* (Lin), *then for every* $\varepsilon \in \,]0,\overline{\varepsilon}]$, *the subspace* H_1^0 *is closed in* $(H_1^\varepsilon, |\cdot|_{H_1^\varepsilon})$. *Let* $Q_\varepsilon\colon H_1^\varepsilon \to H_1^\varepsilon$ *be the* H_1^ε*-orthogonal projection of* H_1^ε *onto* H_1^0. □

Definition 2.3. We say that the family $(H^\varepsilon, \langle\,\cdot\,,\,\cdot\,\rangle_{H^\varepsilon}, A_\varepsilon)_{\varepsilon\in[0,\overline{\varepsilon}]}$ *satisfies condition* (Res) if the following properties are satisfied:

(a) $\overline{\varepsilon} \in \,]0,\infty[$ and for every $\varepsilon \in [0,\overline{\varepsilon}]$, $(H^\varepsilon, \langle\,\cdot\,,\,\cdot\,\rangle_{H^\varepsilon})$ is a Hilbert space and $A_\varepsilon\colon D(A_\varepsilon) \subset H^\varepsilon \to H^\varepsilon$ is a densely defined nonnegative self-adjoint operator on $(H^\varepsilon, \langle\,\cdot\,,\,\cdot\,\rangle_{H^\varepsilon})$. For $\alpha \in [0,\infty[$ write $H_\alpha^\varepsilon := D((A_\varepsilon + I_\varepsilon)^{\alpha/2})$, where $I_\varepsilon = \mathrm{Id}_{H^\varepsilon}$, and $\langle\,\cdot\,,\,\cdot\,\rangle_{H_\alpha^\varepsilon} := \langle\,\cdot\,,\,\cdot\,\rangle_{(A_\varepsilon+I_\varepsilon)^{\alpha/2}}$ with the corresponding norm $|\cdot|_{H_\alpha^\varepsilon}$. In particular, $H_0^\varepsilon = H^\varepsilon$;

(b) for each $\varepsilon \in \,]0,\overline{\varepsilon}]$, H^0 is a linear subspace of H^ε and H_1^0 is a linear subspace of H_1^ε;

(c) there exists a constant $C \in \,]1,\infty[$ such that, for $\varepsilon \in \,]0,\overline{\varepsilon}]$,

$$|u|_{H_1^\varepsilon} \le C|u|_{H_1^0} \quad \text{and} \quad |u|_{H_1^0} \le C|u|_{H_1^\varepsilon}, \quad \text{whenever } u \in H_1^0;$$

and

$$|u|_{H^\varepsilon} \le C|u|_{H^0} \quad \text{and} \quad |u|_{H^0} \le C|u|_{H^\varepsilon}, \quad \text{whenever } u \in H^0;$$

(d) whenever $(\varepsilon_n)_n$ is a sequence in $]0,\overline{\varepsilon}]$ converging to zero, $w \in H^0$ and $(w_n)_n$ is a sequence in H^0 with $|w_n - w|_{H^0} \to 0$ as $n \to \infty$, then $|(A_{\varepsilon_n} + I_{\varepsilon_n})^{-1}w_n - (A_0 + I_0)^{-1}w|_{H_1^{\varepsilon_n}} \to 0$ as $n \to \infty$.

The following result is a rewording of [7, Theorem 3.4].

THEOREM 2.4. *A family* $(H^\varepsilon, \langle\cdot\,,\cdot\rangle_{H^\varepsilon}, A_\varepsilon)_{\varepsilon\in[0,\overline{\varepsilon}]}$ *satisfying condition* (Res) *satisfies condition* (Lin).

DEFINITION 2.5. Let $\overline{\varepsilon} \in\,]0,\infty[$ be arbitrary and $(H^\varepsilon, \langle\cdot\,,\cdot\rangle_{H^\varepsilon}, A_\varepsilon)_{\varepsilon\in[0,\overline{\varepsilon}]}$ be a family satisfying condition (Lin). We say that the family $(f_\varepsilon)_{\varepsilon\in[0,\overline{\varepsilon}]}$ of maps satisfies condition (Conv) if the following properties are satisfied:

(a) $f_\varepsilon: H_1^\varepsilon \to H^\varepsilon$ for every $\varepsilon \in [0,\overline{\varepsilon}]$.

(b) $\lim_{\varepsilon\to 0^+} |e^{-tA_\varepsilon} f_\varepsilon(u) - e^{-tA_0} f_0(u)|_{H_1^\varepsilon} = 0$ for every $u \in H_1^0$ and every $t \in\,]0,\infty[$.

(c) For every $M \in [0,\infty[$ there is an $L = L_M \in [0,\infty[$ such that

$$|f_\varepsilon(u) - f_\varepsilon(v)|_{H^\varepsilon} \le L|u-v|_{H_1^\varepsilon}$$

for all $\varepsilon \in [0,\overline{\varepsilon}]$ and u, $v \in H_1^\varepsilon$ satisfying $|u|_{H_1^\varepsilon}$, $|v|_{H_1^\varepsilon} \le M$.

(d) For every $u \in H_1^0$ there is an $\varepsilon_0' \in\,]0,\overline{\varepsilon}]$ such that

$$\sup_{\varepsilon\in[0,\varepsilon_0']} |f_\varepsilon(u)|_{H^\varepsilon} < \infty.$$

REMARK 2.6. Note that, for α, $t \in\,]0,\infty[$ and $\lambda \in [0,\infty[$

$$\lambda^\alpha e^{-\lambda t} \le C(\alpha) t^{-\alpha} \quad \text{with } C(\alpha) = (\alpha/e)^\alpha.$$

Let $(H^\varepsilon, \langle\cdot\,,\cdot\rangle_{H^\varepsilon}, A_\varepsilon)_{\varepsilon\in[0,\overline{\varepsilon}]}$ satisfy condition (Lin). Let $\varepsilon \in [0,\overline{\varepsilon}]$ and $r \in\,]0,\infty[$. Using the Stone-Neumann operational calculus together with the above estimate with $\alpha = 1/2$ we obtain the estimates

$$|e^{-A_\varepsilon r} u|_{H^\varepsilon} \le |u|_{H^\varepsilon}, \quad u \in H^\varepsilon \tag{2.1}$$

and

$$|e^{-A_\varepsilon r} u|_{H_1^\varepsilon} \le C_0 r^{-1/2} e^r |u|_{H^\varepsilon}, \quad u \in H^\varepsilon \tag{2.2}$$

where $C_0 = C(1/2)$.

The next result shows that the above condition (b) is valid uniformly for t lying in compact subsets of $]0,\infty[$.

PROPOSITION 2.7. *Assume condition* (Conv) *and let* β, $\gamma \in\,]0,\infty]$ *be arbitrary with* $\beta < \gamma$. *Then, for every* $u \in H_1^0$,

$$\lim_{\varepsilon\to 0^+} \sup_{t\in[\beta,\gamma]} |e^{-tA_\varepsilon} f_\varepsilon(u) - e^{-tA_0} f_0(u)|_{H_1^\varepsilon} = 0$$

PROOF. Let $v = e^{-\beta A_0} f_0(u) \in H_1^0$. For every $t \in [\beta,\gamma]$ we have

$$\begin{aligned}
&|e^{-tA_\varepsilon} f_\varepsilon(u) - e^{-tA_0} f_0(u)|_{H_1^\varepsilon} \\
&\le |e^{-(t-\beta)A_\varepsilon}(e^{-\beta A_\varepsilon} f_\varepsilon(u) - e^{-\beta A_0} f_0(u))|_{H_1^\varepsilon} + |e^{-(t-\beta)A_\varepsilon} v - e^{-(t-\beta)A_0} v|_{H_1^\varepsilon} \\
&\le |e^{-\beta A_\varepsilon} f_\varepsilon(u) - e^{-\beta A_0} f_0(u)|_{H_1^\varepsilon} + |e^{-(t-\beta)A_\varepsilon} v - e^{-(t-\beta)A_0} v|_{H_1^\varepsilon}
\end{aligned}$$

Here we have used (2.1). Since, by part (e) of condition (Lin)

$$\lim_{\varepsilon\to 0}\sup_{s\in[0,\gamma-\beta]}|e^{-sA_\varepsilon}v-e^{-sA_0}v|_{H_1^\varepsilon}=0,$$

the assertion follows from condition (Conv) part (b) (with $t=\beta$). $\square$

For the rest of the paper, if $(H^\varepsilon,\langle\,\cdot\,,\,\cdot\,\rangle_{H^\varepsilon},A_\varepsilon)_{\varepsilon\in[0,\overline{\varepsilon}]}$ satisfies condition (Lin) and $(f_\varepsilon)_{\varepsilon\in[0,\overline{\varepsilon}]}$ satisfies condition (Conv) then we will write, for every $\varepsilon\in[0,\overline{\varepsilon}]$, $\pi_\varepsilon:=\pi_{A_\varepsilon,f_\varepsilon}$ to denote the local semiflow on H_1^ε generated by the abstract parabolic equation

$$\dot u=-A_\varepsilon u+f_\varepsilon(u). \tag{2.3}$$

Lemma 2.8. *Suppose that $(H^\varepsilon,\langle\,\cdot\,,\,\cdot\,\rangle_{H^\varepsilon},A_\varepsilon)_{\varepsilon\in[0,\overline{\varepsilon}]}$ satisfies condition* (Lin) *and $(f_\varepsilon)_{\varepsilon\in[0,\overline{\varepsilon}]}$ satisfies condition* (Conv). *Let $\varepsilon_0'\in\,]0,\overline{\varepsilon}]$ be such that*

$$\sup_{\varepsilon\in[0,\varepsilon_0']}|f_\varepsilon(0)|_{H^\varepsilon}<\infty.$$

(ε_0' *exists in view of condition* (Conv).) *For every $R\in\,]0,\infty[$ there exists a $\tau=\tau_R\in\,]0,\infty[$ such that for every $\varepsilon\in[0,\varepsilon_0']$ and every $a\in H_1^\varepsilon$ with $|a|_{H_1^\varepsilon}\le R$, $a\pi_\varepsilon t$ is defined and $|a\pi_\varepsilon t|_{H_1^\varepsilon}\le 4R$ for all $t\in[0,\tau]$.*

Proof. Let C be as in part (c) of condition (Lin), set

$$M':=4R \tag{2.4}$$

and let $L:=L_{M'}$ be as in Condition (Conv) with M replaced by M'. Set

$$C_1=3R\quad\text{and}\quad C_2=\sup_{\varepsilon\in[0,\varepsilon_0']}|f_\varepsilon(0)|_{H^\varepsilon}<\infty. \tag{2.5}$$

Now choose $\tau\in\,]0,\infty[$ with

$$2C_0L\tau^{1/2}e^\tau\le 1/2, \tag{2.6}$$

$$2C_0\tau^{1/2}e^\tau(2LC_1+C_2)\le C_1/4, \tag{2.7}$$

where the constant C_0 is as in Remark 2.6. For every $\varepsilon\in[0,\varepsilon_0']$ and $a\in H_1^\varepsilon$ with

$$|a|_{H_1^\varepsilon}\le R, \tag{2.8}$$

define $S_{\varepsilon,a}:=\{u\mid u\colon[0,\tau]\to H_1^\varepsilon$ is continuous and $|u(t)-a|_{H_1^\varepsilon}\le C_1$ for all $t\in[0,\tau]\}$.

For $u\in S_{\varepsilon,a}$ define the map $T_{\varepsilon,a}(u)\colon[0,\tau]\to H_1^\varepsilon$ by

$$T_{\varepsilon,a}(u)(t):=e^{-tA_\varepsilon}a+\int_0^t e^{-(t-s)A_\varepsilon}f_\varepsilon(u(s))\,ds.$$

The map $T_{\varepsilon,a}(u)$ is continuous. Moreover, by (2.4), whenever $u\in S_{\varepsilon,a}$, then for all $t\in[0,\tau]$

$$|u(t)|_{H_1^\varepsilon}\le C_1+|a|_{H_1^\varepsilon}\le 4R=M',$$

Thus for all $u, v \in S_{\varepsilon,a}$ and for all $t \in [0, \tau]$, we have, by (2.2) and (2.6),

$$
\begin{aligned}
(2.9) \qquad & |T_{\varepsilon,a}(u)(t) - T_{\varepsilon,a}(v)(t)|_{H_1^\varepsilon} \\
&= \left| \int_0^t e^{-(t-s)A_\varepsilon}(f_\varepsilon(u(s)) - f_\varepsilon(v(s)))\, ds \right|_{H_1^\varepsilon} \\
&\le C_0 \int_0^t (t-s)^{-1/2} e^{t-s} |f_\varepsilon(u(s)) - f_\varepsilon(v(s))|_{H^\varepsilon}\, ds \\
&\le C_0 L e^\tau \int_0^t (t-s)^{-1/2}\, ds \sup_{s\in[0,\tau]} |u(s) - v(s)|_{H_1^\varepsilon} \\
&= 2C_0 L \tau^{1/2} e^\tau \sup_{s\in[0,\tau]} |u(s) - v(s)|_{H_1^\varepsilon} \\
&\le 1/2 \sup_{s\in[0,\tau]} |u(s) - v(s)|_{H_1^\varepsilon}.
\end{aligned}
$$

Moreover, for all $u \in S_{\varepsilon,a}$ and $t \in [0, \tau]$,

$$
|T_{\varepsilon,a}(u)(t) - a|_{H_1^\varepsilon} \le |e^{-tA_\varepsilon} a - a|_{H_1^\varepsilon} + \left| \int_0^t e^{-(t-s)A_\varepsilon} f_\varepsilon(u(s))\, ds \right|_{H_1^\varepsilon}.
$$

Since for $\varepsilon \in [0, \varepsilon_0']$ and $s \in [0, \tau]$ we have

$$
\begin{aligned}
|f_\varepsilon(u(s))|_{H^\varepsilon} &\le |f_\varepsilon(u(s)) - f_\varepsilon(a)|_{H^\varepsilon} + |f_\varepsilon(a)|_{H^\varepsilon} \\
&\le L|u(s) - a|_{H_1^\varepsilon} + |f_\varepsilon(a) - f_\varepsilon(0)|_{H^\varepsilon} + |f_\varepsilon(0)|_{H^\varepsilon} \\
&\le LC_1 + LC_1 + C_2 = 2LC_1 + C_2,
\end{aligned}
$$

we obtain, by (2.7),

$$
\begin{aligned}
\left| \int_0^t e^{-(t-s)A_\varepsilon} f_\varepsilon(u(s))\, ds \right|_{H_1^\varepsilon} &\le C_0 \int_0^t (t-s)^{-1/2} e^{(t-s)} |f_\varepsilon(u(s))|_{H^\varepsilon}\, ds \\
&\le 2C_0 \tau^{1/2} e^\tau (2LC_1 + C_2) \le C_1/4 = 3R/4.
\end{aligned}
$$

Moreover, $|e^{-tA_\varepsilon} a - a|_{H_1^\varepsilon} \le |e^{-tA_\varepsilon} a|_{H_1^\varepsilon} + |a|_{H_1^\varepsilon} \le |a|_{H_1^\varepsilon} + |a|_{H_1^\varepsilon} \le 2R$ by (2.1).

Altogether, we obtain

$$
|T_{\varepsilon,a}(u)(t) - a|_{H_1^0} \le 2R + 3R/4 \le 3R = C_1 \quad \text{for all } t \in [0, \tau].
$$

Hence we conclude that $T_{\varepsilon,a}(S_{\varepsilon,a}) \subset S_{\varepsilon,a}$ and so, by (2.9) and Banach Fixed Point Theorem, there is a unique fixed point of $T_{\varepsilon,a}$ in $S_{\varepsilon,a}$. In particular $a\pi_\varepsilon t$ and is defined and $|a\pi_\varepsilon t|_{H_1^\varepsilon} \le 4R$ for all $t \in [0, \tau]$. The lemma is proved. □

We can now state our first singular convergence result for semiflows.

THEOREM 2.9. *Suppose that the family* $(H^\varepsilon, \langle \cdot\,, \cdot \rangle_{H^\varepsilon}, A_\varepsilon)_{\varepsilon\in[0,\overline{\varepsilon}]}$ *satisfies condition* (Lin), *and the family* $(f_\varepsilon)_{\varepsilon\in[0,\overline{\varepsilon}]}$ *satisfies condition* (Conv). *Let* $\varepsilon_0 \in [0, \overline{\varepsilon}]$ *and* $(\varepsilon_n)_n$ *be a sequence in* $[0, \overline{\varepsilon}]$ *with* $\varepsilon_n \to \varepsilon_0$. *We assume that either* $\varepsilon_n = \varepsilon_0$

for all $n \in \mathbb{N}$ or else $\varepsilon_0 = 0$ and $\varepsilon_n > 0$ for all $n \in \mathbb{N}$. Let $u_0 \in H_1^{\varepsilon_0}$ and $(u_n)_n$ be a sequence with $u_n \in H_1^{\varepsilon_n}$ for every $n \in \mathbb{N}$ and

$$|u_n - u_0|_{H^{\varepsilon_n}} \to 0 \quad \text{as } n \to \infty.$$

Let $b \in]0,\infty[$ and suppose that $u_n\pi_{\varepsilon_n}t$ and $u_0\pi_{\varepsilon_0}t$ are defined for all $n \in \mathbb{N}$ and $t \in [0,b]$. Moreover, suppose there exists an $M' \in [0,\infty[$ such that $|u_n\pi_{\varepsilon_n}s|_{H_1^{\varepsilon_n}} \leq M'$ for all $n \in \mathbb{N}$ and for all $s \in [0,b]$. Then for every $t \in]0,b]$ and every sequence $(t_n)_n$ in $]0,b]$ converging to t

$$|u_n\pi_{\varepsilon_n}t_n - u_0\pi_{\varepsilon_0}t_n|_{H_1^{\varepsilon_n}} \to 0 \quad \text{as } n \to \infty.$$

PROOF. Notice that $\widetilde{M} := \sup_{s\in[0,b]} |u_{\varepsilon_0}\pi_{\varepsilon_0}s|_{H_1^0} < \infty$. Hence $|u_{\varepsilon_0}\pi_{\varepsilon_0}s|_{H_1^{\varepsilon_n}} \leq C\widetilde{M}$ for all $s \in [0,b]$. Set $M'' := \max\{M', C\widetilde{M}\}$ and let $L := L_{M''}$ be as in condition (Conv) with M replaced by M''.

By the variation-of-constants formula we have, for all $n \in \mathbb{N}$ and all $t \in [0,b]$,

$$\begin{aligned}(2.10)\quad u_n\pi_{\varepsilon_n}t - u_0\pi_{\varepsilon_0}t &= e^{-tA_{\varepsilon_n}}u_n - e^{-tA_{\varepsilon_0}}u_0 \\ &\quad + \int_0^t e^{-(t-s)A_{\varepsilon_n}}(f_{\varepsilon_n}(u_n\pi_{\varepsilon_n}s) - f_{\varepsilon_n}(u_0\pi_{\varepsilon_0}s))\,ds \\ &\quad + \int_0^t (e^{-(t-s)A_{\varepsilon_n}}f_{\varepsilon_n}(u_0\pi_{\varepsilon_0}s) - e^{-(t-s)A_{\varepsilon_0}}f_{\varepsilon_0}(u_0\pi_{\varepsilon_0}s))\,ds.\end{aligned}$$

Define the function $g_n\colon [0,b]\times[0,b] \to \mathbb{R}$ as follows: If $0 < s < t$ then set

$$g_n(t,s) = |e^{-(t-s)A_{\varepsilon_n}}f_{\varepsilon_n}(u_0\pi_{\varepsilon_0}s) - e^{-(t-s)A_{\varepsilon_0}}f_{\varepsilon_0}(u_0\pi_{\varepsilon_0}s)|_{H_1^{\varepsilon_n}}$$

and set $g_n(t,s) = 0$ otherwise. The function g_n restricted to the set of (s,t) with $0 < s < t$ is continuous. Thus g_n is measurable on $[0,b]\times[0,b]$. By Fubini's theorem the function

$$c_n(t) := \int_0^b g_n(t,s)\,ds = \int_0^t g_n(t,s)\,ds$$

is almost everywhere defined and measurable on $[0,b]$. Moreover, we obtain for $0 < s < t$

$$(2.11)\quad |g_n(t,s)| \leq C_2C_0e^b(t-s)^{-1/2} + CC_2C_0e^b(t-s)^{-1/2} =: C_3(t-s)^{-1/2},$$

where

$$C_2 := \max\Big\{\sup_{s\in[0,b]}\sup_{n\in\mathbb{N}}|f_{\varepsilon_n}(u_0\pi_0 s)|_{H^{\varepsilon_n}},\ \sup_{s\in[0,b]}|f_{\varepsilon_0}(u_0\pi_{\varepsilon_0}s)|_{H^0}\Big\}.$$

Notice that condition (Conv) implies that $C_2 < \infty$. Let $(t_n)_n$ be any sequence in $]0,b]$ converging to some $t \in]0,b]$. If $0 < s < t$ then, then $0 < s < t_n$ for all large n and so Proposition 2.7 implies that $g_n(t_n,s) \to 0$ as $n \to \infty$. If $0 < t < s$, then $0 < t_n < s$ for all large n and so again $g_n(t_n,s) \to 0$ as $n \to \infty$. It follows from (2.11) and the dominated convergence theorem that $c_n(t_n) \to 0$ as $n \to \infty$. Now (2.10) implies for all $r \in \,]0, b]$

$$|u_n\pi_{\varepsilon_n}r - u_0\pi_{\varepsilon_0}r|_{H_1^{\varepsilon_n}} \le |e^{-rA_{\varepsilon_n}}u_n - e^{-rA_{\varepsilon_0}}u_0|_{H_1^{\varepsilon_n}} + c_n(r) \\ + C_0e^bL_{M''}\int_0^r (r-s)^{-1/2}|u_n\pi_{\varepsilon_n}s - u_0\pi_{\varepsilon_0}s|_{H_1^{\varepsilon_n}}\,ds.$$

For $n \in \mathbb{N}$ and $r \in \,]0, b]$ set

$$a_n(r) = |e^{-rA_{\varepsilon_n}}u_n - e^{-rA_{\varepsilon_0}}u_0|_{H_1^{\varepsilon_n}} + c_n(r).$$

It follows that a_n is measurable and bounded on $]0, b]$. Using condition (Lin) we obtain that

$$a_n(t_n) \to 0 \quad \text{as } n \to \infty. \tag{2.12}$$

An application of Henry's Inequality [8, Lemma 7.1.1] now implies that

$$|u_n\pi_{\varepsilon_n}t_n - u_0\pi_{\varepsilon_0}t_n|_{H_1^{\varepsilon_n}} \le a_n(t_n) + \int_0^{t_n} \rho(t_n - s)a_n(s)\,ds,$$

where

$$\rho(x) := \sum_{n=1}^{\infty} \frac{(C_0e^bL_{M''}\Gamma(\beta))^n}{\Gamma(n\beta)}\,x^{n\beta-1}$$

with $\beta := 1/2$.

The function $\rho:]0, \infty[\, \to \,]0, \infty[$ is well defined and continuous on $]0, \infty[$ and it satisfies the estimate

$$\rho(x) \le C_4x^{-1/2} + C_4 \quad \text{for } x \in \,]0, b].$$

Fix a $\delta_0 \in \,]0, t[$ and let $\delta \in \,]0, \delta_0/2[$ be arbitrary. There is an $n_0 = n_0(\delta) \in \mathbb{N}$ such that $|t_n - t| < \delta$ for $n \ge n_0$. Therefore for all such $n \in \mathbb{N}$ and all $s \in [0, t-2\delta]$ it follows that $t_n - s > \delta$ so $\rho(t_n - s) \le C_4\delta^{-1/2} + C_4$. Thus

$$\rho(t_n - s)a_n(s) \le C_5 \quad \text{for } s \in \,]0, t-2\delta].$$

Therefore (2.12) and the dominated convergence theorem show that

$$\int_0^{t-2\delta} \rho(t_n - s)a_n(s)\,ds \to 0 \quad \text{as } n \to \infty.$$

On the other hand,

$$\int_{t-2\delta}^{t_n} \rho(t_n - s)a_n(s)\,ds \le C_6(\delta^{1/2} + \delta).$$

Since $\delta \in \,]0, \delta_0/2[$ is arbitrary, it follows that

$$\int_0^{t_n} \rho(t_n - s)a_n(s)\,ds \to 0 \quad \text{as } n \to \infty.$$

Consequently, $|u_n\pi_{\varepsilon_n}t_n - u_0\pi_{\varepsilon_0}t_n|_{H_1^{\varepsilon_n}} \to 0$ as $n \to \infty$. □

Lemma 2.10. *Suppose that the family* $(H^\varepsilon, \langle\,\cdot\,,\,\cdot\,\rangle_{H^\varepsilon}, A_\varepsilon)_{\varepsilon\in[0,\overline{\varepsilon}]}$ *satisfies condition* (Lin) *and the family* $(f_\varepsilon)_{\varepsilon\in[0,\overline{\varepsilon}]}$ *satisfies condition* (Conv). *Then for every* $\widetilde{R}\in\,]0,\infty[$ *there is a* $\widetilde{\tau}=\widetilde{\tau}_{\widetilde{R}}\in\,]0,\infty[$ *such that whenever* $u_0\in H^0_1$ *is such that* $|u_0|_{H^0_1}\le\widetilde{R}$, $(\varepsilon_n)_n$ *is a sequence in* $]0,\overline{\varepsilon}]$ *with* $\varepsilon_n\to 0$ *and* $(u_n)_n$ *is a sequence with* $u_n\in H^{\varepsilon_n}_1$ *for every* $n\in\mathbb{N}$ *and*

$$|u_n-u_0|_{H^{\varepsilon_n}_1}\to 0\quad \text{as } n\to\infty,$$

then there exist an $n_0\in\mathbb{N}$ *such that* $u_0\pi_0 t$ *and* $u_n\pi_{\varepsilon_n}t$ *are defined for all* $n\ge n_0$ *and* $t\in[0,\widetilde{\tau}]$ *and*

$$\sup_{t\in[0,\widetilde{\tau}]}|u_n\pi_{\varepsilon_n}t-u_0\pi_0 t|_{H^{\varepsilon_n}_1}\to 0\quad \text{as } n\to\infty.$$

Proof. Let ε'_0 be as in Lemma 2.8 and $\widetilde{R}\in\,]0,\infty[$ be arbitrary. Define $R=(1+C)\widetilde{R}$ and $\widetilde{\tau}=\tau_R$, where τ_R be as in Lemma 2.8. Let $u_0\in H^0_1$ be arbitrary such that $|u_0|_{H^0_1}\le\widetilde{R}$. Moreover, let $(\varepsilon_n)_n$ be a sequence in $]0,\overline{\varepsilon}]$ with $\varepsilon_n\to 0$ and $(u_n)_n$ be a sequence with $u_n\in H^{\varepsilon_n}_1$ for every $n\in\mathbb{N}$ and such that $|u_n-u_0|_{H^{\varepsilon_n}_1}\to 0$ as $n\to\infty$. Then there is an $n_0\in\mathbb{N}$ such that, $\varepsilon_n\le\varepsilon'_0$ and $|u_n-u_0|_{H^{\varepsilon_n}_1}\le\widetilde{R}$ for $n\ge n_0$. Thus, for all such n

$$|u_n|_{H^{\varepsilon_n}_1}\le|u_n-u_0|_{H^{\varepsilon_n}_1}+|u_0|_{H^{\varepsilon_n}_1}\le|u_n-u_0|_{H^{\varepsilon_n}_1}+C|u_0|_{H^0_1}\le R.$$

If follows from Lemma 2.8 that $u_0\pi_0 s$ and $u_n\pi_{\varepsilon_n}s$ are defined for $n\ge n_0$ and $s\in[0,\widetilde{\tau}]$ and $|u_n\pi_0 s|_{H^0_1}\le C_1$ and $|u_n\pi_{\varepsilon_n}s|_{H^{\varepsilon_n}_1}\le C_1$ for $n\ge n_0$ and $s\in[0,\widetilde{\tau}]$. Here, $C_1=4R$.

If the lemma does not hold then, taking subsequences if necessary, we may assume that there is a sequence $(t_n)_{n\in\mathbb{N}}$ in $[0,\widetilde{\tau}]$ converging to some $t_0\in[0,\widetilde{\tau}]$ and there is a $\delta\in\,]0,\infty[$ such that

$$|u_n\pi_{\varepsilon_n}t_n-u_0\pi_0 t_n|_{H^{\varepsilon_n}_1}\ge\delta,\quad n\ge n_0. \tag{2.13}$$

If $t_0>0$, then (2.13) contradicts Theorem 2.9. Therefore $t_0=0$. For every $t\in[0,\tau]$ we have

$$u_n\pi_{\varepsilon_n}t-u_0=e^{-tA_{\varepsilon_n}}u_n-u_0+\int_0^t e^{-(t-s)A_{\varepsilon_n}}f_{\varepsilon_n}(u_n\pi_{\varepsilon_n}s)\,ds.$$

Note that $|u_0\pi_0 s|_{H^{\varepsilon_n}_1}\le CC_1$ for all $s\in[0,\tau]$. Let $L:=L_M$ be as in (Conv) with $M=CC_1$. It follows that for all $n\ge n_0$ and for every $s\in[0,\tau]$

$$\begin{aligned}|f_{\varepsilon_n}(u_n\pi_{\varepsilon_n}s)|_{H^{\varepsilon_n}}&\le|f_{\varepsilon_n}(u_n\pi_{\varepsilon_n}s)-f_{\varepsilon_n}(u_0)|_{H^{\varepsilon_n}}+|f_{\varepsilon_n}(u_0)|_{H^{\varepsilon_n}}\\&\le L|u_n\pi_{\varepsilon_n}s-u_0|_{H^{\varepsilon_n}_1}+|f_{\varepsilon_n}(u_0)|_{H^{\varepsilon_n}}\\&\le L(C_1+C|u_0|_{H^0_1})+|f_{\varepsilon_n}(u_0)|_{H^{\varepsilon_n}}.\end{aligned}$$

Part (d) of condition (Conv) now implies

$$|f_{\varepsilon_n}(u_n\pi_{\varepsilon_n}s)|_{H^{\varepsilon_n}}\le\widetilde{C},\quad \text{for all } s\in[0,\tau] \text{ and for all } n\ge n_0,$$

for some $\widetilde{C} \in]0, \infty[$. Therefore for all $n \geq n_0$

$$\begin{aligned}|u_n \pi_{\varepsilon_n} t_n - u_0 \pi_0 t_n|_{H_1^{\varepsilon_n}} &\leq |u_n \pi_{\varepsilon_n} t_n - u_0|_{H_1^{\varepsilon_n}} + C|u_0 \pi_0 t_n - u_0|_{H_1^0} \\ &\leq |e^{-t_n A_{\varepsilon_n}} u_n - e^{-t_n A_0} u_0|_{H_1^{\varepsilon_n}} + C|e^{-t_n A_0} u_0 - u_0|_{H_1^0} \\ &+ C_0 \widetilde{C} \int_0^{t_n} (t_n - s)^{-1/2} \; ds + C|u_0 \pi_0 t_n - u_0|_{H_1^0}.\end{aligned}$$

Since $|e^{-t_n A_0} u_0 - u_0|_{H_1^0} \to 0$ and $|u_0 \pi_0 t_n - u_0|_{H_1^0} \to 0$ as $n \to \infty$, condition (Lin) implies that $|u_n \pi_{\varepsilon_n} t_n - u_0 \pi_0 t_n|_{H_1^{\varepsilon_n}} \to 0$ as $n \to \infty$, but this contradicts (2.13). The lemma is proved. □

We conclude this section proving our main convergence result for semiflows.

THEOREM 2.11. *Suppose the family* $(H^\varepsilon, \langle \cdot\,, \cdot \rangle_{H^\varepsilon}, A_\varepsilon)_{\varepsilon \in [0, \overline{\varepsilon}]}$ *satisfies condition* (Lin) *and the family* $(f_\varepsilon)_{\varepsilon \in [0,\overline{\varepsilon}]}$ *satisfy condition* (Conv). *Let* $(\varepsilon_n)_n$ *be a sequence in* $]0, \overline{\varepsilon}]$ *with* $\varepsilon_n \to 0$ *and let* $(t_n)_n$ *be a sequence in* $[0, \infty[$ *with* $t_n \to t_0$, *for some* $t_0 \in [0, \infty[$. *Let* $u_0 \in H_1^0$ *and* $(u_n)_n$ *be a sequence with* $u_n \in H_1^{\varepsilon_n}$ *for every* $n \in \mathbb{N}$ *and*

$$|u_n - u_0|_{H_1^{\varepsilon_n}} \to 0 \quad \text{as } n \to \infty.$$

Assume $u_0 \pi_0 t_0$ *is defined. Then there exists an* $n_0 \in \mathbb{N}$ *such that* $u_n \pi_{\varepsilon_n} t_n$ *is defined for all* $n \geq n_0$ *and*

$$|u_n \pi_{\varepsilon_n} t_n - u_0 \pi_0 t_0|_{H_1^{\varepsilon_n}} \to 0 \quad \text{as } n \to \infty.$$

PROOF. Since $u_0 \pi_0 t_0$ is defined, there is a $b > t_0$, $b \in]0, \infty[$, such that $u_0 \pi_0 t$ is defined for all $t \in [0, b[$. Define

$$I := \left\{ t \in [0, b[\;\middle|\; \begin{array}{l} \text{there exists an } n_0 \in \mathbb{N} \text{ such that } u_n \pi_{\varepsilon_n} t \text{ is defined for } n \geq n_0 \\ \text{and } \sup_{s \in [0,t]} |u_n \pi_{\varepsilon_n} s - u_0 \pi_0 s|_{H_1^{\varepsilon_n}} \to 0 \text{ as } n \to \infty \end{array} \right\}.$$

It is clear that $0 \in I$. Furthermore if $0 \leq t' < t$ and $t \in I$, then $t' \in I$. Let

$$\overline{t} := \sup I.$$

It follows that $\overline{t} \leq b$ and so $\left[0, \overline{t}\right[\subset I$. An application of Lemma 2.10 with an arbitrary $\widetilde{R}_0 \in]0, \infty[$ with $|u_0|_{H_1^0} \leq \widetilde{R}_0$ and a corresponding $\widetilde{\tau}_0 = \widetilde{\tau}_{\widetilde{R}_0}$ shows that $\overline{t} > \widetilde{\tau}_0 > 0$. We claim that $\overline{t} = b$. Suppose, on the contrary, that $\overline{t} < b$. It follows that $u_0 \pi_0 \overline{t}$ is defined. Let $\widetilde{R}_1 \in]0, \infty[$ be arbitrary with $|u_0 \pi_0 \overline{t}|_{H_1^0} < \widetilde{R}_1$ and $\widetilde{\tau}_1 = \widetilde{\tau}_{\widetilde{R}_1}$ be as in Lemma 2.10 . By continuity of π_0 there is a $t \in \mathbb{R}$ with $0 < t < \overline{t} < t + \widetilde{\tau}_1$ and $|u_0 \pi_0 t|_{H_1^0} < \widetilde{R}_1$. We have that $t \in I$ so there exists an $n_0 \in \mathbb{N}$ such that $u_n \pi_{\varepsilon_n} t$ is defined for all $n \geq n_0$ and

$$\sup_{s \in [0,t]} |u_n \pi_{\varepsilon_n} s - u_0 \pi_0 s|_{H_1^{\varepsilon_n}} \to 0 \quad \text{as } n \to \infty. \tag{2.14}$$

Lemma 2.10 implies that there is an $n_1 \in \mathbb{N}$ with $n_1 \geq n_0$ such that $(u_0\pi_0 t)\pi_0 s$ and $(u_n\pi_{\varepsilon_n} t)\pi_{\varepsilon_n} s$ are defined for all $n \geq n_1$ and $s \in [0, \widetilde{\tau}_1]$ and

$$\sup_{s\in[0,\widetilde{\tau}_1]} |(u_n\pi_{\varepsilon_n} t)\pi_{\varepsilon_n} s - (u_0\pi_0 t)\pi_0 s|_{H_1^{\varepsilon_n}} \to 0 \quad \text{as } n \to \infty. \tag{2.15}$$

Formulas (2.14) and (2.15) imply that $u_0\pi_0(t + \widetilde{\tau}_1)$ is defined, $u_n\pi_{\varepsilon_n}(t + \widetilde{\tau}_1)$ is also defined for all $n \geq n_1$ and

$$\sup_{s\in[0,t+\widetilde{\tau}_1]} |u_n\pi_{\varepsilon_n} s - u_0\pi_0 s|_{H_1^{\varepsilon_n}} \to 0 \quad \text{as } n \to \infty.$$

Thus $t + \widetilde{\tau}_1 \in I$, but $t + \widetilde{\tau}_1 > \bar{t}$, a contradiction, which proves that $\bar{t} = b$.

Since $t_0 \in [0, b[$, it follows that there is a $t \in [0, b[$ with $t_0 < t$ and $t_n < t$ for all n large enough. In particular $u_0\pi_0 t_n$ and $u_n\pi_{\varepsilon_n} t_n$ are defined for all n large enough and

$$|u_n\pi_{\varepsilon_n} t_n - u_0\pi_0 t_n|_{H_1^{\varepsilon_n}} \to 0 \quad \text{as } n \to \infty.$$

Since

$$|u_0\pi_0 t_n - u_0\pi_0 t_0|_{H_1^{\varepsilon_n}} \leq C|u_0\pi_0 t_n - u_0\pi_0 t_0|_{H_1^0}$$

and $|u_0\pi_0 t_n - u_0\pi_0 t_0|_{H_1^0} \to 0$ as $n \to \infty$, the theorem follows. □

Definition 2.12. Suppose that the family $(H^\varepsilon, \langle\cdot,\cdot\rangle_{H^\varepsilon}, A_\varepsilon)_{\varepsilon\in[0,\bar{\varepsilon}]}$ satisfies condition (Lin) and the family $(f_\varepsilon)_{\varepsilon\in[0,\bar{\varepsilon}]}$ satisfies condition (Conv). Let $\beta \in]0, \infty[$ and N be a closed subset of H_1^0. For $\varepsilon \in [0, \bar{\varepsilon}]$ set

$$\begin{aligned} N_{\varepsilon,\beta} &= N, && \text{if } \varepsilon = 0, \\ N_{\varepsilon,\beta} &= \{\, u \in H_1^\varepsilon \mid Q_\varepsilon u \in N \text{ and } |(\mathrm{Id}_{H_1^\varepsilon} - Q_\varepsilon)u|_{H_1^\varepsilon} \leq \beta \,\}, && \text{if } \varepsilon > 0. \end{aligned}$$

We say that N is *strongly admissible rel. to β and the family $(H_1^\varepsilon, \pi_\varepsilon)_{\varepsilon\in[0,\bar{\varepsilon}]}$*, (*resp. the family $(H^\varepsilon, \pi_\varepsilon)_{\varepsilon\in[0,\bar{\varepsilon}]}$*) if the following conditions are satisfied:

(a) for each $\varepsilon \in [0, \bar{\varepsilon}]$ the local semiflow π_ε does not explode in $N_{\varepsilon,\beta}$;

(b) whenever $\varepsilon_0 \in [0, \bar{\varepsilon}]$, $(\varepsilon_n)_n$ is a sequence in $[0, \bar{\varepsilon}]$, $(t_n)_n$ is a sequence in $]0, \infty[$ and $(u_n)_n$ is a sequence such that

$$t_n \to \infty \quad \text{as } n \to \infty,$$

either $\varepsilon_n = \varepsilon_0$ for all $n \in \mathbb{N}$ or else $\varepsilon_0 = 0$ and $\varepsilon_n \to \varepsilon_0$ for $n \to \infty$, and, for each $n \in \mathbb{N}$, $u_n \in H_1^{\varepsilon_n}$ and $u_n\pi_{\varepsilon_n}[0, t_n] \subset N_{\varepsilon_n,\beta}$, then there exist a $v \in H_1^{\varepsilon_0}$ and a subsequence of the sequence $(u_n\pi_{\varepsilon_n} t_n)_{n\in\mathbb{N}}$, denoted again by $(u_n\pi_{\varepsilon_n} t_n)_{n\in\mathbb{N}}$, such that

$$|u_n\pi_{\varepsilon_n} t_n - v|_{H_1^{\varepsilon_n}} \to 0 \quad \text{as } n \to \infty.$$

(resp. $$|u_n\pi_{\varepsilon_n} t_n - v|_{H^{\varepsilon_n}} \to 0 \quad \text{as } n \to \infty.)$$

Remark 2.13. If N is strongly admissible rel. to β and $(H_1^\varepsilon, \pi_\varepsilon)_{\varepsilon\in[0,\bar{\varepsilon}]}$ and v is as in (b) of Definition 2.12, then, as is easily seen, $v \in N_{\varepsilon_0,\beta}$.

Theorem 2.14. *Suppose that the family* $(H^\varepsilon, \langle\,\cdot\,,\,\cdot\,\rangle_{H^\varepsilon}, A_\varepsilon)_{\varepsilon\in[0,\overline{\varepsilon}]}$ *satisfies condition* (Lin) *and the family* $(f_\varepsilon)_{\varepsilon\in[0,\overline{\varepsilon}]}$ *satisfies condition* (Conv). *Let* $\beta \in]0,\infty[$ *and* N *be a closed bounded subset of* H_1^0. *Suppose* N *is strongly admissible rel. to* β *and the family* $(H^\varepsilon,\pi_\varepsilon)_{\varepsilon\in[0,\overline{\varepsilon}]}$. *Then* N *is strongly admissible rel. to* β *and the family* $(H_1^\varepsilon,\pi_\varepsilon)_{\varepsilon\in[0,\overline{\varepsilon}]}$.

Proof. There is an $M \in [0,\infty[$ such that $|w|_{H_1^0} \le M$ so $|w|_{H_1^\varepsilon} \le CM$ for all $w \in N$.

Let $\varepsilon_0 \in [0,\overline{\varepsilon}]$, $(\varepsilon_n)_n$ be a sequence in $[0,\overline{\varepsilon}]$, $(t_n)_n$ be a sequence in $]0,\infty[$ and $(u_n)_n$ be a sequence such that

$$t_n \to \infty \text{ as } n \to \infty,$$
$$\text{either } \varepsilon_n = \varepsilon_0 \text{ for all } n \in \mathbb{N} \text{ or else } \varepsilon_0 = 0 \text{ and } \varepsilon_n \to \varepsilon_0 \text{ for } n \to \infty,$$

and, for each $n \in \mathbb{N}$, $u_n \in H_1^{\varepsilon_n}$ and $u_n\pi_{\varepsilon_n}[0,t_n] \subset N_{\varepsilon_n,\beta}$. There is an $n_0 \in \mathbb{N}$ such that $t_n \ge 1$ for all $n \ge n_0$. Set $t'_n = t_n - 1$ for all $n \ge n_0$. By hypothesis there is a $v \in N_{\varepsilon_0,\beta}$ and a subsequence of the endpoint sequence $(u_n\pi_{\varepsilon_n}t'_n)_{n\ge n_0}$, denoted again by $(u_n\pi_{\varepsilon_n}t'_n)_{n\ge n_0}$, such that

$$|u_n\pi_{\varepsilon_n}t'_n - v|_{H^{\varepsilon_n}} \to 0 \quad \text{as } n \to \infty.$$

Let $v_n = u_n\pi_{\varepsilon_n}t'_n$, $n \ge n_0$. Then, for all $n \ge n_0$ and $s \in [0,1]$, $v_n\pi_{\varepsilon_n}s$ is defined and $v_n\pi_{\varepsilon_n}s \in N_{\varepsilon_n,\beta}$ so

$$|v_n\pi_{\varepsilon_n}s|_{H_1^{\varepsilon_n}} \le |Q_{\varepsilon_n}(v_n\pi_{\varepsilon_n}s)|_{H_1^{\varepsilon_n}} + |(\mathrm{Id}_{H_1^\varepsilon} - Q_\varepsilon)(v_n\pi_{\varepsilon_n}s)|_{H_1^{\varepsilon_n}} \le CM + \beta,$$

so

$$|v_n\pi_{\varepsilon_n}s|_{H_1^{\varepsilon_n}} \le CM + \beta, \quad n \ge n_0,\ s \in [0,1]\,. \tag{2.16}$$

There is an $s_1 \in]0,1[$ such that $v\pi_{\varepsilon_0}s_1$ is defined. Therefore, the estimate (2.16) together with Theorem 2.9 show that

$$|w_n - w|_{H_1^{\varepsilon_n}} \to 0, \quad \text{as } n \to \infty$$

where $w = v\pi_{\varepsilon_0}s_1$ and $w_n = v_n\pi_{\varepsilon_n}s_1$ for $n \ge n_0$. If $\varepsilon_n = \varepsilon_0$ for all $n \ge n_0$, then this implies that $w \in N_{\varepsilon_0,\beta}$. If $\varepsilon_0 = 0$ and $\varepsilon_n \to \varepsilon_0$ as $n \to \infty$, then, as $Q_{\varepsilon_n}w_n \in N$ for all $n \ge n_0$ and $Q_\varepsilon w = w$ for all $\varepsilon \in]0,\overline{\varepsilon}]$, we obtain from the estimates

$$|Q_{\varepsilon_n}w_n - w|_{H_1^0} = |Q_{\varepsilon_n}(w_n - w)|_{H_1^0} \le C|Q_{\varepsilon_n}(w_n - w)|_{H_1^{\varepsilon_n}} \le C|w_n - w|_{H_1^{\varepsilon_n}}$$

that $w \in N = N_{\varepsilon_0,\beta}$. Thus $w \in N_{\varepsilon_0,\beta}$ in both cases. We claim that $w\pi_{\varepsilon_0}(1-s_1)$ is defined. If not, then the fact that π_{ε_0} does not explode in $N_{\varepsilon_0,\beta}$ implies that there is some $s_2 \in]0,1-s_1[$ such that $w\pi_{\varepsilon_0}s_2$ is defined and $w\pi_{\varepsilon_0}s_2 \notin N_{\varepsilon_0,\beta}$. If $\varepsilon_n = \varepsilon_0$ for all $n \ge n_0$, then continuity of π_{ε_0} implies that $w_n\pi_{\varepsilon_n}s_2 \notin N_{\varepsilon_n,\beta}$, for all n large enough, which contradicts the fact that $w_n\pi_{\varepsilon_n}s_2 = v_n\pi_{\varepsilon_n}(s_1+s_2)$ for all $n \geq n_0$ and $s_1 + s_2 < 1$. If $\varepsilon_0 = 0$ and $\varepsilon_n \to \varepsilon_0$ as $n \to \infty$, then an application of Theorem 2.11 shows that

$$|w'_n - w'|_{H_1^{\varepsilon_n}} \to 0, \quad \text{as } n \to \infty$$

where $w' = w\pi_{\varepsilon_0} s_2$ and $w'_n = w_n \pi_{\varepsilon_n} s_2$ for $n \geq n_0$. As above this implies that $w' \in N_{\varepsilon_0,\beta}$, a contradiction. This proves that $w\pi_{\varepsilon_0}(1 - s_1)$ is defined so $v\pi_{\varepsilon_0} s$ is defined for all $s \in [0,1]$. Thus the estimate (2.16) together with Theorem 2.9 show that

$$|u_n \pi_{\varepsilon_n} t_n - v\pi_{\varepsilon_0} 1|_{H_1^{\varepsilon_n}} = |v_n \pi_{\varepsilon_n} 1 - v\pi_{\varepsilon_0} 1|_{H_1^{\varepsilon_n}} \to 0, \quad \text{as } n \to \infty.$$

The theorem is proved. □

Let the family $(H^\varepsilon, \langle\cdot,\cdot\rangle_{H^\varepsilon}, A_\varepsilon)_{\varepsilon\in[0,\overline{\varepsilon}]}$ satisfy condition (Lin) and the family $(f_\varepsilon)_{\varepsilon\in[0,\overline{\varepsilon}]}$ satisfy condition (Conv).

Set $X_0 := H_1^0$. For every $\varepsilon \in \,]0, \varepsilon_0]$, define $Y_\varepsilon := (I - Q_\varepsilon)H_1^\varepsilon$ and endow Y_ε with the norm $|\cdot|_{H_1^\varepsilon}$ restricted to Y_ε. Define on $Z_\varepsilon = X_0 \times Y_\varepsilon$ the following norm:

$$||(u,v)||_\varepsilon := \max\{\, |u|_{H_1^0}, |v|_{H_1^\varepsilon} \,\} \quad \text{for } (u,v) \in Z_\varepsilon.$$

We will denote by $\widetilde{\Gamma}_\varepsilon$ the metric on Z_ε induced by the norm $||\cdot||_\varepsilon$. For each $\varepsilon \in \,]0, \varepsilon_0]$, define $\theta_\varepsilon := 0$.

Let $\Psi_\varepsilon \colon H_1^\varepsilon \to Z_\varepsilon$ be the linear map defined by

$$\Psi_\varepsilon(w) := (Q_\varepsilon w, (I - Q_\varepsilon)w) \quad \text{for } w \in H_1^\varepsilon.$$

It follows that Ψ_ε is a bijective linear map and its inverse map is given by

$$\Psi_\varepsilon{}^{-1}(u,v) = u + v \quad \text{for } (u,v) \in Z_\varepsilon.$$

Moreover, both Ψ_ε and $\Psi_\varepsilon{}^{-1}$ are continuous maps. This fact is a consequence of the following inequalities:

$$\|\Psi_\varepsilon(w)\|_\varepsilon \leq C|w|_{H_1^\varepsilon} \qquad \text{for } w \in H_1^\varepsilon, \tag{2.17}$$

$$|\Psi_\varepsilon{}^{-1}(u,v)|_{H_1^\varepsilon} \leq (1 + C^2)^{1/2}||(u,v)||_\varepsilon \quad \text{for } (u,v) \in Z_\varepsilon, \tag{2.18}$$

where the constant $C \in \,]1, \infty[$ was defined in hypothesis (Lin).

Given $(u,v) \in Z_\varepsilon$ and $t \in [0, \infty[$ define

$$(u,v)\widetilde{\pi}_\varepsilon t := \Psi_\varepsilon(\Psi_\varepsilon{}^{-1}(u,v)\pi_\varepsilon t)$$

whenever $\Psi_\varepsilon{}^{-1}(u,v)\pi_\varepsilon t$ is defined. It follows that $\widetilde{\pi}_\varepsilon$ is a local semiflow on Z_ε, the *conjugate to* π_ε *via* Ψ_ε. Theorem 2.11 and inequalities (2.17) and (2.18) immediately imply the following

COROLLARY 2.15. *Under the above hypotheses the family* $(\widetilde{\pi}_\varepsilon)_{\varepsilon\in]0,\varepsilon_0]}$ *converges singularly to* π_0. □

Theorem 2.14, Remark 2.13 and inequalities (2.17) and (2.18) imply the following:

COROLLARY 2.16. *Under the above hypotheses let* $\beta \in\,]0,\infty[$ *and* N *be a closed bounded subset of* H_1^0. *If* N *is strongly admissible rel. to* β *and the family* $(H^\varepsilon,\pi_\varepsilon)_{\varepsilon\in[0,\overline{\varepsilon}]}$, *then* N *is singularly strongly admissible with respect to* β *and the family* $(\widetilde{\pi}_\varepsilon)_{\varepsilon\in[0,\overline{\varepsilon}]}$, *where* $\widetilde{\pi}_0=\pi_0$.

We can now state the following Conley index continuation principle for singular families of abstract parabolic equations:

THEOREM 2.17. *Suppose that the family* $(H^\varepsilon,\langle\,\cdot\,,\,\cdot\,\rangle_{H^\varepsilon},A_\varepsilon)_{\varepsilon\in[0,\overline{\varepsilon}]}$ *satisfies condition* (Lin) *and the family* $(f_\varepsilon)_{\varepsilon\in[0,\overline{\varepsilon}]}$ *satisfies condition* (Conv). *Let* $\beta\in\,]0,\infty[$ *and* N *be a closed bounded subset of* H_1^0. *Suppose* N *is strongly admissible rel. to* β *and the family* $(H^\varepsilon,\pi_\varepsilon)_{\varepsilon\in[0,\overline{\varepsilon}]}$. *Finally, assume* N *is an isolating neighbourhood of an invariant set* K_0 *relative to* π_0. *For* $\varepsilon\in\,]0,\varepsilon_0]$ *and for every* $\eta\in\,]0,\infty[$ *set*

$$N_{\varepsilon,\eta}:=\{u\in H_1^\varepsilon \mid Q_\varepsilon u\in N \text{ and } |(I_\varepsilon-Q_\varepsilon)u|_{H_1^\varepsilon}\le\eta\}$$

and $K_{\varepsilon,\eta}:=\operatorname{Inv}_{\pi_\varepsilon}(N_{\varepsilon,\eta})$ *i.e.* $K_{\varepsilon,\eta}$ *is the largest* π_ε*-invariant set in* $N_{\varepsilon,\eta}$. *Then for every* $\eta\in\,]0,\beta]$ *there exists an* $\varepsilon^{\mathrm{c}}=\varepsilon^{\mathrm{c}}(\eta)\in\,]0,\varepsilon_0]$ *such that for every* $\varepsilon\in\,]0,\varepsilon^{\mathrm{c}}]$ *the set* $N_{\varepsilon,\eta}$ *is a strongly admissible isolating neighbourhood of* $K_{\varepsilon,\eta}$ *relative to* π_ε *and*

$$h(\pi_\varepsilon,K_{\varepsilon,\eta})=h(\pi_0,K_0).$$

Furthermore, for every $\eta>0$, *the family* $(K_{\varepsilon,\eta})_{\varepsilon\in[0,\varepsilon^{\mathrm{c}}(\eta)]}$ *of invariant sets, where* $K_{0,\eta}=K_0$, *is upper semicontinuous at* $\varepsilon=0$ *with respect to the family* $|\cdot|_{H_1^\varepsilon}$ *of norms i.e.*

$$\lim_{\varepsilon\to0^+}\sup_{w\in K_{\varepsilon,\eta}}\inf_{u\in K_0}|w-u|_{H_1^\varepsilon}=0.$$

PROOF. The isomorphism Ψ_ε conjugates the local semiflow π_ε to the local semiflow $\widetilde{\pi}_\varepsilon$. Thus whenever S is a strongly admissible isolating neighbourhood with respect to π_ε, then $\Psi_\varepsilon(S)$ is a strongly admissible isolating neighbourhood with respect to $\widetilde{\pi}_\varepsilon$ and

$$h(\pi_\varepsilon,S)=h(\widetilde{\pi}_\varepsilon,\Psi_\varepsilon(S)).$$

Corollaries 2.15 and 2.16 imply that the family of semiflows $(\widetilde{\pi}_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ and the set N satisfy the hypotheses of [4, Theorem 4.1]. Notice also that any closed ball in Y_ε is contractible. Hence [4, Theorem 4.1] and [4, Corollary 4.11] completes the proof. □

Remark. The family $(K_{\varepsilon,\eta})_{\varepsilon\in]0,\varepsilon^{c}(\eta)]}$ is asymptotically independent of η i.e. whenever η_1 and $\eta_2 \in\,]0,\infty[$ then there is an $\varepsilon' \in\,]0, \min(\varepsilon^{c}(\eta_1), \varepsilon^{c}(\eta_2))]$ such that $K_{\varepsilon,\eta_1} = K_{\varepsilon,\eta_2}$ for $\varepsilon \in\,]0, \varepsilon']$.

We also state the following (co)homology index continuation principle:

Theorem 2.18. *Assume the hypotheses of Theorem* 2.17 *and for every* $\eta \in\,]0,\infty[$ *let* $\varepsilon^{c}(\eta) \in\,]0, \varepsilon_0]$ *be as in that theorem. Let* $(P, \prec)$ *be a finite poset. Let* $(M_{p,0})_{p\in P}$ *be a* $\prec$*-ordered Morse decomposition of* K_0 *relative to* π_0*. For each* $p \in P$*, let* $V_p \subset N$ *be closed in* X_0 *and such that* $M_{p,0} = \operatorname{Inv}_{\pi_0}(V_p) \subset \operatorname{Int}_{H^0_1}(V_p)$*.* (*Such sets* V_p*,* $p \in P$*, exist.*) *For* $\varepsilon \in\,]0, \varepsilon_0]$*, for every* $\eta \in\,]0,\infty[$ *and* $p \in P$ *set* $M_{p,\varepsilon,\eta} := \operatorname{Inv}_{\pi_\varepsilon}(V_{p,\varepsilon,\eta})$*, where*

$$V_{p,\varepsilon,\eta} := \{u \in H^{\varepsilon}_1 \mid Q_\varepsilon u \in V_p \text{ and } |(I - Q_\varepsilon)u|_{H^{\varepsilon}_1} \le \eta\}.$$

Then for every $\eta \in\,]0,\infty[$ *there is an* $\widetilde{\varepsilon} = \widetilde{\varepsilon}(\eta) \in\,]0, \varepsilon^{c}(\eta)]$ *such that for every* $\varepsilon \in\,]0, \widetilde{\varepsilon}]$ *and* $p \in P$*,* $M_{p,\varepsilon,\eta} \subset \operatorname{Int}_{H^{\varepsilon}_1}(V_{p,\varepsilon,\eta})$ *and the family* $(M_{p,\varepsilon,\eta})_{p\in P}$ *is a* $\prec$*-ordered Morse decomposition of* $K_{\varepsilon,\eta}$ *relative to* π_ε *and the* (*co*)*homology index braids of* $(\pi_0, K_0, (M_{p,0})_{p\in P})$ *and* $(\pi_\varepsilon, K_{\varepsilon,\eta}, (M_{p,\varepsilon,\eta})_{p\in P}))$*,* $\varepsilon \in\,]0, \widetilde{\varepsilon}]$*, are isomorphic and so they determine the same collection of* C*-connection matrices.*

Proof. Since the isomorphism Ψ_ε conjugates the local semiflow π_ε to the local semiflow $\widetilde{\pi}_\varepsilon$, using [6, Proposition 2.7], it follows that whenever S is a strongly admissible isolating neighbourhood with respect to π_ε and $(M_p)_{p\in P}$ is a $\prec$-ordered Morse decomposition of S relative to π_ε, then $\Psi_\varepsilon(S)$ is a strongly admissible isolating neighbourhood with respect to $\widetilde{\pi}_\varepsilon$ and $(\Psi_\varepsilon(M_p))_{p\in P}$ is a $\prec$-ordered Morse decomposition of S relative to $\widetilde{\pi}_\varepsilon$ and the (co)homology index braids of $(\pi_\varepsilon, S, (M_p)_{p\in P})$ and $(\widetilde{\pi}_\varepsilon, \Psi_\varepsilon(S), (\Psi_\varepsilon(M_p))_{p\in P}))$, $\varepsilon \in\,]0, \varepsilon_0]$, are isomorphic.

Corollaries 2.15 and 2.16 imply that the family of semiflows $(\widetilde{\pi}_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ and the set N satisfy the hypotheses of [5, Theorem 3.10]. Since any closed ball in Y_ε is contractible, an application of [5, Theorem 3.10] completes the proof. □

Remark. Again, for each $p \in P$, the family $(M_{p,\varepsilon,\eta})_{\varepsilon\in[0,\widetilde{\varepsilon}(\eta)]}$, where $M_{p,0,\eta} = M_{p,0}$ is upper semicontinuous at $\varepsilon = 0$ with respect to the family $|\cdot|_{H^{\varepsilon}_1}$ of norms and the family $(M_{p,\varepsilon,\eta})_{\varepsilon\in]0,\widetilde{\varepsilon}(\eta)]}$ is asymptotically independent of η.

3. Applications to curved squeezing on unbounded domains

We will now apply the previous results to singularly perturbed equations on curvedly squeezed unbounded domains. We assume the reader's familiarity with the paper [7] and only recall some necessary definitions.

Let ℓ, $\mathbf{k}$ and $\mathbf{r}$ be positive integers with $\mathbf{r} \ge 2$, $\ell \ge 2$ and $\mathbf{k} < \ell$. Let $\langle\,\cdot\,,\,\cdot\,\rangle$ be an inner product on $\mathbb{R}^\ell$ and $\|\cdot\|$ be the corresponding Euclidean norm.

Let $\mathcal{M} \subset \mathbb{R}^\ell$ be a $\mathbf{k}$-dimensional submanifold of $\mathbb{R}^\ell$ of class $C^{\mathbf{r}}$.

For $p \in \mathcal{M}$ let $Q(p): \mathbb{R}^\ell \to \mathbb{R}^\ell$ be the orthogonal projection of $\mathbb{R}^\ell$ onto the tangent space $T_p(\mathcal{M})$ to $\mathcal{M}$ at p and $P(p): \mathbb{R}^\ell \to \mathbb{R}^\ell$ be the orthogonal projection of $\mathbb{R}^\ell$ onto the orthogonal complement $T_p^\perp(\mathcal{M})$ of $T_p(\mathcal{M})$ in $\mathbb{R}^\ell$. We have $P(p) = \mathrm{Id}_{\mathbb{R}^\ell} - Q(p)$.

The map $Q: \mathcal{M} \to \mathcal{L}(\mathbb{R}^\ell, \mathbb{R}^\ell)$ is of class $C^{\mathbf{r}-1}$.

Moreover, $(DQ(p)a)b \in T_p^\perp(\mathcal{M})$ for each $p \in \mathcal{M}$ and all a, $b \in T_p(\mathcal{M})$ and the map

$$\mathrm{II}_p: T_p(\mathcal{M}) \times T_p(\mathcal{M}) \to T_p^\perp(\mathcal{M}), \quad (a,b) \mapsto (DQ(p)a)b$$

is bilinear and symmetric. The map II_p is called the *second fundamental form of $\mathcal{M}$ at p.*

We say that $\mathcal{M}$ *has bounded second fundamental form* if

$$\sup\{\|\mathrm{II}_p(a,b)\| \mid p \in \mathcal{M},\ (a,b) \in T_p(\mathcal{M}) \times T_p(\mathcal{M}),\ \|a\| \le 1,\ \|b\| \le 1\} < \infty.$$

This is equivalent to the requirement that

$$\sup\{\|(DQ(p)a)c\| \mid p \in \mathcal{M},\ (a,c) \in T_p(\mathcal{M}) \times T_p^\perp(\mathcal{M}),\ \|a\| \le 1,\ \|c\| \le 1\} < \infty.$$

DEFINITION 3.1. An open set $\mathcal{U}$ in $\mathbb{R}^\ell$ with $\mathcal{M} \subset \mathcal{U}$ is called a *normal neighbourhood* (or *normal strip*) *of* $\mathcal{M}$ if there is a map $\phi: \mathcal{U} \to \mathcal{M}$ of class $C^{\mathbf{r}-1}$, called an *orthogonal projection of $\mathcal{U}$ onto $\mathcal{M}$* and a continuous function $\delta: \mathcal{M} \to]0, \infty]$, called *the thickness of $\mathcal{U}$* such that:

(a) whenever $x \in \mathcal{U}$ and $p \in \mathcal{M}$ then $\phi(x) = p$ if and only if the vector $x - p \in T_p^\perp \mathcal{M}$ and $\|x - p\| < \delta(p)$;
(b) $\varepsilon x + (1-\varepsilon)\phi(x) \in \mathcal{U}$ for all $x \in \mathcal{U}$ and all $\varepsilon \in [0,1]$.

For the rest of this paper assume that $\mathcal{M}$ has bounded second fundamental form choose $M \in]0, \infty[$ arbitrarily with

$$\sup\big\{\|(DQ(p)a)c\| \mid (p,a,c) \in \mathcal{M} \times T_p(\mathcal{M}) \times T_p^\perp(\mathcal{M}),\ \|a\| \le 1,\ \|c\| \le 1\big\} \le M.$$

PROPOSITION 3.2. *Let $q_0 \in]0,1[$ be arbitrary. There is a normal neighbourhood $\mathcal{U}$ of $\mathcal{M}$ with normal projection ϕ and thickness δ such that $M\delta(p) \le q_0$ for all $p \in \mathcal{M}$.*

For the rest of this paper we fix a $q_0 \in]0,1[$ and a normal neighbourhood $\mathcal{U}$ with normal projection ϕ and thickness δ such that the assertions of Proposition 3.2 are satisfied.

For $\varepsilon \in [0,1]$ define the maps:

- $\Gamma_\varepsilon: \mathcal{U} \to \mathcal{U}$ by $x \mapsto \phi(x) + \varepsilon(x - \phi(x))$,
- $J_\varepsilon: \mathcal{U} \to \mathbb{R}$ by $J_\varepsilon(x) = |\det(D\Gamma_\varepsilon(x)_{|T_{\phi(x)}(\mathcal{M})})|$, $x \in \mathcal{U}$, and
- $S_\varepsilon: \mathcal{U} \to \mathcal{L}(\mathbb{R}^\ell, \mathbb{R}^\ell)$ by

$$S_\varepsilon(x)h = D\phi(\Gamma_\varepsilon(x))h - (DQ(\phi(x))(D\phi(\Gamma_\varepsilon(x))h))(x - \phi(x)) \tag{3.1}$$

for $x \in \mathcal{U}$ and $h \in \mathbb{R}^\ell$.

In the sequel, given a linear map $B: \mathbb{R}^\ell \to \mathbb{R}^\ell$ we denote by B^T the adjoint of B relative to the scalar product $\langle \cdot\,, \cdot \rangle$.

For the rest of this paper we will assume that

(3.2) $\quad \Omega$ is open in $\mathbb{R}^\ell$ with $\mathrm{Cl}(\Omega) \subset \mathcal{U}$. For $\varepsilon \in \,]0,1]$, we write $\Omega_\varepsilon = \Gamma_\varepsilon(\Omega)$.

For $\varepsilon \in \,]0,1]$ define the following bilinear forms:

$$\begin{aligned} \widetilde{a}_\varepsilon &: H^1(\Omega_\varepsilon) \times H^1(\Omega_\varepsilon) \to \mathbb{R}, \quad (\widetilde{u}, \widetilde{v}) \mapsto \int_{\Omega_\varepsilon} \nabla \widetilde{u}(x) \cdot \nabla \widetilde{v}(x)\, dx; \\ \widetilde{b}_\varepsilon &: L^2(\Omega_\varepsilon) \times L^2(\Omega_\varepsilon) \to \mathbb{R}, \quad (\widetilde{u}, \widetilde{v}) \mapsto \int_{\Omega_\varepsilon} \widetilde{u}(x) \widetilde{v}(x)\, dx; \end{aligned}$$

and let $a_\varepsilon: H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ be defined by

$$\begin{aligned} a_\varepsilon(u,v) = &\int_\Omega J_\varepsilon(x) \langle S_\varepsilon(x)^T \nabla u(x), S_\varepsilon(x)^T \nabla v(x) \rangle \, dx \\ &+ \frac{1}{\varepsilon^2} \int_\Omega J_\varepsilon(x) \langle P(x) \nabla u(x), P(x) \nabla v(x) \rangle \, dx, \quad u,\, v \in H^1(\Omega). \end{aligned}$$

For $\varepsilon \in [0,1]$ define the bilinear form $b_\varepsilon: L^2(\Omega) \times L^2(\Omega) \to \mathbb{R}$ by

$$b_\varepsilon(u,v) = \int_\Omega J_\varepsilon(x) u(x) v(x)\, dx, \ u,\, v \in L^2(\Omega).$$

We have

$$\widetilde{a}_\varepsilon(u,u) + \widetilde{b}_\varepsilon(u,u) = |u|^2_{H^1(\Omega_\varepsilon)}, \quad \varepsilon \in \,]0,1]\,, \, u \in H^1(\Omega_\varepsilon). \tag{3.3}$$

Let $\varepsilon \in \,]0,1]$ be arbitrary. Then the pair $(\widetilde{a}_\varepsilon, \widetilde{b}_\varepsilon)$ generates a densely defined selfadjoint operator $\mathbf{B}_\varepsilon$ in $(L^2(\Omega_\varepsilon), \widetilde{b}_\varepsilon)$, which we interpret, as usual, as the operator $-\Delta$ on Ω_ε with Neumann boundary condition on $\partial\Omega_\varepsilon$.

Let us define the space

$$H^1_s(\Omega) := \{\, u \in H^1(\Omega) \mid P(x) \nabla u(x) = 0 \ \text{ a.e.} \,\}.$$

Note that

(3.4) $\quad u \in H^1_s(\Omega)$ iff $u \in H^1(\Omega)$ and $\langle \nabla u(x), \nu \rangle$ for a.a. $x \in \Omega$ and all $\nu \in T^\perp_{\phi(x)}(\mathcal{M})$.

This is a closed linear subspace of the Hilbert space $H^1(\Omega)$. Now define the 'limit' bilinear form

$$a_0: H^1_s(\Omega) \times H^1_s(\Omega) \to \mathbb{R}, \quad (u,v) \mapsto \int_\Omega J_0(x) \langle S_0(x)^T \nabla u(x), S_0(x)^T \nabla v(x) \rangle \, dx.$$

Finally, let $L^2_s(\Omega)$ be the closure of $H^1_s(\Omega)$ in $L^2(\Omega)$. $L^2_s(\Omega)$ is a closed linear subspace of the Hilbert space $L^2(\Omega)$. For $\varepsilon \in \,]0,1]$ and $u,\, v \in L^2(\Omega)$ set

$$\langle u, v \rangle_\varepsilon := b_\varepsilon(u,v).$$

For $\varepsilon \in]0,1]$ and u, $v \in H^1(\Omega)$ set

$$\langle\langle u,v\rangle\rangle_\varepsilon := a_\varepsilon(u,v) + b_\varepsilon(u,v).$$

$\langle\,\cdot\,,\,\cdot\,\rangle_\varepsilon$ (resp. $\langle\langle\,\cdot\,,\,\cdot\,\rangle\rangle_\varepsilon$) is a scalar product on $H^\varepsilon := L^2(\Omega)$ (resp. $H^1(\Omega)$). Let $|\cdot|_\varepsilon$ (resp. $\|\cdot\|_\varepsilon$) be the Euclidean norm on $L^2(\Omega)$ (resp. $H^1(\Omega)$) induced by $\langle\,\cdot\,,\,\cdot\,\rangle_\varepsilon$ (resp. $\langle\langle\,\cdot\,,\,\cdot\,\rangle\rangle_\varepsilon$). Furthermore, for u, $v \in L^2_s(\Omega)$ set

$$\langle u,v\rangle_0 := b_0(u,v).$$

Finally, for u, $v \in H^1_s(\Omega)$ set

$$\langle\langle u,v\rangle\rangle_0 := a_0(u,v) + b_0(u,v).$$

$\langle\,\cdot\,,\,\cdot\,\rangle_0$ (resp. $\langle\langle\,\cdot\,,\,\cdot\,\rangle\rangle_0$) is a scalar product on $H^0 := L^2_s(\Omega)$ (resp. $H^1_s(\Omega)$).

Let $|\cdot|_0$ (resp. $\|\cdot\|_0$) be the Euclidean norm on $L^2_s(\Omega)$ (resp. $H^1_s(\Omega)$) induced by $\langle\,\cdot\,,\,\cdot\,\rangle_0$ (resp. $\langle\langle\,\cdot\,,\,\cdot\,\rangle\rangle_0$).

For $\varepsilon \in [0,1]$, $(H^\varepsilon, \langle\,\cdot\,,\,\cdot\,\rangle_\varepsilon)$ is a Hilbert space.

For $\varepsilon \in [0,1]$, the pair $(a_\varepsilon, \langle\,\cdot\,,\,\cdot\,\rangle_\varepsilon)$ generates a densely defined selfadjoint operator $\mathbf{A}_\varepsilon$ on $(H^\varepsilon, \langle\,\cdot\,,\,\cdot\,\rangle_\varepsilon)$.

The (linear) operators $\mathbf{B}_\varepsilon$ (resp. $\mathbf{A}_\varepsilon$) defined by $(\widetilde{a}_\varepsilon, \widetilde{b}_\varepsilon)$ (resp. $(a_\varepsilon, b_\varepsilon)$) satisfy the following properties:

(a) $u \in D(\mathbf{B}_\varepsilon)$ if and only if $u \circ (\Gamma_\varepsilon)_{|\Omega} \in D(\mathbf{A}_\varepsilon)$;
(b) $\mathbf{A}_\varepsilon(u \circ (\Gamma_\varepsilon)_{|\Omega}) = (\mathbf{B}_\varepsilon u) \circ (\Gamma_\varepsilon)_{|\Omega}$ for $u \in D(\mathbf{B}_\varepsilon)$.

The following result was proved in [7]

Proposition 3.3. [7, Corollary 4.5] *The family $(H^\varepsilon, \langle\,\cdot\,,\,\cdot\,\rangle_{H^\varepsilon}, \mathbf{A}_\varepsilon)_{\varepsilon\in[0,1]}$ defined in this section satisfies hypothesis* (Res).

Now consider the following:

Assumption 3.4. $G\colon [0,1] \times \mathcal{U} \times \mathbb{R} \to \mathbb{R}$, $(\varepsilon, y, s) \mapsto G(\varepsilon, y, s)$ is continuous and such that, for all $(\varepsilon, y) \in [0,1] \times \mathcal{U}$, $G(\varepsilon, y, \,\cdot\,)$ is continuously differentiable in s. Moreover, for $\varepsilon \in [0,1]$, $G(\varepsilon, \Gamma_\varepsilon(\,\cdot\,), 0)_{|\Omega} \in L^2(\Omega)$ and $G(\varepsilon, \Gamma_\varepsilon(\,\cdot\,), 0)_{|\Omega} \to G(0, \phi(\,\cdot\,), 0)_{|\Omega}$ in $L^2(\Omega)$ as $\varepsilon \to 0^+$. Furthermore, there is a constant $C_G \in [0,\infty[$ such that

$$|\partial_s G(\varepsilon, \Gamma_\varepsilon(x), s)| \leq C_G(1 + |s|^\beta)$$

for all $(\varepsilon, x, s) \in [0,1] \times \Omega \times \mathbb{R}$, where $\beta \in]0,\infty[$ is arbitrary for $\ell = 2$ and $\beta = 2/(\ell - 2)$ for $\ell \geq 3$.

Finally, the function $G(0, \,\cdot\,, \,\cdot\,)$ is continuously differentiable in (y,s).

Proposition 3.5. *Given $\varepsilon \in [0,1]$ and $u \in H^1(\Omega)$ define the function $f_\varepsilon(u)\colon \Omega \to \mathbb{R}$ by*

$$f_\varepsilon(u)(x) = G(\varepsilon, \Gamma_\varepsilon(x), u(x)), \quad x \in \Omega. \tag{3.5}$$

For all $\varepsilon \in [0,1]$ the function $f_\varepsilon(u)$ lies in H^ε for $u \in H_1^\varepsilon$ and the induced family $(f_\varepsilon)_{\varepsilon\in[0,1]}$ satisfies condition (Conv).

PROOF. Since there are continuous imbeddings from $H^1(\Omega)$ to $L^2(\Omega)$ and from $H^1(\Omega)$ to $L^{2(\beta+1)}(\Omega)$, it follows that there is a constant $C_1 \in \,]0,\infty[$ such that for all $\varepsilon \in [0,1]$ and all $u \in H^1(\Omega)$, $f_\varepsilon(u)$ lies in $L^2(\Omega)$ and

$$|f_\varepsilon(u)|_{L^2(\Omega)} \le |G(\varepsilon,\Gamma_\varepsilon(\cdot),0)|_{L^2(\Omega)} + C_1|u|^{\beta+1}_{H^1(\Omega)},$$

and for all all u, $v \in H^1(\Omega)$

$$|f_\varepsilon(u) - f_\varepsilon(v)|_{L^2(\Omega)} \le C_1(|u|^\beta_{H^1(\Omega)} + |v|^\beta_{H^1(\Omega)})|u-v|_{H^1(\Omega)}. \tag{3.6}$$

It follows that, for $\varepsilon \in \,]0,1]$, formula (3.5) defines an operator $f_\varepsilon\colon H_1^\varepsilon \to H^\varepsilon$. We will now show that $f_0(u) \in H^0$ if $u \in H_1^0$. To prove this let $(\mathcal{U}_k)_{k\in\mathbb{N}}$ be a covering of $\mathcal{U}$ by a sequence of open sets with $\mathrm{Cl}\,\mathcal{U}_k \subset \mathcal{U}_{k+1}$ and $\mathrm{Cl}\,\mathcal{U}_k$ compact for all $k \in \mathbb{N}$. For $k \in \mathbb{N}$ let $\xi_k\colon \mathcal{U}\times\mathbb{R} \to \mathbb{R}$ be a C^∞-function with $0 \le \xi_k \le 1$, $\xi_k(y,t) = 1$ for $(y,t) \in S_k := \mathrm{Cl}\,\mathcal{U}_k \times [-k,k]$ and $\xi_k(y,t) = 0$ for $(y,t) \notin \mathcal{U}_{k+1} \times \,]-k-1,k+1[$. Since $u \in H^1(\Omega)$, there is a sequence $(u_n)_{n\in\mathbb{N}}$ in $H^1(\Omega) \cap C^\infty(\Omega)$ with $u_n \to u$ in $H^1(\Omega)$. We may assume that $u_n \to u$ and $\partial_j u_n \to \partial_j u$ almost everywhere in Ω for all $j \in [1\,.\,.\,\ell]$. Moreover, we may also assume that there is a $v \in L^2(\Omega)$ such that $|u_n| \le v$ and $|\partial_j u_n| \le v$ almost everywhere on Ω for all $j \in [1\,.\,.\,\ell]$. Let $k \in \mathbb{N}$ be arbitrary. For $n \in \mathbb{N}$ let $w_{k,n}\colon \Omega \to \mathbb{R}$ be defined by

$$w_{k,n}(x) = \xi_k(\phi(x),u_n(x)) \cdot G(0,\phi(x),u_n(x)), \quad x \in \Omega \tag{3.7}$$

It follows that $w_{k,n} \to w_k$ almost everywhere in Ω as $n \to \infty$, where $w_k\colon \Omega \to \mathbb{R}$ is defined by

$$w_k(x) = \xi_k(\phi(x),u(x)) \cdot G(0,\phi(x),u(x)), \quad x \in \Omega \tag{3.8}$$

Moreover, for $x \in \Omega$ we obtain

$$\begin{aligned} |w_{k,n}(x)| &= |\xi_k(\phi(x),u_n(x)) \cdot G(0,\phi(x),u_n(x))| \\ &\le |\xi_k(\phi(x),u_n(x))| \sup_{(y,s)\in S_{k+1}} |G(0,y,s)| \end{aligned} \tag{3.9}$$

as $\xi_k(\phi(x),u_n(x)) \cdot G(0,\phi(x),u_n(x)) = 0$ if $(\phi(x),u_n(x)) \notin S_{k+1}$.

The function $w_{k,n}$ lies in $C^1(\Omega)$ and using the product rule, we obtain for $x \in \Omega$ and $j \in [1\,.\,.\,\ell]$

$$\begin{aligned} \partial_j w_{k,n}(x) = {} & \xi_k(\phi(x),u_n(x)) D_y G(0,\phi(x),u_n(x)) \partial_j \phi(x) \\ & + \xi_k(\phi(x),u_n(x)) \partial_s G(0,\phi(x),u_n(x)) \partial_j u_n(x) \\ & + D_y \xi_k(\phi(x),u_n(x)) \partial_j \phi(x) G(0,\phi(x),u_n(x)) \\ & + \partial_s \xi_k(\phi(x),u_n(x)) \partial_j u_n(x) G(0,\phi(x),u_n(x)). \end{aligned} \tag{3.10}$$

Thus $\partial_j w_{k,n} \to w_k^{(j)}$ almost everywhere in Ω as $n \to \infty$, where $w_k^{(j)}: \Omega \to \mathbb{R}$ is defined by

$$
\begin{aligned}
w_k^{(j)}(x) = {} & \xi_k(\phi(x), u(x)) D_y G(0, \phi(x), u(x)) \partial_j \phi(x) \\
& + \xi_k(\phi(x), u(x)) \partial_s G(0, \phi(x), u(x)) \partial_j u(x) \\
& + D_y \xi_k(\phi(x), u(x)) \partial_j \phi(x) G(0, \phi(x), u(x)) \\
& + \partial_s \xi_k(\phi(x), u(x)) \partial_j u(x) G(0, \phi(x), u(x)).
\end{aligned}
\tag{3.11}
$$

Moreover, for $x \in \Omega$,

$$
\begin{aligned}
|\partial_j w_{k,n}(x)| \le {} & |\xi_k(\phi(x), u_n(x))| \sup_{(y,s)\in S_{k+1}} |D_y G(0,y,s)| \sup_{y\in\mathcal{U}} |\partial_j \phi(y)| \\
& + \sup_{(y,s)\in \mathcal{U}\times\mathbb{R}} |\xi_k(y,s)| \sup_{(y,s)\in S_{k+1}} |\partial_s G(0,y,s)| \cdot |\partial_j u_n(x)| \\
& + |D_y \xi_k(\phi(x), u_n(x))| \sup_{y\in\mathcal{U}} |\partial_j \phi(y)| \sup_{(y,s)\in S_{k+1}} |G(0,y,s)| \\
& + \sup_{(y,s)\in \mathcal{U}\times\mathbb{R}} |\partial_s \xi_k(y,s)| \cdot |\partial_j u_n(x)| \sup_{(y,s)\in S_{k+1}} |G(0,y,s)|.
\end{aligned}
\tag{3.12}
$$

Now by the mean-value theorem we have, for all $(x,s) \in \Omega \times \mathbb{R}$ and all $j \in [1\,.\,.\,\ell]$,

$$
\begin{aligned}
|\xi_k(\phi(x), s)| &\le |\xi_k(\phi(x), 0)| + \sup_{\theta\in[0,1]} |\partial_s \xi_k(\phi(x), \theta s)| \cdot |s|, \\
|D_y \xi_k(\phi(x), s)| &\le |D_y \xi_k(\phi(x), 0)| + \sup_{\theta\in[0,1]} |\partial_s D_y \xi_k(\phi(x), \theta s)| \cdot |s|.
\end{aligned}
$$

Note that $\sup_{y\in\mathcal{U}} |D\phi(y)| < \infty$ by [7, Lemma 4.4]. Consequently, there is a constant $C_2(k) \in]0, \infty[$ such that for all $n \in \mathbb{N}$, all $x \in \Omega$ and all $j \in [1\,.\,.\,\ell]$,

$$
|w_{k,n}(x)| \le (|\xi_k(\phi(x), 0)| + C_2(k)|u_n(x)|) C_2(k)
\tag{3.13}
$$

and

$$
\begin{aligned}
|\partial_j w_{k,n}(x)| \le {} & (|\xi_k(\phi(x), 0)| + C_2(k)|u_n(x)|) C_2(k) \\
& + C_2(k) |\partial_j u_n(x)| \\
& + (|D_y \xi_k(\phi(x), 0)| + C_2(k)|u_n(x)|) C_2(k) \\
& + C_2(k) |\partial_j u_n(x)|.
\end{aligned}
\tag{3.14}
$$

By our choice of M and q_0 (see Proposition 3.2) we see that the functions $|\xi_k(\phi(\,\cdot\,), 0)|$ and $|D_y \xi_k(\phi(\,\cdot\,), 0)|$ lie in $L^2(\Omega)$, since they are continuous hence measurable, bounded and their supports are subsets of the closed q_0/M-neighbourhood of $\mathrm{Cl}\,\mathcal{U}_{k+1}$.

By the dominated convergence theorem we now obtain that $w_{k,n} \to w_k$ and for all $j \in [1\,.\,.\,\ell]$, $\partial_j w_{k,n} \to w_k^{(j)}$ in $L^2(\Omega)$ as $n \to \infty$. Thus $w_k \in H^1(\Omega)$ and $\partial_j w_k = w_k^{(j)}$ for all $j \in [1\,.\,.\,\ell]$.

Now, for all $x \in \mathcal{U}$ and all $\nu \in T^{\perp}_{\phi(x)}(\mathcal{M})$,

$$\sum_{j=1}^{\ell} \partial_j \phi(x)\nu_j = D\phi(x).\nu = 0.$$

Since $u \in H^1_s(\Omega)$, we infer from (3.4) that $\sum_{j=1}^{\ell} \partial_j u(x)\nu_j = 0$ for almost all $x \in \mathcal{U}$ and all $\nu \in T^{\perp}_{\phi(x)}(\mathcal{M})$. Consequently (3.11) implies that for almost all $x \in \mathcal{U}$ and all $\nu \in T^{\perp}_{\phi(x)}(\mathcal{M})$, $\sum_{j=1}^{\ell} \partial_j w_k(x)\nu_j = 0$. Thus $w_k \in H^1_s(\Omega)$ for all $k \in \mathbb{N}$. Now $w_k \to G(0, \phi(\,\cdot\,), u(\,\cdot\,))$ almost everywhere in Ω as $k \to \infty$. Moreover, $|w_k| \le |G(0, \phi(\,\cdot\,), u(\,\cdot\,))|$ almost everywhere in Ω. Since $G(0, \phi(\,\cdot\,), u(\,\cdot\,)) \in L^2(\Omega)$, it follows that $w_k \to G(0, \phi(\,\cdot\,), u(\,\cdot\,))$ in $L^2(\Omega)$ as $k \to \infty$. Hence $G(0, \phi(\,\cdot\,), u(\,\cdot\,)) \in L^2_s(\Omega)$. Our claim is proved. Thus part (a) of condition (Conv) holds with $\overline{\varepsilon} = 1$.

Now (3.6) together with [7, Propositions 2.7 and 2.8] imply that part (c) of condition (Conv) holds. Our assumptions together with dominated convergence theorem imply that whenever $u \in H^1_s(\Omega)$, then $f_\varepsilon(u) \to f_0(u)$ in $L^2(\Omega)$ as $\varepsilon \to 0$. This together with Theorem 2.4 and [7, Proposition 2.7] shows that the remaining parts (b) and (d) of condition (Conv) hold. $\square$

We will now prove that under some additional hypotheses on $(f_\varepsilon)_{\varepsilon \in [0,1]}$, whenever $\beta \in]0, \infty[$ and N is a closed bounded subset of H^0_1, then N is strongly admissible rel. to β and the family $(H^\varepsilon, \pi_\varepsilon)_{\varepsilon \in [0,\overline{\varepsilon}]}$. Here, for $\varepsilon \in [0,1]$, π_ε is the local semiflow generated on H^1_ε by the semilinear differential equation

$$\dot{u} = -\mathbf{A}_\varepsilon u + f_\varepsilon(u).$$

Our guiding principle is the method of tail estimates, cf. [18], [9], [1], [12], which we adapt to the present situation.

Define the function $\lambda: \Omega \to \mathbb{R}$ by

$$\lambda(x) = \langle \phi(x), \phi(x) \rangle, \quad x \in \Omega.$$

Moreover, let $\vartheta: \mathbb{R} \to \mathbb{R}$ be a C^∞-function such that $0 \le \vartheta \le 1$, $\vartheta = 0$ on $]-\infty, 1]$ and $\vartheta = 1$ on $[2, \infty[$. For $k \in \mathbb{N}$ define the function $\theta_k: \Omega \to \mathbb{R}$ by

$$\theta_k(x) = \vartheta(\lambda(x)/k^2), \quad x \in \Omega.$$

Then θ_k is a C^1-function and for all $x \in \Omega$ and all $\nu \in \mathbb{R}^\ell$,

$$\langle \nabla\theta_k(x), \nu \rangle = D\theta_k(x).\nu = (2/k^2)\vartheta'(\lambda(x)/k^2)\langle D\phi(x).\nu, \phi(x) \rangle.$$

In particular,

$$\langle \nabla\theta_k(x), \nu \rangle = 0, \quad x \in \Omega,\ \nu \in T^{\perp}_{\phi(x)}(\mathcal{M}). \tag{3.15}$$

Now, if $u \in H^1(\Omega)$, then $\theta_k \cdot u \in H^1(\Omega)$. For almost all $x \in \Omega$, $\nu_x = P(x)\nabla u(x) \in T^{\perp}_{\phi(x)}(\mathcal{M})$ so

$$\langle P(x)\nabla\theta_k(x), P(x)\nabla u(x)\rangle = 0. \tag{3.16}$$

Now let $\varepsilon \in [0,1]$ and $u \in H_1^{\varepsilon}$ be arbitrary. If $\varepsilon > 0$, then we obtain from (3.16)

$$\begin{aligned}
-a_\varepsilon(\theta_k \cdot u, u) = &- \int_\Omega J_\varepsilon(x)\theta_k(x)\langle S_\varepsilon(x)^T\nabla u(x), S_\varepsilon(x)^T\nabla u(x)\rangle\,dx \\
&- \int_\Omega J_\varepsilon(x)u(x)\langle S_\varepsilon(x)^T\nabla\theta_k(x), S_\varepsilon(x)^T\nabla u(x)\rangle\,dx \\
&- \frac{1}{\varepsilon^2}\int_\Omega J_\varepsilon(x)\theta_k\langle P(x)\nabla u(x), P(x)\nabla u(x)\rangle\,dx \\
&- \frac{1}{\varepsilon^2}\int_\Omega J_\varepsilon(x)u(x)\langle P(x)\nabla\theta_k(x), P(x)\nabla u(x)\rangle\,dx \\
\le &- \int_\Omega J_\varepsilon(x)u(x)\langle S_\varepsilon(x)^T\nabla\theta_k(x), S_\varepsilon(x)^T\nabla u(x)\rangle\,dx.
\end{aligned}$$

If $\varepsilon = 0$ and $u \in H^1_s(\Omega)$ then, by (3.4) and (3.15), $\theta_k \cdot u \in H^1_s(\Omega)$ and we obtain

$$\begin{aligned}
-a_\varepsilon(\theta_k \cdot u, u) = &- \int_\Omega J_\varepsilon(x)\theta_k(x)\langle S_\varepsilon(x)^T\nabla u(x), S_\varepsilon(x)^T\nabla u(x)\rangle\,dx \\
&- \int_\Omega J_\varepsilon(x)u(x)\langle S_\varepsilon(x)^T\nabla\theta_k(x), S_\varepsilon(x)^T\nabla u(x)\rangle\,dx \\
\le &- \int_\Omega J_\varepsilon(x)u(x)\langle S_\varepsilon(x)^T\nabla\theta_k(x), S_\varepsilon(x)^T\nabla u(x)\rangle\,dx.
\end{aligned}$$

Thus, whenever $\varepsilon \in [0,1]$ and $u \in H_1^{\varepsilon}$ then $\theta_k \cdot u \in H_1^{\varepsilon}$ and

$$-a_\varepsilon(\theta_k \cdot u, u) \le -\int_\Omega J_\varepsilon(x)u(x)\langle S_\varepsilon(x)^T\nabla\theta_k(x), S_\varepsilon(x)^T\nabla u(x)\rangle\,dx. \tag{3.17}$$

Now, setting $h(x) = S(x)S^T(x)\nabla u(x)$ we see that

$$\begin{aligned}
\langle S_\varepsilon(x)^T\nabla\theta_k(x), S_\varepsilon(x)^T\nabla u(x)\rangle &= D\theta_k(x).h(x) \\
&= (2/k^2)\vartheta'(\lambda(x)/k^2)\langle\phi(x), D\phi(x).h(x)\rangle.
\end{aligned}$$

Thus, using (3.17) and letting Ω_k be the set of all $x \in \Omega$ with $k^2 \le \lambda(x) \le 2k^2$ we see that

$$-a_\varepsilon(\theta_k \cdot u, u) \le (2/k^2)C'\int_{\Omega_k} |u(x)| \sup_{y\in\Omega_k}|\phi(y)|\,|\nabla u(x)|\,dx$$

where

$$C' = \sup_{\varepsilon\in[0,1], y\in\mathcal{U}} J_\varepsilon(y)\sup_{s\in\mathbb{R}}|\vartheta'(s)|\sup_{y\in\mathcal{U}}|D\phi(y)|\sup_{\varepsilon\in[0,1], y\in\mathcal{U}}|S_\varepsilon(y)S_\varepsilon^T(y)|.$$

Our choice of ϑ and [7, Lemma 4.4] imply that $C' < \infty$. Thus

$$-a_\varepsilon(\theta_k \cdot u, u) \le (C_1/k)\int_\Omega |u(x)|\,|\nabla u(x)|\,dx \le (C_{11}/k)|u|^2_{H^1(\Omega)}.$$

Here, $C_1 = 2\sqrt{2}C'$. We thus obtain

Lemma 3.6. *There is a constant* $C_1 \in \,]0,\infty[$ *such that for all* $k \in \mathbb{N}$*, all* $\varepsilon \in [0,1]$ *and all* $u \in H_1^\varepsilon$ *we have* $\theta_k \cdot u \in H_1^\varepsilon$ *and*

$$-a_\varepsilon(\theta_k \cdot u, u) \leq (C_1/k)|u|^2_{H^1(\Omega)}.$$

Lemma 3.7. *Let* $\varepsilon \in [0,1]$*,* $k \in \mathbb{N}$ *and* $T \in \,]0,\infty[$ *be arbitrary and* $u: [0,T] \to H_1^\varepsilon$ *be a solution of*

$$\dot{u} = -\mathbf{A}_\varepsilon u + f_\varepsilon(u).$$

Then the function $\zeta: [0,T] \to \mathbb{R}$*,* $t \mapsto b_\varepsilon(\theta_k \cdot u(t), u(t))$ *is continuously differentiable on* $[0,T]$ *and*

$$\zeta'(t) = \xi(t) := -2a_\varepsilon(\theta_k \cdot u(t), u(t)) + 2b_\varepsilon(\theta_k \cdot u(t), f_\varepsilon(u(t))), \quad t \in [0,T]\,.$$

For every $\mu \in \,]0,\infty[$

$$\zeta(t) = e^{-2\mu t}\zeta(0) + \int_0^t e^{-2\mu(t-s)}\xi_\mu(s)\,ds, \quad t \in [0,T]\,, \tag{3.18}$$

where $\xi_\mu: [0,T] \to \mathbb{R}$ *is defined by*

$$\xi_\mu(t) = -2a_\varepsilon(\theta_k \cdot u(t), u(t)) + 2b_\varepsilon(\theta_k \cdot u(t), f_\varepsilon(u(t)) + \mu u(t)), \quad t \in [0,T]\,.$$

If $R \in [0,\infty[$ *is such that* $|u(t)|_{H^1(\Omega)} \leq R$ *for all* $t \in [0,T]$ *then*

$$|\theta_k \cdot u(t)|^2_{L^2(\Omega)} \leq C_3^{-1}(e^{-2\mu t}C_2R^2 + (C_1R^2/(\mu k)) + (M_k(R)/\mu)) \tag{3.19}$$

where $C_2 = \sup_{\varepsilon\in[0,1]} \sup_{x\in\mathcal{U}} J_\varepsilon(x) < \infty$*,* $C_3 = \inf_{\varepsilon\in[0,1]} \sup_{x\in\mathcal{U}} J_\varepsilon(x) > 0$ *and*

$$M_k(R) = \sup_{\varepsilon\in[0,1],\, u\in H_1^\varepsilon,\, |u|_{H^1(\Omega)}\leq R} b_\varepsilon(\theta_k \cdot u, f_\varepsilon(u) + \mu u).$$

Proof. Since the map $H^1(\Omega) \to H^1(\Omega)$, $v \mapsto \theta_k \cdot v$ is linear and bounded and since the norm $|\cdot|_{H_1^\varepsilon}$ is equivalent (with bounds depending on ε) to the (restriction of the) norm of $H^1(\Omega)$, it follows that the map $H_1^\varepsilon \to H_1^\varepsilon$, $v \mapsto \theta_k \cdot v$ is well-defined, linear and bounded. Since u is continuous (into H_1^ε) and the restriction of u to $]0,T]$ is differentiable into H^ε, it follows that ζ and ξ are continuous on $[0,T]$. Moreover, ζ differentiable on $]0,T]$ and, for $t \in \,]0,T]$,

$$\begin{aligned}\zeta'(t) &= b_\varepsilon(\theta_k \cdot u'(t), u(t)) + b_\varepsilon(\theta_k \cdot u(t), u'(t)) \\ &= 2b_\varepsilon(\theta_k \cdot u(t), u'(t)) = 2b_\varepsilon(\theta_\varepsilon \cdot u(t), -\mathbf{A}_\varepsilon u(t) + f_\varepsilon(u(t))).\end{aligned}$$

Since, for $w \in H_1^\varepsilon$ and $v \in D(\mathbf{A}_\varepsilon)$,

$$b_\varepsilon(w, \mathbf{A}_\varepsilon v) = a_\varepsilon(w, v),$$

it follows that ζ is differentiable on $]0,T]$ and $\zeta'(t) = \xi(t)$ for $t \in \,]0,T]$. Since ξ is continuous at $t = 0$ it follows that ζ is differentiable at $t = 0$ and $\zeta'(0) = \xi(0)$.

We now conclude that, given $\mu \in]0, \infty[$

$$\zeta'(t) = -2\mu\zeta(t) + \xi_\mu(t), \quad t \in [0, T].$$

Now the variation-of-constants formula proves (3.18).

To prove (3.19), note, that formula (3.18), Lemma 3.6 and the fact that $0 \leq \theta_k \leq 1$ imply

$$C_3|\theta_k \cdot u(t)|_{L^2(\Omega)} \leq \zeta(t) \leq e^{-2\mu t}\zeta(0) + (1/2\mu)\big(2(C_1/k)R^2 + 2M_k(R)\big).$$

Since $\zeta(0) \leq C_2|u(0)|^2_{L^2(\Omega)} \leq C_2 R^2$, estimate (3.19) follows. □

Let us call the family $(f_\varepsilon)_{\varepsilon \in [0,1]}$ *tail admissible* if there is a $\mu \in]0, \infty[$ such that for every $k \in \mathbb{N}$ and every $R \in]0, \infty[$

$$M_k(R) = \sup_{\varepsilon \in [0,1],\, u \in H_1^\varepsilon,\, |u|_{H^1(\Omega)} \leq R} b_\varepsilon(\theta_k \cdot u, f_\varepsilon(u) + \mu u) \to 0, \text{ as } k \to \infty.$$

The following sufficient conditions for tail admissibility are due to Prizzi [9] in the context of reaction-diffusion equations on $\mathbb{R}^\ell$.

LEMMA 3.8. *Assume there are numbers $\mu \in]0, \infty[$, $q \in [2, \infty[$ and $p \in]1, \infty]$ and functions $c \in L^1(\Omega)$, $e \in L^p(\Omega)$ such that*

(a) *$s \cdot G(\varepsilon, \Gamma_\varepsilon(x), s) \leq -\mu|s|^2 + e(x)|s|^q + c(x)$ for all $\varepsilon \in [0, 1]$, $s \in \mathbb{R}$ and a.a. $x \in \Omega$.*
(b) *If $\ell = 2$ and $p = \infty$, then $|\theta_k \cdot e|_{L^\infty(\Omega)} \to 0$ as $k \to \infty$.*
(c) *For $\ell \geq 3$ either $q < 2\ell/(\ell - 2)$ and $2\ell/(2\ell - q(\ell - 2)) \leq p < \infty$ or else $q \leq 2\ell/(\ell - 2)$, $p = \infty$ and $|\theta_k \cdot e|_{L^\infty(\Omega)} \to 0$ as $k \to \infty$.*

Then $(f_\varepsilon)_{\varepsilon \in [0,1]}$ is tail admissible.

PROOF. Let $r = p/(p - 1)$ for $p < \infty$ and $r = 1$ for $p = \infty$. For $\varepsilon \in [0, 1]$, $k \in \mathbb{N}$ and $u \in H_1^\varepsilon$ we have, by Hölder inequality,

$$\begin{aligned}
b_\varepsilon(\theta_k \cdot u, f_\varepsilon(u) + \mu u) &= \int_\Omega J_\varepsilon(x)\theta_k(x)u(x)(G(\varepsilon, \Gamma_\varepsilon(x), u(x)) + \mu u(x))\, dx \\
&\leq \int_\Omega J_\varepsilon(x)\theta_k(x)(e(x)|u(x)|^q + c(x))\, dx \\
&\leq C_2(|\theta_k \cdot e|_{L^p(\Omega)}\, |u|^q_{L^{qr}(\Omega)} + |\theta_k \cdot c|_{L^1(\Omega)})
\end{aligned}$$

where, as before, $C_2 = \sup_{\varepsilon \in [0,1]} \sup_{x \in \mathcal{U}} J_\varepsilon(x) < \infty$. By our assumptions there is a bounded imbedding from $H^1(\Omega)$ to $L^{qr}(\Omega)$. Thus there is a constant $C_4 \in]0, \infty[$, such that for all $\varepsilon \in [0, 1]$, all $k \in \mathbb{N}$ and all $u \in H_1^\varepsilon$

$$b_\varepsilon(\theta_k \cdot u, f_\varepsilon(u) + \mu u) \leq C_2(C_4|\theta_k \cdot e|_{L^p(\Omega)}\, |u|^q_{H^1(\Omega)} + |\theta_k \cdot c|_{L^1(\Omega)}). \tag{3.20}$$

By the properties of ϑ and the dominated convergence theorem we obtain that $|\theta_k \cdot c|_{L^1(\Omega)} \to 0$ and $|\theta_k \cdot e|_{L^p(\Omega)} \to 0$ (for $p < \infty$) for $k \to \infty$. But by our assumption, the latter convergence also holds for $p = \infty$. Together with (3.20), this implies the lemma. $\square$

We can now state the following

Theorem 3.9. *If $(f_\varepsilon)_{\varepsilon\in[0,1]}$ is tail admissible, $\beta \in]0,\infty[$ and N is a closed bounded subset of H_1^0, then N is strongly admissible rel. to β and the family $(H^\varepsilon, \pi_\varepsilon)_{\varepsilon\in[0,\overline{\varepsilon}]}$.*

Proof. Since for each $\varepsilon \in [0,1]$ the map f_ε maps bounded subsets of H_1^ε to bounded subsets of H^ε, it follows that π_ε does not explode in $N_{\varepsilon,\beta}$. There is a bound $R \in]0,\infty[$ such that $|u|_{H^1(\Omega)} \le R$ for all $\varepsilon \in [0,1]$ and $u \in N_{\varepsilon,\beta}$.

Let $\varepsilon_0 \in [0,1]$, $(\varepsilon_n)_n$ be a sequence in $[0,\overline{\varepsilon}]$, $(t_n)_n$ be a sequence in $]0,\infty[$ and $(u_n)_n$ be a sequence such that

$$t_n \to \infty \text{ as } n \to \infty,$$
$$\text{either } \varepsilon_n = \varepsilon_0 \text{ for all } n \in \mathbb{N} \text{ or else } \varepsilon_0 = 0 \text{ and } \varepsilon_n \to \varepsilon_0 \text{ for } n \to \infty,$$

and, for each $n \in \mathbb{N}$, $u_n \in H_1^{\varepsilon_n}$ and $u_n\pi_{\varepsilon_n}[0,t_n] \subset N_{\varepsilon_n,\beta}$.

If $\varepsilon_n = \varepsilon_0$ for all $n \in \mathbb{N}$, then $(u_n\pi_{\varepsilon_0}t_n)_n$ is bounded in the Hilbert space $H_1^{\varepsilon_0}$, so there is a $v \in H_1^{\varepsilon_0} \subset H^1(\Omega)$ and a subsequence of $(u_n\pi_{\varepsilon_0}t_n)_n$, again denoted by $(u_n\pi_{\varepsilon_0}t_n)_n$, such that $(u_n\pi_{\varepsilon_0}t_n)_n$ converges weakly in $H_1^{\varepsilon_0}$, hence in $H^1(\Omega)$ to v. If $\varepsilon_0 = 0$ and $\varepsilon_n \to \varepsilon_0$ for $n \to \infty$, then $(u_n\pi_{\varepsilon_n}t_n)_n$ is bounded in the Hilbert space $H^1(\Omega)$, so there is a $v \in H^1(\Omega)$ and a subsequence of $(u_n\pi_{\varepsilon_n}t_n)_n$, again denoted by $(u_n\pi_{\varepsilon_n}t_n)_n$, such that $(u_n\pi_{\varepsilon_n}t_n)_n$ converges weakly in $H^1(\Omega)$ to v. Thus $(P(,\cdot)\nabla(u_n\pi_{\varepsilon_n}t_n))_n$ converges weakly in $L^2(\Omega,\mathbb{R}^\ell)$ to $P(\cdot)\nabla v$. From the definition of a_ε it follows that $|P(\cdot)\nabla(u_n\pi_{\varepsilon_n}t_n)|_{L^2(\Omega,\mathbb{R}^\ell)} \to 0$. Thus $(P(\cdot)\nabla(u_n\pi_{\varepsilon_n}t_n))_n$ converges to 0 strongly hence weakly in $L^2(\Omega,\mathbb{R}^\ell)$. This shows that $P(\cdot)\nabla v = 0$ almost everywhere in Ω, i.e. $v \in H_s^1(\Omega) = H_1^{\varepsilon_0}$.

The proof will be completed if we can show that there is a subsequence of $(u_n\pi_{\varepsilon_n}t_n)_n$, again denoted by $(u_n\pi_{\varepsilon_n}t_n)_n$, such that $|u_n\pi_{\varepsilon_n}t_n - v|_{H^{\varepsilon_n}} \to 0$ as $n \to \infty$, i.e. equivalently, that $|u_n\pi_{\varepsilon_n}t_n - v|_{L^2(\Omega)} \to 0$ as $n \to \infty$. By the uniqueness of weak limits we thus have to show that the sequence $(u_n\pi_{\varepsilon_n}t_n)_n$ is precompact in $L^2(\Omega)$, i.e. that $\alpha\big(\{\, u_n\pi_{\varepsilon_n}t_n \mid n \in \mathbb{N}\}\big) = 0$, where α is the Kuratowski measure of noncompactness on $L^2(\Omega)$. We have for every $k \in \mathbb{N}$

$$\begin{aligned}\alpha(\{u_n\pi_{\varepsilon_n}t_n \mid n \in \mathbb{N}\}) \le{}& \alpha(\{\theta_k\cdot(u_n\pi_{\varepsilon_n}t_n) \mid n \in \mathbb{N}\}) \\ &+ \alpha(\{(1-\theta_k)\cdot(u_n\pi_{\varepsilon_n}t_n) \mid n \in \mathbb{N}\}).\end{aligned}$$

Let $\delta \in]0,\infty[$ be arbitrary. Then (3.19) together with tail admissibility imply that there is a $k \in \mathbb{N}$ and an $n_0 \in \mathbb{N}$ such that $|\theta_k\cdot(u_n\pi_{\varepsilon_n}t_n)|_{L^2(\Omega)} < \delta$ for all $n \ge n_0$. Thus

$$\alpha(\{\theta_k\cdot(u_n\pi_{\varepsilon_n}t_n) \mid n \in \mathbb{N}\} \le 2\delta.$$

Now define V_k to be the set of all $x \in \Omega$ with $\|x\| < (q_0/M) + k\sqrt{2}$. Let $S = \{(1-\theta_k) \cdot (u_n \pi_{\varepsilon_n} t_n) \mid n \in \mathbb{N}\}$ and $(w_m)_m$ be any sequence in S. For each $m \in \mathbb{N}$ let $\widetilde{w}_m$ be the restriction of w_m to the open set V_k. Then $\widetilde{w}_m \in H^1(V_k)$ and $|\widetilde{w}_m|_{H^1(V_k)} \le R$. Since V_k is bounded, Rellich theorem implies that there is a subsequence $(\widetilde{w}_{m_\ell})_\ell$ of $(\widetilde{w}_m)_m$ converging in $L^2(V_k)$ to some $\widetilde{w} \in L^2(V_k)$. Our choice of ϑ implies that $w_m(x) = 0$, whenever $m \in \mathbb{N}$, $x \in \Omega$ and $x \notin V_k$. Thus $(w_{m_\ell})_\ell$ converges in $L^2(\Omega)$ to the function $w\colon \Omega \to \mathbb{R}$ defined by $w(x) = \widetilde{w}(x)$ if $x \in V_k$ and $w(x) = 0$ otherwise. Altogether we see that every sequence in S has a subsequence which converges in $L^2(\Omega)$. Thus $\alpha(S) = 0$ and so $\alpha(\{u_n \pi_{\varepsilon_n} t_n \mid n \in \mathbb{N}\}) \le 2\delta$. Since $\delta \in]0,\infty[$ is arbitrary, it follows that $\alpha(\{u_n \pi_{\varepsilon_n} t_n \mid n \in \mathbb{N}\}) = 0$. This completes the proof of the theorem. □

We thus obtain the following result.

Theorem 3.10. *For* $(H^\varepsilon, \langle \cdot\,, \cdot \rangle_{H^\varepsilon}, \mathbf{A}_\varepsilon)_{\varepsilon\in[0,1]}$ *and* $(f_\varepsilon)_{\varepsilon\in[0,1]}$ *defined in this section, if* $(f_\varepsilon)_{\varepsilon\in[0,1]}$ *is tail admissible, then the assumptions and hence the conclusions of the Conley index continuation results Theorems* 2.17 *and* 2.18 *hold* (*together with the corresponding remarks*).

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