EXISTENCE AND GLOBAL ATTRACTIVITY
OF THE UNIQUE POSITIVE PERIODIC SOLUTION
FOR DISCRETE HEMATOPOIESIS MODEL

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ABSTRACT. In this paper, a discrete Hematopoiesis model is studied. By using fixed point theorem of decreasing operator, we obtain sufficient conditions for the existence of unique positive periodic solution. Particularly,we give iterative sequence which converges to the positive periodic solution. In addition, the global attractivity of positive periodic solution is also investigated.

1. Introduction

Biological dynamic models are very important and hot research topics. In 1977, Mackey and Glass [14] investigated the Hematopoiesis model

$$(1.1) \qquad x'(t) = -ax(t) + \frac{\beta}{1 + x^n(t - \tau)}$$

which described the production of blood cells. Gyori and Ladas [7] have investigated the global attractivity of positive equilibrium for the above model (1.1). Moreover, the model (1.1) and some generalized models have been investigated by many authors, see [3], [5], [9]–[11], [15]–[17].

The assumption that the environment is constant is rarely the case in real life. When the environmental fluctuation is taken into account, a model must be nonautonomous. Due to the various seasonal effects of the environmental factors in real life situation (e.g. seasonal effects of weather, food supplies, mating habits, harvesting, etc.), the effects of periodically varying environment are important, it is rational and practical to study the biological system with periodic coefficients. Many authors [17], [16], [9] have studied nonautonomous differential equations with periodic coefficients of the above model (1.1).

Though most models are described with differential equations, the discrete-time models are more appropriate than the continuous ones when the size of the population is rarely small or the population has non-overlapping generations [1]. However, the studies in the past literature [7], [5], [11], [15], [10], [3], [17], [16], [9] were concerned with the continuous case of the above model (1.1). To our knowledge, studies on the uniqueness and global attractivity of positive periodic solution for discrete models are scarce.

Motivated by the above facts, in this paper, we investigate the following Hematopoiesis difference equation

$$(1.2) \qquad \Delta x(k) = -a(k)x(k) + \frac{b(k)}{1 + x^n(k - \tau(k))}$$

where

$$\Delta x(k) = x(k+1) - x(k), \quad n > 0,$$

$$a(k): Z \to (0,1), \quad b(k): Z \to (0,+\infty), \quad \tau(k): Z \to Z^+.$$

$a(k)$, $b(k)$, $\tau(k)$ are ω-periodic functions, i.e.

$$a(k) = a(k+\omega), \quad b(k) = b(k+\omega), \quad \tau(k) = \tau(k+\omega), \quad \omega \in Z^+.$$

Let $\tau^* = \max_{0 \le k \le \omega} \tau(k)$, the initial condition of equation (1.2) is $x(k) = \phi(k) > 0$ for $-\tau^* \le k \le 0$.

In the study of biological systems, an important ecological problem concerns the existence of positive periodic solutions. Recently, many authors investigated the existence of positive periodic solution by using Krasnosel'skiĭ cone fixed point theorem and Mawhin's coincidence degree theory [2], [4], [8], [9], [12], [19]. Most of the literature concerned the existence of at least one positive periodic solution [2], [4], [8], [9], [12], [13], [19].

Few papers study the existence and global attractivity of unique positive periodic solution for discrete models. For the existence and uniqueness of positive periodic solution, the method used in most of the past literature is contraction mapping fixed point theorem and Liapunov functionals.

In this paper, different from the past literature, we aim to obtain sufficient conditions that guarantee the existence of unique positive periodic solution of

discrete model (1.2) by using fixed point theorem of decreasing operator. Particularly, we give iterative sequence which converges to the positive periodic solution. We also obtain sufficient conditions for the global attractivity of unique positive periodic solution by means of some analysis tools.

2. Preliminaries

DEFINITION 2.1. Let X be a Banach space and P be a closed nonempty subset of X, P is called a *cone* if

(a) $x \in P$, $\lambda \geq 0$ implies $\lambda x \in P$;
(b) $x \in P$, $-x \in P$ implies $x = 0$.

Every cone $P \subset X$ induces an ordering in X, we define "$\leq$"with respect to P by $x \leq y$ if and only if $y - x \in P$.

DEFINITION 2.2. A cone P of X is called *normal cone* if there exists a positive constant σ, such that $||x + y|| \geq \sigma$ for any $x, y \in P$, $||x|| = ||y|| = 1$.

DEFINITION 2.3. Let P be a cone of X and $A \colon P \to P$ an operator. A is called *decreasing* if $0 \leq x \leq y$ implies $Ax \geq Ay$.

The following fixed point theorem of decreasing operator (see [6]) is an important tool in our proofs.

LEMMA 2.4 ([6]). *Suppose that*:

(a) *P is normal cone of Banach space X, operator $A \colon P \to P$ is decreasing*;
(b) *$A(0) > 0$, $A^2(0) \geq \varepsilon_0 A(0)$, where $\varepsilon_0 > 0$*;
(c) *For any $0 < a < c < 1$, there exists $\eta = \eta(a, c) > 0$ such that $A(\lambda x) \leq [\lambda(1 + \eta)]^{-1} Ax$ for any $a \leq \lambda \leq c$ and $0 < x \leq A(0)$.*

Then A has a unique positive fixed point $x^* > 0$. Moreover, $||x_k - x^*|| \to 0$, $(k \to \infty)$, where $x_k = Ax_{k-1}$, $(k = 1, 2, \ldots)$ for any initial $x_0 \in P$.

In this paper, we will use the above Lemma 2.4 to investigate the existence of unique positive periodic solution of model (1.2).

REMARK 2.5. In Lemma 2.4, the operator A needs neither continuity nor compactness.

Let $X = \{x(k) \mid x(k) = x(k + \omega)\}$, for $x \in X$, we define $||x|| = \max\limits_{k \in Z, 0 \leq k \leq \omega} |x(k)|$, then X is Banach space.

Let $\bar{a} = \max\limits_{0 \leq k \leq \omega - 1} a(k)$, $\underline{a} = \min\limits_{0 \leq k \leq \omega - 1} a(k)$, $\bar{b} = \max\limits_{0 \leq k \leq \omega - 1} b(k)$,

$$m = \frac{\prod\limits_{r=0}^{\omega-1}(1-a(r))}{1-\prod\limits_{r=0}^{\omega-1}(1-a(r))}, \quad M = \frac{1}{1-\prod\limits_{r=0}^{\omega-1}(1-a(r))}, \quad G(k,s) = \frac{\prod\limits_{r=s+1}^{k+\omega-1}(1-a(r))}{1-\prod\limits_{r=0}^{\omega-1}(1-a(r))},$$

for $k \leq s \leq k+\omega-1$. Then we know $G(k,s)$ is ω-periodic in both variables, and

$$m = \frac{\prod\limits_{r=0}^{\omega-1}(1-a(r))}{1-\prod\limits_{r=0}^{\omega-1}(1-a(r))} \leq G(k,s) \leq \frac{1}{1-\prod\limits_{r=0}^{\omega-1}(1-a(r))} = M.$$

LEMMA 2.6. $x(k)$ *is the ω-periodic solution of equation* (1.2) *if and only if* $x(k)$ *is the ω-periodic solution of the following equation*

$$x(k) = \sum_{s=k}^{k+\omega-1}\left(G(k,s)\frac{b(s)}{1+x^n(s-\tau(s))}\right).$$

PROOF. It is easy to see that equation (1.2) is equivalent to

$$(2.1) \qquad x(k+1) - (1-a(k))x(k) = \frac{b(k)}{1+x^n(k-\tau(k))}$$

Multiplying two sides of (2.1) by $\prod\limits_{r=0}^{k}1/(1-a(r))$, we get

$$(2.2) \qquad \Delta\left(x(k)\prod_{r=0}^{k-1}\frac{1}{1-a(r)}\right) = \frac{b(k)}{1+x^n(k-\tau(k))}\prod_{r=0}^{k}\frac{1}{1-a(r)}$$

Summing two sides of (2.2) from k to $k+\omega-1$, we get

$$x(k+\omega)\prod_{r=0}^{k+\omega-1}\frac{1}{1-a(r)} - x(k)\prod_{r=0}^{k-1}\frac{1}{1-a(r)}$$
$$= \sum_{s=k}^{k+\omega-1}\left(\frac{b(s)}{1+x^n(s-\tau(s))}\prod_{r=0}^{s}\frac{1}{1-a(r)}\right)$$

Since $x(k+\omega) = x(k)$, we have

$$x(k)\left(\prod_{r=0}^{k+\omega-1}\frac{1}{1-a(r)} - \prod_{r=0}^{k-1}\frac{1}{1-a(r)}\right) = \sum_{s=k}^{k+\omega-1}\left(\frac{b(s)}{1+x^n(s-\tau(s))}\prod_{r=0}^{s}\frac{1}{1-a(r)}\right)$$

That is

$$x(k)\prod_{r=0}^{k-1}\frac{1}{1-a(r)}\left(\prod_{r=k}^{k+\omega-1}\frac{1}{1-a(r)} - 1\right)$$
$$= \sum_{s=k}^{k+\omega-1}\left(\frac{b(s)}{1+x^n(s-\tau(s))}\prod_{r=0}^{s}\frac{1}{1-a(r)}\right)$$

Hence we get

$$x(k) = \frac{\displaystyle\prod_{r=0}^{k+\omega-1}(1-a(r))}{1-\displaystyle\prod_{r=k}^{k+\omega-1}(1-a(r))}\sum_{s=k}^{k+\omega-1}\left(\frac{b(s)}{1+x^n(s-\tau(s))}\prod_{r=0}^{s}\frac{1}{1-a(r)}\right)$$

$$= \frac{\displaystyle\prod_{r=0}^{k+\omega-1}(1-a(r))}{1-\displaystyle\prod_{r=0}^{\omega-1}(1-a(r))}\sum_{s=k}^{k+\omega-1}\left(\frac{b(s)}{1+x^n(s-\tau(s))}\prod_{r=0}^{s}\frac{1}{1-a(r)}\right)$$

$$= \sum_{s=k}^{k+\omega-1}\left(\frac{\displaystyle\prod_{r=0}^{k+\omega-1}(1-a(r))}{1-\displaystyle\prod_{r=0}^{\omega-1}(1-a(r))}\frac{b(s)}{1+x^n(s-\tau(s))}\prod_{r=0}^{s}\frac{1}{1-a(r)}\right)$$

$$= \sum_{s=k}^{k+\omega-1}\left(\frac{\displaystyle\prod_{r=s+1}^{k+\omega-1}(1-a(r))}{1-\displaystyle\prod_{r=0}^{\omega-1}(1-a(r))}\frac{b(s)}{1+x^n(s-\tau(s))}\right)$$

$$= \sum_{s=k}^{k+\omega-1}\left(G(k,s)\frac{b(s)}{1+x^n(s-\tau(s))}\right).$$

The proof is complete. $\square$

We define operator $A\colon X \to X$,

$$(Ax)(k) = \sum_{s=k}^{k+\omega-1}\left(G(k,s)\frac{b(s)}{1+x^n(s-\tau(s))}\right).$$

Obviously, $x(k) \in X$ is the ω-periodic solution of equation (1.2) if and only if x is the fixed point of operator A.

Define cone $\Omega = \{x \mid x \in X,\ x(k) \geq 0,\ x(k) \geq \delta\|x\|\}$, $\delta = m/M$.

LEMMA 2.7. $A\Omega \subset \Omega$.

PROOF. For any $x \in \Omega$,

$$\|Ax\| = \max_{k\in Z,\,0\leq k\leq\omega}|(Ax)(k)| = \max_{k\in Z,\,0\leq k\leq\omega}\left|\sum_{s=k}^{k+\omega-1}\left(G(k,s)\frac{b(s)}{1+x^n(s-\tau(s))}\right)\right|$$

$$\leq M\max_{k\in Z,\,0\leq k\leq\omega}\left|\sum_{s=k}^{k+\omega-1}\left(\frac{b(s)}{1+x^n(s-\tau(s))}\right)\right|$$

$$= M \max_{k \in Z,\, 0 \le k \le \omega} \left| \sum_{s=0}^{\omega-1} \left(\frac{b(s)}{1 + x^n(s - \tau(s))} \right) \right| = M \sum_{s=0}^{\omega-1} \left(\frac{b(s)}{1 + x^n(s - \tau(s))} \right)$$

On the other hand,

$$(Ax)(k) = \sum_{s=k}^{k+\omega-1} \left(G(k,s) \frac{b(s)}{1 + x^n(s - \tau(s))} \right)$$

$$\ge m \sum_{s=k}^{k+\omega-1} \left(\frac{b(s)}{1 + x^n(s - \tau(s))} \right)$$

$$= m \sum_{s=0}^{\omega-1} \left(\frac{b(s)}{1 + x^n(s - \tau(s))} \right) \ge m \frac{1}{M} \|Ax\| = \delta \|Ax\|$$

which implies $Ax \in \Omega$, hence we have $A\Omega \subset \Omega$. $\square$

LEMMA 2.8. *Every solution $x(k)$ of equation (1.2) is positive.*

PROOF. Let $x(k)$ be the solution of equation (1.2), then we have

$$x(k+1) - [1 - a(k)]x(k) = \frac{b(k)}{1 + x^n(k - \tau(k))}.$$

Hence we get

(E$_1$) $x(k) - [1 - a(k-1)]x(k-1) = \dfrac{b(k-1)}{1 + x^n(k - 1 - \tau(k-1))},$

(E$_2$) $x(k-1) - [1 - a(k-2)]x(k-2) = \dfrac{b(k-2)}{1 + x^n(k - 2 - \tau(k-2))},$

(E$_3$) $x(k-2) - [1 - a(k-3)]x(k-3) = \dfrac{b(k-3)}{1 + x^n(k - 3 - \tau(k-3))},$
....

(E$_{k-1}$) $x(2) - [1 - a(1)]x(1) = \dfrac{b(1)}{1 + x^n(1 - \tau(1))},$

(E$_k$) $x(1) - [1 - a(0)]x(0) = \dfrac{b(0)}{1 + x^n(0 - \tau(0))}.$

Multiplying two sides of (E$_2$), …, (E$_k$) by $1 - a(k-1)$, $[1 - a(k-1)][1 - a(k-2)]$, …, $[1 - a(k-1)][1 - a(k-2)] \ldots [1 - a(2)]$, $[1 - a(k-1)][1 - a(k-2)] \ldots [1 - a(2)][1 - a(1)]$, respectively, we get

(E$_2'$) $[1 - a(k-1)]x(k-1) - [1 - a(k-1)][1 - a(k-2)]x(k-2)$

$$= \frac{b(k-2)}{1 + x^n(k - 2 - \tau(k-2))}[1 - a(k-1)],$$

(E$'_3$) $[1 - a(k - 1)][1 - a(k - 2)]x(k - 2)$

$$- [1 - a(k - 1)][1 - a(k - 2)][1 - a(k - 3)]x(k - 3)$$

$$= \frac{b(k - 3)}{1 + x^n(k - 3 - \tau(k - 3))}[1 - a(k - 1)][1 - a(k - 2)],$$

....

(E$'_{k-1}$) $[1 - a(k - 1)][1 - a(k - 2)]\ldots[1 - a(2)]x(2)$

$$- [1 - a(k - 1)][1 - a(k - 2)]\ldots[1 - a(2)][1 - a(1)]x(1)$$

$$= \frac{b(1)}{1 + x^n(1 - \tau(1))}[1 - a(k - 1)][1 - a(k - 2)]\ldots[1 - a(2)],$$

(E$'_k$) $[1 - a(k - 1)][1 - a(k - 2)]\ldots[1 - a(2)][1 - a(1)]x(1)$

$$- [1 - a(k - 1)][1 - a(k - 2)]\ldots[1 - a(2)][1 - a(1)][1 - a(0)]x(0)$$

$$= \frac{b(0)}{1 + x^n(0 - \tau(0))}[1 - a(k - 1)][1 - a(k - 2)]\ldots[1 - a(2)][1 - a(1)].$$

Summing (E$_1$), (E$'_2$), (E$'_3$), ..., (E$'_{k-1}$), (E$'_k$), we get

$$x(k) - [1 - a(k - 1)][1 - a(k - 2)]\ldots[1 - a(2)][1 - a(1)][1 - a(0)]x(0)$$

$$= \frac{b(k - 1)}{1 + x^n(k - 1 - \tau(k - 1))} + \sum_{s=0}^{k-2}\left(\frac{b(s)}{1 + x^n(s - \tau(s))}\prod_{i=s+1}^{k-1}(1 - a(i))\right).$$

That is

$$x(k) = x(0)\prod_{s=0}^{k-1}(1 - a(s)) + \frac{b(k - 1)}{1 + x^n(k - 1 - \tau(k - 1))}$$

$$+ \sum_{s=0}^{k-2}\left(\frac{b(s)}{1 + x^n(s - \tau(s))}\prod_{i=s+1}^{k-1}(1 - a(i))\right).$$

Since $x(k) = \phi(k) > 0$ for $-\tau^* \leq k \leq 0$, we can deduce that $x(1) > 0$, ... $x(k) > 0$ for all $k \in Z^+$. $\qquad\square$

LEMMA 2.9. *Every solution $x(k)$ of equation* (1.2) *is bounded.*

PROOF. Suppose that equation (1.2) has unbounded solution $x(k)$. Then, for sufficiently large $L > 0$, there exists $K_2 \in Z^+$ and $x(K_2)$ sufficiently large, such that $-\underline{a}x(K_2) + \overline{b} < -L$. Thus we have

$$x(K_2 + 1) - x(K_2) = - a(K_2)x(K_2) + \frac{b(K_2)}{1 + x^n(K_2 - \tau(K_2))}$$

$$\leq - a(K_2)x(K_2) + b(K_2) \leq -\underline{a}x(K_2) + \overline{b} < -L < 0$$

which implies $x(K_2 + 1) < x(K_2)$.

Let $\mu = x(K_2+1)$, $K_1 = \max\{k | x(k) < \mu, \ k < K_2\}$, then we have $x(K_1) < \mu$ and $x(K_1 + 1) \geq \mu$. Thus there exists $K_3 \in Z^+$ and $K_1 + 1 \leq K_3 \leq K_2$, such that $x(K_3) = \max_{K_1+1 \leq k \leq K_2} x(k)$. Hence

$$(2.3) \qquad\qquad x(K_3 - 1) \leq x(K_3).$$

(2.3) can be divided into two cases:

Case 1. If $x(K_2) < x(K_3 - 1) \leq x(K_3)$, then

$$x(K_3) - x(K_3 - 1) = -a(K_3 - 1)x(K_3 - 1) + \frac{b(K_3 - 1)}{1 + x^n(K_3 - 1 - \tau(K_3 - 1))}$$

$$\leq -\underline{a}x(K_3 - 1) + \overline{b} < -\underline{a}x(K_2) + \overline{b} < -L < 0.$$

Thus $x(K_3) < x(K_3 - 1)$, which contradicts with (2.3).

Case 2. If $x(K_3 - 1) \leq x(K_2) \leq x(K_3)$, then

$$x(K_3) - x(K_3 - 1) = -a(K_3 - 1)x(K_3 - 1) + \frac{b(K_3 - 1)}{1 + x^n(K_3 - 1 - \tau(K_3 - 1))}$$

$$\leq -\underline{a}x(K_3 - 1) + \overline{b} < -\underline{a}x(K_3 - 1) + \underline{a}x(K_2) - L$$

$$\leq -\underline{a}x(K_3 - 1) + \underline{a}x(K_3) - L = \underline{a}[x(K_3) - x(K_3 - 1)] - L,$$

Hence we get

$$(1 - \underline{a})[x(K_3) - x(K_3 - 1)] < -L,$$

which yields

$$x(K_3) - x(K_3 - 1) < -\frac{L}{1 - \underline{a}} < 0.$$

Thus $x(K_3) < x(K_3 - 1)$, which contradicts with (2.3). Therefore, every solution of equation (1.2) is bounded. $\qquad\square$

3. Existence of the unique positive periodic solution

Let $\widetilde{b} = \sum\limits_{k=0}^{\omega-1} b(k)$, we make the following assumptions:

(H_1) $0 < n \leq 1$,

(H_2) $n > 1$, $(n-1)(M\widetilde{b})^n \leq 1$.

THEOREM 3.1. *Assume that* (H_1) *or* (H_2) *holds. Then equation* (1.2) *has a unique ω-periodic positive solution $x^*(k)$. Moreover, $\|x_k - x^*\| \to 0$, $(k \to \infty)$, where $x_k = Ax_{k-1}$ $(k = 1, 2, \ldots)$ for any initial $x_0 \in \Omega$.*

PROOF. Firstly, we prove that Ω is normal cone, $A \colon \Omega \to \Omega$ is decreasing operator.

For any $x, y \in \Omega$, $\|x\| = \|y\| = 1$, we have $x(k) \geq 0$, $y(k) \geq 0$, $x(k) \geq \delta\|x\| = \delta$, $y(k) \geq \delta\|y\| = \delta$, here $\delta = m/M$. Hence we get

$$\|x + y\| \geq x(k) + y(k) \geq \delta + \delta = 2\delta.$$

Thus there exists a positive constant $\sigma = 2\delta$, such that $||x + y|| \geq \sigma$ for any $x, y \in \Omega$, $||x|| = ||y|| = 1$. So Ω is normal cone.

For $x, y \in \Omega$, $x \leq y$, then we have

$$(Ax)(k) - (Ay)(k) = \sum_{s=k}^{k+\omega-1} G(k,s) \left(\frac{b(s)}{1 + x^n(s - \tau(s))} - \frac{b(s)}{1 + y^n(s - \tau(s))} \right) \geq 0,$$

which implies $Ax \geq Ay$. So $A \colon \Omega \to \Omega$ is decreasing operator.

Now, we will show that condition (b) of Lemma 2.4 is satisfied.

$$M\widetilde{b} = M \sum_{s=0}^{\omega-1} b(s) = M \sum_{s=k}^{k+\omega-1} b(s) \geq (A(0))(k)$$

$$= \sum_{s=k}^{k+\omega-1} (G(k,s)b(s)) \geq m \sum_{s=k}^{k+\omega-1} b(s) = m \sum_{s=0}^{\omega-1} b(s) = m\widetilde{b} > 0$$

which implies $A(0) > 0$. Again, we have

$$(A^2(0))(k) = \sum_{s=k}^{k+\omega-1} \left(G(k,s) \frac{b(s)}{1 + (A(0))^n(s - \tau(s))} \right)$$

$$\geq \frac{1}{1 + (M\widetilde{b})^n} \sum_{s=k}^{k+\omega-1} (G(k,s)b(s)) = \frac{1}{1 + (M\widetilde{b})^n} (A(0))(k) = \varepsilon_0 (A(0))(k).$$

This implies $A^2(0) \geq \varepsilon_0 A(0)$, here $\varepsilon_0 = 1/(1 + (M\widetilde{b})^n)$.

Finally, we show that condition (c) of Lemma 2.4 is satisfied.

Let for any $0 < a < c < 1$, for any $a \leq \lambda \leq c$ and $0 < x \leq A(0)$, we have $0 < ||x|| \leq ||A(0)|| \leq M\widetilde{b}$.

$$A(\lambda x)(k) = \sum_{s=k}^{k+\omega-1} \left(G(k,s) \frac{b(s)}{1 + \lambda^n x^n(s - \tau(s))} \right)$$

$$= \sum_{s=k}^{k+\omega-1} \left(G(k,s) \frac{b(s)}{1 + x^n(s - \tau(s))} \cdot \frac{1 + x^n(s - \tau(s))}{1 + \lambda^n x^n(s - \tau(s))} \right).$$

Notice that

$$\frac{1 + x^n(s - \tau(s))}{1 + \lambda^n x^n(s - \tau(s))} = \lambda^{-n} \left(1 + \frac{\lambda^n - 1}{1 + \lambda^n x^n(s - \tau(s))} \right)$$

$$\leq \lambda^{-n} \left(1 + \frac{\lambda^n - 1}{1 + \lambda^n (M\widetilde{b})^n} \right) = \frac{1 + (M\widetilde{b})^n}{1 + \lambda^n (M\widetilde{b})^n}.$$

So we have

$$A(\lambda x)(k) \leq \sum_{s=k}^{k+\omega-1} \left(G(k,s) \frac{b(s)}{1 + x^n(s - \tau(s))} \cdot \frac{1 + (M\widetilde{b})^n}{1 + \lambda^n (M\widetilde{b})^n} \right)$$

$$= \frac{1}{\lambda} \sum_{s=k}^{k+\omega-1} \left(G(k,s) \frac{b(s)}{1 + x^n(s - \tau(s))} \cdot \frac{\lambda[1 + (M\widetilde{b})^n]}{1 + \lambda^n (M\widetilde{b})^n} \right).$$

Let

$$f(x) = \frac{[1 + (M\widetilde{b})^n]x}{1 + x^n(M\widetilde{b})^n}.$$

We have

$$f'(x) = \frac{[1 + (M\widetilde{b})^n][1 + (1 - n)x^n(M\widetilde{b})^n]}{[1 + x^n(M\widetilde{b})^n]^2}.$$

Since (H_1) or (H_2) holds, we know that $f'(x) > 0$ for $0 < x < 1$, so we have $0 = f(0) < f(a) \le f(\lambda) \le f(c) < f(1) = 1$. Hence we have

$$A(\lambda x)(k) \le \frac{1}{\lambda} \sum_{s=k}^{k+\omega-1} \left(G(k, s) \frac{b(s)}{1 + x^n(s - \tau(s))} f(\lambda) \right)$$

$$\le \frac{1}{\lambda} \sum_{s=k}^{k+\omega-1} \left(G(k, s) \frac{b(s)}{1 + x^n(s - \tau(s))} f(c) \right)$$

$$= \frac{1}{\lambda} f(c) \sum_{s=k}^{k+\omega-1} \left(G(k, s) \frac{b(s)}{1 + x^n(s - \tau(s))} \right)$$

$$= \frac{1}{\lambda} f(c)(Ax)(k) = \frac{1}{\lambda} \cdot \frac{1}{1 + \eta(c)}(Ax)(k),$$

here $\eta = \eta(c) = 1/f(c) - 1 > 0$.

By Lemma 2.4, we know operator A has a unique positive fixed point $x^* > 0$, $\|x_k - x^*\| \to 0$, $(k \to \infty)$, $x_k = Ax_{k-1}$ $(k = 1, 2, \ldots)$ for any initial $x_0 \in \Omega$. $\square$

REMARK 3.2. The Theorem 3.1 of this paper not only gives sufficient conditions for existence of unique positive periodic solution, but also gives iterative sequence $\{x_k\}$ which converges to x^*.

REMARK 3.3. From the above proof, we have

$$x^*(k) = (Ax^*)(k) = \sum_{s=k}^{k+\omega-1} \left(G(k, s) \frac{b(s)}{1 + x^{*n}(s - \tau(s))} \right)$$

$$\le M \sum_{s=k}^{k+\omega-1} b(s) = M \sum_{s=0}^{\omega-1} b(s) = M\widetilde{b}.$$

We also have

$$x^*(k) = (Ax^*)(k) = \sum_{s=k}^{k+\omega-1} \left(G(k, s) \frac{b(s)}{1 + x^{*n}(s - \tau(s))} \right)$$

$$\ge m \sum_{s=k}^{k+\omega-1} \left(\frac{b(s)}{1 + (M\widetilde{b})^n} \right) = \frac{m}{1 + (M\widetilde{b})^n} \sum_{s=k}^{k+\omega-1} b(s)$$

$$= \frac{m}{1 + (M\widetilde{b})^n} \sum_{s=0}^{\omega-1} b(s) = \frac{m\widetilde{b}}{1 + (M\widetilde{b})^n}.$$

So we get $m\widetilde{b}/(1 + (M\widetilde{b})^n) \le x^*(k) \le M\widetilde{b}$.

4. Global attractivity of positive periodic solution

In this section, we study the global attractivity of positive periodic solution.

THEOREM 4.1. *Assume that* (H$_2$) *holds, and* $(n-1)^{n-1}(M\widetilde{b})^n \leq 1$, *then equation* (1.2) *has a unique globally attractive ω-periodic positive solution.*

PROOF. Since (H$_2$) holds, by Theorem 3.1 we know equation (1.2) has a unique ω-periodic positive solution $x^*(k)$. Now we prove that $x^*(k)$ is globally attractive.

Suppose $x(k)$ is arbitrary solution of equation (1.2). Let $y(k) = x(k) - x^*(k)$, then we get

$$(4.1) \quad \Delta y(k) = \Delta x(k) - \Delta x^*(k)$$

$$= -a(k)y(k) + \frac{b(k)}{1 + x^{*n}(k - \tau(k))}\left[\frac{1 + x^{*n}(k - \tau(k))}{1 + x^n(k - \tau(k))} - 1\right].$$

Now we prove $\lim_{k\to+\infty} y(k) = 0$ in three cases.

Case 1. Suppose $y(k)$ is eventually positive, then we have $y(k) = x(k) - x^*(k) > 0$ for k sufficiently large. This combined with (4.1) leads to $\Delta y(k) < -a(k)y(k) < 0$. Thus we have $y(k+1) < y(k)$, which together with $y(k) > 0$ implies $\lim_{k\to+\infty} y(k) = h \geq 0$.

We claim that $h = 0$. If $h > 0$, then we have $\Delta y(k) < -a(k)y(k) < -ha(k)$ for k sufficiently large, i.e. there exists $N_1 > 0$, such that $\Delta y(k) < -ha(k)$ for all $k \geq N_1$.

Summing two sides of the inequality from N_1 to $+\infty$, we get

$$h - y(N_1) = \sum_{k=N_1}^{+\infty} \Delta y(k) < -h \sum_{k=N_1}^{+\infty} a(k) = -\infty$$

which is a contradiction, so we have $h = 0$.

Case 2. Suppose $y(k)$ is eventually negative. By the similar proof as above, we also can get $h = 0$.

Case 3. Suppose $y(k)$ is oscillatory. By Lemma 2.9, we know $x(k)$ and $x^*(k)$ are all bounded. So $y(k)$ is bounded.

Let $\lim_{k\to+\infty} \sup y(k) = p \geq 0$, $\lim_{k\to+\infty} \inf y(k) = q \leq 0$.

For for any $\varepsilon > 0$ sufficiently small, there exists $N_2 > 0$, such that

$$(4.2) \qquad\qquad q - \varepsilon < y(k) < p + \varepsilon \quad \text{for all } k \geq N_2.$$

From (4.1) we have

$$y(k+1) - (1 - a(k))y(k) = \frac{b(k)}{1 + x^{*n}(k - \tau(k))}\left[\frac{1 + x^{*n}(k - \tau(k))}{1 + x^n(k - \tau(k))} - 1\right].$$

Multiplying the two sides by $\prod_{s=0}^{k} 1/(1 - a(s))$, we get

$$(4.3) \quad \Delta\left(y(k)\prod_{s=0}^{k-1}\frac{1}{1-a(s)}\right)$$

$$= \frac{b(k)}{1+x^{*n}(k-\tau(k))}\left[\frac{1+x^{*n}(k-\tau(k))}{1+x^n(k-\tau(k))}-1\right]\prod_{s=0}^{k}\frac{1}{1-a(s)}$$

$$= \frac{b(k)}{1+x^{*n}(k-\tau(k))}\left[\exp\left(\ln\left[\frac{1+x^{*n}(k-\tau(k))}{1+x^n(k-\tau(k))}\right]\right)-1\right]\prod_{s=0}^{k}\frac{1}{1-a(s)}.$$

By mean value theorem, we have

$$(4.4) \quad \ln\left[\frac{1+x^{*n}(k-\tau(k))}{1+x^n(k-\tau(k))}\right]$$

$$= \ln(1+x^{*n}(k-\tau(k)))-\ln(1+x^n(k-\tau(k)))$$

$$= \frac{n\xi^{n-1}}{1+\xi^n}(x^*(k-\tau(k))-x(k-\tau(k)))=-\frac{n\xi^{n-1}}{1+\xi^n}y(k-\tau(k))$$

where ξ lies between $x^*(k-\tau(k))$ and $x(k-\tau(k))$.

Notice that the function $g(x) = nx^{n-1}/(1+x^n)$ is increasing on $[0, \sqrt[n]{n-1}]$ and decreasing on $[\sqrt[n]{n-1}, +\infty)$. So $g(x) = nx^{n-1}/(1+x^n)$ has maximum $(n-1)^{1-1/n}$ on $[0, +\infty)$. Hence we have $n\xi^{n-1}/(1+\xi^n) \leq (n-1)^{1-1/n}$, which together with (4.2) and (4.4) yields

$$(4.5) \quad \ln\left[\frac{1+x^{*n}(k-\tau(k))}{1+x^n(k-\tau(k))}\right] \leq -(q-\varepsilon)(n-1)^{1-1/n},$$

in which $q - \varepsilon < 0$. Thus, by (4.3) and (4.5), we get

$$(4.6) \quad \Delta\left(y(k)\prod_{s=0}^{k-1}\frac{1}{1-a(s)}\right)$$

$$\leq [\exp(-(q-\varepsilon)(n-1)^{1-1/n})-1]\frac{b(k)}{1+x^{*n}(k-\tau(k))}\prod_{s=0}^{k}\frac{1}{1-a(s)}$$

$$= [\exp(-(q-\varepsilon)(n-1)^{1-1/n})-1]\Delta\left(x^*(k)\prod_{s=0}^{k-1}\frac{1}{1-a(s)}\right),$$

Summing two sides of (4.6) from N_2 to k, $(k > N_2)$, we get

$$y(k+1)\prod_{s=0}^{k}\frac{1}{1-a(s)}-y(N_2)\prod_{s=0}^{N_2-1}\frac{1}{1-a(s)}$$

$$\leq [\exp(-(q-\varepsilon)(n-1)^{1-1/n})-1]$$

$$\cdot\left(x^*(k+1)\prod_{s=0}^{k}\frac{1}{1-a(s)}-x^*(N_2)\prod_{s=0}^{N_2-1}\frac{1}{1-a(s)}\right).$$

This leads to

$$y(k+1) \leq y(N_2) \prod_{s=N_2}^{k} (1-a(s)) + [\exp(-(q-\varepsilon)(n-1)^{1-1/n}) - 1]$$

$$\cdot \left(x^*(k+1) - x^*(N_2) \prod_{s=N_2}^{k} (1-a(s)) \right)$$

$$\leq y(N_2) \prod_{s=N_2}^{k} (1-a(s)) + [\exp(-(q-\varepsilon)(n-1)^{1-1/n}) - 1]$$

$$\cdot \left(M\widetilde{b} - x^*(N_2) \prod_{s=N_2}^{k} (1-a(s)) \right).$$

Let $k \to +\infty$, we get $p \leq M\widetilde{b}[\exp(-(q-\varepsilon)(n-1)^{1-1/n}) - 1]$. As ε is arbitrary, we have

$$(4.7) \qquad\qquad p \leq M\widetilde{b}[\exp(-q(n-1)^{1-1/n}) - 1].$$

Now we prove that the inequality $q \geq M\widetilde{b}[\exp(-p(n-1)^{1-1/n}) - 1]$ holds too. By similar method as above, from $n\xi^{n-1}/(1+\xi^n) \leq (n-1)^{1-1/n}$ and (4.2), (4.4), we get

$$(4.8) \qquad\qquad \ln\left[\frac{1 + x^{*n}(k - \tau(k))}{1 + x^n(k - \tau(k))} \right] \geq -(p+\varepsilon)(n-1)^{1-1/n},$$

Hence, by (4.3) and (4.8), we obtain

$$(4.9) \qquad \Delta\left(y(k) \prod_{s=0}^{k-1} \frac{1}{1-a(s)} \right)$$

$$\geq [\exp(-(p+\varepsilon)(n-1)^{1-1/n}) - 1] \frac{b(k)}{1 + x^{*n}(k-\tau(k))} \prod_{s=0}^{k} \frac{1}{1-a(s)}$$

$$= [\exp(-(p+\varepsilon)(n-1)^{1-1/n}) - 1] \Delta\left(x^*(k) \prod_{s=0}^{k-1} \frac{1}{1-a(s)} \right).$$

Summing two sides of (4.9) from N_2 to k, $(k > N_2)$, we get

$$y(k+1) \prod_{s=0}^{k} \frac{1}{1-a(s)} - y(N_2) \prod_{s=0}^{N_2-1} \frac{1}{1-a(s)}$$

$$\geq [\exp(-(p+\varepsilon)(n-1)^{1-1/n}) - 1]$$

$$\cdot \left(x^*(k+1) \prod_{s=0}^{k} \frac{1}{1-a(s)} - x^*(N_2) \prod_{s=0}^{N_2-1} \frac{1}{1-a(s)} \right).$$

This yields

$$y(k+1) \geq y(N_2) \prod_{s=N_2}^{k} (1-a(s)) + [\exp(-(p+\varepsilon)(n-1)^{1-1/n}) - 1]$$

$$\cdot \left(x^*(k+1) - x^*(N_2) \prod_{s=N_2}^{k} (1-a(s)) \right)$$

$$\geq y(N_2) \prod_{s=N_2}^{k} (1-a(s)) + [\exp(-(p+\varepsilon)(n-1)^{1-1/n}) - 1]$$

$$\cdot \left(M\widetilde{b} - x^*(N_2) \prod_{s=N_2}^{k} (1-a(s)) \right).$$

Let $k \to +\infty$. We get $q \geq M\widetilde{b}[\exp(-(p+\varepsilon)(n-1)^{1-1/n}) - 1]$. As ε is arbitrary, we have

(4.10) $$q \geq M\widetilde{b}[\exp(-p(n-1)^{1-1/n}) - 1].$$

From the condition $(n-1)^{n-1}(M\widetilde{b})^n \leq 1$, we have $M\widetilde{b}(n-1)^{1-1/n} \leq 1$. Since $p \geq 0$, $q \leq 0$, by $M\widetilde{b}(n-1)^{1-1/n} \leq 1$ and the results in [18], we know that (4.7) and (4.10) have a unique solution $p = q = 0$. So we get $\lim_{k \to +\infty} y(k) = 0$. This implies $x^*(k)$ is globally attractive. $\square$

REMARK 4.2. By means of the well-known inequality $x^x \geq x$ for $x > 0$, we get $(n-1)^{n-1} \geq n-1$ for $n > 1$. Hence the inequality $(n-1)^{n-1}(M\widetilde{b})^n \leq 1$ implies $(n-1)(M\widetilde{b})^n \leq 1$. Thus the condition of Theorem 4.1 can be replaced by $n > 1$ and $(n-1)^{n-1}(M\widetilde{b})^n \leq 1$.

So we have the following theorem.

THEOREM 4.3. *Assume that* $n > 1$ *and* $(n-1)^{n-1}(M\widetilde{b})^n \leq 1$, *then equation* (1.2) *has a unique globally attractive* ω-*periodic positive solution.*

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