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THE ASYMPTOTIC BEHAVIOR OF NONLINEAR EIGENVALUES

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ABSTRACT. In this paper we study the asymptotic behavior of eigenvalues of the weighted one dimensional p Laplace operator, by using the Prufer transformation. We found the order of growth of the kth eigenvalue, improving the remainder estimate for regular weights.

1. Introduction. In this paper we study the nonlinear eigenvalue problem:

$$-(|u'(x)|^{p-2}u'(x))' = \lambda r(x)|u(x)|^{p-2}u(x), \tag{1.1}$$

in $[0,1]$, with Dirichlet or Neumann boundary conditions. Here, the weight r is a real-valued, positive continuous function, λ is a real parameter, and $1 < p < +\infty$. The spectrum consists on a countable sequence of nonnegative simple eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots$ tending to $+\infty$, see [**5**]. With the same ideas as in [**1**], it was proved in [**4**] that the sequence $\{\lambda_k\}_k$ coincides with the eigenvalues obtained by the Ljusternik Schnirelmann theory.

We define the spectral counting function $N(\lambda)$ as the number of eigenvalues of problem (1.1) less than a given λ:

$$N(\lambda) = \#\{k : \lambda_k \leq \lambda\}.$$

The problem of estimating the spectral counting function has a long history in the linear case $p = 2$. See, for instance, [**7, 8**] and the references therein. For $p \neq 2$, the asymptotic behavior of $N(\lambda)$ was obtained in [**4**], by using variational arguments, including a suitable extension of the 'Dirichlet-Neumann bracketing' method. In that

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paper, the authors obtained the following expansion for the spectral counting function of problem (1.1),

$$N(\lambda) \sim \frac{\lambda^{1/p}}{\pi_p} \int_0^1 r^{1/p}, \tag{1.2}$$

as $\lambda \to \infty$, where $f \sim g$ means that $f/g \to 1$, and π_p is defined as

$$\pi_p = 2(p-1)^{1/p} \int_0^1 \frac{ds}{(1-s^p)^{1/p}}. \tag{1.3}$$

It is easy to see that this asymptotic expansion is equivalent to

$$\lambda_k \sim ck^p. \tag{1.4}$$

In this work, we give a shorter proof of the previous results, without the use of the variational framework, following the ideas of Hoschtadt [**6**]. By using the Prüfer transformation, we prove the following theorem.

Theorem 1.1. *Let $r(x) \in C^1[0,1]$ be a positive function. Then, the eigenvalues of problem* (1.1) *satisfy*

$$\lambda_k = \left(\frac{\pi_p k}{\int_0^1 r^{1/p}} \right)^p + O(k^{p-1}). \tag{1.5}$$

Remark 1.2. Let us note that this result is better than equation (1.4), which only gives $\lambda_k = \pi_p^p k^p / (\int_0^1 r^{1/p})^p + o(k^p)$. This implies a better remainder estimate in the asymptotic expansion of $N(\lambda)$ for regular weights.

Now, we obtain the asymptotic expansion of $N(\lambda)$ as a corollary.

Corollary 1.3. *Let $r(x) \in C^1[0,1]$ be a positive function. Then,*

$$N(\lambda) = \frac{\lambda^{1/p}}{\pi_p} \int_0^1 r^{1/p} + O(1) \tag{1.6}$$

as $\lambda \to \infty$.

Finally, we extend Theorem 1.1 to continuous weights, by approximating $r(x)$ with regular weights. We prove the following theorem.

Theorem 1.4. *Let $r(x)$ be a real-valued, positive continuous function. Then, the asymptotic behavior of $N(\lambda)$ is given by equation* (1.2). *Also, the eigenvalues of problem* (1.1) *satisfy*

$$\lambda_k = \left(\frac{\pi_p k}{\int_0^1 r^{1/p}} \right)^p + o(k^p). \tag{1.7}$$

Remark 1.5. The remainder estimate cannot be improved. In fact, see [**6**] for $p = 2$, or [**4**] for $1 < p < +\infty$, where the second term of $N(\lambda)$ is related to the regularity of r and the domain.

The paper is organized as follows. In Section 2, we introduce the Prüfer transformation. In Section 3, we prove the main theorems (1.1) and (1.4), and Corollary (1.3).

2. The Prüfer transformation. We need the Prüfer transformation method introduced by Elbert [**2**] for the p-Laplacian. Let us call $S_p(x)$ the generalized sine function, the unique solution of

$$\begin{aligned} -(|u'(x)|^{p-2}u'(x))' &= (p-1)|u(x)|^{p-2}u(x) \\ u(0) &= 0 \\ u'(0) &= 1. \end{aligned} \tag{2.1}$$

The function $S_p(x)$ has a zero if and only if $x = k\hat{\pi}_p$, where

$$\hat{\pi}_p = \frac{2\pi/p}{\sin(\pi/p)}. \tag{2.2}$$

Remark 2.1. Let us stress that $\hat{\pi}_p$ is not the same as π_p, see [**1**], where the authors consider equation (2.1) without the factor $(p-1)$. However, let us note that the kth eigenvalue of the problem,

$$
(2.3)\qquad
\begin{aligned}
-(|u'(x)|^{p-2}u'(x))' &= \lambda|u(x)|^{p-2}u(x)\\
u(0) &= 0\\
u(1) &= 0
\end{aligned}
$$

is $\lambda_k = (p-1)k^p\hat{\pi}_p^p = k^p\pi_p^p$.

Let us call $C_p(x) = {S_p}'(x)$. Both functions are well defined in $\mathbf{R}$, and the following identity could be easily derived from equation (2.1):

$$
(2.4)\qquad |C_p(x)|^p + |S_p(x)|^p = 1.
$$

Also, if $C_p(x) \neq 0$,

$$
(2.5)\qquad |C_p(x)|^{p-2}{C_p}'(x) + |S_p(x)|^{p-2}S_p(x) = 0.
$$

As in [**3**], we define the following Prüfer transformation:

$$
(2.6)\qquad
\begin{aligned}
\left(\frac{\lambda r(x)}{p-1}\right)^{1/p} u(x) &= \rho(x)S_p(\varphi(x)),\\
u'(x) &= \rho(x)C_p(\varphi(x)).
\end{aligned}
$$

A straightforward computation shows that $\rho(x), \varphi(x)$ are continuously differentiable functions satisfying

(2.7)

$$
\begin{cases}
\varphi'(x) = (\lambda r(x)/p-1)^{1/p} + (1/p)\ (r'(x)/r(x))|C_p(\varphi(x))|^{p-2}\\
\qquad\qquad \times C_p(\varphi(x))S_p(\varphi(x))\\
\rho'(x) = (1/p)\ (r'(x)/r(x))\rho(x)|S_p(\varphi(x))|^p
\end{cases}
$$

Remark 2.2. Let us note that $u_k(x) = (\lambda_k r(x)/p-1)^{-1/p}\rho_k(x) \times S_p(\varphi_k(x))$ is an eigenfunction of equation (1.1) corresponding to λ_k with zero Dirichlet boundary conditions if and only if $\varphi_k(0) = 0$ and $\varphi_k(1) = k\hat{\pi}_p$. This is a consequence of the well-known fact that the kth-eigenfunction has exactly k nodal domains, see [**4, 9**] for a proof.

3. Asymptotic behavior of eigenvalues. In this section we prove our main theorems.

Let us rewrite the first equation in (2.7) as an integral equation,

$$\tag{3.1} \begin{aligned} \varphi(x) = &\int_0^x \left(\frac{\lambda r(t)}{p-1}\right)^{1/p} dt \\ &+ \frac{1}{p} \int_0^x \frac{r'(t)}{r(t)} |C_p(\varphi(t))|^{p-2} C_p(\varphi(t)) S_p(\varphi(t))\, dt. \end{aligned}$$

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let us begin with Problem (1.1) with Dirichlet boundary conditions.

Let (λ_k, u_k) be an eigenpair of problem (1.1). Then, the corresponding phase function $\varphi_k(x)$ satisfy:

$$\varphi_k(0) = 0,$$

$$\varphi_k(1) = k\hat{\pi}_p.$$

Replacing in equation (3.1), we obtain

$$\begin{aligned} k\hat{\pi}_p = &\left(\frac{\lambda_k}{p-1}\right)^{1/p} \int_0^1 r^{1/p}(t)\, dt \\ &+ \frac{1}{p} \int_0^1 \frac{r'(t)}{r(t)} |C_p(\varphi(t))|^{p-2} C_p(\varphi(t)) S_p(\varphi(t))\, dt \end{aligned}$$

Now, by using that $|S_p|, |C_p| \leq 1$, the second integral is bounded,

$$\tag{3.2} \begin{aligned} \frac{1}{p} \int_0^1 \frac{r'(t)}{r(t)} &|C_p(\varphi(t))|^{p-2} C_p(\varphi(t)) S_p(\varphi(t))\, dt \\ &\leq \frac{1}{p} \int_0^1 \frac{|r'(t)|}{r(t)}\, dt \leq C, \end{aligned}$$

where C is a positive constant independent of k.

Multiplying by $(p-1)^{1/p} / \int_0^1 r^{1/p}(t)\, dt$, we have

$$\left| \lambda_k^{1/p} - \frac{\pi_p k}{\int_0^1 r^{1/p}(t)\, dt} \right| \leq \frac{(p-1)^{1/p} C}{\int_0^1 r^{1/p}(t)\, dt},$$

and the result follows.

Let us consider now the Neumann boundary condition, which only requires minor changes. Let (λ_k, u_k) be an eigenpair of problem (1.1) with $u'(0) = 0 = u'(1)$. The corresponding phase function $\varphi_k(x)$ satisfy:

$$\varphi_k(0) = \frac{\hat{\pi}_p}{2},$$
$$\varphi_k(1) = \left(k + \frac{1}{2}\right)\hat{\pi}_p.$$

In much the same way, we obtain,

$$\left|\lambda_k^{1/p} - \frac{\pi_p k}{\int_0^1 r^{1/p}(t)\,dt}\right| \le \frac{(p-1)^{1/p}}{\int_0^1 r^{1/p}(t)\,dt}\left(C + \frac{1}{2}\right),$$

and this finishes the proof. □

Now we study the asymptotic expansion of $N(\lambda)$.

Proof of Corollary 1.3. Let us introduce the function $\widehat{N}(\mu) = \#\left\{k : \lambda_k^{1/p} \le \mu\right\}$.

From Theorem 1.1 we know that

$$\frac{\pi_p k}{\int_0^1 r^{1/p}(t)\,dt} - \frac{(p-1)^{1/p}C}{\int_0^1 r^{1/p}(t)\,dt} \le \lambda_k^{1/p} \le \frac{\pi_p k}{\int_0^1 r^{1/p}(t)\,dt} + \frac{(p-1)^{1/p}C}{\int_0^1 r^{1/p}(t)\,dt}$$

Thus, we have

$$\widehat{N}(\mu) \le \#\left\{k : \frac{\pi_p k}{\int_0^1 r^{1/p}(t)\,dt} - \frac{(p-1)^{1/p}C}{\int_0^1 r^{1/p}(t)\,dt} \le \mu\right\},$$
$$\widehat{N}(\mu) \ge \#\left\{k : \frac{\pi_p k}{\int_0^1 r^{1/p}(t)\,dt} + \frac{(p-1)^{1/p}C}{\int_0^1 r^{1/p}(t)\,dt} \le \mu\right\}.$$

This gives

$$\left[\frac{\int_0^1 r^{1/p}(t)\,dt}{\pi_p}\mu - \hat{\pi}_p C\right] \le \widehat{N}(\mu) \le \left[\frac{\int_0^1 r^{1/p}(t)\,dt}{\pi_p}\mu + \hat{\pi}_p C\right].$$

Hence, we obtain,

$$\widehat{N}(\mu) = \frac{\int_0^1 r^{1/p}(t)\,dt}{\pi_p}\mu + O(1)$$

when $\mu \to +\infty$.

Clearly, we have

$$N(\lambda) = \#\{k \ : \ \lambda_k \le \lambda\} = \#\{k \ : \ \lambda_k^{1/p} \le \lambda^{1/p}\} = \widehat{N}(\lambda^{1/p}),$$

and the proof is complete. □

Finally, we arrive at the proof of Theorem 1.4.

Proof of Theorem 1.4. Let r be a positive bounded continuous function. For ε sufficiently small, there exists r_ε, $r^\varepsilon \in C^1[0,1]$ satisfying $0 < r_\varepsilon < r(x) < r^\varepsilon$ and $\|r^\varepsilon - r_\varepsilon\|_\infty < \varepsilon$. Let us write $N(\lambda, r)$ in order to stress the dependence on the weight function. We have

$$N(\lambda, r_\varepsilon) \le N(\lambda, r) \le N(\lambda, r^\varepsilon),$$

which is an easy consequence of the Sturmian comparison principle, see [**9**]. From Theorem 1.3, we obtain an upper bound for $N(\lambda, r)$

$$\begin{aligned} N(\lambda, r^\varepsilon) &= \frac{\lambda^{1/p}}{\pi_p}\int_0^1 (r^\varepsilon(t))^{1/p}\,dt + O(1) \\ &\le \frac{\lambda^{1/p}}{\pi_p}\int_0^1 \left(r(t)+\varepsilon\right)^{1/p}\,dt + O(1), \end{aligned} \tag{3.3}$$

and a lower bound for $N(\lambda, r)$,

$$N(\lambda, r^\varepsilon) \ge \frac{\lambda^{1/p}}{\pi_p}\int_0^1 \left(r(t)-\varepsilon\right)^{1/p}\,dt + O(1). \tag{3.4}$$

From equations (3.3) and (3.4), taking the limit as $\varepsilon \to 0+$ we get

$$N(\lambda, r) = \frac{\lambda^{1/p}}{\pi_p}\int_0^1 r^{1/p}(t)\,dt + o(\lambda^{1/p}). \tag{3.5}$$

Finally, the following asymptotic formula for the eigenvalues:

$$\lambda_k = \left(\frac{\pi_p k}{\int_\Omega r^{1/p}}\right)^p + o(k^p) \tag{3.6}$$

follows immediately from equation (3.5) since $k \sim N(\lambda_k)$, which gives:

$$\lambda_k \sim \left(\frac{\pi_p k}{\int_\Omega r^{1/p}}\right)^p.$$

This completes the proof. □

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