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K-THEORY AND K-HOMOLOGY OF C^*-ALGEBRAS FOR ROW-FINITE GRAPHS

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ABSTRACT. We compute the K-groups and K-homology groups of C^*-algebras of row-finite graphs using the universal covering trees of graphs and Pimsner's six-term exact sequences for KK-groups of crossed products by groups acting on trees.

1. Introduction. Since Cuntz and Krieger introduced a class of C^*-algebras related to subshifts of finite type ([**1**]), these algebras have been generalized in many ways: C^*-algebras of Smale spaces ([**8, 9, 18–20, 23, 26**]) from the viewpoint of hyperbolic dynamics, C^*-algebras of row-finite graphs ([**11–15, 21**]) and C^*-algebras of countably infinite graphs ([**5, 6, 21**]) from the viewpoint of graph representations of subshifts of finite type, and C^*-algebras of continuous graphs and Cuntz-Pimsner algebras ([**2, 10, 17**]) from the viewpoint of Hilbert-bimodules. Many of these algebras with appropriate conditions are contained in the bootstrap category of Rosenberg and Schochet [**22**] so that it is possible to classify these algebras by computing K-groups.

In this paper, we compute the K-groups and K-homology groups of row-finite graph C^*-algebras. Firstly, we remark that Pask, Raeburn and Szymański [**14, 21**] computed the K-groups of C^*-algebras of row-finite graphs using the canonical gauge action of S^1, Pimsner-Voiculescu six-term exact sequence for crossed products by $\mathbf{Z}$ and Takai duality. And Drinen and Tomforde [**4, 25**] computed Ext-groups of C^*-algebras of row-finite graphs with no sinks extending Cuntz and Krieger's method.

We approach the computations of K-groups and K-homology groups from a different direction. The origin of this paper is the author's attempt to understand the works of Kumjian and Pask [**11**] and

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Pimsner [**16**]. By adapting Serre's argument [**3, 24**] for interactions between graphs of groups and their fundamental groups and universal covering trees, we are able to give a new description of K-groups of row-finite graph C^*-algebras. For a row-finite graph E, we follow Kumjian and Pask [**11**] to construct the universal covering tree T. The graph algebra $C^*(E)$ is strongly Morita equivalent to $C^*(T) \rtimes G$ where G is the fundamental group of E [**11**, 4.14]. We remark that our situation, G-action on T and $T/G \cong E$, is the case Pimsner mentioned at [**16**, page 633]. Then we use Pimsner's six-term exact sequence for crossed products by groups acting on trees [**16**, Theorem 18] to compute K-groups and K-homology groups of $C^*(T) \rtimes G$. Since C^*-algebras of trees are AF-algebras, the Pimsner sequence folds into a four-term exact sequence. Therefore K-groups and K-homology groups of $C^*(T) \rtimes G$ are given by cokernel and kernel of vertex matrix of E as in the case of Cuntz-Krieger algebras.

2. Graphs and their universal covering trees. We review the definitions and properties of row-finite graphs, their universal covering trees and fundamental transversals which will be used in the later section. As general references for the notions of graphs and fundamental transversals we refer to [**3, 11, 13**].

2.1 *Graphs and their universal covering trees.* The materials in this section are taken from [**11–14**].

A directed *graph* E consists of countable sets E^0 of vertices and E^1 of edges together with maps $r, s\colon E^1 \to E^0$ describing the range and source of edges. A graph is called *row-finite* if every vertex emits only finitely many edges. A vertex is called a *sink* if it does not emit any edge.

Suppose that E and F are directed graphs. Then a *graph morphism* $\alpha\colon E \to F$ is a pair of maps $\alpha = (\alpha^0, \alpha^1)$ where $\alpha^i\colon E^i \to F^i$ for $i = 0, 1$ satisfies

$$\alpha^0(s(e)) = s(\alpha^1(e)) \quad \text{and} \quad \alpha^0(r(e)) = r(\alpha^1(e)) \quad \text{for every } e \in E^1.$$

There is a natural notion of isomorphisms of directed graphs. The group of automorphisms of a graph E is denoted $\mathrm{Aut}\,(E)$.

Let E be a directed graph. For every $e \in E^1$, we denote the reverse edge by $\bar{e}$ where $s(\bar{e}) = r(e)$ and $r(\bar{e}) = s(e)$. The set of reverse edges is denoted $\overline{E}^1$. It is natural to define $\bar{\bar{e}} = e$ for every $e \in E^1$. A *walk* in E is a product $a = a_1 \cdots a_n$ where $a_i \in E^1 \cup \overline{E}^1$ and $r(a_i) = s(a_{i+1})$ for $i = 1, \ldots, n-1$; we write $s(a) = s(a_1)$ and $r(a) = r(a_n)$. A walk $a = a_1 \cdots a_n$ is called reduced if it does not contain a subwalk $a_i a_{i+1} = e\bar{e}$ for an $e \in E^1 \cup \overline{E}^1$. The set of reduced walks in E is denoted E^{rw}. For $a = a_1 \cdots a_n \in E^{rw}$, the reverse walk is written $\bar{a} = \overline{a_n} \cdots \overline{a_1}$ so that, for $a, b \in E^{rw}$ with $r(a) = s(b)$, ab is understood to be the reduced walk obtained by concatenation and cancelation.

A directed graph E is called *connected* if there is a reduced walk between every two distinct vertices of E. A connected directed graph T is a *tree* if there is precisely one reduced walk between every two distinct vertices of E.

Standing assumption. In this paper every graph is a connected directed row-finite graph.

Definition 2.1 [**11**, 4.4]. Suppose that E is a graph. Fix a vertex $\star \in E^0$, and let $E^{rw}(\star)$ denote the set of all reduced walks in E whose source is $\star$. Define a graph $T = T(E, \star)$ as follows:

$$T^0 = E^{rw}(\star), \quad T^1 = \{(a, e) \in E^{rw}(\star) \times E^1 : r(a) = s(e)\},$$
$$s(a, e) = a \text{ and } r(a, e) = ae.$$

For $(a, e) \in T^1$, we identify $\overline{(a, e)} = (ae, \bar{e})$.

Theorem 2.2 [**11**, 4.5]. *Let E be a graph. Then $T = T(E, \star)$ is a tree, and the isomorphism class of T is independent of the choice of the base point $\star \in E^0$.*

Definition 2.3 [**11**, 4.7]. Let E, F be graphs and $p: E \to F$ a graph morphism. Then p is a *covering map* if

(1) p is onto, that is, p^0 and p^1 are surjective;

(2) for every $v \in E^0$, $p^1 : s^{-1}(u) \to s^{-1}(p^0(u))$ and $p^1 : r^{-1}(u) \to r^{-1}(p^0(u))$ are bijections.

Theorem 2.4 [**11**, 4.8]. *Suppose that E is a graph and that T the corresponding tree as above. Let a graph morphism $p: T \to E$ be defined* *by*

$$p^0(a) = r(a) \quad \textit{and} \quad p^1(a,e) = e.$$

Then p is a covering map, and (T,p) is a universal covering space of E in the sense that if (U,q) is another covering space, then there is a graph morphism $\phi: T \to U$ such that $p = q \circ \phi$.

Definition 2.5 [**11**–**14**]. Let E be a graph. The graph algebra $C^*(E)$ is the universal C^*-algebra generated by a family of mutually orthogonal projections $\{P_v \mid v \in E^0\}$ and a family of partial isometries $\{S_e \mid e \in E^1\}$ satisfying

(1) $S_e^* S_e = P_{r(e)}$ for every $e \in E^1$

(2) $P_v = \sum_{s^{-1}(v)} S_e S_e^*$ for every $v \in s(E^1)$.

Theorem 2.6 [**12**, 2.4]. *If E is a tree, then the graph C^*-algebra $C^*(E)$ is an AF-algebra.*

Remark 2.7. Suppose that E is a tree. Then, by [**11**, Proposition 4.3], $C^*(E)$ is strongly Morita equivalent to $C_0(\partial E)$ where ∂E is a certain locally compact zero-dimensional space canonically attached to the tree. Therefore $K_0(C^*(E))$ is a torsion-free group.

Group actions on graphs. Suppose that E is a graph and G is a group. Let us denote (left) actions of G on E^0 and E^1 by

$$(g,v) \longmapsto gv \quad \text{and} \quad (g,e) \longmapsto ge \quad \text{for } v \in E^0 \text{ and } e \in E^1.$$

We say that G acts on E if there is a group homomorphism $G \to \operatorname{Aut} E$ that satisfies

$$s(ge) = gs(e) \quad \text{and} \quad r(ge) = gr(e).$$

The action of G on E is said to be *free* if it is free on E^0, that is, $gv = v$ for every $v \in E^0$ implies $g = 1_G$. Given a free action of a group G on a graph E, we form the quotient E/G by the equivalence relation $e_1 \sim e_2$ if $e_1 = ge_2$ for some $g \in G$. The quotient E/G is a graph where

$$(E/G)^0 = (E^0)/G, \quad (E/G)^1 = (E^1)/G,$$
$$r([e]) = [r(e)] \quad \text{and} \quad s([e]) = [s(e)].$$

See [**11**, Section 3] for more details.

Lemma 2.8 [**11**, 4.10]. *Suppose that E is a graph, $(T,p) = (T(E,\star),p)$ is its universal covering tree, and $G = \{a \in E^{rw}(\star) : r(a) = \star\}$. Then G is a group under concatenation with cancelation. Let G act on T by*

$$(g,a) \longmapsto ga \quad \text{and} \quad (g,(a,e)) \longmapsto (ga,e).$$

Then G is a free group, the action of G on T is a free action, and T/G is isomorphic to E.

Theorem 2.9 [**11**, 4.13 and 4.14]. *Suppose that E, $T = (T,p)$ and G are as above. Then $C^*(E)$ is strongly Morita equivalent to $C^*(T) \rtimes G$.*

Remark 2.10. By Corollary 3.1 of [**15**], $C^*(T) \rtimes G$ is equal to $C^*(T) \rtimes_r G$, the reduced crossed product.

Fundamental transversals. The material in this subsection is taken from [**3**]. For more details, see [**3**, Chapter I] and [**24**, Chapter I].

Definition and Theorem 2.11 [**3**, I.1.3 and I.2.6]. Suppose that a group G acts on a set X and S is a subset of X. We call S a *G-transversal* if S meets each G-orbit once.

If X is a graph and X/G is connected, then there exist subsets $Y_0 \subseteq Y \subseteq X$ such that

(1) Y is a G-transversal,

(2) Y is bijective to X/G,

(3) $(Y_0 \cap X^0, Y_0 \cap X^1, r, s)$ is a subtree of X,

(4) $Y \cap X^0 = Y_0 \cap X^0$, and

(5) $s(e) \in Y \cap X^0$ for every edge $e \in Y \cap X^1$.

The subset Y is called a *fundamental G-transversal* in X.

Suppose that E, G and (T,p) are as in Lemma 2.8. Then there is a fundamental G-transversal $\mathcal{E}$ in T such that $p|_{\mathcal{E}} : \mathcal{E} \to T/G$ is a bijection. By abuse of notation, when we say $\mathcal{E}^0$ and $\mathcal{E}^1$, respectively, we mean $p^{-1}\left((T/G)^0\right)$ and $p^{-1}\left((T/G)^1\right)$, respectively.

Let r and $\bar{r}$ be the range maps of T and T/G, respectively. For every edge $e \in \mathcal{E}^1$, $p^{-1}(\bar{r}(p(e)))$ is the G-orbit of $r(e)$ in T. Since $\mathcal{E}$ is a G-transversal, there is a unique element $\rho(e) \in \mathcal{E}^0$ such that $\rho(e)$ is contained in the G-orbit of $r(e)$. As G-action on T is a free action, for each $e \in \mathcal{E}^1$, there is a unique element γ_e of G such that $\gamma_e \rho(e) = r(e)$. Because source of each $e \in \mathcal{E}^1$ is already contained in $\mathcal{E}^0$, we have $1_G s(e) = s(e)$. Then we define homomorphisms

$$\sigma_e : G_e \longrightarrow G_{\rho(e)} \quad \text{and} \quad \sigma_{\bar{e}} : G_e \longrightarrow G_{s(e)}$$

by

$$g \longmapsto \gamma_e^{-1} g \gamma_e \quad \text{and} \quad g \longmapsto g$$

where $G_\star$ is the stabilizer of $\star$.

Remarks 2.12. (1) By [**24**, Proposition I.17], the fundamental G-transversal $\mathcal{E}$ in T is not a subgraph of T. But E is a graph isomorphic to $T/G = E$, and its range map and source map are ρ and $s|_{\mathcal{E}}$, respectively.

(2) Since the G-action on T is a free action by Lemma 2.8, it is obvious that $G_v = G_e = \{1_G\}$. We keep G_e and G_v terms to represent index.

3. K-theory and K-homology. In this section, we use Theorem 2.9 and Pimsner's six-term exact sequences for KK-groups of crossed products by groups acting on trees to compute K-groups and K-homology groups of row-finite graph C^*-algebras.

Pimsner's exact sequences. Suppose that E, G and (T, p) are as in Lemma 2.8 and that $\mathcal{E}$ is a fundamental G-transversal in T. We consider the action of G on $C^*(T)$. Let α_e and $\alpha_{\bar{e}}$ be automorphisms of $C^*(T)$ induced from the group actions

$$a \longmapsto \gamma_e^{-1} a \quad \text{and} \quad a \longmapsto a = 1_G^{-1} a, \quad \text{respectively, for every } a \in T.$$

Then there are $*$-homomorphisms on crossed products ([**16**, Section 4]):

$$\alpha_e \times \sigma_e : C^*(T) \rtimes G_e \longrightarrow C^*(T) \rtimes G_{\rho(e)},$$
$$\mathrm{Id}_e = \alpha_{\bar{e}} \times \sigma_{\bar{e}} : C^*(T) \rtimes G_e \longrightarrow C^*(T) \rtimes G_{s(e)}$$

and

$$\alpha^\rho, \mathrm{Id}\colon \bigoplus_{e\in\mathcal{E}^1} C^*(T)\rtimes G_e \longrightarrow \mathcal{K}(l^2(\mathcal{E}^1))\otimes \bigoplus_{v\in\mathcal{E}^0} C^*(T)\rtimes G_v$$

defined by

$$\begin{aligned}\alpha^\rho(\oplus_{e\in\mathcal{E}^1} x_e) &= \sum \varepsilon_{e,e}\otimes(\alpha_e\times\sigma_e)(x_e)\\ \mathrm{Id}\,(\oplus_{e\in\mathcal{E}^1} x_e) &= \sum \varepsilon_{e,e}\otimes \mathrm{Id}_{\,e}(x_e).\end{aligned}$$

Here $\mathcal{K}(l^2(\mathcal{E}^1))$ is the set of compact operators on $l^2(\mathcal{E}^1)$, and $\{\varepsilon_{c,d}\}_{c,d\in\mathcal{E}^1}$ is a system of matrix units in $\mathcal{K}(l^2(\mathcal{E}^1))$. Let α^ρ_* and Id_* denote the induced maps on K-groups, and $\alpha^{\rho *}$ and Id^* the induced maps on K-homology groups.

Theorem 3.1 [**16**, Theorems 16 and 18]. *Suppose that E, G and $T=(T,p)$ are as in Lemma* 2.8 *and that $\mathcal{E}$ is a fundamental G-transversal in T. Then there are exact sequences*

$$\begin{array}{ccc}
\bigoplus_{e\in\mathcal{E}^1} K_0(C^*(T)\rtimes G_e) & \xrightarrow{\alpha^\rho_*-\mathrm{Id}_*} & \bigoplus_{v\in\mathcal{E}^0} K_0(C^*(T)\rtimes G_v)\\
\uparrow & & \downarrow\\
K_1(C^*(T)\rtimes G) & & K_0(C^*(T)\rtimes G)\\
\uparrow & & \downarrow\\
\bigoplus_{v\in\mathcal{E}^0} K_1(C^*(T)\rtimes G_v) & \longleftarrow & \bigoplus_{e\in\mathcal{E}^1} K_1(C^*(T)\rtimes G_e)
\end{array}$$

and

$$\begin{array}{ccc}
K^0\Big(\bigoplus\limits_{e\in\mathcal{E}^1} C^*(T)\rtimes G_e\Big) & \xleftarrow{\alpha^{\rho*}-\mathrm{Id}^*} & K^0\Big(\bigoplus\limits_{v\in\mathcal{E}^0} C^*(T)\rtimes G_v\Big) \\
\downarrow & & \uparrow \\
K^1(C^*(T)\rtimes G) & & K^0(C^*(T)\rtimes G) \\
\downarrow & & \uparrow \\
K^1\Big(\bigoplus\limits_{v\in\mathcal{E}^0} C^*(T)\rtimes G_v\Big) & \longrightarrow & K^1\Big(\bigoplus\limits_{e\in\mathcal{E}^1} C^*(T)\rtimes G_e\Big)
\end{array}$$

where G_v and G_e are stabilizers of v and e, respectively, at the fundamental G-transversal $\mathcal{E}$ in T and $K^i(*) = KK_i(*, \mathbf{C})$ the Kasparov KK-theory.

Recall that $C^*(T)$ is an AF-algebra and $K_0(C^*(T))$ is a torsion-free group by Theorem 2.6 and Remark 2.7. Thus the above six-term sequences become

$$0 \to K_1(C^*(T)\rtimes G) \longrightarrow \bigoplus_{e\in\mathcal{E}^1} K_0(C^*(T)\rtimes G_e) \\
\overset{\alpha^\rho_*-\mathrm{Id}_*}{\longrightarrow} \bigoplus_{v\in\mathcal{E}^0} K_0(C^*(T)\rtimes G_v) \longrightarrow K_0(C^*(T)\rtimes G) \to 0$$

and

$$0 \to K^0(C^*(T)\rtimes G) \longrightarrow K^0(\oplus_{v\in\mathcal{E}^0} C^*(T)\rtimes G_v) \\
\overset{\alpha^{\rho*}-\mathrm{Id}^*}{\longrightarrow} K^0(\oplus_{e\in\mathcal{E}^1} C^*(T)\rtimes G_e) \longrightarrow K^1(C^*(T)\rtimes G) \to 0.$$

And we have the following from Theorem 2.9:

$$K_0(C^*(E)) \cong K_0(C^*(T)\rtimes G) \cong \mathrm{Coker}\,(\alpha^\rho_* - \mathrm{Id}_*)$$
$$K_1(C^*(E)) \cong K_1(C^*(T)\rtimes G) \cong \mathrm{Ker}\,(\alpha^\rho_* - \mathrm{Id}_*)$$

and

$$K^0(C^*(E)) \cong K^0(C^*(T)\rtimes G) \cong \mathrm{Ker}\,(\alpha^{\rho*} - \mathrm{Id}^*)$$
$$K^1(C^*(E)) \cong K^1(C^*(T)\rtimes G) \cong \mathrm{Coker}\,(\alpha^{\rho*} - \mathrm{Id}^*).$$

K-theory. In this subsection, we show

$$\mathrm{Coker}\,(\alpha_*^\rho - \mathrm{Id}_*) \cong \mathrm{Coker}\,(\mathrm{M}^{\mathrm{t}} - \mathrm{I}) \quad \text{and} \quad \mathrm{Ker}\,(\alpha_*^\rho - \mathrm{Id}_*) \cong \mathrm{Ker}\,(\mathrm{M}^{\mathrm{t}} - \mathrm{I})$$

where M is the vertex matrix of E.

We learned the following lemma from [**6**]:

Lemma 3.2 [**6**, 4.9]. *Let*

$$\begin{array}{ccc} G_1 & \xrightarrow{\ M\ } & G_2 \\ {\scriptstyle\phi}\big\downarrow & & \big\downarrow{\scriptstyle\psi} \\ H_1 & \xrightarrow[\ L\ ]{} & H_2 \end{array}$$

be a commuting diagram of abelian groups in which

(1) *the vertical arrows are injective,*

(2) $L(x) \in \mathrm{Im}\,(\psi)$ *implies* $x \in \mathrm{Im}\,(\phi)$ *for every* $x \in H_1$, *and*

(3) *every* $x \in H_2$ *is equivalent to an element of* $\psi(G_2)$ *modulo* $L(H_1)$.

Then $\mathrm{Ker}\,M \cong \mathrm{Ker}\,L$ *and* $\mathrm{Coker}\,M \cong \mathrm{Coker}\,L$.

Suppose that E, G and T are as in Lemma 2.8 and that $\mathcal{E}$ is a fundamental G-transversal in T. Let M be the vertex matrix of E, that is, M is an $E^0 \times E^0$-matrix such that

$$M(u,v) = \begin{cases} 1 & \text{if there is an } e \in E^1 \text{ such that } s(e) = u \text{ and } r(e) = v \\ 0 & \text{otherwise.} \end{cases}$$

Since the graph E is isomorphic to $\mathcal{E}$, M is the vertex matrix of $\mathcal{E}$.

Notation 3.3. Suppose $u \in T^0$, $v \in \mathcal{E}^0$ and $e \in \mathcal{E}^1$. By x_v, we mean the canonical unit vector in $\oplus_{v \in \mathcal{E}^0} \mathbf{Z}$ with 1 at the vth-term and 0 elsewhere. We denote $P_{u,v}$ and $P_{u,e}$ as the equivalence class of the projection P_u of $C^*(T)$ in $K_0(C^*(T) \rtimes G_v)$ and $K_0(C^*(T) \rtimes G_e)$, respectively. Then it is not difficult to obtain $\alpha_*^\rho(P_{u,e}) = P_{\gamma_e^{-1}u,\rho(e)}$ and $\mathrm{Id}_*(P_{u,e}) = P_{u,s(e)}$.

We define maps

$$\begin{aligned}\phi\colon \oplus_{v\in\mathcal{E}^0}\mathbf{Z} &\longrightarrow \oplus_{e\in\mathcal{E}^1}K_0(C^*(T)\rtimes G_e) \quad \text{by}\\ x_v &\longmapsto \oplus_{e\in s^{-1}(v)}P_{r(e),e} \quad \text{and}\\ \psi\colon \oplus_{v\in\mathcal{E}^0}\mathbf{Z} &\longrightarrow \oplus_{v\in\mathcal{E}^0}K_0(C^*(T)\rtimes G_v) \quad \text{by}\\ x_v &\longmapsto P_{v,v}.\end{aligned}$$

Then ψ is a monomorphism.

When E has no sinks. Recall that a vertex is a sink if it does not emit any edge. Then it is straightforward that ϕ is a monomorphism, for every vertex is a source of at least one edge.

Lemma 3.4. *Suppose that the graph E has no sinks and that M is the vertex matrix of E. Then the following diagram commutes*:

$$\begin{array}{ccc}\bigoplus_{v\in\mathcal{E}^0}\mathbf{Z} & \xrightarrow{\;M^t-I\;} & \bigoplus_{v\in\mathcal{E}^0}\mathbf{Z}\\ \Big\downarrow \phi & & \Big\downarrow \psi\\ \bigoplus_{e\in\mathcal{E}^1}K_0(C^*(T)\rtimes G_e) & \xrightarrow[\alpha^\rho_*-\mathrm{Id}_*]{} & \bigoplus_{v\in\mathcal{E}^0}K_0(C^*(T)\rtimes G_v)\end{array}$$

Proof. For a $v\in\mathcal{E}^0$, let $\{e_1,\dots,e_n\}=s^{-1}(v)\cap\mathcal{E}^1$ and $\rho(e_i)=u_i$. Then

$$\psi(M^t-I)x_v=\psi\Big(\sum x_{u_i}-x_v\Big)=P_{u_1,u_1}+\cdots+P_{u_n,u_n}-P_{v,v}.$$

And

$$\begin{aligned}(\alpha^\rho_*-\mathrm{Id}_*)\phi(x_v)&=(\alpha^\rho_*-\mathrm{Id}_*)(P_{r(e_1),e_1}+\cdots+P_{r(e_n),e_n})\\ &=P_{\gamma_{e_1}^{-1}r(e_1),\rho(e_1)}+\cdots+P_{\gamma_{e_n}^{-1}r(e_n),\rho(e_n)}\\ &\quad-(P_{r(e_1),v}+\cdots+P_{r(e_n),v})\\ &=P_{\rho(e_1),\rho(e_1)}+\cdots+P_{\rho(e_n),\rho(e_n)}\\ &\quad-(P_{r(e_1),v}+\cdots+P_{r(e_n),v}).\end{aligned}\tag{1}$$

Since $P_v = \sum_i S_{e_i} S_{e_i}^*$ and $P_{r(e_i)} = S_{e_i}^* S_{e_i}$, we have

$$P_{v,v} = P_{r(e_1),v} + \cdots + P_{r(e_n),v}.$$

Therefore the diagram is commuting. □

When E has sinks. Suppose that J is the set of sinks of the graph E, and F is the maximal subgraph of E that has no sinks. Then the adjacency matrix of E is

$$M = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}$$

where A is the vertex matrix of F and that B is the matrix for the edges whose sources are in F^0 and ranges are in J.

Let $\mathcal{J} = p^{-1}(J) \cap \mathcal{E}^0$ and $\mathcal{F}^0 = p^{-1}(F) \cap \mathcal{E}^0$ where p is the covering map. Then, for every $v \in \mathcal{J}$, there is no edge whose source is $p(v)$. So $\phi(x_v) = 0$ and $\operatorname{Ker}\phi = \oplus_{v\in\mathcal{J}}\mathbf{Z}$. Hence, instead of $\phi\colon \oplus_{v\in\mathcal{E}^0}\mathbf{Z} \to \oplus_{e\in\mathcal{E}^1} K_0(C^*(T) \rtimes G_e)$, we use the monomorphism $\phi|_{\oplus_{v\in\mathcal{F}^0}\mathbf{Z}}$. And the homomorphism $(M^t - I)|_{\bigoplus_{v\in\mathcal{F}^0}\mathbf{Z}}$ is given by $(A - I\ B)^t$. Then the following lemma is straightforward from the proof of Lemma 3.4.

Lemma 3.5. *Suppose that E has sinks, $M = \left(\begin{smallmatrix} A & B \\ 0 & 0 \end{smallmatrix}\right)$ is the vertex matrix of E, and F is the maximal subgraph of E that has no sinks. Let $\mathcal{F}^0 = p^{-1}(F) \cap \mathcal{E}^0$. Then the following diagram commutes*:

$$\begin{array}{ccc}
\bigoplus_{v\in\mathcal{F}^0}\mathbf{Z} & \xrightarrow{\quad N^t \quad} & \bigoplus_{v\in\mathcal{E}^0}\mathbf{Z} \\
\Big\downarrow{\scriptstyle\phi} & & \Big\downarrow{\scriptstyle\psi} \\
\bigoplus_{e\in\mathcal{E}^1} K_0(C^*(T)\rtimes G_e) & \xrightarrow[\alpha_*^\rho - \mathrm{Id}_*]{} & \bigoplus_{v\in\mathcal{E}^0} K_0(C^*(T)\rtimes G_v)
\end{array}$$

where $N = (\,A-I\ \ B\,)$.

Lemma 3.6. *For an $x \in \oplus_{e\in\mathcal{E}^1} K_0(C^*(T) \rtimes G_e)$, if $(\alpha_*^\rho - \mathrm{Id}_*)(x) \in \operatorname{Im}\psi$, then $x \in \operatorname{Im}\phi$.*

Proof. Because $C^*(T)$ is generated by $S_a S_a^*$ and $S_a^* S_a$ for every $a \in T^1$, it suffices to prove the statement for $P_{r(a),v} = [S_a^* S_a] = [S_a S_a^*]$ in $K_0(C^*(T) \rtimes G_v)$ for any $v \in \mathcal{E}^0$.

Recall that, for $x = (P_{v,e}) \in \oplus_{e \in \mathcal{E}^1} K_0(C^*(T) \rtimes G_e)$,

$$(\alpha_*^\rho - \mathrm{Id}_*)(P_{v,e}) = \sum \left(P_{\gamma_e^{-1} v, \rho(e)} - P_{v,s(e)} \right)$$

and

$$\operatorname{Im} \psi = \langle P_{v,v} \rangle_{v \in \mathcal{E}^0}.$$

Now suppose $(\alpha_*^\rho - \mathrm{Id}_*)(P_{v,e}) \in \operatorname{Im} \psi$. It is clear that v has to be $r(e)$ or $s(e)$ so that

$$\begin{aligned}(P_{v,e}) &= (P_{r(a),a}) \oplus (P_{s(b),b}) \\ &\longmapsto \sum \underbrace{P_{\gamma_a^{-1} r(a), \rho(a)}}_{\in \operatorname{Im}\psi} - P_{r(a),s(a)} + \sum P_{\gamma_b^{-1} s(b), \rho(b)} - \underbrace{P_{s(b),s(b)}}_{\in \operatorname{Im}\psi}.\end{aligned}$$

$v = r(a)$: $\sum P_{r(a),s(a)} \in \langle P_{v,v} \rangle$ and $r(a) \neq s(a)$ mean that every $P_{u,s(a)}$ such that $u = r(e)$ and $s(e) = s(a)$ is contained in $\{P_{v,e}\}$ so that

$$\sum P_{u,s(a)} = P_{s(a),s(a)}.$$

Hence, as in (1) of Lemma 3.4,

$$(P_{r(a),a}) = \phi\left((x_{s(a)})\right).$$

$v = s(b)$: Firstly, note that $\gamma_b^{-1} b$ is an edge whose source and range are $\gamma_b^{-1} s(b)$ and $\rho(b)$, respectively. Let $\{\gamma_b^{-1} b, e_1, \dots, e_n\} = s^{-1}(\gamma_b^{-1} s(b))$. Then $P_{\gamma_b^{-1} s(b), \rho(b)} = P_{\rho(b),\rho(b)} + P_{r(e_1),\rho(b)} + \cdots + P_{r(e_n),\rho(b)}$ and

$$\begin{aligned}&\sum P_{\gamma_b^{-1} s(b), \rho(b)} - P_{s(b),s(b)} \\ &= \sum P_{\rho(b),\rho(b)} + P_{r(e_1),\rho(b)} + \cdots + P_{r(e_n),\rho(b)} - P_{s(b),s(b)} \in \langle P_{v,v} \rangle.\end{aligned}$$

Hence it is straightforward that $P_{r(e_1),\rho(b)} = \cdots = P_{r(e_n),\rho(b)} = 0$, and $\gamma_b^{-1} b$ is the unique edge whose source is $\gamma_b^{-1} s(b)$, equivalently b is the unique edge whose source is $s(b)$. Thus we have $P_{s(b),b} = \phi(x_{s(b)})$ and

$$(P_{v,e}) = (P_{r(a),a}) \oplus (P_{s(b),b}) = \phi((x_{s(a)}) \oplus (x_{s(b)})). \qquad \square$$

Lemma 3.7. *For every* $x \in \oplus_{v\in\mathcal{E}^0} K_0(C^*(T)\rtimes G_v)$*, there is an element* y *in* $\oplus_{v\in\mathcal{E}^0}\mathbf{Z}$ *such that*

$$x-\psi(y)\in(\alpha_*^\rho-\mathrm{Id}_*)\left(\oplus_{e\in\mathcal{E}^1}K_0(C^*(T)\rtimes G_e)\right).$$

Proof. For $P_u = S_a^*S_a \in C^*(T)\rtimes G_v$ for some $a\in T^1$ with $u=r(a)$ and $v\in\mathcal{E}^0$, let $e\in\mathcal{E}^1$ be such that $s(e)=v$. Then

$$\alpha_*^\rho-\mathrm{Id}_*\colon K_0(C^*(T)\rtimes G_e)\longrightarrow K_0(C^*(T)\rtimes G_{\rho(e)})\oplus K_0(C^*(T)\rtimes G_{v=s(e)})$$

is given by

$$P_{u,e}\longmapsto P_{\gamma_e^{-1}u,\rho(e)}-P_{u,v},$$

and

$$P_{u,v}=P_{\gamma_e^{-1}u,\rho(e)}-\underbrace{(P_{\gamma_e^{-1}u,\rho(e)}-P_{u,v})}_{\in\mathrm{Im}\,(\alpha_*^\rho-\mathrm{Id}_*)}.$$

We recall that there is a unique reduced walk $w=f_1\cdots f_n$ from $r(e)$ to $r(a)$ where $f_i\in T^1\cup\overline{T}^1$ as T is a tree. For brevity, let us consider the case that $f_1,\dots,f_m\in T^1$, $f_{m+1},\dots,f_{m+k}\in\overline{T}^1$ and $f_{m+k+1},\dots,f_n\in T^1$ for some $m,k\ge 1$.

Because $\mathcal{E}$ is a fundamental G-transversal, there is a unique element $e_1\in\mathcal{E}^1$ such that e_1 is contained in the orbit of f_1. Then $s(f_1)=r(e)=\gamma_e\rho(e)$ and $\rho(e)$ is the unique element of the orbit of $s(f_1)$ in $\mathcal{E}^0$ imply

$$e_1=\gamma_e^{-1}f_1\quad\text{and}\quad s(e_1)=\rho(e).$$

So

$$\begin{aligned}\alpha_*^\rho-\mathrm{Id}_*\colon K_0(C^*(T)\rtimes G_{e_1})&\longrightarrow K_0(C^*(T)\rtimes G_{\rho(e_1)})\\&\qquad\oplus K_0(C^*(T)\rtimes G_{s(e_1)})\\ P_{\gamma_e^{-1}u,e_1}&\longmapsto P_{\gamma_{e_1}^{-1}\gamma_e^{-1}u,\rho(e_1)}-P_{\gamma_e^{-1}u,s(e_1)=\rho(e)}\end{aligned}$$

and that $\gamma_{e_1}^{-1}\gamma_e^{-1}(f_2\cdots f_n)$ is the reduced walk from $\rho(e_1)$ to $r(\gamma_{e_1}^{-1}\gamma_e^{-1}a)$. Hence

$$\begin{aligned}P_{u,v}&=P_{\gamma_e^{-1}u,\rho(e)}-(P_{\gamma_e^{-1}u,\rho(e)}-P_{u,v})\\&=P_{\gamma_{e_1}^{-1}\gamma_e^{-1}u,\rho(e_1)}\\&\quad-\underbrace{\{(P_{\gamma_e^{-1}u,\rho(e)}-P_{u,v})+(P_{\gamma_{e_1}^{-1}\gamma_e^{-1}u,\rho(e_1)}-P_{\gamma_e^{-1}u,s(e_1)})\}}_{\in\mathrm{Im}\,(\alpha_*^\rho-\mathrm{Id}_*)},\end{aligned}$$

and, by induction, we have

$$P_{u,v} = P_{\gamma_{e_m}^{-1}\cdots\gamma_{e_1}^{-1}\gamma_e^{-1}u,\rho(e_m)} + \operatorname{Im}(\alpha_*^\rho - \mathrm{Id}_*)$$

and $\gamma_{e_m}^{-1}\cdots\gamma_e^{-1}(f_{m+1}\cdots f_n)$ is the reduced walk from $\rho(e_m)$ to $\gamma_{e_m}^{-1}\cdots\gamma_e^{-1}u$. Let

$$e_m = \gamma_{e_{m-1}}^{-1}\cdots\gamma_e^{-1}f_m \in \mathcal{E}^1 \quad \text{and} \quad \tilde{e}_{m+1} = \gamma_{e_{m-1}}^{-1}\cdots\gamma_e^{-1}\bar{f}_{m+1}.$$

Then there is a $g_{m+1} \in G$ such that $g_{m+1}\tilde{e}_{m+1} = e_{m+1} \in \mathcal{E}^1$. Since $r(e_m) = r(\tilde{e}_{m+1})$,

$$\rho(e_m) = \gamma_{e_m}^{-1}r(e_m) = \gamma_{e_m}^{-1}r(\tilde{e}_{m+1}) \in \mathcal{E}^0 \quad \text{and}$$
$$\gamma_{e_m}^{-1}g_{m+1}^{-1}r(e_{m+1}) = \gamma_{e_m}^{-1}g_{m+1}^{-1}g_{m+1}r(\tilde{e}_{m+1}) = \gamma_{e_m}^{-1}r(\tilde{e}_{m+1}) = \rho(e_m).$$

Hence, for $\gamma_{e_{m+1}} = g_{m+1}\gamma_{e_m}$, $\rho(e_{m+1}) = \rho(e_m) \in \mathcal{E}^0$ and

$$\begin{aligned} P_{\gamma_{e_m}^{-1}\cdots\gamma_{e_1}^{-1}\gamma_e^{-1}u,\rho(e_m)} &= P_{\gamma_{e_m}^{-1}\cdots\gamma_{e_1}^{-1}\gamma_e^{-1}u,\rho(e_{m+1})} \\ &= P_{\gamma_{e_{m+1}}\gamma_{e_m}^{-1}\cdots\gamma_{e_1}^{-1}\gamma_e^{-1}u,s(e_{m+1})} \\ &\quad + (\alpha_*^\rho - \mathrm{Id}_*)\left(P_{\gamma_{e_{m+1}}\gamma_{e_m}^{-1}\cdots\gamma_{e_1}^{-1}\gamma_e^{-1}u,e_{m+1}}\right). \end{aligned}$$

Thus we have

$$P_{u,v} = P_{\gamma_{e_{m+1}}\gamma_{e_m}^{-1}\cdots\gamma_{e_1}^{-1}\gamma_e^{-1}u,s(e_{m+1})} + \operatorname{Im}(\alpha_*^\rho - \mathrm{Id}_*).$$

We repeat the above process up to $e_{m+k} \in \mathcal{E}^1$ corresponding to $f_{m+k} \in \overline{T}^1$ so that

$$P_{u,v} = P_{\gamma_{e_{m+k}}\cdots\gamma_{e_{m+1}}\gamma_{e_m}^{-1}\cdots\gamma_{e_1}^{-1}\gamma_e^{-1}u,s(e_{m+k})} + \operatorname{Im}(\alpha_*^\rho - \mathrm{Id}_*)$$

and $\gamma_{e_{m+k}}\cdots\gamma_{e_{m+1}}\gamma_{e_m}^{-1}\cdots\gamma_e^{-1}(f_{m+k+1}\cdots f_n)$ is the reduced walk from $s(e_{m+k})$ to $\gamma_{e_{m+k}}\cdots\gamma_{e_{m+1}}\gamma_{e_m}^{-1}\cdots\gamma_e^{-1}u$. Then, for $e_{m+k+1} = \gamma_{e_{m+k}}\cdots\gamma_{e_{m+1}}\gamma_{e_m}^{-1}\cdots\gamma_e^{-1}f_{m+k+1}$ and $\gamma_{e_{m+k+1}} \in G$ such that $\gamma_{e_{m+k+1}}^{-1}r(e_{m+k+1}) = \rho(e_{m+k+1}) \in \mathcal{E}^0$, we get

$$\begin{aligned} &P_{\gamma_{e_{m+k}}\cdots\gamma_{e_{m+1}}\gamma_{e_m}^{-1}\cdots\gamma_{e_1}^{-1}\gamma_e^{-1}u,s(e_{m+k})} \\ &\qquad = P_{\gamma_{e_{m+k+1}}^{-1}\gamma_{e_{m+k}}\cdots\gamma_{e_{m+1}}\gamma_{e_m}^{-1}\cdots\gamma_{e_1}^{-1}\gamma_e^{-1}u,\rho(e_{m+k+1})} \\ &\qquad\quad - (\alpha_*^\rho - \mathrm{Id}_*)P_{\gamma_{e_{m+k}}\cdots\gamma_{e_{m+1}}\gamma_{e_m}^{-1}\cdots\gamma_{e_1}^{-1}\gamma_e^{-1}u,e_{m+k+1}} \end{aligned}$$

and

$$P_{u,v} = P_{\gamma_{e_n}^{-1}\cdots\gamma_{e_{m+k+1}}^{-1}\gamma_{e_{m+k}}\cdots\gamma_{e_{m+1}}\gamma_{e_m}^{-1}\cdots\gamma_{e_1}^{-1}\gamma_e^{-1}u,\rho(e_n)} + \mathrm{Im}\,(\alpha_*^\rho - \mathrm{Id}_*)$$

with

$$\begin{aligned}\rho(e_n) &= r\left(\gamma_{e_n}^{-1}\cdots\gamma_{e_{m+k+1}}^{-1}\gamma_{e_{m+k}}\cdots\gamma_{e_{m+1}}\gamma_{e_m}^{-1}\cdots\gamma_{e_1}^{-1}\gamma_e^{-1}a\right)\\ &= \gamma_{e_n}^{-1}\cdots\gamma_{e_{m+k+1}}^{-1}\gamma_{e_{m+k}}\cdots\gamma_{e_{m+1}}\gamma_{e_m}^{-1}\cdots\gamma_{e_1}^{-1}\gamma_e^{-1}u.\end{aligned}$$

Therefore, we have

$$(2)\qquad P_{\gamma_{e_n}^{-1}\cdots\gamma_{e_{m+k+1}}^{-1}\gamma_{e_{m+k}}\cdots\gamma_{e_{m+1}}\gamma_{e_m}^{-1}\cdots\gamma_{e_1}^{-1}\gamma_e^{-1}u,\rho(e_n)} = \psi(x_{\rho(e_n)})$$

and $P_{u,v} = \psi(x_{\rho(e_n)}) + \mathrm{Im}(\alpha_*^\rho - \mathrm{Id}_*)$.

In general, when some f_i is contained in $\overline{T}^1$, we change the corresponding group element $\gamma_{e_i}^{-1}$ in the equation (2) to γ_{e_i}. □

The previous five lemmas give us the following proposition:

Proposition 3.8 [**14, 21**]. *Suppose that E, G and (T,p) are as in Lemma* 2.8 *and that M is the vertex matrix of E.*

(1) *If E has no sinks, then*

$$\begin{aligned}K_0(C^*(E)) &\cong K_0(C^*(T)\rtimes G) \cong \mathrm{Coker}\,(M^t - I)\\ K_1(C^*(E)) &\cong K_1(C^*(T)\rtimes G) \cong \mathrm{Ker}\,(M^t - I).\end{aligned}$$

(2) *If E has sinks and $M = \begin{pmatrix} A & B \\ 0 & 0\end{pmatrix}$, then*

$$K_0(C^*(E)) \cong \mathrm{Coker}\begin{pmatrix} A^t - I \\ B^t\end{pmatrix} \quad \textit{and} \quad K_1(C^*(E)) \cong \mathrm{Ker}\begin{pmatrix} A^t - I \\ B^t\end{pmatrix}.$$

K-homology. We deduce the following exact sequence from Theorem 3.1:

$$\begin{aligned}0 \to K^0(C^*(T)\rtimes G) &\longrightarrow K^0(\oplus_{v\in\mathcal{E}^0} C^*(T)\rtimes G_v)\\ &\xrightarrow{\alpha^{\rho*}-\mathrm{Id}^*} K^0(\oplus_{e\in\mathcal{E}^1} C^*(T)\rtimes G_e) \longrightarrow K^1(C^*(T)\rtimes G) \to 0\end{aligned}$$

so that

$$K^0(C^*(E)) \cong K^0(C^*(T) \rtimes G) \cong \operatorname{Ker}(\alpha^{\rho *} - \mathrm{Id}^*)$$
$$K^1(C^*(E)) \cong K^1(C^*(T) \rtimes G) \cong \operatorname{Coker}(\alpha^{\rho *} - \mathrm{Id}^*).$$

Lemma 3.9 [**22**, 1.12 and 1.17]. (1) $K^0(\oplus C^*(T)) \cong \prod K^0(C^*(T))$.
(2) *There is a short exact sequence*

$$0 \to \mathrm{Ext}^1_{\mathbf{Z}}(K_1(C^*(T)), \mathbf{Z}) \to K^0(C^*(T)) \\ \to \operatorname{Hom}(K_0(C^*(T)), \beta Z) \to 0.$$

As $C^*(T)$ is an AF-algebra, we infer

$$K^0(\oplus C^*(T)) \cong \prod K^0(C^*(T)) \cong \prod \mathrm{Hom}(\mathrm{K_0(C^*(T))}, \mathbf{Z})$$

from the above lemma. Hence the kernel and cokernel of

$$\alpha^{\rho *} - \mathrm{Id}^*: K^0(\oplus_{v \in \mathcal{E}^0} C^*(T) \rtimes G_v) \to K^0(\oplus_{e \in \mathcal{E}^1} C^*(T) \rtimes G_e)$$

are isomorphic to the kernel and cokernel of

$$\operatorname{Hom}(\alpha^\rho_* - \mathrm{Id}_*): \prod_{v \in \mathcal{E}^0} \operatorname{Hom}(\mathrm{K_0(C^*(T) \rtimes G_v)}, \mathbf{Z}) \\ \longrightarrow \prod_{e \in \mathcal{E}^1} \operatorname{Hom}(\mathrm{K_0(C^*(T) \rtimes G_e)}, \mathbf{Z}),$$

respectively.

Now we apply the Hom functor to the following commuting diagram with exact rows:

$$\begin{array}{ccccccc}
\operatorname{Ker}(L^t - I) & \xrightarrow{\text{one-one}} & \oplus \mathbf{Z} & \xrightarrow{L^t - I} & \oplus \mathbf{Z} & \xrightarrow{\text{onto}} & \operatorname{Coker}(L^t - I) \\
\cong \downarrow & & \phi \downarrow & & \psi \downarrow & & \downarrow \cong \\
K_1(C^*(E)) & \xrightarrow[\text{one-one}]{} & \oplus K_0(C^*(T)) & \xrightarrow[\alpha^\rho_* - \mathrm{Id}_*]{} & \oplus K_0(C^*(T)) & \xrightarrow[\text{onto}]{} & K_0(C^*(E))
\end{array}$$

Here $L - I$ is the matrix $M - I$ in Lemma 3.4 when the graph E has no sinks and the matrix N in Lemma 3.5 when E has sinks. Then we obtain the following exact sequences:

$$\begin{aligned} &0 \to \mathrm{Hom}\,(K_0(C^*(E)), \mathbf{Z}) \\ &\longrightarrow \prod_{\mathcal{E}^0} \mathrm{Hom}\,(K_0(C^*(T)), \mathbf{Z}) \overset{\mathrm{Hom}\,(\alpha_*^\rho - \mathrm{Id}_*)}{\longrightarrow} \prod_{\mathcal{E}^1} \mathrm{Hom}\,(K_0(C^*(T)), \mathbf{Z}) \\ &\qquad \longrightarrow \mathrm{Hom}\,(K_1(C^*(E)), \mathbf{Z}) \longrightarrow \mathrm{Ext}\,(K_0(C^*(E)), \mathbf{Z}) \to 0 \end{aligned}$$

and

$$\begin{aligned} &0 \to \mathrm{Hom}\,(\mathrm{Coker}\,(L^t - I), \mathbf{Z}) \longrightarrow \prod_{\mathcal{E}^0} \mathbf{Z} \overset{L-I}{\longrightarrow} \prod_{\mathcal{E}^0} \mathbf{Z} \\ &\qquad \longrightarrow \mathrm{Hom}\,(\mathrm{Ker}\,(L^t - I), \mathbf{Z}) \longrightarrow \mathrm{Ext}\,(\mathrm{Coker}\,(L^t - I), \mathbf{Z}) \to 0. \end{aligned}$$

Lemma 3.10. $\mathrm{Ker}\,(\alpha^{\rho *} - \mathrm{Id}^*) \cong \mathrm{Ker}\,(\mathrm{Hom}\,(\alpha_*^\rho - \mathrm{Id}_*)) \cong \mathrm{Ker}\,(L - I)$.

Proof.

$$\begin{aligned} \mathrm{Ker}\,(\mathrm{Hom}\,(\alpha_*^\rho - \mathrm{Id}_*)) &\cong \mathrm{Hom}\,(K_0(C^*(E)), \mathbf{Z}) \cong \mathrm{Hom}\,(\mathrm{Coker}\,(L^t - I), \mathbf{Z}) \\ &\cong \mathrm{Ker}\,(L - I). \qquad \square \end{aligned}$$

Lemma 3.11. $\mathrm{Coker}\,(\alpha^{\rho *} - \mathrm{Id}^*) \cong \mathrm{Coker}\,(\mathrm{Hom}\,(\alpha_*^\rho - \mathrm{Id}_*)) \cong \mathrm{Coker}\,(L - I)$.

Proof. We have the following commuting diagram from the above exact sequences:

$$
\begin{array}{ccccc}
0 & \longrightarrow & \operatorname{Coker}(\operatorname{Hom}(\alpha_*^{\rho} - \mathrm{Id}_*)) & \longrightarrow & \operatorname{Hom}(K_1(C^*(E)), \mathbf{Z}) \\
\downarrow \cong & & \downarrow [\operatorname{Hom}(\phi)] & & \downarrow \cong \\
0 & \longrightarrow & \operatorname{Coker}(L - I) & \longrightarrow & \operatorname{Hom}(K_1(C^*(E)), \mathbf{Z})
\end{array}
$$

$$
\begin{array}{cc}
\longrightarrow & \operatorname{Ext}(K_0(C^*(E)), \mathbf{Z})V \\
 & \downarrow \cong \\
\longrightarrow & \operatorname{Ext}(K_0(C^*(E)), \mathbf{Z})
\end{array}
$$

Then the induced quotient map $[\operatorname{Hom}(\phi)]\colon \operatorname{Coker}(\operatorname{Hom}(\alpha_*^{\rho} - \mathrm{Id}_*)) \to \operatorname{Coker}(L - I)$ is an isomorphism by the Four lemma ([**7**, 5.11]). □

Therefore we have the following proposition:

Proposition 3.12 [**25**, 5.16]. *Suppose that E, G and (T, p) are as in Lemma 2.8 and that M is the vertex matrix of E.*

(1) *If E has no sinks, then*

$$
\begin{aligned}
K^0(C^*(E)) &\cong K^0(C^*(T) \rtimes G) \cong \operatorname{Ker}(M - I) \\
K^1(C^*(E)) &\cong K^1(C^*(T) \rtimes G) \cong \operatorname{Coker}(M - I).
\end{aligned}
$$

(2) *If E has sinks and $M = \left(\begin{smallmatrix} A & B \\ 0 & 0 \end{smallmatrix}\right)$, then*

$$
K^0(C^*(E)) \cong \operatorname{Ker}\begin{pmatrix} A - I & B \end{pmatrix}
$$

and

$$
K^1(C^*(E)) \cong \operatorname{Coker}\begin{pmatrix} A - I & B \end{pmatrix}.
$$

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