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A FEW REMARKS ON MIXING PROPERTIES OF C^*-DYNAMICAL SYSTEMS

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ABSTRACT. We consider strictly ergodic and strictly weak mixing C^*-dynamical systems. We prove that the system is strictly weak mixing if and only if its tensor product is strictly ergodic, moreover strictly weak mixing too. We also investigate some other mixing properties of the system.

1. Introduction. It is known [**13, 15**] that a notion of mixing for dynamical systems plays an important role in quantum statistical mechanics. A lot of papers, see, [**5, 6, 9, 10, 16**], were devoted to the investigations of mixing properties of dynamical systems. Very recently in [**11**] certain relations between ergodicity, weak mixing and uniformly weak mixing conditions of C^*-dynamical systems have been investigated. It is known [**8, 16**] that strict ergodicity of a dynamical system is stronger than ergodicity. Therefore, it is natural to ask how this notion is related with mixing conditions. The object of this paper is to investigate this question. Namely, we are going to consider strictly ergodic and strictly weak mixing C^*-dynamical systems. The paper organized as follows. In Section 2 we recall some preliminaries on C^*-algebras and dynamical systems. Section 3 is devoted to the characterization of strictly ergodic C^*-dynamical systems. In Section 4 we prove that the system is strictly weak mixing if and only if its tensor product is so. We also introduce a notion of ϕ-ergodicity and compare it with known mixing conditions.

2. Preliminaries. In this section we recall some preliminaries concerning C^*-dynamical systems.

Let $\mathfrak{A}$ be a C^*-algebra with unit $\mathbf{1}$. An element $x \in \mathfrak{A}$ is called *self-adjoint*, respectively *positive*, if $x = x^*$, respectively there is an element

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$y \in \mathfrak{A}$ such that $x = y^*y$. The set of all self-adjoint, respectively positive, elements will be denoted by $\mathfrak{A}_{sa}$, respectively $\mathfrak{A}_+$. By $\mathfrak{A}^*$ we denote the conjugate space to $\mathfrak{A}$. A linear functional $\varphi \in \mathfrak{A}^*$ is called *Hermitian* if $\varphi(x^*) = \overline{\varphi(x)}$ for every $x \in \mathfrak{A}$. A Hermitian functional φ is called *positive* if $\varphi(x^*x) \geq 0$ for every $x \in \mathfrak{A}$. A positive functional φ is said to be a *state* if $\varphi(\mathbf{1}) = 1$. By S, respectively $\mathfrak{A}_h^*$, we denote the set of all states, respectively Hermitian functionals, on $\mathfrak{A}$. Let $\mathfrak{A} \odot \mathfrak{A}$ be the algebraic tensor product of $\mathfrak{A}$. By $\mathfrak{A} \otimes \mathfrak{A}$ we denote a completion of $\mathfrak{A} \odot \mathfrak{A}$ with respect to the minimal C^*-tensor norm on $\mathfrak{A} \odot \mathfrak{A}$. The set of all states on $\mathfrak{A} \otimes \mathfrak{A}$ we denote by S^2. A linear operator $T : \mathfrak{A} \mapsto \mathfrak{A}$ is called *positive* if $Tx \geq 0$ whenever $x \geq 0$. A positive linear operator T is called a *Markov operator* if $T\mathbf{1} = \mathbf{1}$. A pair $(\mathfrak{A}, T)$ consisting of a C^*-algebra $\mathfrak{A}$ and a Markov operator $T : \mathfrak{A} \mapsto \mathfrak{A}$ is called a *C^*-dynamical system*. In the sequel, we will call any triplet $(\mathfrak{A}, \varphi, T)$ consisting of a C^*-algebra $\mathfrak{A}$, a state φ on $\mathfrak{A}$ and a Markov operator $T : \mathfrak{A} \mapsto \mathfrak{A}$ with $\varphi \circ T = \varphi$, that is a dynamical system with an invariant state, a *state preserving C^*-dynamical system*. A state preserving C^*-dynamical system is a noncommutative C^*-probability space $(\mathfrak{A}, \varphi)$, see [**4**], together with a Markov operator T on $\mathfrak{A}$ preserving the noncommutative probability φ. We say that the state preserving C^*-dynamical system $(\mathfrak{A}, \varphi, T)$ is *ergodic*, respectively *weakly mixing, strictly weak mixing*, with respect to φ if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} (\varphi(yT^k(x)) - \varphi(y)\varphi(x)) = 0, \quad \text{for all } x, y \in \mathfrak{A}. \tag{2.1}$$

(respectively,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} |\varphi(yT^k(x)) - \varphi(y)\varphi(x)| = 0, \quad \text{for all } x, y \in \mathfrak{A}, \tag{2.2}$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} |\psi(T^k(x)) - \varphi(x)| = 0, \quad \text{for all } x \in \mathfrak{A}, \psi \in S.) \tag{2.3}$$

The state preserving C^*-dynamical system $(\mathfrak{A}, \varphi, T)$ is called *strictly ergodic* with respect to φ if φ is the unique invariant state under T.

Given a C^*-algebra $\mathfrak{A}$, by $M_n(\mathfrak{A})$ we denote the set of all $n \times n$-matrices $a = (a_{ij})$ with entries a_{ij} in $\mathfrak{A}$. Recall that a linear mapping

$T : \mathfrak{A} \mapsto \mathfrak{A}$ is called n- *positive* if the linear operator $T_n : M_n(\mathfrak{A}) \mapsto M_n(\mathfrak{A})$ given by $T_n(a_{ij}) = (T(a_{ij}))$ is positive. If T is n-positive for all n, then T is said to be *completely positive*. It is known [**14**] that if T is a completely positive map, then the linear operator $T \otimes T : \mathfrak{A} \otimes \mathfrak{A} \mapsto \mathfrak{A} \otimes \mathfrak{A}$ defined by $(T \otimes T)(x \otimes y) = Tx \otimes Ty$ is also completely positive.

3. Strictly ergodic dynamical systems. In this section we are going to characterize strictly ergodic C^*-dynamical systems. To do it we need the following

Lemma 3.1. *Let $(\mathfrak{A}, \varphi, T)$ be strictly ergodic. If $h \in \mathfrak{A}^*$ is invariant with respect to T, i.e., $h(Tx) = h(x)$ for all $x \in \mathfrak{A}$, then there is a number $\lambda \in \mathbf{C}$ such that $h = \lambda\varphi$.*

Proof. Let us first assume that h is positive, then $\psi = h/h(\mathbf{1})$ is a state. According to the strict ergodicity of $(\mathfrak{A}, \varphi, T)$ we have $\psi = \varphi$, which implies that $h = h(\mathbf{1})\varphi$. Now let h be a Hermitian functional. Then there is a unique Jordan decomposition [**14**] of h such that

$$h = h_+ - h_-, \quad \|h\|_1 = \|h_+\|_1 + \|h_-\|_1, \tag{3.1}$$

where $\|\cdot\|_1$ is the norm on $\mathfrak{A}^*$. The invariance of h implies that

$$h \circ T = h_+ \circ T - h_- \circ T = h_+ - h_-.$$

Using $\|h_+ \circ T\|_1 = h_+(\mathbf{1}) = \|h_+\|_1$, similarly $\|h_+ \circ T\|_1 = \|h_+\|_1$, from uniqueness of the decomposition we find $h_+ \circ T = h_+$ and $h_- \circ T = h_-$. Therefore, by the previous argument, one gets $h = \lambda\varphi$. If h is an arbitrary functional, then there are Hermitian functionals h_1,h_2 such that $h = h_1 + ih_2$. Again, invariance of h implies that $h_i \circ T = h_i$, $i = 1, 2$. Consequently, we obtain that $h = \lambda\varphi$. □

Now we are ready to formulate a criterion for the strict ergodicity of a dynamical system. The proof of the criterion is similar to the proof of [**8**, Theorem 2, Chapter 1, Section 8]. For the sake of completeness we will prove it.

Theorem 3.2. *Let $(\mathfrak{A}, \varphi, T)$ be a state preserving C^*-dynamical system. The following conditions are equivalent*

(i) $(\mathfrak{A}, \varphi, T)$ *is strictly ergodic*;

(ii) *For every* $x \in \mathfrak{A}$ *the following equality holds*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} T^k(x) = \varphi(x)\mathbf{1},$$

where convergence in norm of $\mathfrak{A}$;

(iii) *For every* $x \in \mathfrak{A}$ *and* $\psi \in S$, *the following equality holds*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \psi(T^k(x)) = \varphi(x).$$

Proof. Let us consider the implication (i) $\Rightarrow$ (ii). It is clear that for every element of the form $y = T(x) - x$, $x \in \mathfrak{A}$ we have

$$\left\| \frac{1}{n} \sum_{k=0}^{n-1} T^k(x) \right\| = \left\| \frac{1}{n} (T^n(x) - x) \right\| \leq \frac{2}{n} \|x\| \to 0 \quad \text{as } n \to \infty.$$

So, as $\varphi(y) = 0$ one gets

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} T^k(y) = \varphi(y)\mathbf{1}.$$

It is evident that the set of elements of the form $y = T(x) - x$, $x \in \mathfrak{A}$ forms a linear subspace of $\mathfrak{A}$. By $\mathfrak{B}$ we denote the closure of this linear subspace. Set

$$\mathfrak{B}_0 = \{x \in \mathfrak{A} : \varphi(x) = 0\}.$$

It is clear that $\mathfrak{B} \subseteq \mathfrak{B}_0$. To show $\mathfrak{B} = \mathfrak{B}_0$ assume that $\mathfrak{B} \neq \mathfrak{B}_0$, this means that there is an element $x_0 \in \mathfrak{B}_0$ such that $x_0 \notin \mathfrak{B}$. Then according to the Hahn-Banach theorem there is a functional $h \in \mathfrak{A}^*$ such that $h \restriction \mathfrak{B} = 0$ and $h(x_0) = 1$. The condition $h \restriction \mathfrak{B} = 0$ implies that h is invariant with respect to T. Therefore Lemma 3.1 yields that $h = \lambda\varphi$, which contradicts $\varphi(x_0) = 0$. Hence $\mathfrak{B} = \mathfrak{B}_0$.

Let $y \in \mathfrak{B}_0$. Then for an arbitrary $\varepsilon > 0$ we can find $y_\varepsilon = T(x_\varepsilon) - x_\varepsilon$ such that $\|y - y_\varepsilon\| < \varepsilon/2$. According to the following equality,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} T^k(y_\varepsilon) = 0,$$

there exists $n_0 \in \mathbf{N}$ such that

$$\left\| \frac{1}{n} \sum_{k=0}^{n-1} T^k(y_\varepsilon) \right\| < \varepsilon/2$$

for all $n \geq n_0$. Hence, we have

$$\begin{aligned} \left\| \frac{1}{n} \sum_{k=0}^{n-1} T^k(y) \right\| &\leq \left\| \frac{1}{n} \sum_{k=0}^{n-1} T^k(y - y_\varepsilon) \right\| + \left\| \frac{1}{n} \sum_{k=0}^{n-1} T^k(y_\varepsilon) \right\| \\ &\leq \|y - y_\varepsilon\| + \varepsilon/2 < \varepsilon \quad \text{for all } n \geq n_0. \end{aligned}$$

So,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} T^k(y) = \varphi(y)\mathbf{1}$$

is valid for every $y \in \mathfrak{B}_0$.

Now let $x \in \mathfrak{A}$. Put $y = x - \varphi(x)\mathbf{1}$. Obviously $y \in \mathfrak{B}_0$, and for y the last equality holds, whence we get the required relation.

The implication (ii) $\Rightarrow$ (iii) is evident. Let us prove (iii) $\Rightarrow$ (i). Assume that ν is an invariant state with respect to T. According to condition (iii) we find

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \nu(T^k(x)) = \varphi(x)$$

for every $x \in \mathfrak{A}$. On the other hand, we have

$$\frac{1}{n} \sum_{k=0}^{n-1} \nu(T^k(x)) = \nu(x).$$

Whence $\varphi = \nu$. Thus the theorem is proved. $\square$

From this theorem we immediately infer that strict ergodicity implies ergodicity of the C^*-dynamical system. In the next section we will demonstrate an example of a dynamical system which is ergodic but not strictly ergodic. We mention that, from Theorem 3.2, one gets that strict weak mixing trivially implies strict ergodicity.

4. Strictly weak mixing dynamical systems. In this section we are going to give a criterion characterizing strictly weak mixing C^*-dynamical systems.

Set

$$\mathfrak{A}_1^* = \{g \in \mathfrak{A}^* : \|g\|_1 \leq 1\}, \qquad \mathfrak{A}_{1,h}^* = \mathfrak{A}_1^* \cap \mathfrak{A}_h^*.$$

Before formulating a result we recall a well-known fact, see for example [**15**].

Lemma 4.1. *Let $\{a_n\}$ be a bounded sequence of real numbers. Then the following are equivalent*:

(i) $\displaystyle\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} |a_k| = 0$;

(ii) *There exists a set $J \subset \mathbf{N}$ of density zero, i.e.*,

$$\lim_{n\to\infty} \frac{\text{cardinality}\,(J\cap[1,n])}{n} = 0,$$

such that $\lim_{n\to\infty} a_n = 0$ *provided* $n \notin J$;

(iii) $\displaystyle\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} |a_k|^2 = 0.$

Now we are ready to formulate the following

Theorem 4.2. *Let $(\mathfrak{A}, \varphi, T)$ be a state preserving C^*-dynamical system and T a completely positive map. The following conditions are equivalent*:

(i) $(\mathfrak{A}, \varphi, T)$ *is strictly weak mixing*;

(ii) *The state preserving C^*-dynamical system $(\mathfrak{A}\otimes\mathfrak{A}, \varphi\otimes\varphi, T\otimes T)$ is strictly weak mixing*;

(iii) *The state preserving* C^**-dynamical system* $(\mathfrak{A}\otimes\mathfrak{A}, \varphi\otimes\varphi, T\otimes T)$ *is strictly ergodic*;

(iv) *For every* $x\in\mathfrak{A}$*, the following equality holds*

$$\lim_{n\to\infty}\sup_{\psi\in\mathfrak{A}_1^*}\frac{1}{n}\sum_{k=0}^{n-1}|\psi(T^k(x))-\psi(\mathbf{1})\varphi(x)|=0;$$

(v) *For every* $x\in\mathfrak{A}$ *and* $\psi\in\mathfrak{A}^*$*, the following equality holds*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}|\psi(T^k(x))-\psi(\mathbf{1})\varphi(x)|=0. \tag{4.1}$$

Proof. Consider the implication (i) $\Rightarrow$ (ii). Recall that complete positivity of T implies that $T\otimes T$ is so. It is clear that the state $\varphi\otimes\varphi$ is invariant with respect to $T\otimes T$.

Let $\psi,\phi\in S$ be arbitrary states and $x,y\in\ker\varphi$. Then, according to (i), we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}|\psi(T^k(x))|=0,\qquad \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}|\phi(T^k(y))|=0.$$

So, according to Lemma 4.1, there exist two subsets $J_1,J_2\subset\mathbf{N}$ of density zero such that

$$\lim_{\substack{n\to\infty\\ n\notin J_1}}|\psi(T^k(x))|=0,\qquad \lim_{\substack{n\to\infty\\ n\notin J_2}}|\phi(T^k(y))|=0.$$

Then, for the set $J=J_1\cup J_2$, we have

$$\lim_{\substack{n\to\infty\\ n\notin J}}|\psi(T^k(x))\phi(T^k(y))|=0,$$

and hence again using Lemma 4.1 one gets that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}|\psi(T^k(x))\phi(T^k(y))|=0.$$

Thus,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}|\psi\otimes\phi(T^k\otimes T^k(x\otimes y))| = 0.$$

By G we denote the convex hull of the set $\{\psi\otimes\phi:\ \psi,\phi\in S\}$. It is clear that the $\|\cdot\|_1$-closure of G is S^2. Therefore, given $\varepsilon>0$ and $\omega\in S^2$, there is $\zeta\in G$ such that $\|\omega-\zeta\|_1<\varepsilon$. For ζ there is $n_0\in\mathbf{N}$ such that

$$\frac{1}{n}\sum_{k=0}^{n-1}|\zeta(T^k\otimes T^k(x\otimes y))|<\varepsilon \quad \text{for all } n\ge n_0.$$

Consequently,

$$\begin{aligned}(4.2)\quad &\frac{1}{n}\sum_{k=0}^{n-1}|\omega(T^k\otimes T^k(x\otimes y))|\\ &\le \frac{1}{n}\sum_{k=0}^{n-1}|(\omega-\zeta)(T^k\otimes T^k(x\otimes y))| + \frac{1}{n}\sum_{k=0}^{n-1}|\zeta(T^k\otimes T^k(x\otimes y))|\\ &\le \|\omega-\zeta\|_1\|x\otimes y\| + \varepsilon < \varepsilon(\|x\otimes y\|+1)\end{aligned}$$

for all $n\ge n_0$.

Let $x,y\in\mathfrak{A}$. Denote $x^0=x-\varphi(x)\mathbf{1}$, $y^0=y-\varphi(y)\mathbf{1}$. It is clear that $x^0,y^0\in\ker\varphi$. By means of (4.2), for every $\omega\in S^2$, we have

$$(4.3)\qquad \frac{1}{n}\sum_{k=0}^{n-1}|\omega(T^k\otimes T^k(x^0\otimes y^0))|<\varepsilon \quad \text{for all}\quad n\ge n_1.$$

Denote $\omega_1(x)=\omega(x\otimes\mathbf{1})$, $\omega_2(x)=\omega(\mathbf{1}\otimes x)$, $x\in\mathfrak{A}$. Then, according to condition (i), there exist $N_1,N_2\in\mathbf{N}$ such that

$$(4.4)\qquad \begin{aligned}&\frac{1}{n}\sum_{k=0}^{n-1}|\omega_1(T^k(x))-\varphi(x)|<\varepsilon \quad \text{for all } n\ge N_1,\\ &\frac{1}{n}\sum_{k=0}^{n-1}|\omega_2(T^k(y))-\varphi(y)|<\varepsilon \quad \text{for all } n\ge N_2.\end{aligned}$$

Now using (4.3) and (4.4), we find

$$(4.5)\quad \frac{1}{n}\sum_{k=0}^{n-1}|\omega(T^k\otimes T^k(x\otimes y))-\varphi(x)\varphi(y)|$$
$$\le|\varphi(y)|\left(\frac{1}{n}\sum_{k=0}^{n-1}|\omega_1(T^k(x))-\varphi(x)|\right)$$
$$+|\varphi(x)|\left(\frac{1}{n}\sum_{k=0}^{n-1}|\omega_2(T^k(y))-\varphi(y)|\right)+\frac{1}{n}\sum_{k=0}^{n-1}|\omega(T^k\otimes T^k(x^0\otimes y^0))|$$
$$<\varepsilon(|\varphi(x)|+|\varphi(y)|+1)$$

for all $n\ge\max\{n_1,N_1,N_2\}$.

Now let $z\in\mathfrak{A}\otimes\mathfrak{A}$. Then there exists an element $z_\varepsilon\in\mathfrak{A}\odot\mathfrak{A}$ such that

$$\|z-z_\varepsilon\|<\varepsilon.$$

It follows from (4.5) that

$$\frac{1}{n}\sum_{k=0}^{n-1}|\omega(T^k\otimes T^k(z_\varepsilon))-\varphi\otimes\varphi(z_\varepsilon)|<\varepsilon$$

for all $n\ge n_\varepsilon$. Therefore, we obtain

$$\frac{1}{n}\sum_{k=0}^{n-1}|\omega(T^k\otimes T^k(z))-\varphi\otimes\varphi(z)|$$
$$\le\frac{1}{n}\sum_{k=0}^{n-1}|\omega(T^k\otimes T^k(z-z_\varepsilon))|$$
$$+\frac{1}{n}\sum_{k=0}^{n-1}|\omega(T^k\otimes T^k(z_\varepsilon))-\varphi\otimes\varphi(z_\varepsilon)|+|\varphi\otimes\varphi(z_\varepsilon-z))|$$
$$\le\varepsilon+2\,\|z-z_\varepsilon\|<3\varepsilon$$

for all $n\ge n_\varepsilon$. The last relation implies that $(\mathfrak{A}\otimes\mathfrak{A},\varphi\otimes\varphi,T\otimes T)$ is strictly weak mixing.

The implication (ii) $\Rightarrow$ (iii) is obvious. Let us prove the implication (iii) $\Rightarrow$ (iv). Let $(\mathfrak{A}\otimes\mathfrak{A},\varphi\otimes\varphi,T\otimes T)$ be strictly ergodic. Let $x\in\ker\varphi$,

$x = x^*$. Given $\varepsilon > 0$, strict ergodicity of the dynamical system (see Theorem 3.2) implies that there is $n_{0,x} \in \mathbf{N}$ such that

$$\left\| \frac{1}{n} \sum_{k=0}^{n-1} T^k \otimes T^k (x \otimes x) \right\| < \varepsilon \quad \text{for all } n \geq n_{0,x}.$$

Hence,

$$\left| \frac{1}{n} \sum_{k=0}^{n-1} \psi \otimes \psi (T^k \otimes T^k (x \otimes x)) \right| < \varepsilon \quad \text{for all } n \geq n_{0,x}, \ \forall \psi \in \mathfrak{A}^*_{1,h}.$$

As x is self-adjoint, we get

$$\frac{1}{n} \sum_{k=0}^{n-1} |\psi(T^k(x))|^2 < \varepsilon \quad \text{for all } n \geq n_{0,x}, \ \forall \psi \in \mathfrak{A}^*_{1,h}.$$

According to Lemma 4.1 we infer that there is $n_{1,x} \in \mathbf{N}$ such that

$$\frac{1}{n} \sum_{k=0}^{n-1} |\psi(T^k(x))| < \varepsilon \quad \text{for all } n \geq n_{1,x}, \ \forall \psi \in \mathfrak{A}^*_{1,h}.$$

Consequently,

$$\sup_{\psi \in \mathfrak{A}^*_{1,h}} \frac{1}{n} \sum_{k=0}^{n-1} |\psi(T^k(x))| < \varepsilon \quad \text{for all } n \geq n_{1,x}. \tag{4.6}$$

Let $x \in \ker \varphi$ be an arbitrary element. Then it can be represented as $x = x_1 + ix_2$, where $x_1, x_2 \in \ker \varphi$, $x_j^* = x_j$, $j = 1, 2$. It then follows from (4.6) that

$$\sup_{\psi \in \mathfrak{A}^*_{1,h}} \frac{1}{n} \sum_{k=0}^{n-1} |\psi(T^k(x))| < 2\varepsilon \quad \text{for all } n \geq n_{1,x} := \max\{n_{1,x_1}, n_{1,x_2}\}. \tag{4.7}$$

Let $\psi \in \mathfrak{A}_1^*$. Then $\psi = \psi_1 + i\psi_2$, where $\psi_j \in \mathfrak{A}_{1,h}^*$, $j = 1, 2$. By means of (4.7) one obtains

(4.8)
$$\sup_{\psi \in \mathfrak{A}_1^*} \frac{1}{n} \sum_{k=0}^{n-1} |\psi(T^k(x))| \leq \sup_{\psi_1 \in \mathfrak{A}_{1,h}^*} \frac{1}{n} \sum_{k=0}^{n-1} |\psi_1(T^k(x))| + \sup_{\psi_2 \in \mathfrak{A}_{1,h}^*} \frac{1}{n} \sum_{k=0}^{n-1} |\psi_2(T^k(x))| < 4\varepsilon, \quad \forall n \geq n_{1,x}.$$

Finally let $x \in \mathfrak{A}$. Then we have the last relation (4.8) for the element $x^0 = x - \varphi(x)\mathbf{1}$, which implies that

$$\lim_{n\to\infty} \sup_{\psi \in \mathfrak{A}_1^*} \frac{1}{n} \sum_{k=0}^{n-1} |\psi(T^k(x)) - \psi(\mathbf{1})\varphi(x)| = 0.$$

So the implication (iii) $\Rightarrow$ (iv) is proved. The implications (iv) $\Rightarrow$ (v)$\Rightarrow$ (i) are obvious. □

Remark. The implication (i) $\Leftrightarrow$ (v) can be proved directly using only positivity of the operator T. Indeed, it is enough to prove the implication (i)$\Rightarrow$ (v). Let $x \in \ker \varphi$. Assume that $\psi \in \mathfrak{A}_h^*$ is a positive functional. Then $\tilde{\psi}(x) = 1/(\psi(\mathbf{1}))\psi(x)$ is a state. Hence, using (2.3), we get

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} |\tilde{\psi}(T^k(x))| = 0$$

which means

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} |\psi(T^k(x))| = 0. \tag{4.9}$$

Now let $\psi \in \mathfrak{A}^*$ be an arbitrary functional; then it can be represented as $\psi = \sum_{m=0}^{3} i^m \psi_m$, where $\psi_m \in \mathfrak{A}_h^*$, $m = 0, 1, 2, 3$, are positive functionals. By means of (4.9), we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} |\psi(T^k(x))| \leq \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \sum_{m=0}^{3} |\psi_m(T^k(x))| = 0.$$

Using the same argument as in the final part of the proof (iii) $\Rightarrow$ (iv) we obtain the required assertion. Therefore, if we take $\psi(x) = \varphi(yx), x \in \mathfrak{A}$ in (v), we easily get (2.2); this means that strictly weak mixing implies weak mixing.

Using the same argument as the previous theorem, one can prove the following

Theorem 4.3. *Let $(\mathfrak{A}, \varphi, T)$ be a state preserving C^*-dynamical system and T be a completely positive map. The following conditions are equivalent*:

(i) $(\mathfrak{A}, \varphi, T)$ *is weak mixing*;

(ii) *The state preserving C^*-dynamical system $(\mathfrak{A} \otimes \mathfrak{A}, \varphi \otimes \varphi, T \otimes T)$ is weak mixing.*

(iii) *The state preserving C^*-dynamical system $(\mathfrak{A} \otimes \mathfrak{A}, \varphi \otimes \varphi, T \otimes T)$ is ergodic.*

Remark. It should be noted that Theorem 4.3 adapts Theorem 6.3 of [**16**] to a C^*-algebra setting.

From Lemma 3.1 we infer that 1 is an eigenvalue of multiplicity one for T^* when $(\mathfrak{A}, \phi, T)$ is strictly ergodic. Now what can we say about strictly weak mixing dynamical systems? We have the following

Proposition 4.4. *Let $(\mathfrak{A}, \varphi, T)$ be strictly weak mixing. If there exist a number $\alpha \in \mathbf{C}$ with $|\alpha| = 1$ and $\alpha \neq 1$, and $h \in \mathfrak{A}^*$ such that*

$$h \circ T = \alpha h, \tag{4.10}$$

then $h = 0$.

Proof. Assume that $h \neq 0$. Then $h \neq \mu\varphi$ for all $\mu \in \mathbf{C}$. Now choose $x \in \mathfrak{A}$ such that $h(x)$ is nonzero. Observe that the hypothesis (4.10) with $\alpha \neq 1$ implies that $h(\mathbf{1}) = 0$. Therefore, using $|\alpha| = 1$, one gets

$$\frac{1}{n}\sum_{k=0}^{n-1} |h(T^k(x)) - h(\mathbf{1})\varphi(x)| = \frac{1}{n}\sum_{k=0}^{n-1} |\alpha^k h(x) - h(\mathbf{1})\varphi(x)|$$

$$= |h(x)| > 0 \quad \text{for all } n \in \mathbf{N}$$

which contradicts the strictly weak mixing condition. □

Now we are going to give a concrete example of a strictly weak mixing C^*-dynamical system.

Example 1. Let $\mathfrak{A} = \ell^\infty = \{(x_n) : x_n \in \mathbf{C},\ \sup |x_n| < \infty\}$. Define an operator $T : \ell^\infty \mapsto \ell^\infty$ by means of matrix $(t_{ij})_{i,j\in\mathbf{N}}$ such that $t_{ij} = 1/2^j$, $i, j \geq 1$. It is not hard to check that $\varphi = (1/2, 1/2^2, \ldots, 1/2^n, \ldots)$ is an invariant state with respect to T. It is known from the Theory of Markov Chains with countable state space, see [**12**], that $\psi(T^n x)$ converges to φ in norm of $\mathfrak{A}^*$ for every state $\psi \in \mathfrak{A}^*$. Consequently, T is strictly weak mixing.

The following example shows that strict ergodicity does not imply strict weak mixing.

Example 2. Let $S^1 = \{z \in \mathbf{C} : |z| = 1\}$ and λ be the Lebesgue measure on S^1 such that $\lambda(S^1) = 1$. Fix an element $a = \exp(i2\pi\alpha)$, where $\alpha \in [0, 1)$ is an irrational number. Define a transformation $\tau : S^1 \mapsto S^1$ by $\tau(z) = az$. The measure induces a positive linear functional $\varphi_\lambda(f) = \int_{S^1} f(z)\, d\lambda(z)$ such that $\varphi_\lambda(\mathbf{1}) = 1$. Consider a C^*-algebra $\mathfrak{A} = C(S^1)$, where $C(S^1)$ is the space of all continuous functions on S^1. Now by means of τ define a positive linear operator $T_\tau : C(S^1) \mapsto C(S^1)$ by $(T_\tau(f)(z)) = f(\tau(z))$ for all $f \in C(S^1)$. It is clear that $(C(S^1), \varphi_\lambda, T_\tau)$ is a state preserving C^*-dynamical system. Since α is irrational, then according to [**8**, Theorem 2, Chapter 3] we find that the defined dynamical system is strictly ergodic. On the other hand, it is not strictly weak mixing. Indeed, take a linear functional $h \in C(S^1)^*$ defined by $h(f) = \int_{S^1} z f(z) d\lambda(z)$, $f \in C(S^1)$. Then we have $h(T_\tau(f)) = a^{-2} h(f)$ for all $f \in C(S^1)$. Thus Proposition 4.4 implies that T_τ is not strictly weak mixing. It should be noted that T_τ is also not weakly mixing, see [**15**, Theorem 1.27].

The next example shows that strict ergodicity is stronger that ergodicity of C^*-dynamical systems.

Example 3. Consider C^*-algebra $\mathfrak{A} = \otimes_{\mathbf{Z}} M_2(\mathbf{C})$, where $M_2(\mathbf{C})$ is the algebra of 2×2 matrices over the field $\mathbf{C}$ of complex numbers. By $e_{ij}^{(n)}$, $n \in \mathbf{Z}$, $i,j \in \{1,2\}$, we denote the basis matrices of the algebra $M_2(\mathbf{C})$ sited on nth place in the tensor product $\otimes_{\mathbf{Z}} M_2(\mathbf{C})$. The shift automorphism $\theta : \mathfrak{A} \mapsto \mathfrak{A}$ of the algebra $\mathfrak{A}$ is defined by $\theta(e_{ij}^{(n)}) = e_{ij}^{(n+1)}$ for every $n \in \mathbf{Z}$ and $i,j \in \{1,2\}$.

Let tr be the normalized trace on $M_2(\mathbf{C})$, i.e., $tr(\mathbf{1}) = 1$. Let $\varphi_0(\cdot) = tr(\rho(\cdot))$ be a state on $M_2(\mathbf{C})$, where $\rho \in M_2(\mathbf{C})$ is a positive operator such that $tr(\rho) = 1$. Such kind of ρ is called a *density operator* for φ_0. Now let $K : M_2(\mathbf{C}) \mapsto M_2(\mathbf{C})$ be a completely positive Markov operator such that $\varphi_0(x) = \varphi_0(Kx)$ for every $x \in M_2(\mathbf{C})$. On the algebra $\mathfrak{A}_{[k,n]} = \otimes_{[k,n]} M_2(\mathbf{C})$ define the following linear functional

$$\varphi_{[k,n]}(a_k \otimes a_{k+1} \otimes \cdots \otimes a_n) = \varphi_0(a_k K(a_{k+1}(\cdots K(a_n)\cdots))).$$

The defined functional $\varphi_{[k,n]}$ is a state, see [**2, 3**]. If a compatibility condition holds

$$\varphi_{[k,n]} \restriction \mathfrak{A}_{[k-1,n-1]} = \varphi_{[k-1,n-1]}$$

for the states $\{\varphi_{[k,n]}\}$, then there is a state φ_K on $\mathfrak{A}$ such that $\varphi_K \restriction \mathfrak{A}_{[k,n]} = \varphi_{[k,n]}$, see [**1**], and φ is called a *Markov state*. We note that a more general definition of Markov state was given in [**1, 2**].

It is easy to see that the Markov state is invariant with respect to θ. Define two Markov operators $K_i : M_2(\mathbf{C}) \mapsto M_2(\mathbf{C})$, $i = 1,2$ by

$$K_1 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} p_{11}a + p_{12}d & 0 \\ 0 & p_{21}a + p_{22}d \end{pmatrix},$$
$$K_2 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} q_1 a + q_2 d & 0 \\ 0 & q_1 a + q_2 d \end{pmatrix}.$$

Here $\mathbf{P} = (p_{ij})$ is a stochastic matrix such that $p_{ij} > 0$ for all i,j, and $q_1 + q_2 = 1$, $q_1, q_2 > 0$.

Now consider two states $\varphi_{0,1}$ and $\varphi_{0,2}$ defined on $M_2(\mathbf{C})$, whose density operators are given by

$$\rho_1 = \begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix}, \qquad \rho_2 = \begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix}$$

where $\pi = (p_1, p_2)$ is a vector such that $p_1 + p_2 = 1$, $p_1 \geq 0$, $p_2 \geq 0$ and $\pi\mathbf{P} = \pi$.

Note that for these operators and states the compatibility condition is satisfied; therefore, there are two associated Markov states φ_{K_1} and φ_{K_2}.

Then $(\mathfrak{A}, \varphi_{K_1}, \theta)$ and $(\mathfrak{A}, \varphi_{K_2}, \theta)$ are weak mixing, and hence ergodic, state preserving C^*-dynamical systems, see [**7**, Theorems 4.1 and 4.5]. On the other hand, they are not strictly ergodic because there exist two invariant states with respect to θ.

Remark. From Examples 2 and 3 we conclude that weak mixing and strict ergodicity are not comparable. Therefore, we may formulate the following

Problem 4.5. *Let $(\mathfrak{A}, \varphi, T)$ be a state preserving C^*-dynamical system. Are the following conditions equivalent?*

(i) $(\mathfrak{A}, \varphi, T)$ *is weak mixing and strictly ergodic*;

(ii) $(\mathfrak{A}, \varphi, T)$ *is strictly weak mixing.*

Recall a state preserving dynamical system $(\mathfrak{A}, \varphi, T)$ is called *exact*, see [**9**], if for each $\psi \in \mathfrak{A}^*$

$$\lim_{n\to\infty} \|\psi \circ T^n - \psi(\mathbf{1})\varphi\|_1 = 0$$

is valid, where $\|\cdot\|_1$ is the norm in $\mathfrak{A}^*$. It is not hard to see that the exactness implies strict weak mixing. In [**9**] Łuczak proved that exact and weak mixing conditions, for dynamical semi-groups on von Neumann algebras, are equivalent if and only if the von Neumann algebra is strongly $\mathbf{R}_+$-finite. Regarding this result we can formulate the following

Problem 4.6. *Let $(\mathfrak{A}, \varphi, T)$ be a state preserving C^*-dynamical system. When are the following conditions equivalent?*

(i) $(\mathfrak{A}, \varphi, T)$ *is exact*;

(ii) $(\mathfrak{A}, \varphi, T)$ *is strictly weak mixing.*

Now by S_0 denote the set of all continuous functionals $f : \mathfrak{A}_+ \mapsto \mathbf{R}_+$ such that

$$f(\lambda x) = \lambda f(x) \quad \text{for all } \lambda \in \mathbf{R}_+,\ x \in \mathfrak{A}_+,$$
$$f(\mathbf{1}) = 1.$$

Now we introduce a notion of ϕ-ergodicity. Namely, a state preserving C^*-dynamical system $(\mathfrak{A}, \varphi, T)$ is called *ϕ-ergodic* if the equality

$$f(T(x)) = f(x) \quad \text{for all } x \in \mathfrak{A}_+, \tag{4.11}$$

where $f \in S_0$, implies that $f(x) = \varphi(x)$ for all $x \in \mathfrak{A}_+$.

Theorem 4.7. *Let $(\mathfrak{A}, \varphi, T)$ be a state preserving C^*-dynamical system. Then for the conditions*:

(i) *For every $x \in \mathfrak{A}$ the following equality holds*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \|T^k(x) - \varphi(x)\mathbf{1}\| = 0;$$

(ii) *For every $x \in \mathfrak{A}$ the following equality holds*

$$\lim_{n\to\infty} \|T^n(x) - \varphi(x)\mathbf{1}\| = 0;$$

(iii) $(\mathfrak{A}, \varphi, T)$ *is ϕ-ergodic*;

(iv) *For every $\psi \in S$ and $x \in \mathfrak{A}$ the following equality holds*

$$\lim_{n\to\infty} \psi(T^n(x)) = \varphi(x).$$

The following implications hold: (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv).

Proof. The (i) $\Leftarrow$ (ii) implication is obvious. Consider the implication (i) $\Rightarrow$ (ii). Assume that $x \in \ker \varphi$; then we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \|T^k(x)\| = 0. \tag{4.12}$$

On the other hand, one gets

$$\|T^{n+1}(x)\| \leq \|T^n(x)\|;$$

this means that the sequence $\{\|T^n x\|\}$ is nonincreasing. Hence, we have $\lim_{n\to\infty} \|T^n x\| = \alpha$. It follows from (4.12) that $\alpha = 0$. Let $x \in \mathfrak{A}$; then setting $x^0 = x - \varphi(x)\mathbf{1}$ we find

$$\lim_{n\to\infty} \|T^n(x^0)\| = 0,$$

which implies (ii).

(ii) $\Rightarrow$ (iii). Assume that (4.11) is valid for some $f \in S_0$. According to condition (ii) we have

$$T^n(x) \to \varphi(x)\mathbf{1} \quad \text{as } n \to \infty$$

for $x \in \mathfrak{A}_+$, here the convergence in norm of $\mathfrak{A}$. By means of continuity of f one gets

$$f(T^n(x)) \to f(\varphi(x)\mathbf{1}) = \varphi(x) \quad \text{as } n \to \infty.$$

On the other hand, (4.11) implies that $f(x) = \varphi(x)$, for all $x \in \mathfrak{A}_+$. So $(\mathfrak{A}, \varphi, T)$ is ϕ-ergodic.

(iii) $\Rightarrow$ (iv). Let $\psi \in S$. Define functionals $\hat{f} : \mathfrak{A}_+ \mapsto \mathbf{R}_+$, $\check{f} : \mathfrak{A}_+ \mapsto \mathbf{R}_+$ by

$$\hat{f}(x) = \limsup_{n\to\infty} \psi(T^n(x)), \quad x \in \mathfrak{A}_+,$$
$$\check{f}(x) = \liminf_{n\to\infty} \psi(T^n(x)), \quad x \in \mathfrak{A}_+.$$

It is clear that $\hat{f}, \check{f} \in S_0$. We have

$$\hat{f}(Tx) = \limsup_{n\to\infty} \psi(T^{n+1}(x)) = \hat{f}(x).$$

Similarly, $\check{f}(Tx) = \check{f}(x)$. Hence, ϕ-ergodicity of $(\mathfrak{A}, \varphi, T)$ implies that

$$\hat{f}(x) = \varphi(x), \quad \check{f}(x) = \varphi(x) \quad \forall x \in \mathfrak{A}_+.$$

Consequently, we infer the existence of the following limit

$$\lim_{n\to\infty} \psi(T^n(x)) = \varphi(x), \quad x \in \mathfrak{A}_+.$$

Every $x \in \mathfrak{A}$ can be written as $x = \sum_{m=0}^{3} i^m x_m$, $x_m \in \mathfrak{A}_+$, $m = 0, 1, 2, 3$; therefore, by means of the last equality, we get

$$\lim_{n\to\infty} \psi(T^n(x)) = \varphi(x), \quad x \in \mathfrak{A}.$$

This completes the proof. □

This theorem leads us to ask

Problem 4.8. *Is the implication* (iv) $\Rightarrow$ (iii) *true?*

It is clear that the exactness of a dynamical system implies condition (iv). Therefore, it is natural to formulate the following

Problem 4.9. *Let* $(\mathfrak{A}, \varphi, T)$ *be a state preserving* C^**-dynamical system. How are the following conditions related with each other?*

(i) $(\mathfrak{A}, \varphi, T)$ *is* ϕ*-ergodic*;

(ii) $(\mathfrak{A}, \varphi, T)$ *is exact.*

Remark. If C^*-algebra $\mathfrak{A}$ is finite dimensional, then all conditions in Theorem 4.7 are equivalent.

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