ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 37, Number 5, 2007

GLOBAL ATTRACTORS FOR CROSS DIFFUSION SYSTEMS ON DOMAINS OF ARBITRARY DIMENSION

HENDRIK KUIPER AND LE DUNG

ABSTRACT. A general triangular cross diffusion system given on a domain of arbitrary dimension n is considered. It will be shown that (L^∞, L^p) boundedness implies uniformly boundedness. The general result is then applied to several systems to obtain global existence. In some cases, the existence of a global attractor is also proven.

1. Introduction. Ever since the fundamental work by Amann, see [**2**–**5**], there has been much interest in the study of strongly coupled parabolic systems. The question of local existence of solutions was settled by Amann's work but global existence results seem to be answered in only very few cases.

In this paper we will consider a class of triangular cross diffusion systems given on an open bounded domain Ω in $\mathbb{R}^n$ with $n \geq 1$. Let us consider quasilinear/linear differential operators

$$\begin{aligned}\mathcal{A}_u(u,v) &= \nabla(P(x,u,v)\nabla u + R(x,u,v)\nabla v),\\ \mathcal{A}_v(v) &= \nabla(Q(x,v)\nabla v) + c(x)v,\end{aligned}$$

and the following parabolic system

$$\begin{cases} \partial u/\partial t = \mathcal{A}_u(u,v) + g(u,v) & x \in \Omega,\ t > 0,\\ \partial v/\partial t = \mathcal{A}_v(v) + f(u,v) & x \in \Omega,\ t > 0,\end{cases} \tag{1.1}$$

with mixed boundary conditions for $x \in \partial\Omega$ and $t > 0$

$$\begin{cases} \chi(x)[(\partial v/\partial n)(x,t) + \alpha(x)v(x,t)] + (1-\chi(x))v(x,t) = 0,\\ \chi(x)[(\partial u/\partial n)(x,t) + \beta(x)u(x,t)] + (1-\chi(x))u(x,t) = 0,\end{cases} \tag{1.2}$$

where χ is a given function on $\partial\Omega$ with values in $\{0,1\}$. The initial conditions are described by

$$v(x,0) = v^0(x), \quad u(x,0) = u^0(x), \quad x \in \Omega \tag{1.3}$$

Received by the editors on August 23, 2004, and in revised form on April 19, 2005.

for nonnegative functions v^0, u^0 in $W^{1,p}(\Omega)$ for some $p > n$, see [**2**]. In (1.1), P and Q represent the *self-diffusion* pressures, and R is the *cross-diffusion* pressure acting on the population u by v.

We are interested not only in the question of global existence of solutions to (1.1) but also in long time dynamics of the solutions. The assumptions on the parameters defining (1.1) will be specified later in Section 3 and they are general enough to cover many interesting models investigated in literature. Furthermore, our conclusion is far stronger, in some cases, than what has been known about these systems. To demonstrate this, in the next section, we will first discuss some well-studied systems and state our findings, which are the consequences of our general Theorem 6. Roughly speaking, we establish the following.

A solution (u, v) *of* (1.1) *exists globally in time if the norms* $\|v(\cdot,t)\|_\infty$ *and* $\|u(\cdot,t)\|_{n/2}$, *or even* $\|u(\cdot,t)\|_1$, *do not blow up in finite time. Moreover, if these norms of the solutions are ultimately uniformly bounded, then an absorbing set exists, and therefore there is a compact global attractor, with finite Hausdorff dimension, attracting all solutions.*

Our main vehicle is the L^p estimates proven in Lemma 3.5. This was done, in various applications, by several authors using well-known energy estimates or Lyapunov functional methods, see [**7, 20, 33**]. However, we want to remark that a crucial use of imbedding inequalities in those works forced the assumption on the dimension n being at most 2. In particular, the authors of [**20, 30**] used energy estimate methods and a special version of Gagliardo-Nirenberg's inequality, which is valid only when $n \leq 2$, to establish that L^2 norms of solutions do not blow up in finite time. Similar results were obtained for an electrochemistry model considered in [**7**] using the Lyapunov functional approach. It should be noted that the three-dimensional case was also studied in [**7**[assuming that a priori L^2 estimates were given. This hypothesis is however not established in [**7**] and is difficult to verify. Here, starting with a weaker assumption on a priori L^1 estimates, we combine the energy method with semigroup theory and integro-differential inequalities to obtain L^p estimates for arbitrary dimension n. Furthermore, our approach also gives bounds that are ultimately independent of initial data, a crucial fact in establishing the existence of global attractors. The energy estimate and Lyapunov functional methods mentioned above do not provide such estimates.

Another fact that distinguishes our paper from the other works is that the functional spaces considered here are the Banach spaces $W^{1,p}(\Omega)$, with $p > n$, while other authors usually studied the problems in the Hilbert spaces $H^1(\Omega)$ or $H^2(\Omega)$. This Hilbert space setting is closely linked to the restriction that the dimension of the domain be at most two. Similar to our approach were the works [**12**] where the spatial region is assumed to be a compact Riemannian manifold without boundary and [**14**] where only L^p estimates were derived. We also note the related work [**35**] where the concept of exponential attractors was used in a Hilbert space setting to prove the existence of a global attractor for certain one-dimensional chemotaxis models.

Our assumption on (1.1) and the main results, together with their proofs, will be given in Section 3. Finally, in Section 4, we provide the proof of the theorems for the examples in Section 2.

2. Applications. In this section, we will apply our theorems in Section 3 to several cross diffusion parabolic systems modeling biological and ecological phenomena. Most of them have been studied by authors assuming that the domain Ω is two dimensional. In only very few cases can they establish global existence results. It will be shown in Section 4 that the conditions of our theorems are fulfilled to give global existence results without any restriction on the dimension of Ω or the size of initial data. Moreover, we can further assert the existence of global attractors.

Perhaps the most famous model is the Keller-Segel model of two parabolic partial differential equations, which describes the aggregation of the cellular slime mold *Dictyostelium discoideum*, see [**22**]. After some simplifying assumptions one is led to the following system:

$$
\begin{cases}
\partial u/\partial t = d_1 \Delta u + \alpha_{12} \nabla(u \nabla v), \\
\partial v/\partial t = d_2 \Delta v + bu - cv, \\
\partial u/\partial n = \partial v/\partial n = 0, & x \in \partial\Omega,\ t > 0.
\end{cases}
\tag{2.1}
$$

The constants b, c are taken to be positive. This system has attracted much attention in recent years but many authors assumed that *the dimension of the domain Ω is at most two*. It was conjectured by Nanjundiah, Childress and Percus, [**6**], that there is a threshold number c_* such that, if the L^1 norm of the initial data $\|u(x,0)\|_1 < c_*$, then the solution exists globally in time and, if $\|u(x,0)\|_1 > c_*$, then the solution u can form a delta function singularity in finite time. The latter case is referred to as chemotactic collapse. The arguments were heuristic, making use of numerical computations for the stationary problem, but recent studies (e.g. [**11, 21, 34**]) have confirmed their validity rigorously.

In order to prove global existence result for this type of systems one must obviously introduce some modifications. For example in [**12, 13, 19**] the authors use the choice of the chemotactic sensitivity function to ensure the existence of a global solution. We shall consider a modified version of the Keller-Segel on arbitrary dimensional domains:

$$
\begin{cases} \partial u/\partial t = \nabla[(d_1 + \alpha_{11}u)\nabla u] + \alpha_{12}\nabla(u\nabla v), \\ \partial v/\partial t = d_2\Delta v + bu - cv, \\ \partial u/\partial n = \partial v/\partial n = 0, & x \in \partial\Omega,\ t > 0. \end{cases} \tag{2.2}
$$

Here, being inspired by porous media models and (2.3) below, we introduced the "crowding effect" in the model by adding the term $\alpha_{11}u$ in the diffusion coefficient of u. Another type of "crowding effect" was also introduced in the off diagonal diffusion term, R of (1.1), in [**12**] that allowed an invariant region to exist and gave the global existence. In our case, such an approach does not seem to apply. However, (2.2) is a special case of the general model (1.1), and our Theorem 5 applies here to conclude that

Theorem 1. *Consider the modified Keller-Segel model* (2.2) *on a bounded domain* Ω *of any dimension* n*. For any given initial data in* $W^{1,p}(\Omega)$*,* $p > n$*, the solutions to* (2.2) *exist globally in time.*

It should be noted that our definition of *solution* corresponds to that used by Amann ([**2, 3, 4**]) and will be precisely stated in the next section. It should also be noted that at present there is a wealth of results on finite-time blowup of solutions, e.g., [15, 18, 19, 42]. The reader is referred to the extensive survey articles by Horstmann [**16, 17**].

Our next example is a cross diffusion model in population dynamics. Shigesada, Kawasaki and Teramoto, see [**37**], proposed to study the following nonlinear parabolic system

$$(2.3)\qquad \begin{cases} \partial u/\partial t = \Delta[(d_1 + \alpha_{11}u + \alpha_{12}v)u] + u(a_1 - b_1 u - c_1 v), \\ \partial v/\partial t = \Delta[(d_2 + \alpha_{21}u + \alpha_{22}v)v] + v(a_2 - b_2 u - c_2 v), \\ \partial u/\partial n = \partial v/\partial n = 0, \qquad x \in \partial\Omega, t > 0, \\ u(x,0) = u^0(x), v(x,0) = v^0(x), \qquad x \in \Omega. \end{cases}$$

Here, Ω is a bounded domain in $\mathbb{R}^n$ and the initial data u^0, v^0 are nonnegative functions.

When $\alpha_{ij} = 0$, the above system is the well known Lotka Volterra competition-diffusion system which has been studied intensively. For nonzero α_{ij}, (2.3) is a strongly coupled parabolic system. Yagi, see [**41, 43**], investigated the global existence problem for (2.3) which is given on a two-dimensional domain. Under certain conditions on α_{ij}'s, he proved that solutions to (2.3) cease to exist in finite time if and only if their L^p norms blow up. Recently, Lou, Ni and Wu in [**33**] studied the case when $\alpha_{21} = 0$ and $n = 2$ and established global existence results for the system

$$(2.4)\qquad \begin{cases} \partial u/\partial t = \Delta[(d_1 + \alpha_{11}u + \alpha_{12}v)u] + u(a_1 - b_1 u - c_1 v), \\ \partial v/\partial t = \Delta[(d_2 + \alpha_{22}v)v] + v(a_2 - b_2 u - c_2 v), \\ \partial u/\partial n = \partial v/\partial n = 0, \qquad x \in \partial\Omega, t > 0. \end{cases}$$

In [**27**], we discussed not only global existence but also long time dynamics of solutions to a class of cross diffusion systems which includes (2.4). Again, we had to assume that the dimension of the domain Ω is two. We have here a better result.

Theorem 2. *Consider* (2.4) *on a bounded domain* Ω *of any dimension* n. *The dynamical system associated with* (2.4) *possesses a compact global attractor if either*

i) $\alpha_{22} = 0$ *and* n *is arbitrary*;

ii) *or* $\alpha_{22} > 0$ *and* $n = 2$.

Finally, we look at a system modeling bio-reactors with chemotactic effect.

$$(2.5)\qquad \begin{cases} \partial u/\partial t = \nabla(d_1 \nabla u) + \nabla(u\Phi(S)\nabla S) + u(f(S) - k), \\ \partial S/\partial t = d_2 \Delta S - \gamma u f(S), \\ \partial u/\partial n + \alpha u = 0, \partial S/\partial n + \beta S = S_0, \qquad x \in \partial\Omega, t > 0. \end{cases}$$

Here, k, α, β, γ are positive constants and $\Phi(S), f(S)$ are continuous functions, and $f(S) > 0$. This system was studied in [**28, 40**] where Ω is assumed to be an interval in $\mathbb{R}$. In [**23**], arbitrary dimensional domains are considered, and a Lyapunov function technique is used to establish global existence assuming some condition on the size of initial data for S. However, this method does not seem to cover (1.1). For higher dimensional domains, coexistence problems were investigated in [**31, 32**]. Introducing the "crowding effect" term $\alpha_{11}u$ in the diffusion coefficient for u, see also [**12**], we are led to the following system.

$$
(2.6)\quad \begin{cases} \partial u/\partial t = \nabla[(d_1 + \alpha_{11})u)\nabla u] + \nabla(u\Phi(S)\nabla S) \\ \qquad\qquad +u(f(S) - k), \\ \partial S/\partial t = d_2 \Delta S - \gamma u f(S), \\ \partial u/\partial n + \alpha u = 0, \quad \partial S/\partial n + \beta S = S_0, \qquad x \in \partial\Omega, t > 0. \end{cases}
$$

Although the boundary condition in (2.6) is of Robin type, our proof can be easily modified to cover this case. Our Theorem 6 applies again and asserts that

Theorem 3. *Consider* (2.6) *on a bounded domain* Ω *of any dimension* n. *The dynamical system associated with* (2.6) *possesses a compact global attractor in* $W^{1,p}(\Omega, \mathbb{R}^2)$ *for any* $p > n$.

3. Main results. In this section, we will specify our assumptions on the general system (1.1) and state our main results. First we state precisely what we mean by a solution. We follow Amann [**5**].

Let C^{k-} denote the functions whose derivatives of order $(k-1)$ are Lipschitz continuous. For $1 \le i, j \le m$, we assume

$$
a_{ij}, a_i, b_i \in C^{2-}(\overline{\Omega} \times \mathbf{R}^m, \mathbf{R}^{m\times m}),
$$

$$
a_0, c \in C^{1-}(\overline{\Omega} \times \mathbf{R}^m, \mathbf{R}^{m\times m}).
$$

Using the summation convention we define for each $\eta \in \mathbb{R}^m$ the following elliptic operator and boundary operator:

$$
\mathcal{A}(\eta)u := -\partial_j(a_{jk}(\cdot, \eta)\partial_k u + a_j(\cdot, \eta)u) + b_j(\cdot, \eta)\partial_j u + a_0(\cdot, \eta)u,
$$

and

$$
\mathcal{B}(\eta)u := \nu^j \gamma_0(a_{jk}(\cdot, \eta)\partial_k u + a_j(\cdot, \eta)u) + c(\cdot, \eta)\gamma_0 u,
$$

interpreted in the sense of traces. Their formal adjoints are

$$\mathcal{A}^{\#}(\eta)u := -\partial_j(a^{\#}_{jk}(\cdot,\eta)\partial_k u + a^{\#}_j(\cdot,\eta)u) + b^{\#}_j(\cdot,\eta)\partial_j u + a^{\#}_0(\cdot,\eta)u,$$

and

$$\mathcal{B}^{\#}(\eta)u := \nu^j\gamma_0(a^{\#}_{jk}(\cdot,\eta)\partial_k u + a^{\#}_j(\cdot,\eta)u) + c^{\#}(\cdot,\eta)\gamma_0 u,$$

where, letting the left superscript ^{t} denote transpose,

$$a^{\#}_{jk} := {}^t a_{kj}\,, \quad a^{\#}_j := {}^t b_j\,, \quad b^{\#}_j := {}^t a_j\,, \quad a^{\#}_0 := {}^t a_0\,, \quad c^{\#} := {}^t c.$$

Let a_π and b_π denote the principal symbols for $\mathcal{A}$ and $\mathcal{B}$: $a_\pi(x,\eta,\xi) := a_{ij}(x,\eta)\xi^i\xi^j$, and $b_\pi(x,\eta,\xi) := \nu^i a_{ij}(x,\eta)\xi^j$, where $\xi = (\xi^1,\xi^2,\dots,\xi^n) \in \mathbf{R}^n$. We assume that, for each η, the operator $\mathcal{A}(\eta)$ is *normally elliptic*. By this is meant that for each $x \in \overline{\Omega}$, $\eta \in \mathbf{R}^m$, and $\xi \in \mathbf{R}^n$ with $\|\xi\| = 1$ the spectrum of $a_\pi(x,\eta,\xi) \subset \mathbf{C}_+ := \{z \in \mathbf{C} \mid \operatorname{Re} z > 0\}$. We also assume that $\mathcal{B}$ satisfies the normal complementing condition (Lopatinskii-Shapiro condition) with respect to $\mathcal{A}$. This means that for each (x,ξ) in the tangent bundle of $\partial\Omega$ and each $\lambda \in \mathbf{C}_+$ with $(\xi,\lambda) \neq (0,0)$, 0 is the only exponentially decaying solution on the half line for:

$$[\lambda + a_\pi(x,\xi+\nu(x)i\partial_t)]u = 0, \quad t > 0, \qquad b_\pi(x,\xi+\nu(x)i\partial_t)u(0) = 0.$$

It is not difficult to see that our problems satisfy these restrictions. Consider the problem

$$\begin{aligned} \partial_t u + \mathcal{A}(u)u &= f(\cdot,u) \quad \text{in} \quad \Omega\times(0,\infty) \\ \mathcal{B}(u)u &= g(\cdot,u) \quad \text{on} \quad \partial\Omega\times(0,\infty) \\ u(\cdot,0) &= u_0 \quad \text{on} \quad \Omega, \end{aligned} \tag{3.1}$$

where we assume that f and g are Lipschitz continuous. We define

$$W^s_{q,\mathcal{B}} := \{(w_1,w_2,\dots,w_m) \mid w_i \in W^s_q(\Omega) \quad \text{and} \quad \mathcal{B}(w)w = g(\cdot,w)\,\forall i\}.$$

We say that $u : [0,T] \to W^s_{q,\mathcal{B}}$ is a weak $W^s_{q,\mathcal{B}}$-solution of the above problem on $[0,T]$ if

$$u \in C([0,T], W^{s-2}_{q,\mathcal{B}}) \cap C((0,T), W^s_{q,\mathcal{B}} \cap C^1((0,T), W^{s-2}_{q,\mathcal{B}}),$$

and satisfies $u(0) = u_0$. We then have the following existence theorem:

Theorem (Amann). *Suppose that $n/q < s < (1+1/q) \wedge (2-n/q)$. Then the above boundary value problem has for each $u_0 \in W^s_{q,\mathcal{B}}(\Omega)$ a unique maximal weak $W^s_{q,\mathcal{B}}(\Omega)$-solution. If this solution remains bounded in $W^\rho_{\rho,\mathcal{B}}$ for some $\rho > 1$ then the solution exists on all of $[0,\infty)$. Moreover, if $g \equiv 0$, then the solution is in fact a classical solution. That is to say,*

$$u \in C(\overline{\Omega} \times [0,T]) \cap C^{2,1}(\overline{\Omega} \times (0,T)),$$

$u(0) = u_0$, and u satisfies the parabolic partial differential equation and boundary conditions pointwise.

In order to simplify the statements of our theorems and proofs, we will make use of the following terminology.

Definition 4. Consider the initial-boundary value problem (1.1), (1.2) and (1.3). Assume that there exists a solution (u,v) defined on a subinterval $I = (0,a)$ of $\mathbb{R}_+$, $0 < a \leq \infty$. We define

$$\mathcal{O}_I = \{\omega : I \to \mathbb{R} : \omega(t) \leq C_0,\ \forall t \in I \text{ for some constant } C_0\}. \tag{3.2}$$

(3.3)
$$\mathcal{P}_I = \{\omega : I \to \mathbb{R} :\ \omega \in \mathcal{O}_{(0,b)} \text{ for each } b < a. \text{ Moreover, if } I = (0,\infty), \text{ then for some constant } C \text{ independent of } u^0,\ v^0 \text{ we have } \limsup_{t\to\infty} \omega(t) \leq C\}.$$

Examples of functions in $\mathcal{P}_I$ include $\omega(t) = e^{-t}\|u^0\|_\infty$. On the other hand, if $\|u(\cdot,t)\|_\infty$, $\|v(\cdot,t)\|_\infty$, as functions in t, belong to $\mathcal{O}_I$, then (3.2) says that the supremum norms of the solutions to (1.1) do not blow up in any finite time interval. This implies that the solution exists globally, see [**2**]. We remark that the constant C_0 in (3.2) may depend on the initial data u^0, v^0. On the other hand, if $\|u(\cdot,t)\|_\infty$ and $\|v(\cdot,t)\|_\infty$ are in $\mathcal{P}_I$, then again $I = (0,\infty)$. Moreover, (3.3) says that these supremum norms can be majorized eventually by a universal constant independent of the initial data. This property implies that there is an absorbing ball for the solution and therefore shows the existence of the global attractor if certain compactness is proven, see [**9**].

If $\omega \in \mathcal{P}_I$, then we will also say that ω is *ultimately uniformly bounded.*

We will consider the following conditions on the parameters of the system.

(H1) There are differentiable functions $P(u,v), R(u,v)$ such that $\mathcal{A}_u$ is given by

$$\mathcal{A}_u(u,v) = \nabla(P(u,v)\nabla u + R(u,v)\nabla v).$$

There exist a continuous function Φ and positive constants C, d such that

$$P(u,v) \geq d(1+u) > 0, \quad \forall u \geq 0, \tag{3.4}$$

$$|R(u,v)| \leq \Phi(v)u. \tag{3.5}$$

Moreover, the partial derivatives of P, R with respect to u, v can be majorized by some powers of u, v.

The operator $\mathcal{A}_v$ is regular linear elliptic in divergence form. That is, for some functions $Q(x,t)$ and $c(x,t)$,

$$\mathcal{A}_v(v) = \nabla(Q(x,t)\nabla v) + c(x,t)v, \quad Q(x,t) \geq d > 0, \ c(x,t) \leq 0. \tag{3.6}$$

We assume that $\nabla Q(x,t)$ and $c(x,t)$ are Hölder continuous in (x,t).

We will be interested only in nonnegative solutions, which are relevant in many applications. Therefore, we will assume that the solution u, v stay nonnegative if the initial data u^0, v^0 are nonnegative functions. Conditions on f, g that guarantee such positive invariance can be found in [**24**]. Moreover, we will impose the following assumption on the reaction terms.

(H2) There exists a nonnegative continuous function $C(v)$ such that

$$\begin{gathered} |f(u,v)| \leq C(v)(1+u), \quad g(u,v)u^p \leq C(v)(1+u^{p+1}), \\ \text{for all } u, v \geq 0 \text{ and } p > 0. \end{gathered} \tag{3.7}$$

Local existence of solutions for (1.1) was established, in its most general setting, in [**3**]. Our first result is the following regularity result that can be used to obtain global existence.

Theorem 5. *Assume* (H1) *and* (H2). *Let* (u,v) *be a nonnegative solution to* (1.1) *with maximal existence interval* I. *If* $\|v(\cdot,t)\|_\infty$ *and* $\|u(\cdot,t)\|_1$ *are in* $\mathcal{O}_I$, *then there exists* $\nu > 1$ *such that*

$$\|v(\cdot,t)\|_{C^\nu(\Omega)}, \quad \|u(\cdot,t)\|_{C^\nu(\Omega)} \in \mathcal{O}_I. \tag{3.8}$$

If we have better bounds on the norms of the solutions, then a stronger conclusion follows.

Theorem 6. *Assume* (H1) *and* (H2). *Let* (u,v) *be a nonnegative solution to* (1.1) *with its maximal existence interval* I. *If the Holder norms of* $\nabla Q, c$, $\|v(\cdot,t)\|_\infty$ *and* $\|u(\cdot,t)\|_1$ *are in* $\mathcal{P}_I$, *then there exists* $\nu > 1$ *such that*

$$\|v(\cdot,t)\|_{C^\nu(\Omega)}, \quad \|u(\cdot,t)\|_{C^\nu(\Omega)} \in \mathcal{P}_I. \tag{3.9}$$

It is now standard that the above theorem immediately gives the following.

Theorem 7. *Assume the conditions of Theorem* 6. *Suppose that, for every solution* (u,v) *of* (1.1), *with its maximal existence interval* I, *we have* $\|v(\cdot,t)\|_\infty$ *and* $\|u(\cdot,t)\|_1$ *are in* $\mathcal{P}_I$. *Then the solutions of* (1.1) *exist globally. Moreover, there exists an absorbing ball where all solutions will enter eventually. Thus, if system* (1.1) *is autonomous, then there is a compact global attractor in* $W^{1,p}(\Omega,\mathbb{R}^2)$ *with finite Hausdorff dimension which attracts all solutions.*

The reader is referred to [**9**] for the definition and further properties of global attractors. The estimates in (3.9) also provided the compactness needed in proving the existence of global attractors, see [**9, 39**]. Furthermore, when the system is autonomous, this uniform estimate of higher norms and the general theory in [**3**] shows that the semiflow is smooth (at least C^1) and its linearization is compact. The fact that the global attractor has finite fractal dimension then immediately follows from the well-known theory, e.g., see [**39**].

We can also allow $\mathcal{A}_v$ to be a quasilinear operator given by

$$\mathcal{A}_v(v) = \nabla(Q(v)\nabla v) + c(x,t)v, \quad Q(v) \geq d > 0, \tag{3.10}$$

for some differentiable function Q. However, we can only assert the following

Theorem 8. *Assume as in Theorem* 5, *respectively Theorem* 6, *but with* $\mathcal{A}_v$ *described as in* (3.10). *The conclusions of Theorem* 5, *respectively Theorem* 6, *continue to hold if* $\|v(\cdot,t)\|_\infty$ *and* $\|u(\cdot,t)\|_p$ *are in* $\mathcal{O}_I$, *respectively* $\mathcal{P}_I$, *for some* $p > n/2$.

This theorem improves our previous result [**27**] where we had to assume that $\|u(\cdot,t)\|_p$ are in $\mathcal{P}_I$ for some $p > n$.

Remark 3.1. The assumption $\|v(\cdot,t)\|_\infty \in \mathcal{P}_I$ can be weakened by assuming only that $\|v(\cdot,t)\|_r \in \mathcal{P}_I$ for some r sufficiently large such that $\|f(u,v)(\cdot,t)\|_\infty \in \mathcal{P}_I$ for some $q > n/2$. This is due to the results of [**25**] which assert that the weaker assumption implies the stronger one. We also remark that the assumption on g in (3.7) could be relaxed to $g(u,v)u^p \leq C(v)(1+u^{p+1+\lambda})$ for some appropriate $\lambda > 0$. A simple use of Sobolev imbedding inequality in the proof of Lemma 3.5 will cover this case.

In the proof we will use $\omega(t), \omega_1(t), \dots$ to denote various functions in $\mathcal{O}_I$ or $\mathcal{P}_I$. Moreover, as the interval I is understood, we will also omit them in the dependencies of $\mathcal{O}_I$, $\mathcal{P}_I$.

The proof of Theorems 5 and 6 will be based on several lemmas. We first state some standard facts from the theory of parabolic equations.

For any $t > \tau \geq 0$, we denote $Q_t = \Omega \times [0,t]$ and $Q_{\tau,t} = \Omega \times [\tau,t]$. For $r \in (1,\infty)$ and Q as one of the cylinders $Q_t, Q_{\tau,t}$, let $W_r^{2,1}(Q)$ be the Banach space of functions $u \in L^r(Q)$ having generalized derivatives $u_t, \partial_x u, \partial_{xx} u$ with finite $L^r(Q)$ norms, see [**25**, page 5].

Let us consider the parabolic equation

$$(3.11)\qquad \begin{cases} \partial v/\partial t = A(t)v + f_0(x,t) & x \in \Omega,\ t > 0, \\ \partial v/\partial n(x,t) = 0 & x \in \partial\Omega,\ t > 0, \\ v(x,0) = v_0(x) & x \in \Omega, \end{cases}$$

here $A(t)$ is a uniformly regular elliptic operator (with domain of definition $W_r^2(\Omega)$)

$$(3.12)\qquad A(t)v = a_{ij}(x,t)\partial_{ij}v + b_i(x,t)\partial_i v + c(x,t)v.$$

For simplicity, we consider in (3.11) the Neumann boundary condition. The discussion below holds equally well for Dirichlet or Robin boundary conditions. If the coefficients of the operator $A(t)$ are uniformly Hölder continuous in a cylinder $Q_{\tau,t}$ and $(\lambda I + A(s))^{-1}$ exists for all $\lambda \geq 0$ and $s \in [\tau, t]$ then it is well known that (see, e.g., [**8**, Sections I.19 and II.16-17]) there exists an evolution operator $U(t,s)$ for (3.11) such that the abstract integral version of (3.11) in L^r is

$$(3.13)\qquad v(t) = U(t,\tau)v(\tau) + \int_\tau^t U(t,s)F(s)\,ds,$$

where $F(s)(x) = f_0(x,t)$. Moreover, for each $t > 0$, $r > 1$ and any $\beta \geq 0$, the fractional power $A^\beta(t)$, with its domain of definition $D(A_r^\beta(t))$ in $L^r(\Omega)$, of $A(t)$ is well defined [**8**]. We recall the following imbeddings, see [**10**].

$$(3.14)\qquad D(A_r^\beta(t)) \subset C^\mu(\Omega), \quad \text{if } 2\beta > \mu + n/r,$$

and

$$(3.15)\qquad D(A_r^\beta(t)) \subset W^{1,q}(\Omega), \quad \text{if } 2\beta \geq 1 - n/q + n/r.$$

Next, we collect some well-known facts about (3.11).

Lemma 3.2. *Let $r \in (1,\infty)$ and $f(\cdot,t) \in L^r(\Omega)$. Assume that the coefficients of the operator $A(t)$ are in $C^\alpha(\Omega \times (0,\infty))$ for some $\alpha > 0$.*

Moreover, there exists $\delta_0 > 0$ such that $(\lambda I + A(t))^{-1}$ exists for all $\lambda \ge -\delta_0$ and all $t > 0$. For any $\beta \in [0,1]$, we have

(3.16)

$$\|A^{\beta}(t)v(t)\|_r \le C_\beta t^{-\beta}e^{-\delta t}\|v_0\|_r + C_\beta \int_0^t (t-s)^{-\beta}e^{-\delta(t-s)}\|f_0(\cdot,s)\|_r\,ds$$

for some constants $\delta, C_\beta > 0$ depending on the $C^\alpha(\Omega \times (0,\infty))$ norms of the coefficients of $A(t)$.

Proof. We apply $A^\beta(t)$ to both sides of (3.13), take the L^r norm and then make use the inequality [**8**, (16.38)]. □

Remark 3.3. The inequality (16.38) appears in [**8**] where the condition (13.2) [**8**, page 153] was assumed. However, the reader may refer to [**38**, Theorem 2] where the same inequality is obtained without condition [**8**, (13.2)]. In our case, the assumption on the Hölder norms of the coefficients of $A(t)$ guarantees the validity of the key smoothness assumption [**38**, 0.11].

We now go back to the solutions of Theorem 6. For brevity, we will write $\mathcal{P}$ for $\mathcal{P}_{I,C'}$ with various C' dependent on the constant C stated in Theorem 6.

We first have the following estimates for the component v and its spatial derivative.

Lemma 3.4. *There exist nonnegative functions ω_0, ω defined on the maximal interval of existence I of v such that $\omega_0 \in \mathcal{P}_I$ and the followings hold for v.*

i) *For some $\delta > 0$, $r > 1$, $\beta \in (0,1)$ such that $2\beta > \mu + n/r$, we have*

$$\|v(\cdot,t)\|_{C^\mu(\Omega)} \le \omega_0(t) + \int_0^t (t-s)^{-\beta}e^{-\delta(t-s)}\omega(s)\|u(\cdot,s)\|_r\,ds. \tag{3.17}$$

ii) *For some $\delta > 0$, $r > 1$, $\beta \in (0,1)$ such that $2\beta > 1 - n/q + n/r$, we have*

$$(3.18)\quad \|v(\cdot,t)\|_{W^{1,q}(\Omega)} \le \omega_0(t) + \int_0^t (t-s)^{-\beta} e^{-\delta(t-s)} \omega(s) \|u(\cdot,s)\|_r \, ds.$$

Moreover, ω belongs to $\mathcal{O}_I$, respectively $\mathcal{P}_I$, if $\|v(\cdot,t)\|_\infty$ does.

Proof. Setting $A(t) = \mathcal{A}_v(v) - kv$ and $\hat{f}_0(x,t) = f(u,v) + kv$ for $k > 0$ sufficiently large, we see that v satisfies (3.11). We find that the conditions of Lemma 3.2 are verified. If $\|v(\cdot,t)\|_\infty \in \mathcal{O}_I$, respectively $\mathcal{P}_I$, we have $\|\hat{f}_0\|_r \le \omega(t)(1 + \|u(\cdot,s)\|_r)$, for some function $\omega(t) \in \mathcal{O}_I$, respectively $\mathcal{P}_I$. Hence, (3.16) of Lemma 3.2 gives

$$\|A(t)^\beta v(t)\|_r \le C_\beta t^{-\beta} e^{-\delta t} \|v_0\|_r + C_\beta \int_0^t (t-s)^{-\beta} e^{-\delta(t-s)} \omega(s)(1 + \|u(\cdot,s)\|_r) \, ds.$$

From the imbedding (3.14), (3.17) now follows. The proof of (3.18) is similar as one uses the imbedding (3.15). □

From now on we will suppress I from $\mathcal{O}_I$ and $\mathcal{P}_I$. We will show that the L^p norm of u is in the class $\mathcal{O}$ or $\mathcal{P}$ for any $p \ge 1$. In fact, this is the crucial step in proving Theorem 6.

Lemma 3.5. *Given the conditions of Theorem 5, respectively Theorem 6, for any finite $p \ge 1$, there exists a function $\omega_p \in \mathcal{O}$, respectively $\mathcal{P}$, such that*

$$(3.19)\qquad \|u(\cdot,t)\|_p \le \omega_p(t).$$

The idea of the proof is to derive certain differential inequalities for the L^p norm of u. To this end, we have to control the norm of ∇v that occurs in the equation of u by using the equation for v. This requires that we first study certain functional differential inequalities before giving a proof of this lemma.

For a function $y : \mathbb{R}^+ \to \mathbb{R}$, let us consider the following inequality

$$(3.20)\qquad y'(t) \le \mathcal{F}(t,y), \quad y(0) = y_0, \quad t \in (0,\infty),$$

where $\mathcal{F}$ is a functional from $\mathbb{R}^+ \times C(\mathbb{R}^+, \mathbb{R})$ into $\mathbb{R}$. The following lemma gives a global estimate for y but the estimate is still dependent on the initial data.

Lemma 3.6. *Assume* (3.20) *and*

(F1) *Suppose that there is a function* $F(y, Y) : \mathbb{R}^2 \to \mathbb{R}$ *such that* $\mathcal{F}(t, y) \le F(y(t), Y)$ *if* $y(s) \le Y$ *for all* $s \in [0, t]$.

(F2) *There exists a real* M *such that* $F(y, Y) < 0$ *if* $y, Y \ge M$.

Then there exists a finite M_0 *such that* $y(t) \le M_0$ *for all* $t \ge 0$.

Remark 3.7. In (F1), the inequality $\mathcal{F}(t, y) \le F(y(t), Y)$ is not pointwise. It requires that $y(s) \le Y$ on the interval $s \in [0, t]$ not just that $y(t) \le Y$. Such situation usually happens when $f(t, y)$ contains integrals of $y(t)$ over $[0, t]$.

The above constant M_0 still depends on the initial data y_0. Moreover, the function F may depend on y_0 too. Next, we consider conditions which guarantee uniform estimates for $y(t)$.

Proposition 9. *Assume* (3.20) *and assume that*

(G1) *There exists a continuous function* $G(y, Y) : \mathbb{R}^2 \to \mathbb{R}$ *such that for* τ *sufficiently large, if* $t > \tau$ *and* $y(s) \le Y$ *for every* $s \in [\tau, t]$ *then there exists* $\tau' \ge \tau$ *such that*

$$\mathcal{F}(t, y) \le G(y(t), Y), \quad \text{if} \quad t \ge \tau' \ge \tau. \tag{3.21}$$

(G2) *The set* $\{z : G(z, z) = 0\}$ *is not empty and* $z_* = \sup\{z : G(z, z) = 0\} < \infty$. *Moreover,* $G(M, M) < 0$ *for all* $M > z_*$.

(G3) *For* $y, Y \ge z_*$, $G(y, Y)$ *is increasing in* Y *and decreasing in* y.

If $\limsup_{t\to\infty} y(t) < \infty$, *then*

$$\limsup_{t\to\infty} y(t) \le z_*. \tag{3.22}$$

The proofs of the above statements are elementary and can be found in [**27**].

Remark 3.8. Examples of functions F, G satisfying the conditions of the above two lemmas includes

$$F(y(t),Y),\ G(y(t),Y) = -Ay^{\eta}(t)+D(y^{\gamma}+1)+y^{\theta}(B+CY^{\vartheta})^{k}, \tag{3.23}$$

with positive constants A, B, C, D, η, θ, ϑ, k satisfies $\eta > \theta + k\vartheta$ and $\eta > \gamma$.

Lemma 3.9. *Given the conditions of Theorem* 5*, respectively Theorem* 6*. For any* $p > \max\{n/2, 1\}$*, we set* $y(t) = \int_\Omega u^p\,dx$*. We can find* $\beta \in (0,1)$ *and positive constants* A, B, C *and functions* $\omega_i \in \mathcal{O}$*, respectively* $\mathcal{P}$*, such that the following inequality holds*

$$\begin{aligned}\frac{d}{dt}y \le -Ay^{\eta} + (\omega_0(t) + \|u(\cdot,t)\|_1)y + B\omega(t)\\ + Cy^{\theta}\left\{\omega_1(t) + \int_0^t (t-s)^{-\beta}e^{-\delta(t-s)}\omega_2(s)\|u(\cdot,s)\|_1^{\zeta}y^{\vartheta}(s)\,ds\right\}^2.\end{aligned} \tag{3.24}$$

Here, $\eta = (p+1)/p$*,* $\theta = (p-1)/p$ *and* $\vartheta = (r-1)/r(p-1)$*,* $\zeta = (p-r)/r(p-1)$*. Moreover,* $\eta > \theta + 2\vartheta$*.*

Proof. We assume the conditions of Theorem 6 as the proof for the other case is identical. We multiply the equation for u by u^{p-1} and integrate over Ω. Using integration by parts and noting that the boundary integrals are all zero thanks to the boundary condition on u, we see that

$$\begin{aligned}\int_\Omega u^{p-1}\frac{d}{dt}u\,dx + \int_\Omega P(u,v)\nabla u\nabla(u^{p-1})\\ \le \int_\Omega(-R(u,v)\nabla(u^{p-1})\nabla v + g(u,v)u^{p-1})\,dx.\end{aligned}$$

Using the conditions (3.4) and (3.5), we derive (for some positive constants $C(d,p)$, ε, $C(\varepsilon,d,p)$)

$$
\begin{aligned}
\int_\Omega P(u,v)\nabla u\nabla(u^{p-1})\,dx &\geq C(d,p)\int_\Omega u^{p-1}|\nabla u|^2\,dx,\\
&-\int_\Omega R(u,v)\nabla(u^{p-1})\nabla v\,dx\\
&\leq C(d,p)\int_\Omega u^{p-1}\Phi(v)\nabla u\nabla v\,dx\\
&\leq \varepsilon\int_\Omega u^{p-1}|\nabla u|^2\,dx\\
&\quad + C(\varepsilon,d,p)\int_\Omega u^{p-1}\Phi^2(v)|\nabla v|^2\,dx.
\end{aligned}
$$

Together with (3.7), we obtain

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega u^p\,dx + C_1(\varepsilon,d,p)\int_\Omega u^{p-1}|\nabla u|^2\,dx\\
&\qquad \leq C_2(\varepsilon,d,p)\int_\Omega (u^{p-1}\Phi^2(v)|\nabla v|^2 + C(v)(u^p+1))\,dx.
\end{aligned}
$$

Furthermore, the second term on the left can be estimated by

$$
\begin{aligned}
\int_\Omega u^{p-1}|\nabla u|^2\,dx &= C(p)\int_\Omega |\nabla(u^{(p+1)/2})|^2\,dx\\
&\geq C\int_\Omega u^{p+1}\,dx - C\biggl(\int_\Omega u^{(p+1)/2}\,dx\biggr)^2\\
&\geq C\biggl(\int_\Omega u^p\,dx\biggr)^{(p+1)/p} - C\|u\|_1\int_\Omega u^p\,dx.
\end{aligned}
$$

Here, we have used the Hölder's inequality

$$
\biggl(\int_\Omega u^{(p+1)/2}\,dx\biggr)^2 = \biggl(\int_\Omega u^{1/2}u^{p/2}\,dx\biggr)^2 \leq \|u\|_1\int_\Omega u^p\,dx.
$$

We next consider the first integral on the right of (3.25). By our assumption on L^∞ norm of v, $\Phi(v)\leq\omega_1(t)$ for some $\omega_1\in\mathcal{P}$. Using the Hölder inequality, we have

$$\int_\Omega u^{p-1}\Phi^2(v)|\nabla v|^2\,dx \le \omega_1^2(t)\Big(\int_\Omega u^p\,dx\Big)^{(p-1)/p}\Big(\int_\Omega |\nabla v|^{2p}\,dx\Big)^{1/p}$$
$$= \omega_1^2(t)y^{(p-1)/p}\|\nabla v\|_{2p}^2.$$

As $p > \max\{n/2, 1\}$, there exists $r \in (1, p)$ such that

$$\frac{1}{n} + \frac{1}{2p} > \frac{1}{r} > \frac{1}{p}. \tag{3.26}$$

This implies $2 > 1 - n/2p + n/r$. Hence, we can find $\beta \in (0, 1)$ such that $2\beta > 1 - n/2p + n/r$. From (3.18), with $q = 2p > r$, we have

$$\|\nabla v\|_{2p} \le \omega_0(t) + \int_0^t (t-s)^{-\beta}e^{-\delta(t-s)}\omega(s)\|u(\cdot, s)\|_r\,ds.$$

Applying the above estimates in (3.25), we derive the following inequality for $y(t)$

$$\frac{d}{dt}y + C(d,p)y^{(p+1)/p} \tag{3.27}$$
$$\le Cy^{(p-1)/p}\omega_1(t)\Big\{\omega_0(t) + \int_0^t (t-s)^{-\beta}e^{-\delta(t-s)}\omega(s)\|u(\cdot,s)\|_r\,ds\Big\}^2$$
$$+ C(\omega_2(t) + \|u\|_1)y + B\omega_2(t).$$

As $1 < r < p$, we can use Hölder's inequality

$$\|u\|_r \le \|u\|_1^{1-\lambda}\|u\|_p^\lambda = \|u\|_1^{1-\lambda}y^{\lambda/p}$$

with $\lambda = (1 - 1/r)/(1 - 1/p) = p(r-1)/r(p-1)$. Applying this in (3.27) and re-indexing the functions ω_i, we prove (3.24). The last assertion of the lemma follows from the following equivalent inequalities

$$\eta > \theta + 2\vartheta \Longleftrightarrow \frac{p+1}{p} > \frac{p-1}{p} + \frac{2(r-1)}{r(p-1)} \Longleftrightarrow \frac{1}{p} > \frac{(r-1)}{r(p-1)}$$
$$\Longleftrightarrow rp - r > pr - p \Longleftrightarrow p > r.$$

This completes the proof. □

We are now ready to give the proof of Lemma 3.5.

Proof. Assume first the conditions of Theorem 5. From (3.24), we deduce the following integro-differential inequality

$$\frac{d}{dt}y \leq -Ay^{\eta} + \omega_1(t)y + B\omega_2(t) + Cy^{\theta}\{\omega_0(t) + K(t)\}^2, \tag{3.28}$$

where

$$K(t) := \int_0^t (t-s)^{-\beta} e^{-\delta(t-s)} \omega(s) y^{\vartheta}(s)\, ds$$

for some $\omega_0, \omega_1, \omega \in \mathcal{O}$ (because $\|u(\cdot,t)\|_1 \in \mathcal{O}$). We will show that Lemma 3.6 can be used here to assert that $y(t)$ is bounded in any finite interval. This means $\|u\|_p \in \mathcal{O}$. We define the functional

$$\mathcal{F}(t,y) = -Ay^{\eta} + \omega_1(t)y + B + Cy^{\theta}\{\omega_0(t) + K(t)\}^2. \tag{3.29}$$

Since $\omega_i \in \mathcal{O}$, we can find a positive constant C_ω, which may still depend on the initial data, such that $\omega_i(t) \leq C_\omega$ for all $t > 0$. Let

$$C_1 := \sup_{t>0} \int_0^t (t-s)^{-\beta} e^{-\delta(t-s)}\, ds \leq \int_0^\infty s^{-\beta} e^{-\delta s}\, ds < \infty,$$

because $\beta \in (0,1)$ and $\delta > 0$. We then set

$$F(y,Y) = -Ay^{\eta} + C_\omega(y+B) + Cy^{\theta}(C_\omega + C_\omega C_1 Y^{\vartheta})^2.$$

Because $\eta > \theta + 2\vartheta$, by Lemma 3.9 and Remark 3.8, the functionals $\mathcal{F}, F$ satisfy conditions (F1) and (F2). Hence, Lemma 3.6 applies and gives

$$y(t) \leq C_0(v^0, u^0), \quad \forall t > 0, \tag{3.30}$$

for some constant $C_0(v^0, u^0)$ which may still depend on the initial data since F does. We have shown that $y(t) \in \mathcal{O}$.

We now seek for uniform estimates and assume the conditions of Theorem 6. From Lemma 3.9 we again obtain (3.28) with ω_i are now in $\mathcal{P}$. If a function ω belongs to $\mathcal{P}$, by Definition 4, we can find $\tau_1 > 0$ such that $\omega(s) \leq \overline{C}_\infty = C_\infty + 1$ if $s > \tau_1$. We emphasize the fact that

$\overline{C}_\infty$ is independent of the initial data. Let $t > \tau \geq \tau_1$ and assume that $y(s) \leq Y$ for all $s \in [\tau, t]$. Let us write

$$K(t) = \int_0^\tau (t-s)^{-\beta} e^{-\delta(t-s)} \omega(s) y^\vartheta(s)\, ds + \int_\tau^t (t-s)^{-\beta} e^{-\delta(t-s)} \omega(s) y^\vartheta(s)\, ds = J_1 + J_2.$$

By (3.30), there exists some constant $C(v^0, u^0)$ such that $\omega(s)y^\vartheta(s) \leq C(v^0, u^0)$ for every s. Hence, we can find $\tau' > \tau$ such that $J_1 \leq 1$ if $t > \tau'$. Thus,

$$K(t) \leq 1 + \overline{C}_\infty C_* Y^\vartheta, \quad \text{where}$$
$$C_* = \sup_{t>\tau, \tau>0} \int_\tau^t (t-s)^{-\beta} e^{-\delta(t-s)}\, ds < \infty.$$

Therefore, for $t > \tau'$ we have $f(t, y) \leq G(y(t), Y)$ with

$$G(y(t), Y) = -Ay^\eta(t) + \overline{C}_\infty (y + B) + y^\theta (\overline{C}_\infty + 1 + \overline{C}_\infty C_* Y^\vartheta)^2. \tag{3.31}$$

We see that G is independent of the initial data and satisfies (G.1)–(G.3) as $\eta > \theta + 2\vartheta$, see Remark 3.8. Therefore, Proposition 9 applies here to conclude the proof. □

We now give

Proof of Theorem 5 *and Theorem* 6. Having established the fact that $\|u(\cdot, t)\|_p \in \mathcal{O}$, respectively $\|u(\cdot, t)\|_p \in \mathcal{P}$, for any $p > 1$, we can follow the proof of [**27**, Theorem 2] to assert (3.8), respectively (3.9). □

Proof of Theorem 8. We set

$$V(x, t) = \int_0^{v(x,t)} Q(s)\, ds,$$

and multiply the equation for v by $Q(v)$ to find that

$$\frac{\partial V}{\partial t} = Q(v(x, t)) \Delta V + Q(v(x, t)) f(u, v).$$

Since we assume that $\|v(\cdot,t)\|_\infty \in \mathcal{P}$ and (3.7) holds, we see that $f(u,v) \in L_p(\Omega)$ for $p = n > n/2$. Moreover, $\|f(u(\cdot,t),v(\cdot,t))\|_p \in \mathcal{P}$. Regularity theory for quasilinear parabolic equations, see [**25, 29**], asserts that there is an $\alpha > 0$ such that $v \in C^{\alpha,\alpha/2}(\Omega \times (0,\infty))$ with uniformly bounded norm. So is $Q(v(x,t))$. Now, we can regard $Q(v(x,t))\Delta$ as a *linear* regular elliptic operator with Hölder continuous coefficient (whose norm is also ultimately uniformly bounded) so that Lemma 3.4 is applicable. We then follow the proof of Theorem 6 to complete our proof. □

4. Proof of examples. We conclude our paper with the proof of our theorems stated in Section 2. The proof is mainly the verification of the assumptions required by our main results. In some cases, we also need certain minor modifications.

4.1 *Proof of Theorem* 1. By integrating the equation for u we easily see that the L^1 norm of u is conserved so that $\|u(\cdot,t)\|_1 \in \mathcal{O}$ but $\|u(\cdot,t)\|_1$ is not in $\mathcal{P}$. Using this in the equation of v, we find that $\|v(\cdot,t)\|_1 \in \mathcal{P}$. However, it is not so easy to show that $\|v(\cdot,t)_\infty \in \mathcal{O}$ and therefore our theorems, as they were stated, are not immediately applicable here.

However, a careful inspection of the proof of Lemma 3.9 reveals that the only places where we make use of the assumption $\|v(\cdot,t)\|_\infty \in \mathcal{O}$, respectively $\|v(\cdot,t)\|_\infty \in \mathcal{P}$, to derive (3.24) are when we estimate $\Phi(v)$ and $\|\nabla v\|_{2p}$, in (3.18). In this case, $\Phi(v) = \alpha_{12}$, a constant. Moreover, if we define $\mathcal{A}_v(v) = d_2\Delta v - cv$ and $f(u,v) = bu$, which is independent of v and thus $C(v) = \text{constant}$ in (3.7), we then find that the proof of Lemma 3.9 is still in force to get (3.24). Thus, $\|u(\cdot,t)\|_1 \in \mathcal{O}$ implies that $\|u(\cdot,t)\|_\infty \in \mathcal{O}$. Using this in the equation for v, we can easily prove that $\|v(\cdot,t)\|_\infty \in \mathcal{O}$.

4.2 *Proof of Theorem* 2. It is easy to see that (2.4) is a special case of (1.1) with $P(u,v) = d_1 + 2\alpha_{11}u + \alpha_{12}v$, $R(u,v) = \alpha_{12}u$ and $Q(v) = d_2 + 2\alpha_{22}$. The fact that $\|v(\cdot,t)\|_\infty$ and $\|u(\cdot,t)\|_1$ are in $\mathcal{P}$ is easy to show, see [**27**]. If $\alpha_{22} = 0$, then the conditions of Theorem 6 are fulfilled. When $n = 2$ and $\alpha_{22} \neq 0$, it is also proven in [**27**] that $\|u(\cdot,t)\|_2 \in \mathcal{P}$ and Theorem 8 can apply.

4.3 *Proof of Theorem* 3. By comparison principles, one can show easily that $\|S(\cdot,t)\|_\infty \in \mathcal{P}$. Multiplying the equation of u by γ and adding the result to the equation of S, we can easily prove that $\|u(\cdot,t)\|_1 \in \mathcal{P}$ by integrating over Ω. This fact has been proven in [**28**] where we assumed that $\alpha_{11} = 0$ and $n = 1$. Applying our general result to the case $\alpha_{11} > 0$, we obtain the theorem.

REFERENCES

1. R.A. Adams, *Sobolev spaces*, Academic Press, New York, 1975.

2. H. Amann, *Dynamic theory of quasilinear parabolic equations* I. *Abstract evolution equations*, Nonlinear Anal. T.M.A. **12** (1988), 895–919.

3. ———, *Dynamic theory of quasilinear parabolic systems*–III. *Global existence*, Math. Z. **202** (1989), 219–250.

4. ———, *Dynamic theory of quasilinear parabolic equations*–II. *Reaction-diffusion systems*, Differential Integral Equations **3** (1990), 13–75.

5. ———, *Nonhomogeneous linear and quasilinear elliptic and parabolic boundary value problems*, in *Function spaces, differential operators and nonlinear analysis*, H. Schmeisser and H. Triebel, eds., Teubner-Texte Math. **133**, Stuttgart and Leipzig, 1993.

6. S. Childress and J.K. Percus, *Nonlinear aspects of chemotaxis*, Math. Biosc. **856** (1981), 217–237.

7. Y.S. Choi and R. Lui, *Multi-dimensional electrochemistry model*, Arch. Rational Mech. Anal. **130** (1995), 315–342.

8. A. Friedman, *Partial differential equations*, Holt, Rinehart and Winston, New York, 1969.

9. J. Hale, *Asymptotic behavior of dissipative systems*, Amer. Math. Soc. Math. Surveys Mono. **25**, 1988.

10. D. Henry, *Geometric theory of semilinear parabolic equations*, Lecture Notes Math. **840**, Springer-Verlag, Berlin, 1981.

11. M. Herrero and J. Velázquez, *Chemotactic collapse for the Keller-Segel model*, J. Math. Biol. **35** (1996), 177–194.

12. T. Hillen and K. Painter, *Global existence for a parabolic chemotaxis model with prevention of overcrowding*, J. Adv. Appl. Math. **26** (2001), 280–301.

13. T. Hillen and K.J. Painter, *Volume-filling and quorum-sensing in models for chemosensitive movement*, Canad. Appl. Math. Quart. **10** (2002), 501–543.

14. D. Horstmann, *Lyapunov functionals and L^p estimates for a class of reaction diffusion systems*, Adv. Appl. Math. **26** (2001), 280–301.

15. ———, *The nonsymmetric case of the Keller-Segel model in chemotaxis*: *Some recent results*, Nonlinear Differential Equations Appl. **8** (2001), 399–423.

16. ———, *From* 1970 *until present*: *The Keller-Segel model in chemotaxis and its consequences* I, Jahresberichte der DMV **105** (2003), 103–165.

17. D. Horstmann, *From* 1970 *until present*: *The Keller-Segel model in chemotaxis and its consequences* II, Jahresberichte der DMV, (2003).

18. D. Horstmann and G. Wang, *Blow-up in a chemotaxis model without symmetry assumptions*, preprint.

19. D. Horstmann and M. Winkler, *Boundedness vs. blow-up in a chemotaxis system*, preprint.

20. T. Ichikawa and Y. Yamada, *Some remarks on global solutions to quasilinear parabolic system with cross diffusion*, Funk. Ekv. **43** (2000), 285–301.

21. W. Jäger and S. Luckhaus, *On explosion of solutions to a system of partial differential equations modelling chemotaxis*, Trans. Amer. Math. Soc. **329** (1992), 819–824.

22. E.F. Keller and L.A. Segel, *Initiation of slime mold aggregation viewed as an instability*, J. Theor. Biol. **26** (1970), 399–415.

23. H.J. Kuiper, *A priori bounds and global existence for a strongly coupled quasilinear parabolic system modeling chemotaxis*, Electronic J. Differential Equations **2001** (2001), 1–18.

24. K.H.W. Küfner, *Global existence for a certain strongly coupled quasilinear parabolic system in population dynamics*, Analysis **15** (1995), 343–357.

25. O.A. Ladyzenskaja, V.A. Solonnikov and N.N. Ural'tseva, *Linear and quasilinear equations of parabolic type*, AMS Transl. Monographs **23**, 1968.

26. D. Lauffenburger, *Quantitative studies of bacterial chemotaxis and microbial population dynamics*, Microbial Ecol. **22** (1991), 175–185.

27. D. Le, *Cross diffusion systems on n spatial dimensional domains*, Indiana Univ. Math. J., to appear.

28. ———, *On a time dependent chemotaxis system*, J. Appl. Comp. Math., to appear.

29. ———, *Remark on Hölder continuity for parabolic equations and the convergence to global attractors*, Nonlinear Anal. T.M.A. **41** (2000), 921–941.

30. ———, *Global attractors and steady state solutions for a class of reaction diffusion systems*, J. Differential Equations **147** (1998), 1–29.

31. ———, *Coexistence with chemotaxis*, SIAM J. Math. Anal. **32** (2000), 504–521.

32. D. Le and H.L. Smith, *Steady states of models of microbial growth and competition with chemotaxis*, J. Math. Anal. Appl. **229** (1999), 295–318.

33. Y. Lou, W. Ni and Y. Wu, *On the global existence of a cross-diffusion system*, Discrete Continuous Dynamical Systems **4** (1998), 193–203.

34. T. Nagai, T. Senba and T. Suzuki, *Chemotactic collapse in a parabolic system of mathematical biology*, Hiroshima Math. J. **30** (2000), 463–497.

35. K. Osaki and A. Yagi, *Finite dimensional attractors for one-dimensional Keller-Segel equations*, Funk. Ekv. **44** (2001), 441–469.

36. R. Redlinger, *Existence of the global attractor for a strongly coupled parabolic system arising in population dynamics*, J. Differential Equations **118** (1995), 219–252.

37. N. Shigesada, K. Kawasaki and E. Teramoto, *Spatial segregation of interacting species*, J. Theoret. Biol. **79** (1979), 83–99.

38. P.E. Sobolevskii, *On the equations of parabolic type in a Banach space*, Trudy Moscov. Mat. Obsc. **10** (1961), 297–350.

39. R. Temam, *Infinite-dimensional dynamical systems in mechanics and physics*, 2nd ed., Appl. Math. Sci. **68**, Springer-Verlag, New York, 1997.

40. X. Wang, *Qualitative behavior of solutions of a chemotactic diffusion system: Effects of motility and chemotaxis and dynamics*, SIAM J. Math. Anal. **31** (2000), 535–560.

41. A. Yagi, *Global solution to some quasilinear parabolic system in population dynamics*, Nonlinear Anal. T.M.A. **21** (1993), 531–556.

42. ———, *Norm behavior of solutions to a parabolic system of chemotaxis*, Math. J. **45** (1997), 241–265.

43. ———, *A priori estimates for some quasilinear parabolic system in population dynamics*, Kobe J. Math. **14** (1997), 91–108.

Department of Mathematics, Arizona State University, Tempe, AZ 85287
Email address: kuiper@asu.edu

Department of Applied Mathematics, University of Texas at San Antonio, 6900 North Loop 1604 West, San Antonio, TX 78249
Email address: dle@math.utsa.edu