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DISTRIBUTION OF ARITHMETIC FUNCTIONS ON CERTAIN SUBSETS OF INTEGERS

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ABSTRACT. Let $d(n)$, respectively $\sigma(n)$, stand for the number, respectively the sum, of the positive divisors of n, and let φ be Euler's totient function. Also let $d_3(n)$ be the number of solutions of $a_1a_2a_3 = n$ in positive integers a_1, a_2, a_3. We determine the order of the set of positive integers $n \le x$ for which $(nd(n), \varphi(n))$ is a power of 2. We do the same for the set of positive integers $n \le x$ for which $(nd_3(n), \varphi(n)) = 1$ and for the set of positive integers $n \le x$ for which $U(n) := (nd_3(n), \sigma(n)) = 1$. We also show that $\sum_{p\le x,\ U(p+a)=1} d(p+a)$ is of order $\mathrm{li}\,(x)/\log\log\log x$. Moreover, generalizing an approach used by Erdős to prove that $\#\{n \le x : (n, \varphi(n)) = 1\} \sim \#\{n \le x : p(n) > \log\log x\}$ (where $p(n)$ stands for the smallest prime factor of n), we show that the same result holds when we add the condition $\omega(n) = r$ in each of these two sets, where $\omega(n)$ is the number of distinct prime divisors of n. Finally, we estimate the size of $\#\{n \le x : (n, \varphi(n)) = 1,\ \omega(n) = r\}$ uniformly for $r = r(x) = (1 + o(1)) \log\log x$.

1. Introduction. Let φ stand for Euler's totient function, $\sigma(n)$ for the sum of the positive divisors of n and $d(n)$ for the number of positive divisors of n. Moreover, for each integer $n \ge 2$, let $p(n)$ stand for the smallest prime factor of n with $p(1) = 1$ and $\omega(n)$ for the number of distinct prime factors of n.

In 1958, Kanold [**4**] showed that, if $E(x) := \#\{n \le x : (nd(n), \sigma(n)) = 1\}$, then there exist positive constants $C_1 < C_2$ and a positive number x_0 such that

$$(1) \qquad C_1 < E(x)/\sqrt{x/\log x} < C_2, \quad x \ge x_0.$$

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Recently, we showed [**2**] that there exists a positive constant C such that

$$E(x) = C(1+o(1))\sqrt{\frac{x}{\log x}}.$$

Also recently, Kátai and Subbarao [**5**] obtained various results regarding the function φ_k, that is, the k-fold iterate of φ; they showed in particular that, for most integers $n \le x$,

$$(n, \varphi_k(n)) = \prod_{\substack{p^\alpha \| n \\ p < x_2^k}} p^\alpha$$

and that, for any given integer $a \neq 0$, for most primes $q \le x$,

$$(q+a, \varphi_k(q+a)) = \prod_{\substack{p^\alpha \| q+a \\ p < x_2^k}} p^\alpha.$$

Here and hereafter, $x_1 = \max(1, \log x)$ and $x_i = \max(1, \log x_{i-1})$ for each integer $i \ge 2$.

Consider the sets $A_x := \{n \le x : (n, \varphi(n)) = 1\}$ and $B_x := \{n \le x : p(n) > x_2\}$. In 1948, Erdős [**3**] proved that

$$\#A_x = (1+o(1))e^{-\gamma}\frac{x}{x_3}, \quad x \to \infty, \tag{2}$$

where γ stands for Euler's constant, and somewhat surprisingly, that

$$\#A_x = (1+o(1))\#B_x, \quad x \to \infty. \tag{3}$$

In 2001, Beguns [**1**] improved (2) by showing that

$$\#A_x = e^{-\gamma}\frac{x}{x_3}\left(1 + O\left(\frac{x_4}{x_3}\right)\right).$$

In this paper, we establish various results analogous to those mentioned above.

In particular, after observing that $(nd(n), \varphi(n)) = 1$ if and only if $n = 1$ or 2, we estimate the size of the set $\{n \le x : (nd(n), \varphi(n)) = \text{a power of } 2\}$.

Letting $d_3(n)$ stand for the number of solutions of $a_1a_2a_3 = n$ in positive integers a_1, a_2, a_3 and setting $F(n) := (nd_3(n), \varphi(n))$ and $U(n) := (nd_3(n), \sigma(n))$, we estimate the size of the sets $\{n \le x : F(n) = 1\}$ and $\{n \le x : U(n) = 1\}$ and of the sum $\sum_{p\le x,\ U(p+a)=1} d(p+a)$.

Then, by using the method of Erdős, we prove a result similar to (3) with the additional restriction that the integers n in A_x and B_x satisfy $\omega(n) = r$ with $r = r(x) = (1 + o(1))x_2$.

Finally, we show that $\#\{n \le x : (n, \varphi(n)) = 1,\ \omega(n) = r\} \sim \prod_{p<x_2}(1 - (1/p)) \cdot \#\{n \le x : \omega(n) = r\}$ uniformly for $r = r(x) = (1 + o(1))x_2$.

2. Main results. Observe that $E(n) := (nd(n), \varphi(n)) = 1$ if and only if $n = 1$ or 2. Indeed, assuming that $n \ge 3$ with $E(n) = 1$, it is clear that n must be squarefree; therefore, n must be divisible by an odd prime p, in which case $\varphi(p)$ is even and $d(p) = 2$, implying that $2|E(n)$, thus establishing our claim. On the other hand, it turns out that $E(n)$ is quite often a power of 2, as is shown by the following result.

Theorem 1. *As $x \to \infty$,*

$$\#\{n \le x : (nd(n), \varphi(n)) = \text{a power of } 2\} = (1 + o(1))c_1\frac{x}{x_3},\ \ x \to \infty,$$

where $c_1 = e^{-\gamma}\left(1 + 2\sum_{\ell=1}^{\infty}(1/2^{2^\ell})\right)$.

Theorem 2. *There exists a positive constant c_2 such that*

$$\#\{n \le x : F(n) = 1\} = (1 + o(1))c_2\frac{x}{\sqrt{x_1x_3}}, \quad x \to \infty.$$

Theorem 3. *There exists a positive constant c_3 such that*

$$\#\{n \le x : U(n) = 1\} = (1 + o(1))c_3\frac{x}{\sqrt{x_1x_3}}, \quad x \to \infty.$$

Let χ be the Dirichlet character

$$\chi(n) = \begin{cases} 1 & \text{if } n \equiv 1 \pmod 3, \\ -1 & \text{if } n \equiv -1 \pmod 3, \\ 0 & \text{if } 3|n. \end{cases}$$

Let

$$r(n) = \sum_{d|n} \chi(d) = \prod_{p^\alpha \| n} \left(1 + \chi(p) + \cdots + \chi(p^\alpha)\right).$$

It is clear that for squarefree integers n, we have $r(n) > 0$ if and only if every prime factor of n is $\equiv 1 \pmod 3$. Moreover if $r(n) > 0$ and n is squarefree, then $r(n) = d(n)$. By using the Bombieri-Vinogradov theorem, one can deduce that, given any integer $a \neq 0$, there exists a positive constant c such that

$$\sum_{p \le x} r(p+a) = c(1+o(1))x, \quad x \to \infty.$$

Here, setting as usual $\mathrm{li}\,(x) := \int_0^x (dt/\log t)$, we shall prove the following result.

Theorem 4. *Given any nonzero integer a, there exists a positive constant c_4 such that*

$$\sum_{\substack{p \le x \\ U(p+a)=1}} d(p+a) = (1+o(1)) c_4 \frac{\mathrm{li}(x)}{x_3}, \quad x \to \infty.$$

For each positive integer r, let

$$A_x(r) := \{n \le x : (n, \varphi(n)) = 1,\ \omega(n) = r\}$$

and

$$B_x(r) := \{n \le x : p(n) > x_2,\ \omega(n) = r\}.$$

Theorem 5. *Uniformly for* $r = r(x) = (1 + o(1))x_2$, *as* $x \to \infty$,

(i) $\#A_x(r) = (1 + o(1))\#B_x(r)$,

(ii) $\#B_x(r) = (1 + o(1)) \prod_{p<x_2} (1 - (1/p)) \cdot (x/x_1)(x_2^{r-1}/(r-1)!)$.

3. Further notations and preliminary results. Letting $P(n)$ stand for the largest prime factor of n with $P(1) = 1$, we set as is customary

$$\Psi(x, y) := \#\{n \le x : P(n) \le y\}.$$

We shall be using the well-known estimate

$$\Psi(x, y) := \#\{n \le x : P(n) \le y\} \ll x \exp\left\{-\frac{\log x}{2\log y}\right\}, \tag{4}$$

see, for instance, Tenenbaum [**6**].

Throughout this paper, the letters p, q and π always stand for prime numbers. For each positive integer r, let

$$\pi_r(x) := \#\{n \le x : \omega(n) = r\}.$$

Theorem A. *If* $x \to \infty$, $1 \le y \le x_1$, $(1 - \varepsilon_x)x_2 \le r \le (1 + \varepsilon_x)x_2$, *where* $\varepsilon_x \to 0$, *then*

$$\#\{n \le x : \omega(n) = r,\ p(n) > y\} = (1 + o(1)) \prod_{p<y} \left(1 - \frac{1}{p}\right) \cdot \pi_r(x).$$

Proof. This result can be proved by using the now classical Selberg-Delange method applied to the study of the function

$$F_y(s, z) = \prod_{p \ge y} \left(1 + \frac{z}{p^s} + \frac{z}{p^{2s}} + \cdots\right).$$

We shall therefore omit the details. □

Lemma 1. *Let* A *be a positive constant, and let*

$$\mathcal{F}_r(x, y, Q) := \{n \le x : n = p_1 \cdots p_r,\ y < p_1 < \cdots < p_r,\ p_i \not\equiv 1 \pmod Q\},$$

where Q is an odd prime number less than x_2^A, and assume that $y < x_2^A$ and that $r = r(x) = (1+o(1))x_2$. Then

$$\#\mathcal{F}_r(x,y,Q) \ll \frac{x \cdot x_2^{r-1}}{x_1 \cdot (r-1)!}\left(1-\frac{1}{Q-1}\right)^r \left(1-\frac{\log\log y}{x_2}\right)^{r-1}.$$

Proof. We may assume that $P(n) = p_r \geq x^{1/(2x_2)}$, since the contribution of those integers $n \in \mathcal{F}_r(x,y,Q)$ for which $P(n) < x^{1/(2x_2)}$ is smaller than

$$\Psi(x, x^{1/(2x_2)}) \ll x\exp\{-x_2\} = O(x/x_1),$$

where we used (4). From this observation and taking into account that $\log x \leq 2\log n$ for $\sqrt{x} \leq n \leq x$, it follows that

$$x_1 \cdot \#\mathcal{F}_r(x,y,Q) \ll x + \sqrt{x}\cdot x_1 + S^*, \tag{5}$$

where

$$S^* := \sum_{\substack{n = p_1\cdots p_r \leq x \\ p_1 > y,\ p_r \geq x^{1/(2x_2)} \\ p_i \not\equiv 1 \pmod{Q}}} \log n.$$

Now, using the Prime Number theorem for arithmetic progressions in the weak form

$$\sum_{\substack{p \leq x \\ p \neq 1 \pmod{Q}}} \log p \ll \left(1-\frac{1}{Q-1}\right)x,$$

we have

$$\begin{aligned} S^* &\ll \sum_{\substack{\nu p \le x \\ \omega(\nu)=r-1 \\ y<P(\nu)\le p,\ p\ge x^{1/(2x_2)},\ p\not\equiv 1 \pmod Q \\ p_i|\nu \Rightarrow p_i\not\equiv 1 \pmod Q}} \log p \\ &\ll \sum_{\substack{\nu \le x/x^{1/(2x_2)} \\ \omega(\nu)=r-1 \\ p_i|\nu \Rightarrow p_i \not\equiv 1 \pmod Q}} \sum_{\substack{y<p\le x/\nu \\ p\not\equiv 1 \pmod Q}} \log p \\ &\ll \left(1-\frac{1}{Q-1}\right) x \sum_{\substack{\nu \le x \\ \omega(\nu)=r-1 \\ p_i|\nu\Rightarrow p_i\not\equiv 1 \pmod Q}} \frac{1}{\nu} \\ &\ll \left(1-\frac{1}{Q-1}\right) \frac{x}{(r-1)!} \left(\sum_{\substack{y<q<x \\ q\not\equiv 1 \pmod Q}} \frac{1}{q}\right)^{r-1} \\ &\ll \frac{x}{(r-1)!}\left(1-\frac{1}{Q-1}\right)^r (x_2-\log\log y)^{r-1}, \end{aligned}$$

which combined with (5) completes the proof of Lemma 1.

Corollary. *Let $R_x(r|Q)$ be the set of those integers $n \le x$ such that $(n,\varphi(n))=1$, $\omega(n)=r$ and $p(n)=Q$. Then*

$$\sum_{Q<r/\log^2 r} \#R_x(r|Q) \ll \frac{x}{x_1}\frac{x_2^{r-1}}{(r-1)!}\exp\left\{-\frac{1}{2}\log^2 r\right\}.$$

Proof. If $n\le x$, $(n,\varphi(n))=1$, $\omega(n)=r$ and $p(n)=Q$, then $n=Qm\le x$, $m=p_1\cdots p_{r-1}$, $Q<p_1<\cdots<p_{r-1}$ and $p_i\not\equiv 1 \pmod Q$. Therefore, it follows from Lemma 1 that

$$\begin{aligned} \#R_x(r|Q) &= \#\mathcal{F}_{r-1}\left(\frac{x}{Q}, Q, Q\right) \\ &\ll \frac{x}{Qx_1}\frac{x_2^{r-2}}{(r-2)!}\left(1-\frac{1}{Q-1}\right)^{r-1}\left(1-\frac{\log\log Q}{x_2}\right)^{r-1}. \end{aligned}$$

From this, it follows that

$$\sum_{Q<r/(\log r)^2} \#R_x(r|Q)$$
$$\ll \frac{x}{x_1}\frac{x_2^{r-1}}{(r-1)!}\sum_{Q<r/(\log r)^2}\frac{1}{Q}\exp\left\{\left(-\frac{r}{Q}-\frac{r\log\log Q}{x_2}\right)\right\}$$
$$\ll \frac{x}{x_1}\frac{x_2^{r-1}}{(r-1)!}\exp\{-\frac{1}{2}\log^2 r\},$$

thus completing the proof of the corollary.

4. The proof of Theorem 1. Let

$$\mathcal{E} := \{n : E(n) = \text{ power of } 2\} \quad \text{and} \quad \mathcal{E}(x) := \{n \le x : n \in \mathcal{E}\}.$$

In order to estimate the size of $\mathcal{E}(x)$, we shall examine the size of its subsets

$$\mathcal{E}_t(x) := \{2^t m \le x : \ (m,2)=1,\ 2^t m \in \mathcal{E}\}, \quad t = 0,1,2,\dots .$$

Let us first consider $\mathcal{E}_0(x)$. It is clear that $n \in \mathcal{E}_0(x)$ if and only if $(n,\varphi(n)) = 1$. Hence, by Erdős' estimate (2), it follows that

$$\#\mathcal{E}_0(x) = (1+o(1))e^{-\gamma}\frac{x}{x_3}, \quad x \to \infty. \tag{6}$$

Let us now consider the case where $1 \le t \le x_3 - 1$. If $n = 2^t m \in \mathcal{E}_t(x)$, then $m \in \mathcal{E}_0(x/2^t)$ and $(d(2^t),\varphi(m)) = (t+1,\varphi(m))$ is a power of 2. We shall see that $\#\mathcal{E}_t(x)$ is negligible if $t+1$ is not a power of 2. Indeed, consider a fixed integer $t \le x_3 - 1$ such that $t+1$ is not a power of 2; such a number t must have an odd prime divisor $q \le x_3$, in which case $q \nmid \varphi(m)$. In this case, by a simple sieve argument, we have

$$S_q\left(\frac{x}{2^t}\right) := \#\left\{m \le \frac{x}{2^t} : p \mid m \quad \text{implies that} \quad p \not\equiv 1 \pmod q\right\}$$
$$\ll \frac{x}{2^t}\prod_{\substack{\pi<x\\ \pi\equiv 1 \pmod q}}\left(1-\frac{1}{\pi}\right) \ll \frac{x}{2^t}\exp\left\{-\frac{1}{2}\frac{x_2}{q}\right\}$$
$$\ll \frac{x}{2^t}\exp\left\{-\frac{1}{2}\frac{x_2}{x_3}\right\}.$$

From this, it follows that

$$\sum_{q<t} S_q\left(\frac{x}{2^t}\right) \ll \frac{xt}{2^t}\exp\left\{-\frac{1}{2}\frac{x_2}{x_3}\right\} \ll \frac{x}{2^t x_2^B}, \tag{7}$$

where B is an arbitrarily large constant.

Now if $t+1$ is a power of 2, then $m \in \mathcal{E}_0(x/2^t)$, which implies that $n = 2^t m \in \mathcal{E}_t(x)$. From this observation, it follows using (6) that

$$\begin{aligned} \#\mathcal{E}_t(x) = \#\mathcal{E}_o(x/2^t) &= (1+o(1))e^{-\gamma}\frac{x}{2^t \log\log\log(x/2^t)} \\ &= (1+o(1))e^{-\gamma}\frac{x}{2^t x_3} \end{aligned} \tag{8}$$

uniformly for $t+1 \le x_3$.

It remains to consider the situation where $t \ge x_3$. But the contribution of these t's is small since

$$\sum_{t\ge x_3} \#\mathcal{E}_t(x) \ll \sum_{t \ge x_3} x/2^t \ll \frac{x}{x_2}. \tag{9}$$

In view of (6), (7), (8) and (9), we may conclude that

$$\begin{aligned} \#\mathcal{E}(x) &= \#\mathcal{E}_0(x) + \sum_{\substack{t\le x_3-1\\ t+1\ne \text{power of } 2}} \#\mathcal{E}_t(x) \\ &\quad + \sum_{\substack{t\le x_3-1\\ t+1= \text{power of } 2}} \#\mathcal{E}_t(x) + \sum_{t\ge x_3}\#\mathcal{E}_t(x) \\ &= (1+o(1))e^{-\gamma}\frac{x}{x_3} + O\left(\sum_{\substack{t\le x_3-1\\ t+1\ne \text{power of } 2}} \frac{x}{2^t x_2^B}\right) \\ &\quad + \sum_{\substack{t\le x_3-1\\ t+1= \text{power of } 2}} (1+o(1))e^{-\gamma}\frac{x}{x_3}\sum_{t+1=2^\ell}\frac{1}{2^t} + O\left(\frac{x}{x_2}\right) \\ &= (1+o(1))e^{-\gamma}\frac{x}{x_3} + (1+o(1))2e^{-\gamma}\frac{x}{x_3}\sum_{\ell=1}^{\infty}\frac{1}{2^{2^\ell}} + O\left(\frac{x}{x_2}\right), \end{aligned}$$

which completes the proof of Theorem 1. □

5. The proofs of Theorems 2 and 3. Let $\mathcal{L}_x := \{n \le x : F(n) = 1\}$ and

$$\mathcal{F}_x(y) := \{n \le x : \mu^2(n) = 1,\ p \mid n \Longrightarrow p \equiv 1 \pmod 3,\ p(n) > y\}.$$

Let $\varepsilon > 0$ be fixed, q a prime smaller than $x_2^{1-\varepsilon}$ and $\mathcal{L}_x(q) := \{n = q\nu \le x : F(n) = 1\}$.

Given $n = q\nu \in \mathcal{L}_x(q)$, it is clear that $q\nu$ is squarefree, and since $d_3(p) = 3$, all primes $p \mid \nu$ must satisfy $p \equiv -1 \pmod 3$ and $p \not\equiv 1 \pmod q$. Therefore,

$$\begin{aligned}\#\mathcal{L}_x(q) &\ll \frac{x}{q} \prod_{\substack{\pi < x \\ \pi \equiv -1 \pmod 3}} \left(1 - \frac{1}{\pi}\right) \cdot \prod_{\substack{p \equiv 1 \pmod 3 \\ p \equiv 1 \pmod q}} \left(1 - \frac{1}{p}\right) \\ &\ll \frac{x}{q\sqrt{x_1}} \exp\left\{-\frac{1}{3}\frac{x_2}{q}\right\},\end{aligned}$$

so that

$$\sum_{q < x_2^{1-\varepsilon}} \#\mathcal{L}_q(x) \ll \frac{x}{\sqrt{x_1}} \frac{1}{x_2^B},$$

where B is an arbitrarily large constant.

It follows from this that

$$\mathcal{L}_x \subseteq \mathcal{F}_x(x_2^{1-\varepsilon}) \cup \text{ a set of size } \frac{x}{\sqrt{x_1}} \frac{1}{x_2^B}.$$

Hence, it is clear that

$$\mathcal{F}_x(x_2) \subseteq \mathcal{L}_x \cup \mathcal{T},$$

where

$$\begin{aligned}\mathcal{T} := \{n \le x : p(n) \ge x_2,\ p \mid n \Longrightarrow p \equiv 1 \pmod 3 \\ \text{and there exists } q \mid n,\ p \mid n,\ p \equiv 1 \pmod q\}.\end{aligned}$$

Now

$$\#\mathcal{T} \le \sum_{x_2 \le q < x} \sum_{p \equiv 1 \pmod q} \#K_{p,q} = S_1 + S_2 + S_3, \tag{10}$$

where

$$K_{p,q} := \#\{\nu \le x/pq : \mu^2(\nu) = 1,\ p(\nu) > x_2,\ p \mid \nu \Longrightarrow p \equiv 1 \pmod 3\}$$

and

$$S_1 = \sum_{x_2 < q < x_1} \sum_{\substack{p \le x/x_1 \\ p \equiv 1 \pmod q}} K_{p,q},$$
$$S_2 = \sum_{x_2 < q < x_1} \sum_{\substack{x/x_1 < p \le x \\ p \equiv 1 \pmod q}} K_{p,q},$$
$$S_3 = \sum_{q \ge x_1} \sum_{\substack{p \le x \\ p \equiv 1 \pmod q}} K_{p,q}.$$

For S_1, using the inequality

$$K_{p,q} \ll \frac{x}{pq(\log(x/pq))^{1/2}\sqrt{x_3}},$$

we easily conclude that

$$\text{(11)} \qquad \begin{aligned} S_1 &\ll \frac{x}{\sqrt{x_1 x_3}} \sum_{x_2 < q < x_1} \frac{1}{q} \sum_{\substack{p < x \\ p \equiv 1 \pmod q}} \frac{1}{p} \\ &\ll \frac{x x_2}{\sqrt{x_1 x_3}} \sum_{q > x_2} \frac{1}{q^2} \ll \frac{x}{\sqrt{x_1 x_3}} \cdot \frac{1}{x_3}. \end{aligned}$$

To estimate S_2, we use the fact that $K_{p,q} \le x/pq$, so that since

$$\sum_{\substack{x/x_1 \le p < x_2 \\ p \equiv 1 \pmod q}} \frac{1}{p} = (1 + o(1)) \frac{1}{q} \int_{x/x_1}^{x} \frac{du}{u \log u} = \frac{1}{q} \int_{x_1 - x_2}^{x_1} \frac{dv}{v} \ll \frac{x_2}{q x_1},$$

it follows that

$$\text{(12)} \qquad S_2 \ll \frac{x x_2}{x_1} \sum_{q > x_2} \frac{1}{q^2} \ll \frac{x x_2}{x_1 x_2 x_3} = \frac{x}{x_1 x_3}.$$

Finally, using again the fact that $K_{p,q} \le x/pq$, it follows that

$$S_3 \ll \sum_{q \ge x_1} \frac{x}{q} \sum_{\substack{p \le x \\ p \equiv 1 \pmod q}} \frac{1}{p} \ll x x_2 \sum_{q \ge x_1} \frac{1}{q^2} \ll \frac{x}{x_1}. \tag{13}$$

Gathering (11), (12) and (13), it follows from (10) that

$$\#\mathcal{T} \ll \frac{x}{\sqrt{x_1}} \cdot \frac{1}{x_3^{3/2}}. \tag{14}$$

Let us now estimate the size of the difference

$$\mathcal{D}_x := \mathcal{F}_x(x_2^{1-\varepsilon}) - \mathcal{F}_x(x_2).$$

If $n \in \mathcal{D}_x$, then n is squarefree and can be written as $n = q\nu$, where $p(n) = q \in [x_2^{1-\varepsilon}, x_2]$, and all prime factors of n are $\equiv 1 \pmod 3$. The number of these $\nu \le x/q$ is less than

$$\frac{x}{q} \cdot \frac{1}{\sqrt{x_1 x_3}}.$$

Since

$$\sum_{x_2^{1-\varepsilon} < q < x_2} \frac{1}{q} \ll \log \frac{1}{1-\varepsilon} \ll \varepsilon,$$

it follows that

$$\#\mathcal{D}_x \ll \frac{\varepsilon x}{\sqrt{x_1 x_3}}. \tag{15}$$

Combining (14) and (15), the proof of Theorem 2 is complete.

The proof of Theorem 3 can be obtained in a similar manner. Therefore we shall only give a sketch of it.

Assume that $m > 1$ is squarefree and that $U(m) = 1$. Then $d_3(m) = 3^{\omega(m)}$, so that $p \mid m$ implies that $3 \nmid p + 1 = \sigma(p)$, that is, $p \equiv 1 \pmod 3$. Similarly, one can show that m cannot be a multiple of 3. Then, as in the proof of Theorem 2, we can deduce that there exists a positive constant c_3 such that

$$(16)\quad \begin{aligned} &\#\{m \le x : \mu^2(m) = 1,\ U(m) = 1\} \\ &\quad = (1 + o(1))\#\{n \le x : p(n) > x_2,\ p \mid n \Longrightarrow p \equiv 2 \pmod 3\} \\ &\quad = (1 + o(1))c_3 \frac{x}{\sqrt{x_1 x_3}}. \end{aligned}$$

Now, let $n \le x$ with $U(n) = 1$. We can write $n = km$, where k is squarefull, m is squarefree and $(k, m) = 1$. We have $(kd_3(k) \cdot 3^{\omega(m)}, \sigma_3(k)\sigma(m)) = 1$. Thus, $U(m) = 1$, $U(k) = 1$, $3 \nmid \sigma_3(k)$ and $(kd_3(k), \sigma(m)) = 1$.

In light of (16), the proof of Theorem 3 will be complete if we can show that

$$(17)\quad \#\{n \le x : U(n) = 1\} = (1 + o(1))\#\{m \le x : \mu^2(m) = 1,\ U(m) = 1\}.$$

To do so, we first observe that we can drop all those integers $n = km \le x$ such that $k > x_1^2$, since their number is clearly $O(x/x_1)$. Letting $T = T_x$ be a function slowly tending to $+\infty$ with x, we then have

$$(18)\quad \begin{aligned} &\sum_{\substack{T_x < k \le x_1^2 \\ k \text{ squarefull}}} \#\{n = km \le x : U(n) = 1\} \\ &\qquad \le \sum_{\substack{T_x < k \le x_1^2 \\ k \text{ squarefull}}} \#\{m \le x/k : \mu^2(m) = 1,\ U(m) = 1\} \\ &\qquad \ll \frac{x}{\sqrt{x_1 x_3}} \sum_{\substack{T_x < k \le x_1^2 \\ k \text{ squarefull}}} \frac{1}{k} = o(1)\frac{x}{\sqrt{x_1 x_3}}. \end{aligned}$$

So let us assume that $2 \le k \le T_x$, $U(k) = 1$ and $T_x \ll x_5$, say. We shall obtain an upper bound for the number $D(x)$ of positive integers $n = km \le x$ such that $U(n) = 1$ and $m \ge 2$. Then $3 \nmid \sigma(k)$ and $2 \nmid kd_3(k)$. Assume that there exists a prime $\pi \mid kd_3(k)$, $\pi \ne 3$. Then, for this fixed number k, the corresponding integer m $(\le x/k)$ has to satisfy the four conditions $p \mid m$, $\pi \nmid p + 1$, $p \not\equiv 1 \pmod 3$ and

$p(m) > x_2^{1-\varepsilon}$. But the number of these integers is less than

$$\frac{x}{k\sqrt{x_1 x_3}} \prod_{\substack{x_2 < p < x \\ p \equiv -1 \pmod{\pi} \\ p \equiv 1 \pmod 3}} \left(1 - \frac{1}{p}\right) \ll \frac{x}{k\sqrt{x_1 x_3}} \exp\left\{-\frac{1}{2}\frac{x_2}{2(\pi - 1)}\right\}. \tag{19}$$

Since $\pi \le T_x \ll x_5$, this far right side of (19) is less than

$$\frac{x}{k\sqrt{x_1}} \cdot \frac{1}{x_2^B},$$

where $B > 0$ is an arbitrary constant.

Therefore it remains to consider the case when $kd_3(k) = 3^\beta$ for some positive integer β, in which case $k = 3^\alpha$ for some integer $\alpha \ge 2$. This implies that $d_3(3^\alpha) = ((\alpha + 1)(\alpha + 2))/2$ and therefore that $3^{\beta - \alpha} = ((\alpha + 1)(\alpha + 2))/2$. But no integer $\alpha \ge 2$ exists with this condition.

We have thus established that

$$\begin{aligned} D(x) &= \sum_{\substack{2 \le k \le T_x \\ k \text{ squarefull}}} \#\{n = km \le x : U(n) = 1,\, m \ge 2\} \\ &\ll \sum_{\substack{2 \le k \le T_x \\ k \text{ squarefull}}} \frac{x}{k\sqrt{x_1}} \frac{1}{x_2^B} = o\left(\frac{x}{\sqrt{x_1 x_3}}\right). \end{aligned}$$

Therefore, in view of (18), estimate (17) is proved. Hence, in light of (16) and (17), the proof of Theorem 3 is complete.

6. The proof of Theorem 4. Let $\mathcal{L}$ be the set of squarefree numbers for which all the prime factors are $\equiv 1 \pmod 3$, and let $\mathcal{L}(y)$ ($\subseteq \mathcal{L}$) be the set of those integers in $\mathcal{L}$ with the additional condition that all their prime divisors are $\ge y$.

Now let k run over the squarefull numbers (including $k = 1$) and assume that $U(p + a) = 1$ with $p + a = km$, where $\mu^2(m) = 1$ and $(k, m) = 1$. Then $(kmd_3(k) \cdot 3^{\omega(m)}, \sigma(k)\sigma(m)) = 1$, so that $U(m) = 1$ and $U(k) = 1$, and consequently $(3, \sigma(m)) = 1$, in which case $m \in \mathcal{L}$, $(kd_3(k), \sigma(m)) = 1$ and $(m, \sigma(k)) = 1$.

Let

$$E_k(x) := \sum_{\substack{p\le x\\ U(p+a)=1\\ k\mid p+a,\ (p+a)/k\in\mathcal{L}}} d(p+a) \quad \text{and} \quad E(x) = \sum_{\substack{k\le x\\ k \text{ squarefull}}} E_k(x).$$

Since $\sum_{n\le y} d(n) < 3y\log y$ for each $y\ge 2$, we have

$$E_k(x) \le d(k)\sum_{\nu\le x/k} d(\nu) < 3\frac{d(k)}{k}xx_1,$$

thus implying that

$$\sum_{\substack{k>x_1^A\\ k \text{ squarefull}}} E_k(x) \ll xx_1 \sum_{\substack{k\ge x_1^A\\ k \text{ squarefull}}} \frac{d(k)}{k} \ll \frac{xx_1x_2}{x_1^{A/2}}.$$

Consequently,

$$E(x) = \sum_{\substack{k\le x_1^A\\ k \text{ squarefull}}} E_k(x) + O\left(xx_2x_1^{1-A/2}\right). \tag{20}$$

Let us first consider the case $k=1$.

For this, we set

$$S_1(x,y) := \sum_{p+a\in\mathcal{L}(y)} d(p+a).$$

Fix $\varepsilon>0$. Our plan is to show that

$$(1+o(1))S_1(x,x_2) \le E_1(x) \le (1+o(1))S_1(x,x_2^{1-\varepsilon}), \quad x\to\infty,$$

and that

$$S_1(x,x_2^{1-\varepsilon}) = (1+o(1))S_1(x,x_2), \quad x\to\infty.$$

and finally that, for some constant C_0,

$$S_1(x, x_2) = (1 + o(1))C_0 \frac{\mathrm{li}\,(x)}{x_3},$$

thereby completing the proof of Theorem 4. $\square$

We start by estimating the sum of $d(p+a)$ for those $p+a \in \mathcal{L}$ for which the smallest prime divisor of $p+a$ is smaller than $x_2^{1-\varepsilon}$, in which case we may write

$$p + a = q\pi_1\pi_2 \cdots \pi_s.$$

If $U(p+a) = 1$, then clearly $q \nmid (\pi_j + 1)$ for $j = 1, \dots, s$.

Now let $q \neq 3$, and let $\mathcal{L}_q \subseteq \mathcal{L}$ be the set of those integers n for which $\pi \mid n$ implies that $\pi + 1 \not\equiv 0 \pmod q$. Moreover, let $\mathcal{L}_q(y) = \mathcal{L}(y) \cap \mathcal{L}_q$.

Set

$$T_1^{(q)}(x) := \sum_{\substack{p \leq x \\ (p+a/q) \in \mathcal{L}_q(q)}} d(p+a).$$

We may obtain an upper bound for $T_1^{(q)}(x)$ by counting the number of couples (u, v) for which

$$p + a = quv, \quad u < v, \qquad uv \leq \frac{x+a}{q}, \quad u, v \in \mathcal{L}_q(q).$$

By classical sieve theory, this quantity is less than

$$R_q := \sum_{\substack{u < \sqrt{x} \\ u \in \mathcal{L}_q(q)}} \frac{\mathrm{li}\,(x)}{q\varphi(u)} \prod_{\pi \in \mathcal{T}_q} \left(1 - \frac{1}{\pi}\right), \tag{21}$$

where

$$\mathcal{T}_q := \{\pi \leq q \text{ or } \pi \equiv -1 \pmod 3 \text{ or } \pi \equiv -1 \pmod q\}.$$

Thus,

$$\prod_{\pi \in \mathcal{T}_q} \left(1 - \frac{1}{\pi}\right) \ll \left(\frac{\log q}{x_1}\right)^{(1/2)+(1/2(q-1))} \cdot \frac{1}{\log q} \tag{22}$$

and

$$\sum_{\substack{u<\sqrt{x}\\ u\in\mathcal{L}_q(q)}} \frac{1}{\varphi(u)} \le \prod_{\substack{q<\pi<\sqrt{x}\\ \pi\equiv 1 \pmod 3\\ \pi+1\not\equiv 0 \pmod q}} \left(1+\frac{1}{\pi}\right) \ll \exp\left\{ \sum_{\substack{q<\pi<\sqrt{x}\\ \pi\equiv 1 \pmod 3\\ \pi+1\not\equiv 0 \pmod q}} \frac{1}{\pi}\right\} \tag{23}$$

$$= \exp\left\{\left(\frac{1}{2}-\frac{1}{2(q-1)}\right)(x_2-\log\log q)\right\}$$

$$= (\log q)^{-1/2+(1/2(q-1))}\cdot x_1^{1/2-(1/2(q-1))}.$$

Using (22) and (23) in (21), we get that

$$T_1^{(q)}(x) \ll R_q \ll \frac{\mathrm{li}\,(x)}{q\log q}(\log q)^{1/(q-1)}\cdot\frac{1}{x_1^{1/(q-1)}},$$

so that

$$\sum_{q<x_2^{1-\varepsilon}} T_1^{(q)}(x) \ll \sum_{q<x_2^{1-\varepsilon}} \frac{\mathrm{li}\,(x)}{q\log q}\exp\left\{-\frac{x_2}{q-1}\right\} \ll \mathrm{li}\,(x)\cdot\exp(-x_2^{1-\varepsilon}),$$

from which it follows that

$$E_1(x) \le S_1(x, x_2^{1-\varepsilon}) + O\left(\mathrm{li}\,(x)\cdot\exp\{-x_2^{1-\varepsilon}\}\right). \tag{24}$$

Now we estimate the sum of $d(p+a)$ over all those primes p for which $p+a\in\mathcal{L}(x_2)$ and $U(p+a)\ne 1$. If p is such a prime, then there exist prime divisors π_1,π_2 of $p+a$ such that $\pi_2\equiv 1 \pmod{\pi_1}$. For fixed π_1,π_2, the contribution is less than

$$\#\{p+a=\pi_1\pi_2uv : u\le v,\ (u,v)=1,\ uv<x/(\pi_1\pi_2),\ u,v\in\mathcal{L}(x_2)\}. \tag{25}$$

Since

$$\sum_{\pi_1>x_1^B}\ \sum_{\pi_2\equiv 1 \pmod{\pi_1}}\ \sum_{\pi_1\pi_2\mid p+a} d(p+a) \ll \sum_{\pi_1}\sum_{\pi_2}\sum_{\nu\le(x/\pi_1\pi_2)} d(\nu)$$

$$\ll \sum_{\pi_1}\sum_{\pi_2}\frac{xx_1}{\pi_1\pi_2} \ll xx_1x_2\sum_{\pi_1}\frac{1}{\pi_1^2} \ll xx_1^{1-B/2},$$

we may assume that $\pi_1 \le x_1^B$. Using this fact, we shall now estimate from above the expression (25) separating the set into two subsets, namely

C1. those elements for which $\pi_1 \le x_1^B$, $\pi_2 \equiv 1 \pmod{\pi_1}$ and $\pi_2 > x^{2/3}$;

C2. those elements for which $\pi_1 \le x_1^B$, $\pi_2 \equiv 1 \pmod{\pi_1}$ and $\pi_2 \le x^{2/3}$.

We start with case C1 and set $\alpha := \pi_1 uv$. The number of solutions of the equation $p + a = \alpha\pi_2$ in primes $p \le x$ and $\pi_2 \equiv 1 \pmod{\pi_1}$ is less than

$$\frac{x/\log^2 x}{\pi_1\varphi(\alpha)} \ll \frac{x}{x_1^2\pi_1^2\varphi(uv)}.$$

Summing up over u, v, the total number of solutions is therefore

$$\begin{aligned} &\ll \frac{x}{x_1^2}\sum_{\pi > x_2}\frac{1}{\pi^2}\sum_{\substack{n\le x\\ n\in\mathcal{L}(x_2)}}\frac{d(n)}{\varphi(n)} \ll \frac{x}{x_1^2 x_3}\prod_{\substack{x_2<\pi<x\\ \pi\equiv 1 \pmod 3}}\left(1+\frac{2}{\pi}\right) \\ &\ll \frac{x}{x_1^2 x_3}\exp\left\{2\sum_{\substack{x_2<\pi<x\\ \pi\equiv 1 \pmod 3}}\frac{1}{\pi}\right\} \ll \frac{x}{x_1^2 x_3}\exp\{x_2 - x_4\} = \frac{x}{x_1 x_3^2}. \end{aligned} \tag{26}$$

To consider case C2, we fix π_1, π_2 and u $(\le \sqrt{x/(\pi_1\pi_2)})$. By using standard sieve theory, the number of solutions of $p + a = \pi_1\pi_2 uv \le x$ in p and $v \in \mathcal{L}(x_2)$ is less than

$$\frac{x/x_1}{\pi_1\pi_2\varphi(u)}\prod_{\substack{x_2<\pi<x\\ \pi\equiv -1 \pmod 3}}\left(1-\frac{1}{\pi}\right) = \frac{x}{x_1\pi_1\pi_2}\left(\frac{x_1}{x_3}\right)^{-1/2}\frac{1}{\varphi(u)}.$$

Summing up this last expression on u, it becomes

$$\begin{aligned} &\ll Q(\pi_1,\pi_2) := \frac{x}{x_1\pi_1\pi_2}\left(\frac{x_1}{x_3}\right)^{-1/2}\sum_{u\le\sqrt{\frac{x}{\pi_1\pi_2}}}\frac{1}{\varphi(u)} \\ &\ll \frac{x}{x_1\pi_1\pi_2}\left(\frac{x_1}{x_3}\right)^{-1/2}\prod_{\substack{x_2\pi<x\\ \pi\equiv 1 \pmod 3}}\left(1+\frac{1}{p-1}\right) \\ &\ll \frac{x}{x_1\pi_1\pi_2}\left(\frac{x_1}{x_3}\right)^{-1/2}\left(\frac{x_1}{x_3}\right)^{1/2} = \frac{x}{x_1\pi_1\pi_2}. \end{aligned}$$

Since

$$\sum_{\pi_1 > x_2} \frac{1}{\pi_1} \sum_{\pi_2 \equiv 1 \pmod{\pi_1}} \frac{1}{\pi_2} \ll x_2 \sum_{\pi_1 > x_2} \frac{1}{\pi_1^2} \ll \frac{1}{x_3},$$

it follows that

$$\sum_{\pi_1 > x_2} \sum_{\pi_2 \equiv 1 \pmod{\pi_1}} Q(\pi_1, \pi_2) \ll \frac{x}{x_1 x_3}. \tag{27}$$

Gathering the bounds (26) and (27) obtained from cases C1 and C2, we have thus proved that

$$E_1(x) \geq S_1(x, x_2) - O\left(\frac{x}{x_1 x_3}\right). \tag{28}$$

In order to take advantage of estimates (24) and (28), let us now count $S_1(x, y)$ for $y = x_2^{1-\varepsilon}$ and $y = x_2$.

By using a method of Hooley and the Bombieri-Vinogradov theorem, one can show that, for some positive constant C_0,

$$S_1(x, y) = (1 + o(1)) C_0 \frac{\operatorname{li}(x)}{\log y}, \quad x \to \infty.$$

Therefore,

$$S_1(x, x_2^{1-\varepsilon}) - S_1(x, x_2) = C_0 \frac{\operatorname{li}(x)}{x_3} \left(\frac{1}{1-\varepsilon} - 1\right) + o\left(\frac{\operatorname{li}(x)}{x_3}\right).$$

By replacing ε by a function $\varepsilon(x)$ which tends to 0 very slowly as $x \to \infty$, we may conclude from (24) and (28) that

$$E_1(x) = (1 + o(1)) C_0 \frac{\operatorname{li}(x)}{x_3}, \quad x \to \infty. \tag{29}$$

As we shall now see,

$$\sum_{\substack{1 < k \leq x_1^A \\ k \text{ squarefull}}} E_k(x) = o\left(\frac{\operatorname{li}(x)}{x_3}\right), \tag{30}$$

which in light of (20) and (29) will complete the proof of Theorem 4.

We shall first consider the case of squarefull $k > 1$.

First observe that, since $d_3(\pi^\alpha) = ((\alpha+1)(\alpha+2))/2$, it follows that if $2|d_3(\pi^\alpha)$ for some prime power $\pi^\alpha \| k$, then $U(p+a) \neq 1$ unless $\phi(m)$ is odd, which only occurs for $m = 1$. Whence, one can obtain $U(p+a) = 1$ only when $p + a = k$.

Since $\alpha \geq 2$, it follows that $((\alpha+1)(\alpha+2))/2$ is odd only if $\alpha \equiv 0 \pmod 4$.

If $d_3(\pi^\alpha)$ is not divisible by any prime other than 3, then

$$\frac{(\alpha+1)(\alpha+2)}{2} = 3^\delta$$

for some integer $\delta \geq 1$, an equation whose only solution is $\alpha = 1$, $\delta = 1$. But this is impossible since k is squarefull.

So let $q(k)$ be the smallest prime divisor of $kd_3(k)$. For each squarefull number $k > 1$, we shall consider separately the two cases:

D1. $q(k) \leq x_2^{1-\varepsilon}$;

D2. $q(k) > x_2^{1-\varepsilon}$.

In case D1, we have

$$E_k(x) \ll \#\{p \leq x : p + a = kuv,\ u < v,\ u, v \in \mathcal{L}_{q(k)}\}.$$

Using basic sieve theory, we obtain that

$$\begin{aligned} E_k(x) &\ll \frac{1}{k} \sum_{\substack{u \leq \sqrt{x/k} \\ u \in \mathcal{L}_{q(k)}}} \frac{\operatorname{li}(x)}{\varphi(u)} \prod_{\substack{\pi+1 \not\equiv 0 \pmod{q(k)} \\ \text{or } \pi \equiv -1 \pmod 3}} \left(1 - \frac{1}{\pi}\right) \\ &\ll \frac{\operatorname{li}(x)}{k} \cdot x_1^{1/2-(1/2(q(k)-1))} \cdot \frac{1}{x_1^{1/2+(1/2(q(k)-1))}} \\ &\ll \frac{\operatorname{li}(x)}{k} \exp\left\{-\frac{x_2}{q(k)-1}\right\} \ll \frac{\operatorname{li}(x)}{k} \exp\{-x_2^{\varepsilon}\}. \end{aligned}$$

It follows from this that, in case D1, we have

$$\sum_{\substack{1 < k \leq x_1^A \\ k \text{ squarefull} \\ q(k) \leq x_2^{1-\varepsilon}}} E_k(x) = o\left(\frac{\operatorname{li}(x)}{x_3}\right). \tag{31}$$

We now move to case D2. Given a squarefull number $k > 1$, let $\pi^\alpha \| k$ (with $\alpha \geq 2$). In this case, we have $\pi > x_2^{1-\varepsilon}$, $(\alpha+2)/2 > x_2^{1-\varepsilon}$ and therefore $k \geq \pi^\alpha > \exp\{x_2^{1-\varepsilon}\}$. For such squarefull $k > 1$, we have that

$$\begin{aligned} E_k(x) &\ll \#\Big\{p+1 = kuv,\ u < v,\ uv \in \mathcal{L},\ uv < \frac{x}{k}\Big\} \\ &\ll \frac{\mathrm{li}\,(x)}{\varphi(k)} \sum_{u < \sqrt{\frac{x}{k}}} \frac{1}{\varphi(u)} \prod_{\pi \equiv -1 \pmod 3} \left(1 + \frac{1}{\pi}\right) \\ &\ll \frac{\mathrm{li}\,(x)}{\varphi(k)}. \end{aligned} \tag{32}$$

Since we clearly have

$$\sum_{\substack{\exp\{x_2^{1-\varepsilon}\} < k \leq x_1^A \\ k \text{ squarefull}}} \frac{1}{\varphi(k)} = o\left(\frac{1}{x_3}\right),$$

it follows from (32) that

$$\sum_{\substack{1 < k \leq x_1^A \\ k \text{ squarefull} \\ q(k) > x_2^{1-\varepsilon}}} E_k(x) = o\left(\frac{\mathrm{li}\,(x)}{x_3}\right). \tag{33}$$

Gathering (31) and (33), then (30) follows, thus completing the proof of Theorem 4. □

7. The proof of Theorem 5. We start by proving that, for most $n \in B_x(r)$, we have $(n, \varphi(n)) = 1$.

In order to do so, first observe that the number of non squarefree numbers belonging to $B_x(r)$ is negligible. Assume now that $n \in B_x(r)$, $\mu^2(n) = 1$ and $n \notin A_x(r)$. Then clearly there exist two prime numbers $p_1 < p_2$ with $p_2 \equiv 1 \pmod{p_1}$ such that $n = p_1 p_2 \nu$, with $\nu \in B_{x/(p_1 p_2)}(r-2)$.

We shall now consider the three possible cases:

(i) $p_1 > x_1$;

(ii) $p_1 \leq x_1$ and $p_2 > x^{2/3}$;

(iii) $p_1 \le x_1$ and $p_2 \le x^{2/3}$.

Then

$$\sum_{\substack{n\in B_x(r)\setminus A_x(r)\\ \text{Case (i)}}} \mu^2(n) \ll \sum_{x_1<p_1<x} \sum_{\substack{p_2<x\\ p_2\equiv 1 \pmod{p_1}}} \frac{x}{p_1p_2} \ll \sum_{p_1>x_1} \frac{xx_2}{p_1^2} \ll \frac{xx_2}{x_1x_2} = \frac{x}{x_1}. \tag{34}$$

When considering those integers n satisfying case (ii), we may assume that $\nu \le x^{0.35}$, say. Then, for fixed p_1, the number of p_2's satisfying $p_2 \le x/(p_1\nu)$, $p_2 \equiv 1 \pmod{p_1}$, the number of squarefree $n = p_1p_2\nu \le x$ is no larger than

$$\frac{x}{x_1p_1^2} \sum_{\substack{\omega(\nu)=r-2\\ p(\nu)>x_2}} \frac{\mu^2(\nu)}{\varphi(\nu)} \ll \frac{x}{x_1p_1^2}\frac{1}{(r-3)!}\left(\sum_{x_2<q<x}\frac{1}{q-1}\right)^{r-3}.$$

Summing up this last quantity over $p_1 \le x_1$, we may conclude that

$$\sum_{\substack{n\in B_x(r)\setminus A_x(r)\\ \text{Case (ii)}}} \mu^2(n) \ll \frac{x}{x_1}\frac{1}{(r-3)!}(x_2-x_4)^{r-3} \sum_{x_2<p_1\le x_1} \frac{1}{p_1^2} \ll \frac{x}{x_1x_2x_3}\frac{1}{(r-3)!}(x_2-x_4)^{r-3}. \tag{35}$$

In case (iii), we have $\nu \le x/(p_1p_2)$ with $x/(p_1p_2) > x^{0.3}$, say. Then using a result similar to Lemma 1 but where we drop the condition $p_i \not\equiv 1 \pmod{Q}$, and choosing $y = x_2$ and taking $r-2$ instead of r, we get that

(36)
$$\sum_{\substack{n\in B_x(r)\setminus A_x(r)\\ \text{Case (iii)}}} \mu^2(n) \ll \frac{x}{x_1}\sum_{x_2<p_1\le x_1}\frac{1}{p_1}\sum_{\substack{p_1<p_2\le x^{2/3}\\ p_2\equiv 1 \pmod{p_1}}}\frac{1}{p_2}\frac{1}{(r-3)!}(x_2-x_4)^{r-3}$$
$$\ll \frac{x}{x_1}\frac{x_2^{r-3}}{(r-3)!}\left(1-\frac{x_4}{x_2}\right)^{r-3}\sum_{x_2<p_1\le x_1}\frac{1}{p_1^2} \ll \frac{x}{x_1x_2x_3}\frac{x_2^{r-3}}{(r-3)!}\left(1-\frac{x_4}{x_2}\right)^{r-3}.$$

Let $\varepsilon > 0$ be a small constant. We shall prove that

$$\#\{n \le x : \omega(n) = r,\ p(n) \le x_2^{1-\varepsilon},\ (n,\varphi(n)) = 1\} = o(\#B_x(r)). \tag{37}$$

First we drop all the integers $n \le x$ for which $P(n) \le \exp\{x_1/x_3^2\} := Y$. In light of (4), the quantity of these numbers n is less than $\Psi(x,Y) \ll x\exp\{-1/2(\log x/\log Y)\} \ll x\exp\{-x_3^2/2\}$, a quantity which is so small that we can indeed drop this category of integers $n \le x$. Hence, we only need to consider those integers $n \le x$ such that $\omega(n) = r$, $p(n) = q \le x_2^{1-\varepsilon}$, $(n, \varphi(n)) = 1$ and $P(n) > Y$. Writing these numbers n as $n = qp_1 \cdots p_{r-1}$ with $p_{r-1} > Y$, then, using a classical Hardy and Ramanujan approach, we get that the number of these integers $n \le x$, for a fixed prime q, is less than

$$\frac{c_5 x}{qx_1(r-2)!}\left\{\sum_{\substack{q<p<x\\ p-1\not\equiv 0 \pmod q}} \frac{1}{p}\right\}^{r-2} \le \frac{c_6 x}{qx_1(r-2)!}\exp\{(r-2)(x_2 - \log\log q)\}$$

for some positive constants c_5 and c_6. Summing up over $q \le x_2^{1-\varepsilon}$, (37) follows.

Finally, it follows from Theorem A that

$$\begin{aligned}\#\{n \le x : \omega(n) = r,\ p(n) > x_2^{1-\varepsilon}\} \\ - \#\{n \le x : \omega(n) = r,\ p(n) > x_2\} \\ = O\left((\varepsilon + o(1))\frac{x}{x_1}\frac{x_2^{r-1}}{(r-1)!}\prod_{p<x_2}\left(1-\frac{1}{p}\right)\right).\end{aligned}$$

Hence, in light of (34), (35), (36) and (37), the proof of Theorem 5 is complete.

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