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ON THE SUM FORMULA FOR MULTIPLE q-ZETA VALUES

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ABSTRACT. Multiple q-zeta values are a one-parameter generalization (in fact, a q-analog) of the multiple harmonic sums commonly referred to as multiple zeta values. These latter are obtained from the multiple q-zeta values in the limit as $q \to 1$. Here, we discuss the sum formula for multiple q-zeta values, and provide a direct, self-contained proof. As a consequence, we also derive a q-analog of Euler's evaluation of the double zeta function $\zeta(m,1)$.

1. Introduction. Sums of the form

$$\zeta(n_1, n_2, \dots, n_r) := \sum_{k_1>k_2>\cdots>k_r>0} \prod_{j=1}^{r} \frac{1}{k_j^{n_j}} \tag{1}$$

have attracted increasing attention in recent years, see e.g., [**1**–**4, 6**–**8, 10, 11**]. The survey articles [**5, 19, 20**] provide an extensive list of references. Here and throughout, $n_1, \dots, n_r$ are positive integers with $n_1 > 1$, and we sum over all positive integers $k_1, \dots, k_r$ satisfying the indicated inequalities. Note that, with positive integer arguments, $n_1 > 1$ is necessary and sufficient for convergence. The sums (1) are sometimes referred to as Euler sums, because they were first studied by Euler [**12**] in the case $r = 2$. In general, they may be profitably viewed as instances of the multiple polylogarithm [**2, 5, 18**], and are now more commonly referred to as multiple zeta values, reducing to the Riemann zeta function in the case $r = 1$. The present author introduced a q-analog of (1) in [**9**] as

$$\zeta[n_1, n_2, \dots, n_r] := \sum_{k_1>k_2>\cdots>k_r>0} \prod_{j=1}^{r} \frac{q^{(n_j-1)k_j}}{[k_j]_q^{n_j}}, \tag{2}$$

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where

$$[k]_q := \sum_{j=0}^{k-1} q^j = \frac{1-q^k}{1-q}, \quad 0 < q < 1.$$

Observe that we now have

$$\zeta(n_1, \dots, n_r) = \lim_{q \to 1} \zeta[n_1, \dots, n_r],$$

so that (2) represents a generalization of (1). In this note, we prove an identity for (2), the $q = 1$ case of which was originally conjectured by Moen [**14**] and Markett [**16**].

It is convenient to state results in terms of the shifted multiple q-zeta function defined by

$$\begin{aligned}\zeta^*[n_1, \dots, n_r] &:= \zeta[1 + n_1, n_2, \dots, n_r] \\ &= \sum_{k_1 > \cdots > k_r > 0} \frac{q^{k_1}}{[k_1]_q} \prod_{j=1}^{r} \frac{q^{(n_j-1)k_j}}{[k_j]_q^{n_j}}.\end{aligned}$$

The main focus of our discussion is the following result.

Theorem 1. *If N and r are positive integers with $N \geq r$, then*

$$\sum_{\substack{n_1 + \cdots + n_r = N \\ \forall j,\, n_j \geq 1}} \zeta^*[n_1, n_2, \dots, n_r] = \zeta^*[N],$$

where the sum is over all positive integers $n_1, n_2, \dots, n_r$ such that $\sum_{j=1}^{r} n_j = N$.

The $q = 1$ case of Theorem 1 was proved for $r = 2$ by Euler, for $r = 3$ by Hoffman and Moen [**15**], and for general r by Granville [**13**]. The first proof of Theorem 1 in full generality, i.e., for general positive integer r and arbitrary $0 < q \leq 1$, was given by the present author [**9**], where it is derived as a consequence of other, deeper results for the multiple q-zeta function. Our purpose here is to give a simpler, more direct and self-contained proof of Theorem 1, and to exhibit a consequence of this result which generalizes another formula of Euler for the multiple zeta function.

2. Self-Contained proof of Theorem 1. By expanding both sides in powers of z and comparing coefficients, one readily sees that Theorem 1 is equivalent to the following result.

Theorem 2. *If r is a positive integer and $z \in \mathbf{C} \setminus \{q^{-m}[m]_q : m \in \mathbf{Z}^+\}$, then*

$$\sum_{k_1 > \cdots > k_r > 0} \frac{q^{k_1}}{[k_1]_q} \prod_{j=1}^{r} \frac{1}{[k_j]_q - zq^{k_j}} = \sum_{m=1}^{\infty} \frac{q^{rm}}{[m]_q^r \left([m]_q - zq^m\right)}. \tag{3}$$

Proof of Theorem 2. Let $L_r = L_r(z)$ denote the lefthand side of (3). By partial fractions,

$$L_r = \sum_{j=1}^{r} S_j \tag{4}$$

where

$$S_j = S_{j,r}(z) := \sum_{k_1 > \cdots > k_r > 0} \frac{q^{k_1}}{[k_1]_q \left([k_j]_q - zq^{k_j}\right)} \prod_{\substack{i=1 \\ i \neq j}}^{r} \frac{1}{[k_i - k_j]_q}.$$

Now rename $k_j = m$ and sum first on m, so that

$$S_j = \sum_{m=1}^{\infty} \frac{A(m, j-1)B(m, r-j)}{[m]_q - zq^m}, \tag{5}$$

where $A(m, 0) := q^m/[m]_q$,

$$A(m, j-1) := \sum_{k_1 > \cdots > k_{j-1} > m} \frac{q^{k_1}}{[k_1]_q} \prod_{i=1}^{j-1} \frac{1}{[k_i - m]_q} \quad \text{for} \quad 2 \le j \le r,$$

$B(m, 0) := 1$ and for $1 \le j \le r-1$,

$$\begin{aligned} B(m, r-j) &:= \sum_{m > k_{j+1} > \cdots > k_r > 0} \prod_{i=j+1}^{r} \frac{1}{[k_i - m]_q} \\ &= (-1)^{r-j} \sum_{m > k_{j+1} > \cdots > k_r > 0} \prod_{i=j+1}^{r} \frac{q^{m-k_i}}{[m - k_i]_q}. \end{aligned}$$

From (4) and (5) we now get that

$$L_r = \sum_{j=0}^{r-1} S_{j+1} = \sum_{m=1}^{\infty} \frac{1}{[m]_q - zq^m} \sum_{j=0}^{r-1} A(m,j)B(m,r-1-j),$$

and hence

$$\sum_{r=1}^{\infty} x^{r-1} L_r = \sum_{m=1}^{\infty} \frac{A_m(x)B_m(x)}{[m]_q - zq^m}, \tag{6}$$

where the generating functions A_m and B_m are defined by

$$A_m(x) := \sum_{n=0}^{\infty} x^n A(m,n), \qquad B_m(x) := \sum_{n=0}^{\infty} x^n B(m,n).$$

The proof of Theorem 2 now follows more or less immediately from the representations

$$A_m(x) = \frac{q^m}{[m]_q} \prod_{c=1}^{m} \left(1 - \frac{xq^c}{[c]_q}\right)^{-1} \quad \text{and} \quad B_m(x) = \prod_{b=1}^{m-1} \left(1 - \frac{xq^b}{[b]_q}\right). \tag{7}$$

To see this, observe that (7) gives

$$A_m(x)B_m(x) = \frac{q^m}{[m]_q} \left(1 - \frac{xq^m}{[m]_q}\right)^{-1} = \sum_{r=1}^{\infty} x^{r-1} \frac{q^{rm}}{[m]_q^r},$$

and hence from (6),

$$\sum_{r=1}^{\infty} x^{r-1} L_r = \sum_{r=1}^{\infty} x^{r-1} \sum_{m=1}^{\infty} \frac{q^{rm}}{[m]_q^r\,([m]_q - zq^m)}.$$

It now remains only to prove the representations (7). First, note that

$$\begin{aligned} B_m(x) &= \sum_{n=0}^{\infty} x^n (-1)^n \sum_{m>k_1>\cdots>k_n>0} \prod_{j=1}^{n} \frac{q^{m-k_j}}{[m-k_j]_q} \\ &= \sum_{n=0}^{\infty} (-x)^n \sum_{m>b_n>\cdots>b_1>0} \prod_{j=1}^{n} \frac{q^{b_j}}{[b_j]_q} \\ &= \prod_{b=1}^{m-1} \left(1 - \frac{xq^b}{[b]_q}\right). \end{aligned}$$

Next, we define

$$A(m,n,k) := \sum_{b_1>\cdots>b_n>k} \frac{q^{m+b_1}}{[m+b_1]_q} \prod_{j=1}^{n} \frac{1}{[b_j]_q},$$

and note that $A(m,n) = A(m,n,0)$. We have

$$\begin{aligned} A(m,1,k) &= \sum_{b>k} \frac{q^{m+b}}{[m+b]_q[b]_q} \\ &= \frac{q^m}{[m]_q} \sum_{b>k} \left(\frac{q^b}{[b]_q} - \frac{q^{m+b}}{[m+b]_q} \right) \\ &= \frac{q^m}{[m]_q} \sum_{m\ge c\ge 1} \frac{q^{c+k}}{[c+k]_q}, \end{aligned}$$

and if, for some positive integer n,

$$A(m,n,k) = \frac{q^m}{[m]_q} \sum_{m\ge c_1\ge\cdots\ge c_n\ge 1} \frac{q^{c_n+k}}{[c_n+k]_q} \prod_{j=1}^{n-1} \frac{q^{c_j}}{[c_j]_q},$$

then

$$\begin{aligned} A(m,n+1,k) &= \sum_{b_1>\cdots>b_{n+1}>k} \frac{q^{m+b_1}}{[m+b_1]_q} \prod_{j=1}^{n+1} \frac{1}{[b_j]_q} \\ &= \sum_{b_2>\cdots>b_{n+1}>k} \left(\prod_{j=2}^{n+1} \frac{1}{[b_j]_q} \right) \sum_{b_1>b_2} \frac{q^{m+b_1}}{[m+b_1]_q[b_1]_q} \\ &= \sum_{b_2>\cdots>b_{n+1}>k} \left(\prod_{j=2}^{n+1} \frac{1}{[b_j]_q} \right) A(m,1,b_2) \\ &= \sum_{b_2>\cdots>b_{n+1}>k} \left(\prod_{j=2}^{n+1} \frac{1}{[b_j]_q} \right) \frac{q^m}{[m]_q} \sum_{c_0=1}^{m} \frac{q^{c_0+b_2}}{[c_0+b_2]_q} \\ &= \frac{q^m}{[m]_q} \sum_{c_0=1}^{m} \sum_{b_2>\cdots>b_{n+1}>k} \frac{q^{c_0+b_2}}{[c_0+b_2]_q} \prod_{j=2}^{n+1} \frac{1}{[b_j]_q} \end{aligned}$$

$$= \frac{q^m}{[m]_q} \sum_{c_0=1}^{m} A(c_0, n, k)$$

$$= \frac{q^m}{[m]_q} \sum_{m \ge c_0 \ge \cdots \ge c_n \ge 1} \frac{q^{c_n+k}}{[c_n+k]_q} \prod_{j=0}^{n-1} \frac{q^{c_j}}{[c_j]_q},$$

by the induction hypothesis. It follows that

$$A(m,n) = A(m,n,0) = \frac{q^m}{[m]_q} \sum_{m \ge c_1 \ge \cdots \ge c_n \ge 1} \prod_{j=1}^{n} \frac{q^{c_j}}{[c_j]_q},$$

and hence

$$A_m(x) = \frac{q^m}{[m]_q} \prod_{c=1}^{m} \left(1 + \frac{xq^c}{[c]_q} + \left(\frac{xq^c}{[c]_q}\right)^2 + \left(\frac{xq^c}{[c]_q}\right)^3 + \cdots\right)$$
$$= \frac{q^m}{[m]_q} \prod_{c=1}^{m} \left(1 - \frac{xq^c}{[c]_q}\right)^{-1}. \qquad \square$$

3. Evaluation of $\zeta[m,1]$. Euler [**12, 17**] (see also [**1**, equation (31)]) proved that, for all integers $m \ge 2$,

$$2\zeta(m,1) = m\zeta(m+1) - \sum_{k=1}^{m-2} \zeta(m-k)\zeta(k+1),$$

thereby expressing $\zeta(m,1)$ in terms of values of the Riemann zeta function. The following q-analog of Euler's formula, which the author proved in [**9**] using generating function techniques, is an easy consequence of the $r=2$ case of Theorem 1 and the q-stuffle multiplication rule [**9**].

Corollary 1. *Let* $2 \le m \in \mathbf{Z}$. *Then*

$$2\zeta[m,1] = m\zeta[m+1] + (1-q)(m-2)\zeta[m] - \sum_{k=1}^{m-2} \zeta[m-k]\,\zeta[k+1].$$

In particular, when $m = 2$ we get $\zeta[2,1] = \zeta[3]$, which is probably the simplest nontrivial identity satisfied by the multiple q-zeta function.

Proof of Corollary 1. For $1 \le k \le m-2$ the q-stuffle multiplication rule [**9**] implies that

$$\zeta[m-k]\zeta[k+1] = \zeta[m+1] + (1-q)\zeta[m] + \zeta[m-k, k+1] + \zeta[k+1, m-k].$$

Summing on k, we find that

$$\sum_{k=1}^{m-2} \zeta[m-k]\zeta[k+1] = (m-2)\left(\zeta[m+1] + (1-q)\zeta[m]\right) + 2 \sum_{\substack{s+t=m+1\\ s,t\ge 2}} \zeta[s,t].$$

But Theorem 1 gives

$$\sum_{\substack{s+t=m+1\\ s,\, t\ge 2}} \zeta[s,t] = \sum_{\substack{s+t=m+1\\ s\ge 2,\, t\ge 1}} \zeta[s,t] - \zeta[m,1] = \zeta[m+1] - \zeta[m,1].$$

It follows that

$$\sum_{k=1}^{m-2} \zeta[m-k]\zeta[k+1] = m\zeta[m+1] + (1-q)(m-2)\zeta[m] - 2\zeta[m,1]. \qquad \square$$

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