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NEAR CONVEXITY, METRIC CONVEXITY AND CONVEXITY

FRED RICHMAN

ABSTRACT. It is shown that a subset of a uniformly convex normed space is nearly convex if and only if its closure is convex. Also, a normed space satisfying a mild completeness property is strictly convex if and only if every metrically convex subset is convex.

1. Classical and constructive mathematics. The arguments in this paper conform to constructive mathematics in the sense of Errett Bishop. This means roughly that they do not depend on the general law of excluded middle. More precisely, the arguments take place in the context of intuitionistic logic. Arguments in the context of ordinary logic will be referred to as *classical*. As intuitionistic logic is a fragment of ordinary logic, our arguments should be valid from a classical point of view, although some of the maneuvers to avoid invoking the law of excluded middle may seem puzzling.

I had intended to write this paper primarily to be read classically, at least in its positive aspects, allowing constructive mathematicians to see for themselves that the arguments were constructively valid. But for those classical mathematicians who want to follow some of the constructive fine points, I include here (at the suggestion of the referee) two constructive principles about real numbers that are used instead of the classical trichotomy, $a < b$ or $a = b$ or $a > b$, which is not constructively valid.

• To deny $a > b$ is to affirm $a \leq b$. Note that "$a \leq b$" is not an abbreviation for "$a < b$ or $a = b$"; in fact, it is defined to be the negation of $a > b$. Moreover, $a \neq b$ is defined to be $a < b$ or $a > b$ (a positive notion), and $a = b$ is the denial of $a \neq b$.

• If $a < b$, then for any c, either $a < c$ or $c < b$. This is sometimes called *cotransitivity*. The argument for it is that if you have close enough rational approximations to a, b, and c, you can figure out either that $a < c$ holds or that $c < b$ holds.

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2. Strict and uniform convexity. We will be working in real normed spaces, denoting the closed ball of radius r centered at x by $B_r(x)$.

Following [**1**] we say that a normed space (or its norm) is *uniformly convex* if for each $\varepsilon > 0$ there exists $q < 1$ so that if u and v are unit vectors, and $\|u - v\| \geq \varepsilon$, then $\|(u+v)/2\| \leq q$. The spaces L_p with $1 < p < \infty$ are uniformly convex [**1**, VII.3.22]. However, $\mathbf{R}^2$ equipped with either of the norms $\|(x,y)\| = |x|+|y|$ or $\|(x,y)\| = \sup(|x|, |y|)$ is not. It will be convenient, see Lemma 3, to allow u and v to be arbitrary vectors in $B_1(0)$ and to state the defining property of uniform convexity in terms of $B_r(0)$ with $r > 0$.

Lemma 1. *Let V be a uniformly convex normed space. For all $\varepsilon > 0$ there exists $q < 1$ such that if u and v are in $B_r(0)$, for some $r > 0$, and $\|u - v\| \geq r\varepsilon$, then $\|(u+v)/2\| \leq qr$.*

Proof. It suffices to prove the lemma for $r = 1$. Given ε, choose q' as in the definition of uniformly convex and let $q = (q'+2)/3$, so $q' = 3q-2$. If $\inf(\|u\|, \|v\|) \leq 2q-1$, then $\|(u+v)/2\| \leq (1+(2q-1))/2 = q$. If $\inf(\|u\|, \|v\|) \geq 2q-1$, then

$$\begin{aligned}\left\|\frac{u+v}{2} - \frac{1}{2}\left(\frac{u}{\|u\|} + \frac{v}{\|v\|}\right)\right\| &= \frac{1}{2}\left\|u - \frac{u}{\|u\|} + v - \frac{v}{\|v\|}\right\| \\ &\leq \frac{1}{2}\left\|u - \frac{u}{\|u\|}\right\| + \frac{1}{2}\left\|v - \frac{v}{\|v\|}\right\| \\ &= \frac{1}{2}(1-\|u\|) + \frac{1}{2}(1-\|v\|) \\ &\leq 1-(2q-1) = 2-2q,\end{aligned}$$

so $\|(u+v)/2\| \leq 2-2q+q' = q$ because $u/\|u\|$ and $v/\|v\|$ are unit vectors. So, in either case, $\|(u+v)/2\| \leq q$. $\square$

For those worried about the constructivity of separating into the two cases, $\inf(\|u\|, \|v\|) \leq 2q-1$ and $\inf(\|u\|, \|v\|) \geq 2q-1$, note that the conclusion $\|(u+v)/2\| \leq q$ is the denial of $\|(u+v)/2\| > q$, and use the theorem of intuitionistic propositional logic that

$$((P \vee \neg P) \Longrightarrow \neg Q) \Longrightarrow \neg Q.$$

(Can we somehow take $q = q'$ in this proof?)

We will also consider a weaker condition: A normed space is *strictly convex* if the convex hull of $\{x, y\}$ is contained in the boundary of $B_1(0)$ only when $x = y$. Classically this is equivalent to requiring that $\|(x+y)/2\| < 1$ whenever x and y are distinct unit vectors. For finite-dimensional spaces, strict convexity is classically equivalent to uniform convexity because the set U of unit vectors is compact, as is the subset $S_\varepsilon = \{(x, y) \in U \times U : \|x - y\| \geq \varepsilon\}$, so the function $\|(x + y)/2\|$ on a nonempty S_ε achieves its supremum q, whence $q < 1$.

The condition that $\|(x+y)/2\| < 1$ whenever x and y are distinct unit vectors is stronger, from a constructive point of view, than what we have called "strict convexity." One has a choice here of calling this condition "strict convexity" and the other "weak strict convexity," or calling this one "strong strict convexity." Although the stronger notion has a better feel to it, because it is more positive, I suspect that it takes a back seat to uniform convexity. In any case, it plays no role in this paper, so I'm not going to call it anything.

Uniform convexity is stronger than strict convexity, even classically. Let V_n be $\mathbf{R}^2$ with the ℓ_n-norm $\|(x, y)\|_n = \sqrt{[n]|x|^n + |y|^n}$ and equip the (algebraic) direct sum V of the V_n with the norm $\|(v_1, v_2, \ldots)\| = \sum_{i=1}^{\infty} \|v_n\|_n$. Then V is strictly convex but not uniformly convex.

Uniform convexity is constructively stronger than strict convexity even for $\mathbf{R}^2$. To establish that, we will show that the equivalence of these two notions implies *Markov's principle*: if a binary sequence cannot be all zeros, then it must contain a one.

Theorem 2. *If every strictly convex norm on* $\mathbf{R}^2$ *is uniformly convex, then Markov's principle holds.*

Proof. Let a_n be an increasing binary sequence that can't be all zeros, and equip $V = \mathbf{R}^2$ with the norm

$$\|(x, y)\| = \lim b_n$$

where b_n is defined inductively as

$$b_n = \|(x,y)\|_n = (|x|^n + |y|^n)^{1/n} \quad \text{if} \quad a_n = 0$$
$$b_n = b_{n-1} \quad \text{if} \quad a_n = 1.$$

The limit exists because $(|x|^n + |y|^n)^{1/n}$ converges to $\sup(|x|, |y|)$, so forms a Cauchy sequence, whence b_n is also a Cauchy sequence.

To see that V is strictly convex, suppose the convex hull of $\{u, v\}$ is contained in the boundary of the unit ball. If $a_n = 1$ for some n, then $u = v$ because the norms $\|(x,y)\|_n$ are strictly convex. So if $u \neq v$, then $a_n = 0$ for all n, a contradiction.

Now suppose V is uniformly convex. Then there exists $q < 1$ so that if u and v are unit vectors with $\|u - v\| \geq 1$, then $\|(u+v)/2\| \leq q$. Let $u = (1,1)/\|(1,1)\|$ and $v = (1,-1)/\|(1,-1)\|$. Then $u - v = (0,2)/\|(1,1)\|$ and $\|(1,1)\| \leq 2$ so $\|u - v\| \geq 1$. Now $(u+v)/2 = (1,0)/\|(1,1)\|$ so $\|(u+v)/2\| = 1/\|(1,1)\| \leq q$ whence $\|(1,1)\| \geq 1/q$. So if $a_n = 0$, then $2^{1/n} \geq 1/q$. This puts an upper limit on n such that $a_n = 0$, so we can find n such that $a_n = 1$. □

3. Near convexity. In [**2**], Mandelkern defined a subset S of a metric space to be *nearly convex* if, for each pair of elements $x, y \in S$, and positive real numbers λ and μ such that $d(x,y) < \lambda + \mu$, there exists $z \in S$ such that $d(x,z) < \lambda$ and $d(z,y) < \mu$. Note that near convexity is a metric notion; it does not refer to any linear structure the space may have. Convexity, on the other hand, is purely linear; it makes no reference to a metric. In [**3**], Schuster gave a simple example of a closed connected subset of the Euclidean plane that is not nearly convex. We will show that a subset of a uniformly convex normed space is nearly convex if and only if its closure is convex. This theorem fails for the supremum norm on $\mathbf{R}^2$.

First we show that, as you pull intersecting balls apart in a uniformly convex normed space, the diameter of their intersection goes to zero.

Lemma 3. *Let V be a uniformly convex normed space. For any distinct vectors x and y in V, the diameter of $B_r(x) \cap B_s(y)$ goes to zero as $r + s$ approaches $\|x - y\|$.*

Proof. We may assume that r and s are positive because the diameter of $B_r(x) \cap B_s(y)$ is at most $2\inf(r,s)$. So we may assume that $r+s \leq \|x-y\| + \inf(r,s)$, in which case $\sup(r,s) = r+s-\inf(r,s) \leq \|x-y\|$. Given $\varepsilon > 0$, choose $q < 1$ as in Lemma 1. Then for u and v in $B_r(x) \cap B_s(y)$, and $\|u-v\| \geq \varepsilon \|x-y\| \geq \varepsilon \sup(r,s)$, we have $\|(u+v)/2 - x\| \leq qr$ and $\|(u+v)/2 - y\| \leq qs$. It follows that $\|x-y\| \leq q(r+s)$ so

$$r + s - \|x-y\| \geq (r+s)(1-q) \geq \|x-y\|(1-q).$$

Thus, contrapositively, if $r + s - \|x-y\| < \|x-y\|(1-q)$, then $\|u-v\| \leq \varepsilon \|x-y\|$ for all u and v in $B_r(x) \cap B_s(y)$. □

Theorem 4. *A subset of a uniformly convex normed space is nearly convex if and only if its closure is convex.*

Proof. Let S be the subset and $\overline{S}$ its closure. Suppose that $\overline{S}$ is convex, that $x, y \in S$, and that λ and μ are positive real numbers such that $d(x,y) < \lambda + \mu$. Define

$$z' = \frac{\mu}{\lambda+\mu}\, x + \frac{\lambda}{\lambda+\mu}\, y.$$

Then $z' \in \overline{S}$ because $\overline{S}$ is convex. Moreover

$$d(x,z') = \frac{\lambda}{\lambda+\mu}\, d(x,y)$$

and

$$d(z',y) = \frac{\mu}{\lambda+\mu}\, d(x,y).$$

Choose $z \in S$ such that

$$d(z,z') < \min\left(\lambda - \frac{\lambda}{\lambda+\mu}\, d(x,y),\ \mu - \frac{\mu}{\lambda+\mu}\, d(x,y)\right).$$

Then $d(x,z) < \lambda$ and $d(z,y) < \mu$.

Conversely, suppose that S is nearly convex, and that we are given $x', y' \in \overline{S}$ and nonnegative real numbers α and β such that $\alpha + \beta = 1$.

We must show that $\alpha x' + \beta y' \in \overline{S}$, that is, for each $\varepsilon > 0$ we must find $z \in S$ such that $d(z, \alpha x' + \beta y') < \varepsilon$. If $d(x', y') < \varepsilon/2$, then simply choose z within $\varepsilon/2$ of x'. So we may assume that $d(x', y') > 0$.

Choose $\lambda = (\beta+\delta)d(x', y')$ and $\mu = (\alpha+\delta)d(x', y')$, where δ is a small positive number to be determined later. Choose $x, y \in S$ such that $d(x, x'), d(y, y') < \delta d(x', y')$. Then $d(x, y) < (1 + 2\delta)d(x', y') = \lambda + \mu$, so there exists $z \in S$ such that $d(z, x) < \lambda$ and $d(z, y) < \mu$. This gives us $d(z, x') < (\beta + 2\delta)d(x', y')$ and $d(z, y') < (\alpha + 2\delta)d(x', y')$. That is, $z \in B_r(x') \cap B_s(y')$ where $r = (\beta+2\delta)d(x', y')$ and $s = (\alpha+2\delta)d(x', y')$. So $r + s - d(x', y') = 4\delta d(x', y')$.

Thus by choosing δ sufficiently small, we can assure, by Lemma 3, that the diameter of $B_r(x') \cap B_s(y')$ is less than ε. But $\alpha x' + \beta y'$ and z are both in $B_r(x') \cap B_s(y')$, so $d(z, \alpha x' + \beta y') < \varepsilon$. □

Under the supremum norm in the plane, the balls are squares with sides parallel to the coordinate axes so are not uniformly convex (or even strictly convex). Indeed, Theorem 4 fails there as the following example shows.

Example. Consider the norm on the plane given by $\|(x, y)\| = \sup(|x|, |y|)$. Let S be the closure of the union of the line segments from $(0, 0)$ to $(1, 1)$ and from $(0, 0)$ to $(1, -1)$. The distance between two points in S is the distance between their second coordinates, so S is isometric to the interval $[-1, 1]$. That makes S nearly convex. So S is closed and nearly convex, but not convex.

4. Metric convexity. A subset S of a metric space is *metrically convex* if, for each pair of elements $x, y \in S$, and nonnegative real numbers λ and μ such that $d(x, y) = \lambda + \mu$, there exists $z \in S$ such that $d(x, z) = \lambda$ and $d(z, y) = \mu$. Note that metric convexity implies near convexity by decreasing the given λ and μ in the definition of nearly convex. The example in the preceding section is metrically convex, not just nearly convex. Convex subsets of a normed space are clearly metrically convex. For what normed spaces is the convexity of a subset determined by its metric structure, that is, when does metric convexity imply convexity?

Theorem 5. *The first five of the following conditions on a normed space are equivalent and imply the sixth.*

1. **Osculating balls.** *Any two closed balls of radii r_1 and r_2 whose centers are a distance $r_1 + r_2$ from each other intersect at precisely one point.*

2. **Nondegenerate osculating balls.** *Same as* (1) *but $r_1 + r_2 > 0$.*

3. **Osculating unit balls.** *Same as* (1) *but $r_1 = r_2 = 1$.*

4. *Strict convexity.*

5. **Strict triangle inequality.** *If $\|x + y\| = \|x\| + \|y\| \neq 0$, then x and y are linearly dependent.*

6. *Any metrically convex subset is convex.*

Proof. Clearly (1) implies (2) implies (3). To see that (3) implies (4), suppose the convex hull of x and y is contained in the boundary of $B_1(0)$. If $\lambda \in [0, 1]$, then

$$\|\lambda x + (1 - \lambda)y - (x + y)\| = \|-((1 - \lambda)x + \lambda y)\| = 1$$

so the convex hull of $\{x, y\}$ is contained in the intersection of the balls $B_1(0)$ and $B_1(x + y)$. Moreover, setting $\lambda = 1/2$ in the displayed equation we see that $\|x + y\| = 2$, so these are osculating unit balls. Therefore, $x = y$.

To see that (4) implies (2), first note that (4) implies the strict convexity of any ball of positive radius. If $\{x, y\}$ is contained in the intersection of the osculating balls, then so is the convex hull H of $\{x, y\}$. If t is in the intersection, and the centers of the balls are c_1 and c_2, then from the inequalities

$$r_1 + r_2 = \|c_1 - c_2\| \leq \|t - c_1\| + \|t - c_2\| \leq r_1 + r_2$$

it follows that $\|t - c_i\| = r_i$ so H must be contained in the boundary of each ball. One of the balls has positive radius, so $x = y$ by strict convexity.

To see that (2) implies (1), suppose $\|c_1 - c_2\| = r_1 + r_2$ and $\{x, y\} \subset B_{r_1}(c_1) \cap B_{r_2}(c_2)$. Then $\|x - y\| \leq 2 \inf(r_1, r_2) \leq \|c_1 - c_2\|$. If $x \neq y$, then $\|c_1 - c_2\| \neq 0$, so $x = y$ by (2), a contradiction. So $x = y$.

To prove that (2) implies (5), suppose $\|x+y\| = \|x\| + \|y\| \neq 0$. Consider the nondegenerate osculating balls $B_{\|x\|}(x)$ and $B_{\|y\|}(-y)$. Then 0 is the unique element of their intersection, so 0 is on the line through x and $-y$. Thus $\lambda x + (\lambda - 1)y = 0$ for some λ, whence x and y are linearly dependent.

To prove that (5) implies (2), suppose $v \in B_{r_1}(c_1) \cap B_{r_2}(c_2)$ where $r_1 + r_2 = \|c_1 - c_2\| > 0$. It suffices to show that v lies on the line through c_1 and c_2, for then v will be unique. Now $\|c_1 - c_2\| = \|c_1 - v\| + \|c_2 - v\|$ so $c_1 - v$ and $c_2 - v$ are dependent. Write $\lambda(c_1 - v) + \mu(c_2 - v) = 0$, that is, $\lambda c_1 + \mu c_2 = (\lambda + \mu)v$, with either $\lambda \neq 0$ or $\mu \neq 0$. The problem is to show that $\lambda + \mu \neq 0$. We may assume that $\mu \neq 0$. So

$$\begin{aligned} |\lambda + \mu| \, \|v\| = \|\lambda c_1 + \mu c_2\| &= \|(\lambda + \mu)c_1 - \mu(c_1 - c_2)\| \\ &\geq |\mu| \, \|c_1 - c_2\| - |\lambda + \mu| \, \|c_1\| . \end{aligned}$$

But $\mu \neq 0$, so $\lambda + \mu \neq 0$.

To show that (1) implies (6), suppose that S is metrically convex, $\{x, y\} \subset S$, and $\alpha \in [0, 1]$. Let $\lambda = \alpha \|x - y\|$ and $\mu = (1 - \alpha)\|x - y\|$. As S is metrically convex, there is $s \in S$ such that $\|s - x\| = \lambda$ and $\|s - y\| = \mu$. Those equations also hold with s replaced by $(1-\alpha)x + \alpha y$. So $s = (1 - \alpha)x + \alpha y$ by (1). □

The next theorem shows that condition (6) implies the other five conditions if we put the following mild completeness condition on the normed space:

(∗) If $\|x\| + \|y\| \neq 0$, then the closure of the union of the convex hulls of $\{x, 0\}$ and $\{0, y\}$ is complete.

This condition is so mild that it holds classically for any normed space.

Theorem 6. *Let V be a normed space satisfying condition* (∗). *If every metrically convex subset of V is convex, then V is strictly convex.*

Proof. It suffices to show that condition (6) of the preceding theorem implies condition (5) if V satisfies (∗). Suppose that $\|x - y\| = \|x\| + \|y\| \neq 0$. This is the hypothesis of (5) with y replaced by $-y$.

Let U be the union of the convex hulls of $\{x, 0\}$ and $\{0, y\}$. We claim that the closure $\overline{U}$ of U is metrically convex. We will prove this by showing that $\overline{U}$ is isometric to the closed interval $[-\|x\|, \|y\|]$ under a map taking x to $-\|x\|$ and y to $\|y\|$.

Define $\varphi : U \to [-\|x\|, \|y\|]$ by $\varphi(\lambda x) = -\|\lambda x\|$ and $\varphi(\mu y) = \|\mu y\|$ where $\lambda, \mu \in [0,1]$. We first show that φ preserves distance. We have

$$|\varphi(\lambda x) - \varphi(\lambda' x)| = |-\|\lambda x\| + \|\lambda' x\|| = |\lambda - \lambda'| \|x\| = \|\lambda x - \lambda' x\|$$

and similarly for $|\varphi(\mu y) - \varphi(\mu' y)|$. Also

$$(1-\lambda)\|x\| + \|\lambda x - \mu y\| + (1-\mu)\|y\| \geq \|x - y\| = \|x\| + \|y\|$$

so

$$\|\lambda x - \mu y\| \geq \lambda \|x\| + \mu \|y\|,$$

from which equality follows. So

$$|\varphi(\lambda x) - \varphi(\mu y)| = |-\|\lambda x\| - \|\mu y\|| = \|\lambda x - \mu y\|.$$

Condition $(*)$ says that $\overline{U}$ is complete, so the isometry φ extends uniquely to an isometry of $\overline{U}$ with the completion $[-\|x\|, \|y\|]$ of $\varphi(U)$.

Thus $\overline{U}$ is metrically convex, hence convex, and each point of $\overline{U}$ is determined by its distance from x. Now if $\alpha \in [0,1]$, then the distance from $\alpha x + (1-\alpha)y$ to x is $(1-\alpha)\|x - y\|$. If $1 - \alpha = \|x\| / \|x - y\|$, then this distance is $\|x\|$, the same as the distance from 0 to x. But $0 \in U \subset \overline{U}$, so $0 = \alpha x + (1-\alpha)y$, a linear dependence relation. □

We observed that condition $(*)$ holds classically in any normed space. However, it is not provable constructively. To see this, let $x = (1,0)$ and $y = (0,a)$ in $\mathbf{R}^2$ and let $V = \mathbf{R}x + \mathbf{R}y$ with norm inherited from $\mathbf{R}^2$. Let C be the closure of the union of the convex hulls of $\{x, 0\}$ and $\{0, y\}$. Given $\varepsilon > 0$, we can approximate a point in the completion of C within ε by $y/3$ if $a > 0$ and by $2y/3$ if $a < \varepsilon$. If C were complete, then there would be a point $sx + ty \in C$ so that $t = 1/3$ if $a > 0$ and $t = 2/3$ if $a < 0$. So if $t > 1/3$, then $a \leq 0$ while if $t < 2/3$, then $a \geq 0$. That would enable us to say, of any real number a, that either $a \leq 0$ or $a \geq 0$. But clearly that information need not be available to us if all we have are arbitrarily close rational approximations to a. Of course this argument does not rule out the possibility that Theorem 6 admits a constructive proof without the hypothesis that V satisfies $(*)$.

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Florida Atlantic University, Boca Raton, FL 33431
E-mail address: richman@fau.edu