ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 37, Number 4, 2007

ON CONSTRUCTING ORTHOGONAL IDEMPOTENTS

S. MOUTON

ABSTRACT. Given a finite-dimensional, semi-simple, commutative algebra A over an algebraically closed field K, and $n-1$ orthogonal idempotents different from 0 and 1, of which at least $n-2$ are minimal, we construct explicitly n orthogonal idempotents different from 0 and 1, of which at least $n-1$ are minimal, using the given idempotents, in the case that n is not larger than the dimension of A.

1. Introduction. If A is a finite-dimensional, semi-simple, commutative algebra over an algebraically closed field K, then A is isomorphic to K^n, where $n = \dim A$. This follows, for instance, from the Wedderburn-Artin theorem, see e.g., [**2**, Theorem 2.1.6]. From this fact it follows immediately that A has a basis of orthogonal idempotents. It is, however, interesting to consider different ways of constructing explicitly such a basis. In this note we consider, in particular, a method to use $n-1$ given orthogonal idempotents to construct n orthogonal idempotents, for $n \leq \dim A$. For this construction we use the properties of the socle of an algebra.

2. Preliminaries. Throughout, A will be a unital algebra over a field K. We recall the following definitions and basic facts. A *minimal left ideal* of A is a nonzero left ideal L such that $\{0\}$ and L are the only left ideals contained in L. An element $p \in A$ is called *idempotent* if $p^2 = p$, and $p \neq 0$ is a *minimal idempotent* if the algebra pAp (with unit p) is a division algebra. If A is finite-dimensional and commutative, and K is algebraically closed, then a nonzero idempotent p is minimal if and only if $Ap = Kp$. If A is semi-simple, then L is a minimal left ideal in A if and only if $L = Ap$ where p is a minimal idempotent in A, [**1**, Proposition 30.6].

If A is semi-simple, then its *socle* $\operatorname{Soc} A$ is defined as the sum of the minimal left ideals in A. (It is also equal to the sum of the minimal

2000 AMS *Mathematics Subject Classification.* Primary 16B99, 46H05, 46J05.
Key words and phrases. Semi-simple algebra, orthogonal idempotent.
Received by the editors on January 3, 2005.

right ideals, so it is a two-sided ideal.) If A is semi-simple and finite-dimensional, then $A = \operatorname{Soc} A$, [**1**, Corollary 32.6].

3. Construction of orthogonal idempotents. The following properties of idempotents are well known and very easy to prove. We supply these properties in the interest of self-containedness.

Lemma 3.1. *Let A be an algebra.*

1. *If $p \in A$ is an idempotent, then $1 - p$ is an idempotent.*

2. *The sum of any finite number of orthogonal idempotents in A is an idempotent.*

3. *The sum of any finite number of orthogonal idempotents is nonzero, if at least one of them is nonzero.*

Lemma 3.2. *Let A be an m-dimensional algebra. If $p_1, \dots, p_m$ are linearly independent idempotents and e is a nonzero idempotent in A, then there exists an $N \in \{1, \dots, m\}$ such that $p_N e \neq 0$.*

Lemma 3.3. *Let A be a finite-dimensional algebra over an algebraically closed field K.*

1. *If $\dim A \geq 2$, then 1 is not a minimal idempotent.*

2. *Suppose, in addition, that A is commutative. If $\dim A = m$, $n < m$ and $p_1, \dots, p_n$ are minimal idempotents, then $\sum_{k=1}^{n} p_k \neq 1$.*

Lemma 3.4. *Let A be a commutative algebra. Then the following holds*:

1. *If p and q are idempotents, then pq is an idempotent, and if $p \neq 1$, then $pq \neq 1$.*

2. *If p is a minimal idempotent and q an idempotent in A such that $pq \neq 0$, then pq is a minimal idempotent.*

Using the properties of the socle, we now prove that a finite-dimensional, semi-simple, commutative algebra over an algebraically closed field has a basis consisting of minimal idempotents.

Proposition 3.5. *Let A be a finite-dimensional, semi-simple, commutative algebra over an algebraically closed field K. Then A has a basis consisting of minimal idempotents.*

Proof. Since A is semi-simple and finite-dimensional, $A = \operatorname{Soc} A$, and each element of $\operatorname{Soc} A$ is a finite sum of elements of the form yp, with p a minimal idempotent and $y \in A$. Let $\dim A = m$, and let $\{a_1, \dots, a_m\}$ be a basis for A with $a_i = \sum_{j=1}^{N_i} y_{ij} p_{ij}$ for all $i = 1, \dots, m$. Since A is finite-dimensional and commutative, K is algebraically closed and each p_{ij} is a minimal idempotent, $Ap_{ij} = Kp_{ij}$, so that $a_i = \sum_{j=1}^{N_i} \lambda_{ij} p_{ij}$ with $\lambda_{ij} \in K$ for all $i = 1, \dots, m$. Therefore, $\{p_{ij} : i = 1, \dots, m,\ j = 1, \dots, N_i\}$ forms a generating set for A, so that a basis $p_1, \dots, p_m$ for A can be chosen from this set. □

We now formulate our main theorem. In this theorem we use $n-1$ given orthogonal idempotents different from 0 and 1, of which at least $n-2$ are minimal, to construct n orthogonal idempotents different from 0 and 1, of which at least $n-1$ are minimal, in the case that n is not larger than the dimension of A.

Theorem 3.6. *Let A be a semi-simple commutative algebra over an algebraically closed field K, with $\dim A = m \geq 2$, and let $3 \leq n \leq m$. If $e_1, \dots, e_{n-1}$ are orthogonal idempotents different from* 0 *and* 1 *with $e_1, \dots, e_{n-2}$ minimal idempotents, then there exist orthogonal idempotents $q_1, \dots, q_n$ different from* 0 *and* 1 *with $q_1, \dots, q_{n-1}$ minimal idempotents.*

Proof. Let $\{p_1, \dots, p_m\}$ be a basis of minimal idempotents of A, see Proposition 3.5. By Lemma 3.2 there exists an $N \in \{1, \dots, m\}$ such that $e_{n-1}p_N \neq 0$. Let $k \in \{1, \dots, n-1\}$ be such that

$$e_{n-j}p_N \neq 0 \quad \text{for all} \quad j = 1, \dots, k$$

and

$$e_{n-j}p_N = 0 \quad \text{for} \quad j = k+1, \dots, n-1, \tag{3.7}$$

if $k < n-1$. Choose $q_j = e_{n-j}p_N$ for $j = 1, \dots, k$. If $k < n-1$, choose $q_{k+1} = e_{n-(k+1)}, \dots, q_{n-1} = e_{n-(n-1)} = e_1$ and $q_n = 1 -$

$\sum_{i=k+1}^{n-1} e_{n-i} - p_N$, and if $k = n-1$, choose $q_n = 1 - p_N$. We prove that $q_1, \dots, q_n$ are orthogonal idempotents different from 0 and 1 with $q_1, \dots, q_{n-1}$ minimal.

First consider the case $k < n-1$, i.e.,

$$\{q_1, \dots, q_n\} = \left\{e_{n-1}p_N, \dots, e_{n-k}p_N, e_{n-(k+1)}, \dots, e_1, 1 - \sum_{i=k+1}^{n-1} e_{n-i} - p_N\right\}.$$

Clearly $q_1, \dots, q_{n-1} \neq 0$. If $q_n = 0$, then $\sum_{i=k+1}^{n-1} e_{n-i} + p_N = 1$. But there are at most $n-1$ terms in this sum and all of them are minimal idempotents, so that this contradicts Lemma 3.3.2. So $q_n \neq 0$.

It follows from Lemma 3.4.1 that $q_j \neq 1$ for $j = 1, \dots, k$. It is clear that $q_{k+1}, \dots, q_{n-1} \neq 1$. Since $e_{n-(k+1)}, \dots, e_1$ and p_N are orthogonal, by (3.7), it follows from Lemma 3.1.3 that $\sum_{i=k+1}^{n-1} e_{n-i} + p_N \neq 0$, so that $q_n \neq 1$.

Clearly, $q_1, \dots, q_{n-1}$ are idempotents. Furthermore,

$$\begin{aligned}
{q_n}^2 &= \left(1 - \sum_{i=k+1}^{n-1} e_{n-i} - p_N\right)\left(1 - \sum_{i=k+1}^{n-1} e_{n-i} - p_N\right) \\
&= 1 - \sum_{i=k+1}^{n-1} e_{n-i} - p_N - \sum_{i=k+1}^{n-1} e_{n-i} \\
&\quad + \left(\sum_{i=k+1}^{n-1} e_{n-i}\right)^2 + \left(\sum_{i=k+1}^{n-1} e_{n-i}\right)p_N \\
&\quad - p_N + p_N\left(\sum_{i=k+1}^{n-1} e_{n-i}\right) + p_N \\
&= q_n + 2p_N\left(\sum_{i=k+1}^{n-1} e_{n-i}\right) \quad \text{by Lemma 3.1.2} \\
&= q_n \quad \text{by (3.7)},
\end{aligned}$$

so that q_n is idempotent.

To prove orthogonality, let $j_1 \neq j_2 \in \{1, \dots, k\}$. Then

$$q_{j_1}q_{j_2} = e_{n-j_1}e_{n-j_2}p_N = 0.$$

Clearly $q_{k+1}, \ldots, q_{n-1}$ are orthogonal. Now let $j \in \{1, \ldots, k\}$ and $l \in \{k+1, \ldots, n-1\}$. Then $q_j q_l = e_{n-j} p_N e_{n-l} = 0$, since $j \neq l$. Furthermore,

$$\begin{aligned} q_j q_n &= e_{n-j} p_N \left(1 - \sum_{i=k+1}^{n-1} e_{n-i} - p_N\right) \\ &= e_{n-j} p_N - e_{n-j} \left(\sum_{i=k+1}^{n-1} e_{n-i}\right) p_N - e_{n-j} p_N \\ &= 0, \end{aligned}$$

and, by (3.7),

$$\begin{aligned} q_l q_n &= e_{n-l} \left(1 - \sum_{i=k+1}^{n-1} e_{n-i} - p_N\right) \\ &= e_{n-l} - e_{n-l} \left(\sum_{i=k+1}^{n-1} e_{n-i}\right) - e_{n-l} p_N \\ &= e_{n-l} - {e_{n-l}}^2 \\ &= 0, \end{aligned}$$

so that $q_1, \ldots, q_n$ are orthogonal.

Since p_N is minimal, $q_1, \ldots, q_k$ are minimal, by Lemma 3.4.2. If $j \in \{k+1, \ldots, n-1\}$, then $q_j \in \{e_1, \ldots, e_{n-k-1}\}$, and since $k \geq 1$, $n-k-1 \leq n-2$, so that q_j is minimal. This proves the case $k < n-1$.

Now consider the case $k = n-1$, i.e.,

$$\{q_1, \ldots, q_n\} = \{e_{n-1} p_N, e_{n-2} p_N, \ldots, e_1 p_N, 1 - p_N\}.$$

Since, by construction, $e_{n-j} p_N \neq 0$ for $j = 1, \ldots, n-1$, it follows that $q_1, \ldots, q_{n-1} \neq 0$. By Lemma 3.3.1 we have that $p_N \neq 1$, so that $q_n \neq 0$. It follows from Lemma 3.4.1 that $q_j \neq 1$ for $j = 1, \ldots, n-1$. Since $p_N \neq 0$, $q_n \neq 1$.

Lemma 3.4.1 implies that $q_1, \ldots, q_{n-1}$ are idempotents, and Lemma 3.1.1 implies that q_n is idempotent. If $j_1 \neq j_2 \in \{1, \ldots, n-1\}$, then $q_{j_1} q_{j_2} = e_{n-j_1} e_{n-j_2} p_N = 0$. Also, if $j \in \{1, \ldots, n-1\}$, then

$q_j q_n = e_{n-j} p_N (1 - p_N) = e_{n-j}(p_N - {p_N}^2) = 0$. Finally, since p_N is minimal, it follows from Lemma 3.4.2 that $q_j = e_{n-j} p_N$ is minimal, for $j = 1, \dots, n-1$. This proves the case $k = n - 1$. □

If A is finite-dimensional, semi-simple and commutative and K is algebraically closed, then A has a basis of orthogonal idempotents different from 0 and 1. This is a well-known fact, following, for instance, from [**2**, Theorem 2.1.6]. It can also be obtained as a corollary of Theorem 3.6.

Corollary 3.8. *Let A be a finite-dimensional, semi-simple, commutative algebra over an algebraically closed field K, with* $\dim A = m \geq 2$. *Then A has a basis $q_1, \dots, q_m$ of orthogonal idempotents, different from* 0 *and* 1.

Proof. By Proposition 3.5 and Lemma 3.3.1, A has a basis $p_1, \dots, p_m$ of minimal idempotents different from (0 and) 1. Then p_1 and $1 - p_1$ are two orthogonal idempotents different from 0 and 1 with p_1 minimal. If $m = 2$, then we can take $q_1 = p_1$, $q_2 = 1 - p_1$.

Suppose $m \geq 3$. Then it follows from Theorem 3.6 that there exist three orthogonal idempotents different from 0 and 1 with two of them minimal, say e_1, e_2, e_3. If $m = 3$, then we can take $q_i = e_i$, $i = 1, 2, 3$.

Repeating this procedure, after $m - 2$ applications of Theorem 3.6, we obtain m orthogonal idempotents different from 0 and 1 (with $m - 1$ of them minimal)—call them $q_1, \dots, q_m$. This is the required basis. □

The *spectrum* of an element a in an algebra A over a field K is defined by

$$\mathrm{Sp}\,(a) = \{\lambda \in K : \lambda 1 - a \text{ is not invertible in } A\}.$$

If K is algebraically closed and $\dim A = m < \infty$, then if $a \in A$, a is algebraic of degree $\leq m$, so that $\mathrm{Sp}\,(a)$ contains at most m elements. Let $\#X$ denote the number of elements in a set X.

Corollary 3.9. *Let A be a finite-dimensional, semi-simple, commutative algebra over an algebraically closed field K, with* $\dim A = m \geq 2$. *Then A contains an element a such that $\#\mathrm{Sp}\, a = m$. In fact, given* *any different* $\alpha_1, \dots, \alpha_m \in K$*, there exists an* $a \in A$ *such that* $\operatorname{Sp} a = \{\alpha_1, \dots, \alpha_m\}$.

Proof. Let $q_1, \dots, q_m$ be the basis of orthogonal idempotents, different from 0 and 1, which exists by Corollary 3.8, and let $\{\alpha_1, \dots, \alpha_m\}$ be a set of different elements in K. If $a = \alpha_1 q_1 + \dots + \alpha_m q_m$, then $(a - \alpha_k)q_k = \alpha_k q_k - \alpha_k q_k = 0$ for all $k \in \{1, \dots, m\}$. Since $q_k \neq 0$, it follows that $a - \alpha_k$ is not invertible, for all $k \in \{1, \dots, m\}$. Hence, $\{\alpha_1, \dots, \alpha_m\} \subset \operatorname{Sp}(a)$. Since $\dim A = m$, we must have $\#\operatorname{Sp}(a) \leq m$. Consequently, $\operatorname{Sp}(a) = \{\alpha_1, \dots, \alpha_m\}$. □

Corollary 3.9 is in particular useful if A is a complex Banach algebra.

REFERENCES

1. F.F. Bonsall and J. Duncan, *Complete normed algebras*, Springer-Verlag, New York, 1973.

2. I.N. Herstein, *Noncommutative rings*, Math. Assoc. America, Washington, D.C., 1968.

Department of Mathematical Sciences, University of Stellenbosch, Private Bag X1, Matieland 7602, South Africa
E-mail address: smo@sun.ac.za