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LIMITS OF VERTEX REPLACEMENT RULES

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ABSTRACT. In an earlier paper [**10**], J. Previte developed a framework for studying iterated replacements of certain vertices in a graph G by a finite replacement graph H. He showed that the normalized sequence of iterated graphs converges in the Gromov-Hausdorff metric (except for special cases). In this paper, we extend the framework in [**10**] to iterated vertex replacements where there are at least two replacement graphs and prove a convergence result. We also give examples of vertex replacement rules that yield convergent sequences of graphs.

1. Introduction. The notion of vertex replacement rules was motivated by studying the horospheres of the geodesic flow on a two-dimensional singular space X of nonpositive curvature, see [**1**].

For two-dimensional singular spaces of nonpositive curvature, the horospheres of X are graphs. The work in this paper is also related to a class of iterative systems, introduced by Aristid Lindenmayer, see [**12, 13**], which is used to model the growth of plants and simple multicellular organisms. Lindenmayer theorized that the development of a complex object, such as a plant, must be governed by a relatively simple set of production rules. His approach created a new branch of biomathematics. Lindenmayer systems were later used in the areas of data and image compression. Since the systems introduced in this paper are more natural and geometric, they promise applications in the same fields that Lindenmayer impacted.

A vertex replacement rule $\mathcal{R}$ is a rule for substituting copies of finite graphs, called replacement graphs, for certain vertices in a given graph G. The result is a new graph $\mathcal{R}(G)$. Iterating $\mathcal{R}$ produces a sequence of graphs $\mathcal{R}^n(G)$. By letting $(\mathcal{R}^n(G), 1)$ be the metric space $\mathcal{R}^n(G)$ normalized to have diameter 1, the sequence of the normalized graphs can be studied.

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Vertex replacement rules with one replacement graph were examined in [**10, 11**]. Necessary and sufficient conditions were found for the sequence $\{(\mathcal{R}^n(G), 1)\}$ to converge in the Gromov-Hausdorff metric. Furthermore, these limit spaces have topological dimension one and, except for special cases, Hausdorff dimension greater than one. It was also shown that many standard examples of fractals, like the Sierpinski triangle, are limit spaces of iterated vertex replacements.

In this paper, we examine the asymptotic behavior of iterated vertex replacements when the replacement rule is given by more than one replacement graph. Our main result, Theorem 3.5, gives sufficient conditions for the convergence of these iterated vertex replacements. Several examples of limit objects of such iterated vertex replacements can be found in the final section.

Understanding how the lengths of certain paths grow under the replacement rule is essential to the proof of convergence of iterated vertex replacements. Not surprisingly, it is considerably more difficult to describe the growth rate of paths when there is more than one replacement graph. Thus we will restrict our attention to vertex replacement rules where the growth rate can be described using primitive matrices.

2. Preliminaries. In this section we define and provide some basic examples of vertex replacements. Throughout this paper we will assume that all graphs are connected, locally finite, unit metric graphs, i.e., each graph is a metric space and every edge has length one. Furthermore, the distance between two points in a graph will be measured by the shortest path in the graph between the two points. In particular, in the graph shown in Figure 1, the distance between vertices a and b is 7.

Definition 2.1. A graph H with a designated set of vertices $\{v_1, \ldots, v_k\}$ is called *symmetric about* $\{v_1, \ldots, v_k\}$ if every permutation of $\{v_1, \ldots, v_k\}$ can be realized by an isometry of H. The vertices in such a designated set are called *boundary vertices* of H and are denoted by ∂H.

Definition 2.2. A *vertex replacement rule* $\mathcal{R}$ consists of a finite list of finite graphs, called *replacement graphs*, $\{H_1, \ldots, H_p\}$, each with a

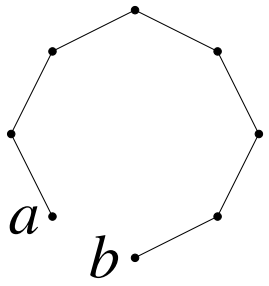


FIGURE 1.

set ∂H_i of boundary vertices, so that $|\partial H_i| \neq |\partial H_j|$ for $i \neq j$, where $|\cdot|$ denotes the cardinality of a set.

Let G be a graph and let $\mathcal{R}$ be a vertex replacement rule given by the replacement graphs $H_1, \ldots, H_p$. Recall that the degree of a vertex v in G, denoted $\deg(v)$, is the number of edges in G adjacent to v.

Definition 2.3. A vertex v in G is called *replaceable* if $\deg(v) = |\partial H_i|$ for some replacement graph H_i in the replacement rule.

The replacement rule $\mathcal{R}$ acts on G by substituting each replaceable vertex in G with its corresponding replacement graph so that the $\deg(v)$ edges previously attached to v in G are attached to the $|\partial H_i|$ vertices of H_i. Since $|\partial H_i| \neq |\partial H_j|$ for $i \neq j$, each replaceable vertex has a unique corresponding replacement graph. Also, since each replacement graph H_i is symmetric about ∂H_i, it is irrelevant how the edges previously adjacent to v are attached to ∂H_i. Thus, vertex replacement is a well-defined procedure.

For example, we may define a vertex replacement rule $\mathcal{R}$ by the replacement graphs H_1 and H_2 depicted in Figure 2.

The boundary vertices of the replacement graphs are shown with circles. Note that each replacement graph is symmetric about its set of boundary vertices. Let G be as depicted in Figure 3. Vertices w_1, w_2, and w_3 are replaceable by H_1, and vertices v_1, v_2, and v_3 are replaceable by H_2, but vertices x_1, x_2, and x_3 are not replaceable. Figure 4 shows $\mathcal{R}(G)$.

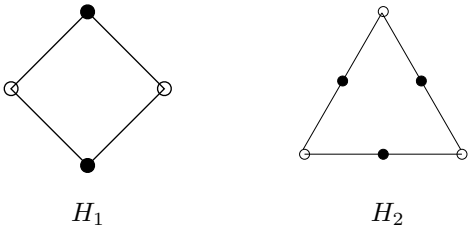


FIGURE 2. A replacement rule $\mathcal{R}$.

We extend the idea of a replaceable vertex to include the vertices of the replacement graphs themselves, but only after the replacement graphs have replaced some vertices. That is, one should not treat a replacement graph H_i as an initial graph G, but always view it as having already replaced some vertex. Hence, we view each boundary vertex as having another edge attached.

Definition 2.4. A boundary vertex v is called *replaceable* if $\deg(v) = |\partial H_i| - 1$ for some replacement graph H_i in the replacement rule.

Notice that for the replacement rule in Figure 2, the boundary vertices of H_1 are replaceable by H_2 (each such vertex will have three edges adjacent after being inserted into a graph G) while the remaining vertices of H_1 are replaceable by H_1. Likewise, the boundary vertices of H_2 are replaceable by H_2 while the remaining vertices of H_2 are

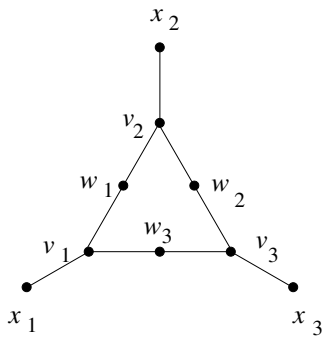


FIGURE 3. A graph G.

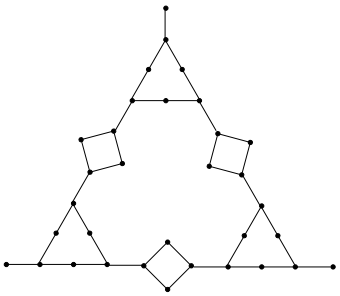


FIGURE 4. The graph $\mathcal{R}(G)$.

replaceable by H_1. Thus, the replacement rule $\mathcal{R}$ may be iterated to create a sequence of graphs $\mathcal{R}^n(G)$. When each graph in this sequence is scaled to have diameter one, we obtain the sequence $\{(\mathcal{R}^n(G), 1)\}$ which, according to our main result (Theorem 3.5), will converge in the Gromov-Hausdorff metric. Figure 5 shows the next two graphs in the sequence and the limit space of this sequence.

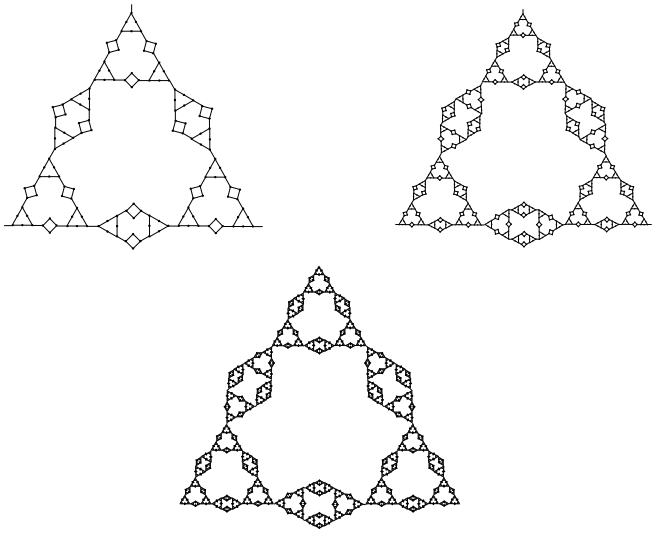


FIGURE 5. $(\mathcal{R}^2(G), 1)$, $(\mathcal{R}^3(G), 1)$, and the limit of $\{(\mathcal{R}^n(G), 1)\}$.

We now select some notation. There exists a pointwise map $\pi : \mathcal{R}(G) \to G$ which undoes replacement by crushing the inserted replacement graphs to the vertices they replaced. In general, for any set F in G, let $\mathcal{R}(F)$ be $\pi^{-1}(F)$. If $F \subset G$ contains no replaceable vertices, then $\mathcal{R}^n(F)$ can be identified with F, and we label $\mathcal{R}^n(F)$ as $F \subset \mathcal{R}^n(G)$. Similarly, if $z \in G$ is not replaceable, label $\mathcal{R}^n(z)$ as $z \in \mathcal{R}^n(G)$. Let F be any finite graph, and let γ be a simple path in F. Let $N_i(F)$ denote the number of vertices in F replaceable by H_i, and let $L(\gamma)$ denote the length of γ. For a replacement graph H_i, we define $N_j(H_i)$ to be the number of vertices in H_i replaceable by H_j when one regards H_i as a subset of $\mathcal{R}(G)$. That is, $N_j(H_i)$ is the number of vertices v in H_i such that $\deg(v) = |\partial H_j| - 1$ if v is a boundary vertex or $\deg(v) = |\partial H_j|$ if v is not a boundary vertex.

Let H_i be a replacement graph in the replacement rule $\mathcal{R}$, and let v_i be a vertex in a graph G replaceable by H_i. Define the set $\partial\mathcal{R}^n(v_i)$ to be all vertices $w \in \mathcal{R}^n(v_i)$ that are adjacent to one of the $\deg(v_i)$ edges outside of $\mathcal{R}^n(v_i)$ that were adjacent to v_i. So $\partial\mathcal{R}^n(v_i)$ is the set of vertices through which a path in $\mathcal{R}^n(G)$ passes when entering or exiting $\mathcal{R}^n(v_i) \subset \mathcal{R}^n(G)$. Note that $|\partial\mathcal{R}^n(v_i)| = |\partial H_i|$. For example, if $\mathcal{R}$ is the replacement rule given in Figure 2 and w_1 is as in Figure 3, then Figure 6 depicts the two vertices in $\partial\mathcal{R}^3(w_1)$ with circles. To determine the growth of $\operatorname{diam}(\mathcal{R}^n(G))$ (and the growth in complexity of $(\mathcal{R}^n(G), 1)$), we need to measure the distance between points in $\partial\mathcal{R}^n(v_i)$. Hence we define the function

$$a_i(n) = \operatorname{dist}_{\mathcal{R}^n(v_i)}(u, u'),$$

where $u, u' \in \partial\mathcal{R}^n(v_i)$ for $u \neq u'$. By the symmetry of each H_i about ∂H_i, the above definition is independent of the choices of u and u' in $\partial\mathcal{R}^n(v_i)$.

Definition 2.5. A path σ in a replacement graph is called a *simple boundary connecting path* if σ is a simple path with boundary vertices for endpoints and no boundary vertices on its interior.

For each n, there is a path in $\mathcal{R}^n(v_i)$ that realizes $a_i(n)$ and projects via π^{n-1} to a simple boundary connecting path $\sigma_i(n)$ in H_i. It is extremely difficult to combinatorially determine the $\sigma_i(n)$ given an arbitrary replacement rule $\mathcal{R}$ since, in general, $\sigma_i(n) \neq \sigma_i(m)$ for

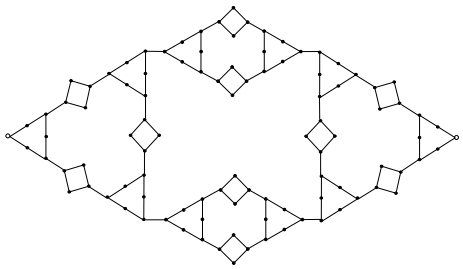


FIGURE 6. The graph $\mathcal{R}^3(w_1)$.

$n \neq m$. However, when the replacement rule is simple, $a_i(n)$ is given by matrix multiplication using the matrix $\widetilde{A}$ defined below. (See Lemma 2.7.)

Definition 2.6. A replacement rule $\mathcal{R}$ given by the graphs $H_1, \ldots, H_p$ is *simple* if there is a unique matrix $\widetilde{A}$ such that, for any set of simple boundary connecting paths $\{\sigma_1, \ldots, \sigma_p\}$, where $\sigma_i \subset H_i$, we have

$$\widetilde{A} = \begin{bmatrix} A & \mathbf{L} \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} N_1(\sigma_1) & \cdots & N_p(\sigma_1) & L(\sigma_1) \\ \vdots & \ddots & \vdots & \vdots \\ N_1(\sigma_p) & \cdots & N_p(\sigma_p) & L(\sigma_p) \\ 0 & \cdots & 0 & 1 \end{bmatrix}.$$

Let A denote the upper left $p \times p$ block of $\widetilde{A}$. We call the matrix A a *path matrix* of $\mathcal{R}$. If A is primitive, i.e., $A^k > 0$ for some power k, then $\mathcal{R}$ is called *primitive*.

In the case where a replacement graph H_i has only one boundary vertex, and hence it has no path between distinct boundary vertices, then the row $N_1(\sigma_i), \ldots, N_p(\sigma_i), L(\sigma_i)$ in the matrix $\widetilde{A}$ above is either a row of zeros (when the boundary vertex of H_i is nonreplaceable) or else a row in which all but one entry is a zero (when the boundary vertex of H_i is replaceable).

The replacement rule in Figure 2 is simple and primitive. So if σ_1 and σ_2 are simple boundary connecting paths in H_1 and H_2, respectively,

then

$$\widetilde{A} = \begin{bmatrix} N_1(\sigma_1) & N_2(\sigma_1) & L(\sigma_1) \\ N_1(\sigma_2) & N_2(\sigma_2) & L(\sigma_2) \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Lemma 2.7. *For a simple replacement rule, we have*

$$\mathbf{a}(n) = \widetilde{A}^n \mathbf{e}_{p+1}, \tag{1}$$

where $\mathbf{a}(n) = [a_1(n), \ldots, a_p(n), 1]^T$ *and* $\mathbf{e}_{p+1}$ *is the last column of the* $(p+1) \times (p+1)$ *identity matrix.*

Proof. Since $\mathcal{R}$ is simple, there is a unique matrix $\widetilde{A}$ such that for any set of simple boundary connecting paths $\{\sigma_1, \ldots, \sigma_p\}$, where $\sigma_i \subset H_i$, we have

$$\widetilde{A} = \begin{bmatrix} N_1(\sigma_1) & \cdots & N_p(\sigma_1) & L(\sigma_1) \\ \vdots & \ddots & \vdots & \vdots \\ N_1(\sigma_p) & \cdots & N_p(\sigma_p) & L(\sigma_p) \\ 0 & \cdots & 0 & 1 \end{bmatrix}.$$

For $i = 1, \ldots, p$, let $\alpha_i(n)$ be a path in $\mathcal{R}^n(v_i)$ that realizes $a_i(n)$, where v_i is a vertex replaceable by H_i. Then

$$a_i(2) = L(\alpha_i(2)) = L(\sigma_i) + \sum_{I=1}^{p} N_I(\sigma_i) L(\sigma_I)$$

and

$$N_k(\alpha_i(2)) = \sum_{I=1}^{p} N_I(\sigma_i) N_k(\sigma_I).$$

This implies

$$\begin{bmatrix} N_1(\alpha_1(2)) & \cdots & N_p(\alpha_1(2)) & L(\alpha_1(2)) \\ \vdots & \ddots & \vdots & \vdots \\ N_1(\alpha_p(2)) & \cdots & N_p(\alpha_p(2)) & L(\alpha_p(2)) \\ 0 & \cdots & 0 & 1 \end{bmatrix} = \widetilde{A}^2.$$

Similarly,

$$\begin{bmatrix} N_1(\alpha_1(n)) & \cdots & N_p(\alpha_1(n)) & L(\alpha_1(n)) \\ \vdots & \ddots & \vdots & \vdots \\ N_1(\alpha_p(n)) & \cdots & N_p(\alpha_p(n)) & L(\alpha_p(n)) \\ 0 & \cdots & 0 & 1 \end{bmatrix} = \widetilde{A}^n.$$

Hence $\mathbf{a}(n) = \widetilde{A}^n \mathbf{e}_{p+1}$. □

For the replacement rule given in Figure 2, the cube of the corresponding matrix $\widetilde{A}$ given on page 7 is $\begin{bmatrix} 9 & 18 & 26 \\ 9 & 18 & 26 \\ 0 & 0 & 1 \end{bmatrix}$. One may verify that $a_1(3) = 26$ by measuring the distance between the two vertices of $\partial\mathcal{R}^3(w_1)$ in Figure 6.

Definition 2.8. Suppose $A = [a_{ij}]$ and $B = [b_{ij}]$ are $p \times p$ matrices. We write

$$A \geq B \text{ (resp. } A > B) \text{ if } a_{ij} \geq b_{ij} \text{ for all } i \text{ and } j \text{ (resp. } a_{ij} > b_{ij}).$$

We note that imposing simplicity on the replacement rule $\mathcal{R}$ is stronger than we really need for the proof of the main result. Our proof extends to rules $\mathcal{R}$ that are eventually simple:

Definition 2.9. Denote by $\mathcal{M}(\mathcal{R})$ the set of all possible path matrices of a replacement rule $\mathcal{R}$. We say $\mathcal{R}$ is *eventually simple* if there exists an integer m and a unique path matrix $A \in \mathcal{M}(\mathcal{R})$ with the property that $B_1 \cdots B_m \geq A^m$ for all $B_i \in \mathcal{M}(\mathcal{R})$.

However, to simplify the argument, we will restrict to simple, primitive replacement rules.

3. Convergence results. Before we state our main result, let us recall some facts about the Gromov-Hausdorff metric. For any metric space X, dist_X will denote the metric on X. Let Z be a metric space. For $C \subset Z$ and $\varepsilon > 0$, let $C_\varepsilon = \{z \in Z : \text{dist}_Z(z, C) < \varepsilon\}$.

Definition 3.1. The *Hausdorff distance* between two nonempty compact subsets A and B of Z is defined by

$$\mathrm{dist}_Z^{\mathrm{Haus}}(A,B) = \inf\{\varepsilon > 0 : A \subseteq B_\varepsilon \text{ and } B \subseteq A_\varepsilon\}.$$

The Hausdorff distance defines a metric on the set of all compact subsets of Z.

We are now able to define the Gromov-Hausdorff distance. Informally, when measuring the Gromov-Hausdorff distance between spaces X and X', we place X and X' into some space in such a way that they are as close together as possible and then measure the resulting Hausdorff distance. Let $\mathcal{S}$ denote the collection of all isometry classes of compact metric spaces.

Definition 3.2. The *Gromov-Hausdorff distance* between two compact metric spaces X and X' is defined by

$$\mathrm{dist}_{\mathcal{S}}^{GH}(X,X') = \inf_{\substack{Z\in\mathcal{S}\\ I,J}}\{\varepsilon > 0 : \mathrm{dist}_Z^{\mathrm{Haus}}(I(X),\ J(X')) < \varepsilon\},$$

where I and J are isometric embeddings of X and X' into Z, respectively.

The space $(\mathcal{S}, \mathrm{dist}_{\mathcal{S}}^{GH})$ is a complete metric space. Moreover,

$$\mathrm{dist}_{\mathcal{S}}^{GH}(X,X') = 0$$

if and only if X is isometric to X'.

Definition 3.3. Let $\varepsilon > 0$. A finite subset S of a metric space X is an *ε-net* of X if $X = S_\varepsilon$.

In the proof of our main result (Theorem 3.5), we will use the following:

Lemma 3.4. *If X and X' are compact metric spaces having ε-nets $\{x_1,\dots,x_n\}$ and $\{x'_1,\dots,x'_n\}$, respectively, and $|\mathrm{dist}_X(x_i,x_j) - \mathrm{dist}_{X'}(x'_i,x'_j)| < \varepsilon$ for all i,j, then $\mathrm{dist}_{\mathcal{S}}^{GH}(X,X') \le 2\varepsilon$.*

Recall that $(\mathcal{R}^n(G), 1)$ denotes the metric space $\mathcal{R}^n(G)$ normalized to have diameter 1, i.e., every edge in $(\mathcal{R}^n(G), 1)$ has length $1/\text{diam}\,(\mathcal{R}^n(G))$. In this section we prove the following theorem.

Theorem 3.5. *Let $H_1, \ldots, H_p$ define a simple, primitive vertex replacement rule $\mathcal{R}$ with $p \geq 2$, and let G be a finite graph with at least one replaceable vertex. Then the normalized sequence $\{(\mathcal{R}^n(G), 1)\}$ converges in the Gromov-Hausdorff metric.*

The proof of our theorem relies on Perron-Frobenius theory. We state the needed results below. See [**9**] for details.

Theorem 3.6 (Perron-Frobenius). *If A is a $p \times p$ nonnegative primitive matrix, then the following hold*:

(1) *The matrix A has a real positive eigenvalue of algebraic multiplicity one which is greater than (in magnitude) its remaining $p - 1$ eigenvalues, and*

(2) *the matrix A has a positive eigenvector corresponding to that eigenvalue.*

Definition 3.7. The eigenvalue of the matrix A described in the Perron-Frobenius theorem is called the *maximal eigenvalue* of A.

Theorem 3.8 (Frobenius). *If A is a nonnegative primitive matrix with maximal eigenvalue λ and row sums $r_1, \ldots, r_p$, then*

$$\rho \leq \lambda \leq R, \tag{2}$$

where $\rho = \min_i r_i$ and $R = \max_i r_i$. Equality holds on either side of inequality (2) *if and only if all row sums of A are equal.*

Corollary 3.9. *Let A be a nonnegative primitive $p \times p$ matrix with integer entries and $p \geq 2$. Then the maximal eigenvalue of A is greater than* 1.

Throughout the remainder of the paper $p \geq 2$. The next lemma shows that for $\mathcal{R}$ simple and primitive, the a_i's grow at the same rate.

Lemma 3.10. *If the replacement rule $\mathcal{R}$ is simple and primitive, then there are positive constants $C_{i,j}$, $i,j = 1,\dots,p$ (depending on $\mathcal{R}$) so that*

$$\lim_{n\to\infty} \frac{a_i(n)}{a_j(n+m)} = \frac{C_{i,j}}{r^m},$$

where r denotes the maximal eigenvalue of the path matrix of $\mathcal{R}$.

Proof. Since $\mathcal{R}$ is simple, then there is a unique matrix $\widetilde{A}$ such that for any set of simple boundary connecting paths $\{\sigma_1,\dots,\sigma_p\}$, where $\sigma_i \subset H_i$, we have

$$\widetilde{A} = \begin{bmatrix} A & \mathbf{L} \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} N_1(\sigma_1) & \cdots & N_p(\sigma_1) & L(\sigma_1) \\ \vdots & \ddots & \vdots & \vdots \\ N_1(\sigma_p) & \cdots & N_p(\sigma_p) & L(\sigma_p) \\ 0 & \cdots & 0 & 1 \end{bmatrix}.$$

The eigenvalues of $\widetilde{A}$ are the eigenvalues $\lambda_1,\dots,\lambda_p$ of the path matrix A together with $\lambda_{p+1} = 1$. Since $\mathcal{R}$ is primitive, by the Perron-Frobenius theorem (Theorem 3.6), A has an eigenvalue, say λ_1, of algebraic multiplicity 1 that is greater than (in modulus) all of its other eigenvalues and has associated positive unit eigenvector $\mathbf{v}_1'$.

Recall, $\mathbf{a}(n) = [a_1(n),\dots,a_p(n),1]^T$. If $\widetilde{A}$ has a set of $(p+1)$-linearly independent real unit eigenvectors $\mathbf{v}_1 = [\mathbf{v}_1', 0], \mathbf{v}_2, \dots, \mathbf{v}_{p+1}$ corresponding to $\lambda_1, \lambda_2, \dots, \lambda_{p+1}$, respectively, then there exist fixed constants c_i so that

$$\mathbf{a}(n) = c_1\lambda_1^n\mathbf{v}_1 + \cdots + c_{p+1}\lambda_{p+1}^n\mathbf{v}_{p+1}. \tag{3}$$

Let $v_{j,k}$ denote the k-th component of $\mathbf{v}_j$. Then for $i,j \in \{1,\dots,p\}$,

$$\frac{a_i(n)}{a_j(n+m)} = \frac{c_1\lambda_1^n v_{1,i} + \cdots + c_{p+1}\lambda_{p+1}^n v_{(p+1),i}}{c_1\lambda_1^{n+m} v_{1,j} + \cdots + c_{p+1}\lambda_{p+1}^{n+m} v_{(p+1),j}}.$$

Dividing by λ_1^{n+m} yields

$$\frac{a_i(n)}{a_j(n+m)} = \frac{c_1(1/\lambda_1^m)v_{1,i} + \cdots + c_{p+1}\left(\lambda_{p+1}/\lambda_1^m\right)^n v_{(p+1),i}}{c_1 v_{1,j} + \cdots + c_{p+1}\left(\lambda_{p+1}/\lambda_1\right)^{n+m} v_{(p+1),j}}.$$

Since $\lambda_1 > 1$ and $\lambda_1 > |\lambda_\ell|$ for $\ell = 2, \ldots, p+1$, we have that

$$\lim_{n\to\infty} \frac{a_i(n)}{a_j(n+m)} = \frac{C_{i,j}}{\lambda_1^m},$$

where $C_{i,j} = v_{1,i}/v_{1,j}$. Since $\mathbf{v}_1'$ is positive, each $C_{i,j} > 0$.

In the general case, one can use complex and generalized eigenvectors to obtain a similar proof. □

Let G be a graph containing a vertex v_i which is replaceable by H_i. Recall the set $\partial\mathcal{R}^n(v_i)$ is the set of all vertices $w \in \mathcal{R}^n(v_i)$ that are adjacent to one of the $\deg(v_i)$ edges outside of $\mathcal{R}^n(v_i)$ that were adjacent to v_i, and the function a_i is given by

$$a_i(n) = \mathrm{dist}_{\mathcal{R}^n(v_i)}(u, u'),$$

where $u, u' \in \partial\mathcal{R}^n(v_i)$ for $u \neq u'$. Define

$$b_i(n) = \sup_{z\in\mathcal{R}^n(v_i)} \{\mathrm{dist}_{\mathcal{R}^n(v_i)}(u, z) \mid u \in \partial\mathcal{R}^n(v_i)\}.$$

By the symmetry of each H_i about ∂H_i, the above definition is independent of the choices of $u, u' \in \partial\mathcal{R}^n(v_i)$. Clearly, $a_i(n) \leq b_i(n)$. However, the lemma below shows that for $\mathcal{R}$ simple and primitive, all of the a_i's and b_i's grow at the same rate.

Lemma 3.11. *If the replacement rule $\mathcal{R}$ is simple and primitive and r is the maximal eigenvalue of the path matrix of $\mathcal{R}$, then there exist positive constants $\widetilde{K}$, κ_1, and κ_2 such that*

$$\kappa_1 \leq \frac{a_i(n)}{b_j(n)} \leq \kappa_2, \tag{4}$$

$$\frac{\kappa_1}{r^m} \leq \frac{b_i(n)}{b_j(n+m)} \leq \frac{\kappa_2}{r^m}, \tag{5}$$

and

$$\frac{\kappa_1}{r^m} \leq \frac{a_i(n)}{b_j(n+m)} \leq \frac{\kappa_2}{r^m} \tag{6}$$

for all $i, j = 1, \ldots, p$ and for all $n > \widetilde{K}$.

Proof. From Lemma 3.10, we have that

$$\lim_{n\to\infty} \frac{a_i(n)}{a_j(n)} = C_{i,j}, \tag{7}$$

where $C_{i,j}$ is a positive constant. Let $k_2 = \max_{i,j} C_{i,j} + 1$. The fact that $a_i(n) \leq b_i(n)$ for all $i = 1, \ldots, p$ together with equation (7) imply that there is an integer K such that

$$\frac{a_i(n)}{b_j(n)} \leq k_2$$

for all $i, j = 1, \ldots, p$ and all $n > K$.

We now show the left inequality in inequality (4). Let $\beta_j(n)$ be a path in $\mathcal{R}^n(v_j)$ which realizes $b_j(n)$. Note that $\pi^{n-1}(\beta_j(n)) \subset H_j$ passes through at most $M = \max_\ell N(H_\ell)$ replaceable vertices. If a replaceable vertex w on $\pi^{n-1}(\beta_j(n)) \subset H_j$ is in the interior of $\pi^{n-1}(\beta_j(n))$, then w corresponds to a portion of $\beta_j(n)$ in $\mathcal{R}^{n-1}(w)$ of length $a_i(n-1)$ for some $i = 1, \ldots, p$. Likewise, if the endpoint w of $\pi^{n-1}(\beta_j(n))$ which is in $\partial\mathcal{R}^n(v_j)$ is replaceable, then it corresponds to a portion of $\beta_j(n)$ in $\mathcal{R}^{n-1}(w)$ that also has length $a_i(n-1)$ for some $i = 1, \ldots, p$. On the other hand, if the endpoint w of $\pi^{n-1}(\beta_j(n))$ which is not in $\partial\mathcal{R}^n(v_j)$ is replaceable, then it will give rise to a portion of $\beta_j(n)$ in $\mathcal{R}^{n-1}(w)$ of length $b_i(n-1)$ for some $i = 1, \ldots, p$.

Therefore, for all $j = 1, \ldots, p$, we have

$$L(\beta_j(n)) = b_j(n) \leq \operatorname{diam}(H_j) + (M-1)\max_\ell a_\ell(n-1) + \max_\ell b_\ell(n-1).$$

This implies that, for all $j = 1, \ldots, p$,

$$\begin{aligned} b_j(n) \leq{}& (n-K)\max_\ell \operatorname{diam}(H_\ell) + \max_\ell b_\ell(K) \\ &+ (M-1)\left(\max_\ell a_\ell(n-1) + \cdots + \max_\ell a_\ell(K)\right). \end{aligned}$$

By Lemma 3.10 there exists a constant W and an integer K so that, for all $n > K$, we have

$$a_i(n) \leq W a_1(n)$$

for all $i = 1, \ldots, p$. So

$$\begin{aligned} b_j(n) \leq{}& (n-K)\max_\ell \operatorname{diam}(H_\ell) + \max_\ell b_\ell(K) \\ &+ W(M-1)\left(a_1(n-1) + \cdots + a_1(K)\right). \end{aligned}$$

As in Lemma 3.10, we may assume without loss of generality that the path matrix A of $\mathcal{R}$ has a set of $p-$linearly independent real unit eigenvectors. Equation (3) in the proof of Lemma 3.10 gives

$$a_\ell(j) = c_1\lambda_1^j v_{1,\ell} + \cdots + c_{p+1}\lambda_{p+1}^j v_{(p+1),\ell}, \tag{8}$$

where the c_i are fixed and $r = \lambda_1$ is the maximal eigenvalue of A. Then, for $n > K$, we have

$$\begin{aligned}
&\sum_{j=K}^{n-1} a_1(j) \\
&\quad = c_1(\lambda_1^{n-1} + \cdots + \lambda_1^K)v_{1,1} + \cdots + c_{p+1}(\lambda_{p+1}^{n-1} + \cdots + \lambda_{p+1}^K)v_{(p+1),1} \\
&\quad \le (p+1)(\max_\ell |c_\ell|)\,\frac{\lambda_1^n - \lambda_1^K}{\lambda_1 - 1}.
\end{aligned}$$

Hence for $n > K$,

$$\begin{aligned}
b_j(n) \le (n-K)\max_\ell \operatorname{diam}(H_\ell) + \max_\ell b_\ell(K) \\
+ W(M-1)(p+1)(\max_\ell |c_\ell|)\,\frac{\lambda_1^n - \lambda_1^K}{\lambda_1 - 1}.
\end{aligned} \tag{9}$$

Equation (8) and inequality (9) imply that there exist positive constants k_1 and K so that for all $n > K$,

$$k_1 \le \frac{a_i(n)}{b_j(n)} \le k_2$$

for all $i, j = 1, \ldots, p$.

The above inequality along with Lemma 3.10 imply that there are positive constants K', k_1', and k_2' such that

$$\frac{k_1'}{r^m} \le \frac{b_i(n)}{b_j(n+m)} \le \frac{k_2'}{r^m}$$

and

$$\frac{k_1'}{r^m} \le \frac{a_i(n)}{b_j(n+m)} \le \frac{k_2'}{r^m}$$

for all $i, j = 1, \ldots, p$ and for all $n > K'$, where r is the maximal eigenvalue of the path matrix of $\mathcal{R}$. Let $\widetilde{K} = \max(K, K')$, $\kappa_1 = \min(k_1, k_1')$, and $\kappa_2 = \max(k_2, k_2')$. □

For large n, we can use Lemma 3.10 to normalize each $a_i(n)$ to $a_1(n)$. In particular, for any path $\eta \subset G$, define

$$f(\eta) = N_1(\eta) + N_2(\eta)C_{2,1} + \cdots + N_p(\eta)C_{p,1},$$

where $C_{i,1}$ are the positive constants from Lemma 3.10. Using f, we can identify those paths in G that give rise to distance minimizing paths in $\mathcal{R}^n(G)$ when $\mathcal{R}$ is simple and primitive.

Definition 3.12. A path η in a graph G is called *preminimizing* if, for any path η' connecting the endpoints of η, we have $f(\eta) \leq f(\eta')$.

Note that if η is preminimizing, then any subpath of η is preminimizing. Lemma 3.13 below shows that preminimizing paths give rise to distance minimizing paths.

Lemma 3.13. *Let G be a finite graph. There is a number $K(G)$ so that if ξ is a distance minimizing path in $\mathcal{R}^n(G)$ and $n > K(G)$, then $\pi^n(\xi)$ is preminimizing in G.*

Proof. Let ξ be a distance minimizing path in $\mathcal{R}^n(G)$ such that the endpoints w_1 and w_2 of $\eta' = \pi^n(\xi) \subset G$ are nonreplaceable (or not vertices). Suppose η' is not preminimizing. Then there exists a preminimizing path $\eta \subset G$ having endpoints w_1 and w_2 and such that $f(\eta) < f(\eta')$, and there is a path in $\mathcal{R}^n(\eta)$ connecting w_1 and w_2 having length $L(\eta) + \sum_i N_i(\eta)a_i(n)$.

By Lemma 3.10,

$$\lim_{n\to\infty} \frac{L(\eta) + \sum_i N_i(\eta)a_i(n)}{a_1(n)} = f(\eta).$$

Similarly, $\lim_{n\to\infty}(L(\eta') + \sum_i N_i(\eta')a_i(n)/a_1(n)) = f(\eta')$. Thus, there exists a constant K depending only on G and $\mathcal{R}$ so that for all $n > K$,

we have $L(\eta) + \sum_i N_i(\eta)a_i(n) < L(\eta') + \sum_i N_i(\eta')a_i(n) = L(\xi)$, which contradicts the fact that ξ is distance minimizing. Therefore, $\eta' = \pi^n(\xi)$ is preminimizing.

If the endpoints of $\pi^n(\xi)$ are replaceable, apply the above argument to the path $\pi^n(\xi)$ with the endpoints removed. The resulting path is preminimizing. Hence the entire path $\pi^n(\xi)$ is preminimizing. □

Remark 3.14. A consequence of the above proof is that if η and η' are paths in G with common endpoints and $f(\eta) = f(\eta')$, then

$$\lim_{n\to\infty} \frac{L(\eta) + \sum_i N_i(\eta)a_i(n)}{L(\eta') + \sum_i N_i(\eta')a_i(n)} = 1.$$

The idea in the proof of our main result is to use Lemma 3.4 to prove that the sequence $\{(\mathcal{R}^\ell(G), 1)\}$ is Cauchy in the Gromov-Hausdorff metric by constructing an ε-net for each graph $(\mathcal{R}^\ell(G), 1)$ and showing that for large ℓ the nets do not change very much. Since $(\mathcal{S}, \mathrm{dist}_S^{GH})$ is a complete metric space, where $\mathcal{S}$ is the collection of all isometry classes of compact metric spaces, then $\{(\mathcal{R}^\ell(G), 1)\}$ converges in the Gromov-Hausdorff metric.

For any finite graph G, define $\Delta_0(G)$ to be the set of all midpoints of edges that are adjacent to replaceable vertices in G. Clearly $x \in \Delta_0(G)$ is not a vertex. Since $x \in \Delta_0(G)$ is not a replaceable vertex, then for $n \geq 0$ we can identify $x \in \Delta_0(G)$ with $\mathcal{R}^n(x) \in \mathcal{R}^n(G)$. For notational purposes, we write $\mathcal{R}^n(x)$ as $x(n)$ and $\mathcal{R}^n(\Delta_0(G))$ as $\Delta_n(G)$. Note that $|\Delta_n(G)| = |\Delta_0(G)|$ for all n. For example, if the replacement rule $\mathcal{R}$ is as in Figure 2 on page 3 and initial graph G is as in Figure 3, then Figure 7 points out the elements of $\Delta_0(G) \subset (G, 1), \Delta_1(G) \subset (\mathcal{R}(G), 1)$ and $\Delta_2(G) \subset (\mathcal{R}^2(G), 1)$ with arrows.

Before we begin, we illustrate the proof that $\{(\mathcal{R}^\ell(G), 1)\}$ is Cauchy using the replacement rule $\mathcal{R}$ and initial graph G from Figures 2 and 3, respectively. We first construct a net for each graph $(\mathcal{R}^\ell(G), 1)$ for all very large ℓ.

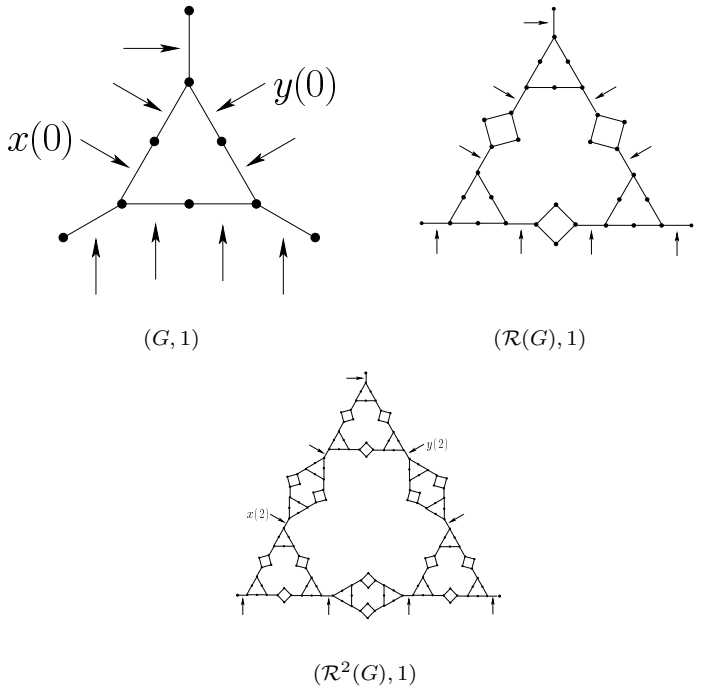
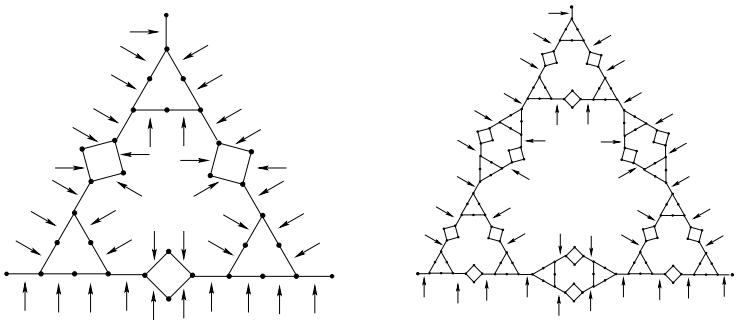


FIGURE 7. The elements of $\Delta_0(G)$, $\Delta_1(G)$ and $\Delta_2(G)$ pointed out with arrows.

Since each point $x \in \Delta_0(G)$ is within a distance of $1/3$ to $\Delta_0(G)\backslash\{x\}$, then $\Delta_0(G)$ forms a $1/3$-net of $(G, 1)$. Similarly, $\Delta_1(G)$ and $\Delta_2(G)$ form $1/3$-nets of $(\mathcal{R}(G), 1)$ and $(\mathcal{R}^2(G), 1)$, respectively. In fact, for all $n > 0$, $\Delta_n(G)$ is a $1/3-$net of $(\mathcal{R}^n(G), 1)$. If we require finer nets, then notice from Figure 8 that $\Delta_0(\mathcal{R}(G))$ is a $1/10$-net of $(\mathcal{R}(G), 1)$ and for all $n > 0$, $\Delta_n(\mathcal{R}(G))$ is a $1/10$-net for $(\mathcal{R}^{n+1}(G), 1)$. If we require a sequence of still finer nets, then we simply fix m sufficiently large and use $\{\Delta_0(\mathcal{R}^m(G)), \Delta_1(\mathcal{R}^m(G)), \Delta_2(\mathcal{R}^m(G)), \dots\}$ as nets for the graphs in the sequence $\{(\mathcal{R}^m(G), 1), (\mathcal{R}^{1+m}(G), 1), (\mathcal{R}^{2+m}(G), 1), \dots\}$.

Once we have a sufficiently fine net for each graph $(\mathcal{R}^\ell(G), 1)$, where ℓ is very large, we then show that these nets do not change very much as $\ell \to \infty$. Figure 7 illustrates that, for any pair $x(0)$, $y(0)$ in $\Delta_0(G)$, there is a corresponding pair $x(n)$, $y(n)$ in $\Delta_n(G)$ and that, for all $n > 0$, the distance between $x(n)$ and $y(n)$ is well approximated by the



$(\mathcal{R}(G), 1)$ $(\mathcal{R}^2(G), 1)$

FIGURE 8. The elements of $\Delta_0(\mathcal{R}(G))$ and $\Delta_1(\mathcal{R}(G))$ are pointed out.

distance between $x(0)$ and $y(0)$. In the proof below we show that, for fixed m large enough and for all large n and n', we have that

$$\left|\mathrm{dist}_{(\mathcal{R}^{n+m}(G),1)}(x(n), y(n)) - \mathrm{dist}_{(\mathcal{R}^{n'+m}(G),1)}(x(n'), y(n'))\right|$$

is very small. By Lemma 3.4, the sequence $\{(\mathcal{R}^\ell(G), 1)\}$ is Cauchy in the Gromov-Hausdorff metric, and hence converges.

Theorem 3.15 (Theorem 3.5). *Let $H_1, \ldots, H_p$ define a simple, primitive vertex replacement rule $\mathcal{R}$, and let G be a finite graph with at least one replaceable vertex. Then the normalized sequence $\{(\mathcal{R}^n(G), 1)\}$ converges in the Gromov-Hausdorff metric.*

Proof. Let $\varepsilon > 0$ be given. We first show that we may fix m large enough so that $\Delta_n(\mathcal{R}^m(G))$ forms an ε-net of $(\mathcal{R}^{m+n}(G), 1)$ for all sufficiently large n.

Let $b_{\max}(n) = \max_{i=1,\ldots,p} b_i(n)$. For each $x(n) \in \Delta_n(\mathcal{R}^m(G))$, consider the ball $B(x(n), b_{\max}(n)+1)$ of radius $b_{\max}(n)+1$ centered at $x(n)$ in the (unscaled) graph $\mathcal{R}^{m+n}(G)$. Let $V(\mathcal{R}^m(G))$ denote the set of all replaceable vertices in $\mathcal{R}^m(G)$. Since $2b_{\max}(n) \geq \mathrm{diam}\,(\mathcal{R}^n(v))$ for any replaceable vertex v, the union of these balls covers $\mathcal{R}^n(V(\mathcal{R}^m(G)))$ for all $n \geq 1$. Moreover, since $\mathcal{R}^{n+m}(G) \setminus \mathcal{R}^n(V(\mathcal{R}^m(G)))$ contains no replaceable vertices for all n and $\mathrm{diam}\,(\mathcal{R}^{n+m}(G)) \to \infty$ as $n \to \infty$,

then for large enough n, the balls $B(x_m(n), b_{\max}(n)+1)$ form a cover of all of $\mathcal{R}^{n+m}(G)$.

Now in order to prove that $\Delta_n(\mathcal{R}^m(G))$ forms an ε-net of the normalized graph $(\mathcal{R}^{n+m}(G), 1)$, it remains to show that the scaled balls $B(x(n), (b_{\max}(n)+1/\operatorname{diam}(\mathcal{R}^{n+m}(G))))$ have positive radius less than ε. Since $\mathcal{R}$ is primitive, we may assume without loss of generality that the initial graph G contains at least two replaceable vertices of each type. Therefore, $2b_{\max}(n+m) \leq \operatorname{diam}(\mathcal{R}^{n+m}(G))$. Hence

$$\frac{b_{\max}(n)+1}{\operatorname{diam}(\mathcal{R}^{n+m}(G))} \leq \frac{b_{\max}(n)+1}{2b_{\max}(n+m)}.$$

Therefore, by Lemma 3.11, we may choose m large enough so that

$$0 < \frac{b_{\max}(n)+1}{2b_{\max}(n+m)} < \varepsilon$$

for all n greater than some constant K_1. Thus, we may fix m large enough so that the set $\Delta_n(\mathcal{R}^m(G))$ forms an ε-net of $(\mathcal{R}^{n+m}(G), 1)$ for all $n > K_1$.

Let x and y be any arbitrary pair of points in $\Delta_0(\mathcal{R}^m(G))$. We now show that when m is fixed as above, then for all large n and n', we have

$$\left|\operatorname{dist}_{(\mathcal{R}^{n+m}(G),1)}(x(n), y(n)) - \operatorname{dist}_{(\mathcal{R}^{n'+m}(G),1)}(x(n'), y(n'))\right| < \varepsilon.$$

Let K_2 be the constant $K(\mathcal{R}^m(G))$ from Lemma 3.13. Suppose $n > \max(K_1, K_2)$. Let $\gamma \subset \mathcal{R}^m(G)$ be the projection via π^n of a path realizing the diameter of $\mathcal{R}^{n+m}(G)$, and let η be the projection of a distance minimizing path between $x(n)$ and $y(n)$. In general, these paths are preminimizing, by Lemma 3.13, but not necessarily unique. However, in light of Remark 3.14, we may as well assume that for any two points $x = x(0)$ and $y = y(0)$ in $\Delta_0(\mathcal{R}^m(G))$, there is a unique preminimizing path η in $\mathcal{R}^m(G)$ between them such that for n large enough, any path realizing $\operatorname{dist}_{\mathcal{R}^{n+m}(G)}(x(n), y(n))$ projects via π^n to η. Similarly, we may assume that γ is unique.

Since γ has at most two replaceable endpoints, then we have

$$\begin{aligned} L(\gamma) + \sum_i N_i(\gamma) a_i(n) &\leq \operatorname{diam} \mathcal{R}^{n+m}(G) \\ &\leq L(\gamma) + \sum_i N_i(\gamma) a_i(n) + 2b_{\max}(n). \end{aligned}$$

Hence,

(10)
$$\frac{L(\eta)+\sum_i N_i(\eta)a_i(n)}{L(\gamma)+\sum_i N_i(\gamma)a_i(n)+2b_{\max}(n)} \le \operatorname{dist}_{(\mathcal{R}^{n+m}(G),1)}(x(n),y(n)) \le \frac{L(\eta)+\sum_i N_i(\eta)a_i(n)}{L(\gamma)+\sum_i N_i(\gamma)a_i(n)}.$$

So Lemmas 10 and 11 imply

(11)
$$\limsup_{n\to\infty} \operatorname{dist}_{(\mathcal{R}^{n+m}(G),1)}(x(n),y(n)) \le \limsup_{n\to\infty} \frac{\left(\frac{L(\eta)+\sum_i N_i(\eta)a_i(n)}{a_1(n)}\right)}{\left(\frac{L(\gamma)+\sum_i N_i(\gamma)a_i(n)}{a_1(n)}\right)} = \frac{f(\eta)}{f(\gamma)}$$

and

(12)
$$\liminf_{n\to\infty} \operatorname{dist}_{(\mathcal{R}^{n+m}(G),1)}(x(n),y(n)) \ge \liminf_{n\to\infty} \frac{\left(\frac{L(\eta)+\sum_i N_i(\eta)a_i(n)}{a_1(n)}\right)}{\left(\frac{L(\gamma)+\sum_i N_i(\gamma)a_i(n)+2b_{\max}(n)}{a_1(n)}\right)} = \frac{f(\eta)}{f(\gamma)+\frac{1}{\kappa_1}}.$$

Recall that as m increases $f(\gamma)\to\infty$. Therefore, inequalities (11) and (12) imply that for fixed m large enough there exists a constant K_3 (depending only on ε) so that for all $n, n' > K_3$, we have

(13) $$|\operatorname{dist}_{(\mathcal{R}^{n+m}(G),1)}(x(n),y(n)) - \operatorname{dist}_{(\mathcal{R}^{n'+m}(G),1)}(x(n'),y(n'))| < \varepsilon.$$

Hence, by Lemma 3.4,

$$\operatorname{dist}_{\mathcal{S}}^{GH}\left[(\mathcal{R}^{\ell}(G),1),(\mathcal{R}^{\ell'}(G),1)\right] \le 2\varepsilon$$

for all $\ell, \ell' > m + \max(K_1, K_2, K_3)$. In other words, $\{(\mathcal{R}^\ell(G), 1)\}$ is Cauchy. Since $(\mathcal{S}, \mathrm{dist}_{\mathcal{S}}^{GH})$ is complete, there is a compact metric space X to which $(\mathcal{R}^\ell(G), 1)$ converges as $\ell \to \infty$. □

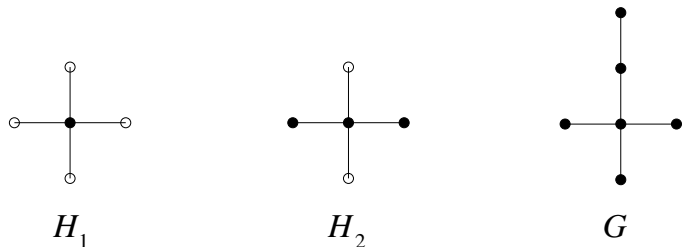


FIGURE 9. A replacement rule $\mathcal{R} = \{H_1, H_2\}$ and an initial graph G.

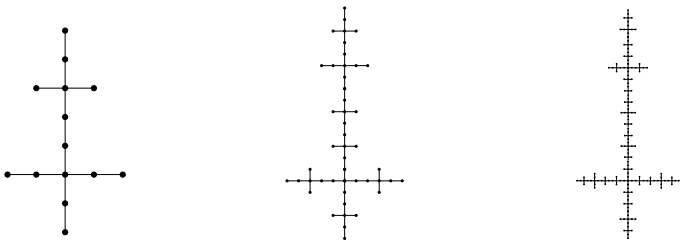


FIGURE 10. $(\mathcal{R}(G), 1)$, $(\mathcal{R}^2(G), 1)$, and $(\mathcal{R}^3(G), 1)$.

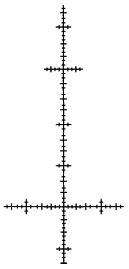


FIGURE 11. Peter's cross.

4. Examples.

Example 1. In our first example, the replacement rule $\mathcal{R}$ (Figure 9) has path matrix $\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$. A few iterations of the replacement rule are shown in Figure 10 and the limit space of the sequence $\{(\mathcal{R}^n(G), 1)\}$ is depicted in Figure 11.

Example 2. For our second example, the replacement rule $\mathcal{R}$ (Figure 12) has path matrix $\begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}$. A few iterations of the replacement rule are shown in Figure 13 and the limit space of the sequence $\{(\mathcal{R}^n(G), 1)\}$ is depicted in Figure 14.

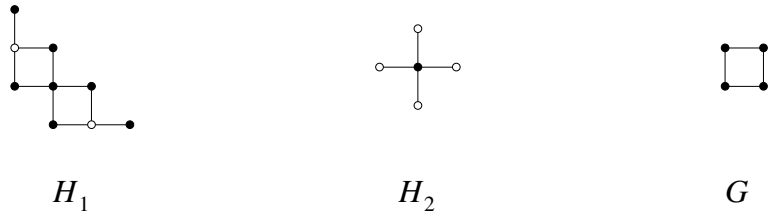


FIGURE 12. A replacement rule $\mathcal{R} = \{H_1, H_2\}$ and an initial graph G.

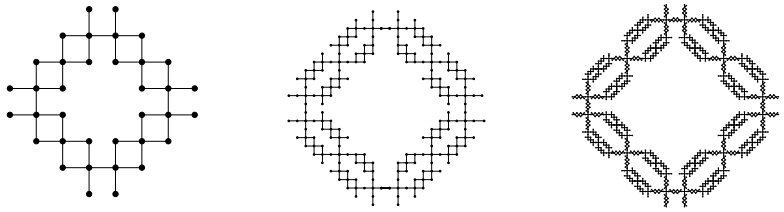


FIGURE 13. $(\mathcal{R}(G), 1)$, $(\mathcal{R}^2(G), 1)$, and $(\mathcal{R}^3(G), 1)$.

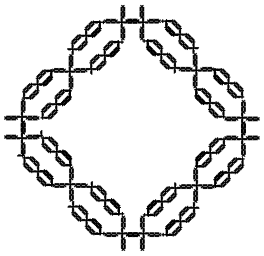


FIGURE 14. A doily.

Example 3. In this example, the replacement rule $\mathcal{R}$ (Figure 15) has path matrix $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. A few iterations of the replacement rule are shown in Figure 16 and the limit space of the sequence $\{(\mathcal{R}^n(G), 1)\}$ is depicted in Figure 17.

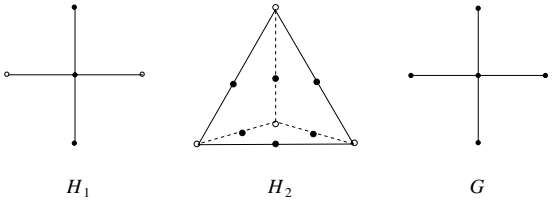


FIGURE 15. A replacement rule $\mathcal{R} = \{H_1, H_2\}$ and an initial graph G.

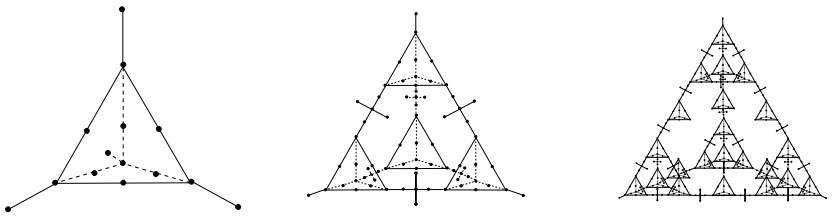


FIGURE 16. $(\mathcal{R}(G), 1)$, $(\mathcal{R}^2(G), 1)$, and $(\mathcal{R}^3(G), 1)$.

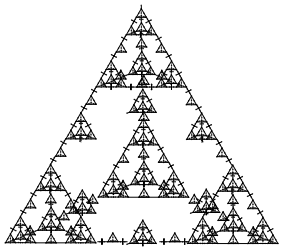


FIGURE 17. A modified Sierpinski tetrahedron.

Example 4. In our final example, the replacement rule $\mathcal{R}$ (Figure 18) has path matrix $A = \begin{bmatrix} 0 & 2 \\ 1 & 2 \end{bmatrix}$. Note that, although A is not positive, it is primitive since A^2 is positive. Figure 19 shows a few iterations of the replacement.

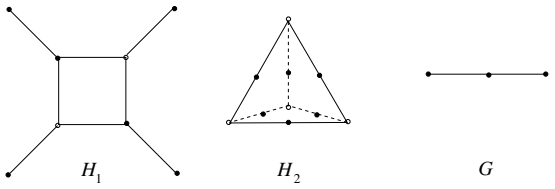


FIGURE 18. A replacement rule $\mathcal{R} = \{H_1, H_2\}$ and an initial graph G.

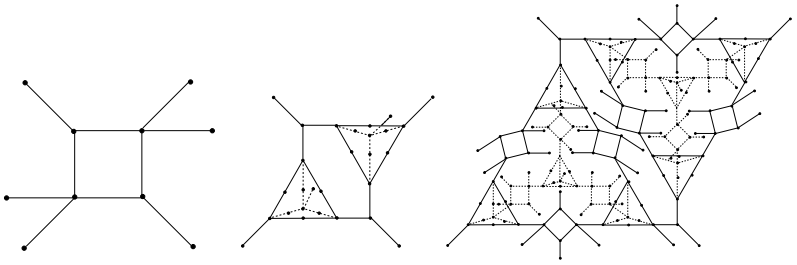


FIGURE 19. $(\mathcal{R}(G), 1)$, $(\mathcal{R}^2(G), 1)$, and $(\mathcal{R}^3(G), 1)$.

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