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ON HYPERBOLICITY AND TAUTNESS OF CERTAIN HARTOGS TYPE DOMAINS

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ABSTRACT. We study the hyperbolicity, tautness, hyperconvexity of Hartogs domains with balanced fibers and Hartogs-Laurent domains. In particular, we shall compare k-, $\tilde{k}$-hyperbolicity and Brody hyperbolicity in the class of such Hartogs type domains. To study tautness, we shall use so-called Royden's criterion for taut domains.

1. Introduction. The purpose of this article is to establish the characterizations of hyperbolicity, tautness, and hyperconvexity in the classes of the following two typical Hartogs type domains, see also [**1, 2, 6, 9, 12, 15, 16**]. Throughout this paper, $G \subset \mathbf{C}^n$ is a domain, $H : G \times \mathbf{C}^m \to [-\infty, \infty)$ is upper semi-continuous (shortly $H \in \mathcal{C}^{\uparrow}(G \times \mathbf{C}^m)$) such that $H(z,w) \geq 0$, $H(z, \lambda w) = |\lambda| H(z,w)$, $\lambda \in \mathbf{C}$, $z \in G$, $w \in \mathbf{C}^m$, and $u, v \in \mathcal{C}^{\uparrow}(G)$ with $u + v < 0$ on G. Put

$$\Omega \equiv \Omega_H(G) := \{(z,w) \in G \times \mathbf{C}^m : H(z,w) < 1\},$$
$$\Sigma \equiv \Sigma_{u,v}(G) := \{(z,\lambda) \in G \times \mathbf{C} : e^{v(z)} < |\lambda| < e^{-u(z)}\}.$$

Here, if $H(z,w) := h(w)e^{u(z)}$, $z \in G$, $w \in \mathbf{C}^m$, where $h \in \mathcal{C}^{\uparrow}(\mathbf{C}^m)$, $h \not\equiv 0$, $h(\lambda w) = |\lambda| h(w)$, $\lambda \in \mathbf{C}$, $w \in \mathbf{C}^m$, we denote $\Omega_H(G)$ by $\Omega_{u,h}(G)$. We say that Ω is a *Hartogs domain over the base G with m-dimensional balanced fibers* and Σ is a *Hartogs-Laurent domain over the base G*.

There are various notions of hyperbolicity of a given domain in $\mathbf{C}^n$, and the relationships between them are investigated by many authors. For example, it is known that any domain G in $\mathbf{C}^n$ which is hyperbolic with respect to the Kobayashi pseudodistance k_G (shortly *k-hyperbolic*) is automatically *Brody hyperbolic*, which means that it does not contain nontrivial entire curve. Analogous to the k-hyperbolicity, we can define the notion of hyperbolicity with respect to the Lempert function $\tilde{k}_G$

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(shortly *$\tilde{k}$-hyperbolic*) of a given domain G in $\mathbf{C}^n$. Here, the latter term seems to be a simpler notion than the former one. We denote the family of all k-, $\tilde{k}$-hyperbolic, Brody hyperbolic domains in all $\mathbf{C}^n$'s by $\mathfrak{G}_K$, $\mathfrak{G}_L$, $\mathfrak{G}_B$, respectively. Obviously, $\mathfrak{G}_K \subset \mathfrak{G}_L \subset \mathfrak{G}_B$. In [**20**], Zwonek proved that $\mathfrak{G}_K \cap \mathcal{PR} = \mathfrak{G}_L \cap \mathcal{PR} = \mathfrak{G}_B \cap \mathcal{PR}$, where $\mathcal{PR}$ is the family of all pseudoconvex Reinhardt domains in all $\mathbf{C}^n$'s. On the other hand, it is known that there is a non-pseudoconvex balanced Hartogs domain G_{ET} of type $\Omega_{u,|\cdot|}(E) \subset \mathbf{C}^2$ which belongs to $\mathfrak{G}_B \setminus \mathfrak{G}_K$, due to Eisenman and Tayler, see, e.g., [**10**, p. 104] or Example 3.6. Therefore, $\mathfrak{G}_B \setminus \mathfrak{G}_L \neq \varnothing$ or $\mathfrak{G}_L \setminus \mathfrak{G}_K \neq \varnothing$. As a main purpose of our work, we shall give some examples of domains which belong to $\mathfrak{G}_B \setminus \mathfrak{G}_L$ and $\mathfrak{G}_L \setminus \mathfrak{G}_K$, respectively. As a first step, we shall give:

- a complete characterization of k-hyperbolic $\Omega_{u,h}(G)$, see Proposition 3.2;
- some sufficient conditions for $\Omega_{u,h}(G)$ to be $\tilde{k}$-hyperbolic, see Theorem 3.8.

These results allow us to find examples of (pseudoconvex) Hartogs domains $\Omega_{u,|\cdot|}(E)$ which belong to $\mathfrak{G}_L \setminus \mathfrak{G}_K$, see, e.g., Example 3.11. In particular, we also get that the domain G_{ET} belongs to $\mathfrak{G}_B \setminus \mathfrak{G}_L$, see Example 3.6. Consequently, we obtain

$$\mathfrak{G}_K \cap \mathcal{H} \subsetneqq \mathfrak{G}_L \cap \mathcal{H} \subsetneqq \mathfrak{G}_B \cap \mathcal{H} \tag{1}$$

where $\mathcal{H}$ is the family of all Hartogs type domains with balanced fibers in all $\mathbf{C}^n$'s.

Moreover, in Section 4 we shall discuss the hyperbolicities of Σ. Since $\Sigma = \Sigma_{u,v}(G) \subset \Omega_{u,|\cdot|}(G) =: \Omega'$, we often get hyperbolicities of Σ from the corresponding characteristics of Ω'. From this point of view, a second purpose of the paper is to find the differences between the hyperbolicities of those domains. First, we see that the $\tilde{k}$-hyperbolicity of Σ implies that $\max\{u, v\}$ is locally bounded on G, see Lemma 4.1, but we do not know whether its converse also holds. On the other hand, it is clear that the hyperbolicities of Ω' implies those of these base G. So it is natural to ask whether this phenomenon remains true for Σ, i.e.,

(2) *"if Σ is hyperbolic, so is the base G?".*

We shall prove that, in general, the answer to (2) is 'NO' for all hyperbolicities. In fact, there is a pseudoconvex Reinhardt Hartogs-

Laurent domain Σ which is hyperbolic, but its base G is not hyperbolic, see Example 4.8. Nevertheless, we shall show that there is a certain significant subclass of Hartogs-Laurent domains for which the answer to (2) is always positive, see Theorem 4.9.

Next, in Section 5 we shall discuss the tautness of Ω and Σ. In [**8**], we can find a characterization for the tautness of a 'bounded' Hartogs domain $\Omega_H(G)$ with m-dimensional balanced fibers. In the case $m = 1$ without boundedness, the tautness of $\Omega_{u,|\cdot|}(G)$ was also studied in [**15**]. It is based on the following result [**16**]:

(3) A holomorphic fiber bundle is taut iff both the fiber and the base are taut.

A third purpose of the paper is to give some general versions of the previous results. First we shall give, applying Royden's criterion, see Proposition 2.1, a full characterization of a taut Hartogs domain $\Omega_{u,h}(G)$ (without the assumption of boundedness) with m-dimensional balanced fibers, see Theorem 5.2. As a consequence, we also get a sufficient condition for the Hartogs-Laurent domain Σ to be taut, see Corollary 5.4; cf. Lemma 4.10. Notice that the original proof of (3) is based on Zorn's lemma and is not elementary. In [**14**], or [**10**, Theorem 5.1], we also find a simple proof of the sufficiency of (3) under the additional assumption, that π is a proper map. In Theorem 5.5, we shall state the sufficiency of (3) for domains in $\mathbf{C}^n$ and give an elementary and direct proof by using Royden's criterion.

Finally, we shall shortly make a comparison between tautness and hyperconvexity of Ω and Σ, see Propositions 5.6 and 5.7.

2. Preliminaries. Let E be the unit disk in the complex plane. For domains $G \subset \mathbf{C}^n$, $S \subset \mathbf{C}^m$, let us denote by $\mathcal{O}(G, S)$ the set of all holomorphic maps from G to S, $\mathcal{O}(G) := \mathcal{O}(G, \mathbf{C})$, and by $\mathcal{PSH}(G)$ the set of all plurisubharmonic functions on G, $\mathcal{SH}(G) := \mathcal{PSH}(G)$ if $G \subset \mathbf{C}^1$. By $\|\cdot\| = \|\cdot\|_n$ we denote the Euclidean norm on $\mathbf{C}^n$, $|\cdot| := \|\cdot\|_1$, and by $\mathbf{B}_n(z, r)$ the n-dimensional Euclidean open ball with center z and radius $r > 0$.

In 1967, Kobayashi, cf. [**10**], defined

$$k_G := \text{the largest pseudodistance not exceeding } \tilde{k}_G,$$

where $\tilde{k}_G(a,z) := \inf\{p(0,\lambda) : \exists_{\varphi\in\mathcal{O}(E,G)},\, \varphi(0)=a,\, \varphi(\lambda)=z\}$ for any $a, z \in G$, and $p(\lambda,\zeta) := \tanh^{-1}(|\lambda-\zeta|/|1-\bar{\lambda}\zeta|)$ is the *Poincaré distance*. We say that k_G, respectively $\tilde{k}_G$, is the *Kobayashi pseudodistance* (*Lempert function*) on G. We write d in all cases where one can take k as well as $\tilde{k}$. It is known that $d_E = p$ and the family $\underline{d} := (d_G)_{G:\text{domain}}$ has the *decreasing property*, i.e., for any domain $S \subset \mathbf{C}^m$ one has

$$d_S(f(a), f(z)) \le d_G(a,z), \quad f \in \mathcal{O}(G,S), \quad a, z \in G;$$

moreover, $k_G \in \mathcal{C}(G\times G)$, $\tilde{k}_G \in \mathcal{C}^{\uparrow}(G\times G)$.

A domain $G \subset \mathbf{C}^n$ is said to be *d-hyperbolic* if $d_G(a,z) > 0$ for $a,\, z \in G,\, a \ne z$. For example, any bounded domain in $\mathbf{C}^n$ is d-hyperbolic, but its converse does not hold in general, e.g. $\mathbf{C}\setminus\{0,1\}$ is d-hyperbolic.

Let us recall the concept of taut domains introduced by Wu [**18**]. A domain $G \subset \mathbf{C}^n$ is said to be *taut* if $\mathcal{O}(E,G)$ is a *normal family*, which means that for any sequence $(f_\nu)_{\nu\ge1} \subset \mathcal{O}(E,G)$ there is a subsequence $(f_{\nu_j})_{j\ge1}$ which is either *normally convergent in* $\mathcal{O}(E,G)$, i.e., it converges uniformly on compact subsets to a map f in $\mathcal{O}(E,G)$ (shortly $f_{\nu_j} \overset{\mathrm{K}}{\rightrightarrows} f$ as $j \to \infty$), or *compactly divergent*, i.e., for any compact subsets $K \subset E$, $L \subset G$, the set $f_{\nu_j}(K)\cap L$ is empty for all sufficiently large j. Wu proved that any taut domain in $\mathbf{C}^n$ is pseudoconvex. It is known that any taut domain in $\mathbf{C}^n$ is d-hyperbolic, but its converse does not hold in general, e.g., $\mathbf{B}_n(0,1)\setminus\{0\}$ is not taut for $n \ge 2$.

We recall that a bounded domain $G \subset \mathbf{C}^n$ is said to be *hyperconvex* if there exists a continuous *bounded plurisubharmonic* exhaustion function on G. This notion was introduced by Stehlé [**13**]. Kerzman and Rosay [**9**] proved that any hyperconvex domain is taut, but its converse does not hold in general, e.g., $E\setminus\{0\}$ is not hyperconvex.

Related to tautness, there is a function $k_G^{(2)}$ defined by

$$\begin{aligned} k_G^{(2)}(a,z) := \inf\{&p(0,\lambda)+p(0,\zeta) : \varphi,\, \psi \in \mathcal{O}(E,G),\ \lambda,\, \zeta \in E,\\ &\varphi(0)=a,\ \varphi(\lambda)=\psi(0),\ \psi(\zeta)=z\}, \quad a,z\in G. \end{aligned}$$

Obviously, $\tilde{k}_G \ge k_G^{(2)} \ge k_G$. The following criterion is due to Royden [**11**].

Proposition 2.1 (A criterion for taut domains). *A domain $G \subset \mathbf{C}^n$ is taut if and only if $\mathbf{B}_{k_G^{(2)}}(a,R) := \{z \in G : k_G^{(2)}(a,z) < R\} \Subset G$ for any $R > 0$ and $a \in G$.*

For a proof, see [**6**]. Some other criteria for tautness can be found in e.g. [**10**].

A domain $G \subset \mathbf{C}^m$ is called *Reinhardt* if $(\lambda_1 w_1, \dots, \lambda_m w_m) \in G$ for any $\lambda_1, \dots, \lambda_m \in \partial E$ and $(w_1, \dots, w_m) \in G$. Obviously, $\Sigma_{u,v}(G)$ is Reinhardt if and only if G is Reinhardt, $u(z) = u(|z_1|, \dots, |z_n|)$, $v(z) = v(|z_1|, \dots, |z_n|)$, $z \in G$. Put $D = D_h := \{w \in \mathbf{C}^m : h(w) < 1\}$. The following properties are known:

- $D \Subset \mathbf{C}^m \Leftrightarrow \exists_{C>0} : h(w) \geq C\|w\|$, $w \in \mathbf{C}^m$;
- $h \in \mathcal{PSH}(\mathbf{C}^m) \Leftrightarrow \log h \in \mathcal{PSH}(\mathbf{C}^m) \Leftrightarrow D$ is pseudoconvex;
- D is convex or bounded $\Rightarrow h$ is a *quasinorm*, i.e.,

$$\exists_{C \geq 1} : h(b+w) \leq C(h(b) + h(w)), \quad b,\, w \in \mathbf{C}^m;$$

- (due to Barth [**1**]) D is taut $\Leftrightarrow D \Subset \mathbf{C}^m$ and $h \in (\mathcal{C} \cap \mathcal{PSH})(\mathbf{C}^m)$.
- $\Omega_H(G)$ is pseudoconvex $\Leftrightarrow$ G is pseudoconvex and $\log H \in \mathcal{PSH}(G \times \mathbf{C}^m)$;
- $\Sigma_{u,v}(G)$ is pseudoconvex $\Leftrightarrow G$ is pseudoconvex and $u, v \in \mathcal{PSH}(G)$.

We refer to, e.g., [**6, 7**] for more information.

The following result, due to Fu [**5**] and Zwonek [**20**], gives that all notions of hyperbolicity coincide in the class of pseudoconvex Reinhardt domains.

Theorem 2.2. *The following implications for a pseudoconvex Reinhardt domain in $\mathbf{C}^n$ are true*:

$$\begin{aligned} k\text{-hyperbolic} &\iff \tilde{k}\text{-hyperbolic} \iff \text{Brody hyperbolic} \iff \text{taut} \\ &\iff \text{biholomorphic to a bounded Reinhardt domain.} \end{aligned}$$

Therefore, we will speak only on hyperbolic pseudoconvex Reinhardt domains.

3. Hyperbolicity of the Hartogs domain $\Omega_{u,h}(G)$. Using a similar argument as in the proofs of Remark 3.1.7 and Proposition 3.1.10 in [**6**], it is easy to see that

Lemma 3.1. *Let $\Omega = \Omega_H(G)$. Then $\tilde{k}_\Omega((z,0),(z,w)) \le p(0, H(z,w))$ for any $(z,w) \in \Omega$. Here, the equality holds if $H \in \mathcal{PSH}(G \times \mathbf{C}^m)$.*

Clearly, if $\Omega_{u,h}(G)$ is d-hyperbolic, so is G and u is real-valued. Moreover,

Proposition 3.2. *$\Omega = \Omega_{u,h}(G)$ is k-hyperbolic if and only if G is k-hyperbolic, $D_h \Subset \mathbf{C}^m$, u is locally bounded on G.*

In case $m = 1$ the above result was already investigated in [**3, 15, 17, 19**]. To verify the sufficiency of Proposition 3.2 we need the following result by Eastwood [**4**].

Theorem 3.3. *Let $\pi : G \to S$ be a holomorphic map of domains. If S is k-hyperbolic and has an open covering (U_j) such that $\pi^{-1}(U_j)$ is k-hyperbolic, then G is also k-hyperbolic.*

A version of this result for the tautness can be found in Theorem 5.5.

Proof of Proposition 3.2 ($\Leftarrow$). Obviously, for every $z \in G$ we may choose a k-hyperbolic open neighborhood $U(z)$ of z in G, so that $(U(z))_{z \in G}$ is an open covering of G and $R(z) := \inf_{z' \in U(z)} u(z') > -\infty$. Observe that

$$\pi^{-1}(U(z)) = \Omega_{u,h}(U(z)) \subset U(z) \times \{w \in \mathbf{C}^m : h(w) < e^{-R(z)}\},$$

where $\pi : \Omega \to G$ is defined by $\pi(z,w) := z$ for $(z,w) \in (G \times \mathbf{C}^m) \cap \Omega$. Since $D_h \Subset \mathbf{C}^m$, there is a $C > 0$ such that $h(w) \ge C\|w\|$, $w \in \mathbf{C}^m$, so $\{w \in \mathbf{C}^m : h(w) < e^{-R(z)}\} \subset \mathbf{B}_m(0, e^{-R(z)}/C)$. Therefore, $\pi^{-1}(U(z))$ is k-hyperbolic, and thus our assertion follows directly from Theorem 3.3.

($\Rightarrow$). Suppose Ω is k-hyperbolic and put $H(z,w) := h(w)e^{u(z)}$, $(z,w) \in G \times \mathbf{C}^n$.

1°. *u is locally bounded on G.* Suppose that there is a point $z_0 \in G$ and a sequence $(z_j)_{j\geq 1} \subset G$ such that $\lim_{j\to\infty} z_j = z_0$ and $\lim_{j\to\infty} u(z_j) = -\infty$. Now we take a point $w_0 \in \mathbf{C}^m \setminus \{0\}$ with $(z_0, w_0) \in \Omega$. Without loss of generality, we may assume that $\{(z_j, w_0)\}_{j\geq 1} \subset \Omega$. So it follows from Lemma 3.1 that

$$0 \leq k_\Omega((z_j, 0), (z_j, w_0)) \leq p(0, H(z_j, w_0)), \quad j \geq 1.$$

But since $\lim_{j\to\infty} H(z_j, w_0) = 0$, the continuity of k_Ω implies that $k_\Omega((z_0, 0), (z_0, w_0)) = 0$; a contradiction to the fact that $w_0 \neq 0$.

2°. *$D = D_h$ is bounded in $\mathbf{C}^m$.* Suppose the contrary. Let $R > 0$ be so small that $\mathbf{B}_m(0, R) \Subset D$. Choose a sequence $(w_j)_{j\geq 1} \subset D$ with $\max\{R, 1\} < \|w_j\| \to \infty$ as $j \to \infty$. Fix $z^0 \in G$. Observe that $u(z^0) > -\infty$ and $(z_0, Re^{-u(z_0)} w_j/\|w_j\|) \in \Omega$ for $j \geq 1$. By Lemma 3.1, one has

$$0 \leq k_\Omega\left((z_0, 0), \left(z_0, Re^{-u(z_0)} \frac{w_j}{\|w_j\|}\right)\right) \leq p\left(0, H\left(z_0, Re^{-u(z_0)} \frac{w_j}{\|w_j\|}\right)\right), \quad j \geq 1.$$

However,

$$0 \leq H\left(z_0, Re^{-u(z_0)} \frac{w_j}{\|w_j\|}\right) = \frac{R}{\|w_j\|}\, h(w_j) < \frac{R}{\|w_j\|} \overset{j\to\infty}{\longrightarrow} 0.$$

On the other hand, we may assume, without loss of generality, that there exists a point $w_0 \in \mathbf{C}^m$ with $\|w_0\| = 1$ such that $\lim_{j\to\infty}(w_j/\|w_j\|) = w_0$. In particular, $(z_0, Re^{-u(z_0)} w_0) \in \Omega$ by the choice of R. Therefore, the continuity of k_Ω gives that

$$\begin{aligned} 0 &\leq k_\Omega((z_0, 0), (z_0, Re^{-u(z_0)} w_0)) \\ &= \lim_{j\to\infty} k_\Omega\left((z_0, 0), \left(z_0, \frac{Rw_j/\|w_j\|}{e^{u(z_j)}}\right)\right) \\ &\leq \lim_{j\to\infty} p\left(0, H\left(z_0, \frac{Rw_j/\|w_j\|}{e^{u(z_j)}}\right)\right) = 0, \end{aligned}$$

which is a contradiction to the fact that $e^{-u(z_0)} w_0 \neq 0$. □

Remark 3.4. In the case $m = 1$, $h(\lambda) = |\lambda|$, there are some differences between the previous proof and the proof in [**3**]. For the latter, to show the fact that u is locally bounded on G, the authors did not use Lemma 3.1. For more details, see [**3**].

Now we shall discuss the $\tilde{k}$-hyperbolicity of $\Omega_{u,h}(G)$.

Lemma 3.5. *Let* $\Omega = \Omega_{u,h}(G)$. *Assume that* $\lim_{z\to z_0, z\neq z_0} u(z) = -\infty$ *for some* $z_0 \in G$. *Then* $\tilde{k}_\Omega = 0$ *on* $\mathcal{M}\times\mathcal{M}$, *where* $\mathcal{M} := (\{z_0\}\times \mathbf{C}^m)\cap\Omega$.

Proof. Let $w_1, w_2 \in \mathbf{C}^m$ be not all zero with $\max\{h(w_1), h(w_2)\} < e^{-u(z_0)}$. For $j \geq 1$, put $M_j := \max_{\|w\|\leq r_j} h(w)$ where $r_j := (1+j)\|w_1\|+j\|w_2\|$. Since $h \not\equiv 0$, there exists $j_0 \geq 1$ such that $0 < M_j < \infty$ for $j \gg j_0$. By the assumption, for any $j \geq j_0$, we may take $\delta_j > 0$ such that: $0 < \|z - z_0\| < \delta_j \Longrightarrow z \in G$, $u(z) < -\log M_j$. Fix $j \geq j_0$ and choose $\alpha_j > 0$ so small that $\alpha_j(j^2 + j)\sqrt{n} < \delta_j$. Put $\mathbf{I} := (1, \dots, 1) \in \mathbf{C}^n$ and define two mappings $f_j : E \to \mathbf{C}^n$ and $g_j : E \to \mathbf{C}^m$ by

$$f_j(\lambda) := z_0 + \alpha_j(1-j\lambda)j\lambda\mathbf{I}, \quad g_j(\lambda) := (1-j\lambda)w_1 + j\lambda w_2, \quad \lambda\in E.$$

Then $\Psi_j := (f_j, g_j) \in \mathcal{O}(E,\Omega)$ with $\Psi_j(0) = (z_0, w_1)$, $\Psi_j(1/j) = (z_0, w_2)$; moreover,

$$0 \leq \tilde{k}_\Omega((z_0,w_1),(z_0,w_2)) = \tilde{k}_\Omega(\Psi_j(0),\Psi_j(1/j)) \leq p(0,1/j) \overset{j\to\infty}{\longrightarrow} 0. \quad \square$$

Example 3.6. Let $G := E$ and define $u(\lambda) := \log|\lambda|$ for $\lambda \in E\setminus\{0\}$ and $u(0) := 0$. The domain $G_{ET} := \Omega_{u,|\cdot|}(E)$ was first studied by Eisenman and Taylor. They showed that G_{ET} belongs to $\mathfrak{G}_B \setminus \mathfrak{G}_K$, see, e.g., [**10**, p. 104]. In fact, by Lemma 3.5, $\tilde{k}_{G_{ET}} = 0$ on $\mathcal{M}\times\mathcal{M}$, where $\mathcal{M} := (\{0\}\times\mathbf{C})\cap\Omega$, cf. Remark 3.2.16-(i) in [**10**], so $G_{ET} \in \mathfrak{G}_B\setminus\mathfrak{G}_L$. However, by using the Montel theorem, it is easy to check that $\tilde{k}_\Omega((a,z),(a,w)) > 0$ for $(a,z),(a,w) \in \Omega\cap(\mathbf{C}\setminus\{0\}\times\mathbf{C})$, $z\neq w$.

Lemma 3.7. *Let* $\Omega = \Omega_{u,h}(G)$. *Assume that* G *is Brody hyperbolic,* $u \in \mathcal{PSH}(G,\mathbf{R})$, $h \in \mathcal{PSH}(\mathbf{C}^m)$ *is a quasinorm on* $\mathbf{C}^m$ *with* $h^{-1}(0) = \{0\}$. *Moreover, assume that*

($\star$) *any sequence* $(f_\nu)_{\nu\geq 1}$ *of holomorphic functions* $f_\nu \in \mathcal{O}(r_\nu E, G)$ *with* $f_1(0) = f_\nu(0) = f_\nu(1)$ *for* $\nu \geq 1$*, where* $(r_\nu)_{\nu\geq 1}$ *is a sequence in* $\mathbf{R}_{>0}$ *with* $1 < r_\nu < r_{\nu+1} \nearrow \infty$ *as* $\nu \to \infty$*, has a subsequence* $(f_{\nu_j})_{j\geq 1}$ *converging to an* $f \in \mathcal{O}(\mathbf{C}, G)$ *uniformly on every compact subset of* $\mathbf{C}$*.*

Then for any $(a, z), (a, w) \in \Omega \cap (G \times \mathbf{C}^m)$

$$(\star\star) \qquad \tilde{k}_\Omega((a, z), (a, w)) = 0 \quad \Longleftrightarrow \quad z = w.$$

Proof. Fix $(a, z), (a, w) \in \Omega\cap(G\times\mathbf{C}^m)$. Assume that $\tilde{k}_\Omega((a, z), (a, w)) = 0$. Then there are two sequences $(r_j)_{j\geq 1} \subset \mathbf{R}$ and $(\varphi_j)_{j\geq 1} \subset \mathcal{O}(r_j E, \Omega)$ such that $\varphi_j(0) = (a, z)$, $\varphi_j(1) = (a, w)$, and $1 < r_j < r_{j+1} \nearrow \infty$ as $j \to \infty$. Let $j \geq 1$ and put $\varphi_j := (f_j, g_j)$, where $f_j \in \mathcal{O}(r_j E, \mathbf{C}^n)$ and $g_j \in \mathcal{O}(r_j E, \mathbf{C}^m)$. Note that the mapping g_j can be written in the form $g_j(\lambda) = z + \lambda\tilde{g}_j(\lambda)$ for some $\tilde{g}_j \in \mathcal{O}(r_j E, \mathbf{C}^m)$. Because of $\varphi_j(r_j E) \subset \Omega$, one has $H(f_j(\lambda), g_j(\lambda)) < 1$ for any $\lambda \in r_j E$. Since h is a quasinorm on $\mathbf{C}^m$, there is a $C > 0$ such that for any $\lambda \in r_j E$

$$\begin{aligned}|\lambda|\, H(f_j(\lambda), \tilde{g}_j(\lambda)) &= H(f_j(\lambda), \lambda\tilde{g}_j(\lambda))\\ &\leq C\,(H(f_j(\lambda), g_j(\lambda)) + H(f_j(\lambda), z))\,.\end{aligned}$$

Now, put $\varepsilon := \operatorname{dist}(a, \partial G)/2 > 0$. Obviously, $\mathbf{B}_n(a, \varepsilon) \Subset G$ and also $\log M := \max_{\|\zeta-a\|\leq\varepsilon} u(\zeta) < \infty$, because of $u \in \mathcal{C}^\uparrow(G)$. By the condition ($\star$) and the fact that G is Brody hyperbolic, without loss of generality we may assume that $f_j \overset{\mathrm{K}}{\rightrightarrows} a$ on $\mathbf{C}$. Thus, for $R > 1$, we may choose $j_R \geq 1$ such that $r_{j_R} > R$ and $f_j(\lambda) \in \mathbf{B}_n(a, \varepsilon)$ for $|\lambda| < R$ and $j \geq j_R$. This implies that

$$|\lambda|\, H(f_j(\lambda), \tilde{g}_j(\lambda)) \leq C(1 + h(z)M), \quad |\lambda| \leq R, \quad j \geq j_R.$$

Then it follows from the maximum principle for subharmonic functions that

$$H(f_j(\lambda), \tilde{g}_j(\lambda)) \leq \frac{C}{R}\,(1 + Mh(z)), \quad |\lambda| \leq R, \quad j \geq j_R.$$

On the other hand, $f_j(1) = a$, $\tilde{g}_j(1) = g_j(1)-z = w-z$ for any $j \geq 1$, so the previous inequality tells us that $h(w - z)e^{u(a)} \leq C(1 + Mh(z))/R$.

Since h is nonnegative and $R > 1$ is arbitrary, we then get that $h(w-z)e^{u(a)} = 0$ by letting $R \to \infty$. Because u is real-valued and $h^{-1}(0) = \{0\}$, it must be $z - w = 0$. □

Now we are able to give some sufficient conditions for Ω to be $\tilde{k}$-hyperbolic.

Theorem 3.8. *Let $u \in \mathcal{PSH}(G, \mathbf{R})$, and $h \in \mathcal{PSH}(\mathbf{C}^m)$ is a quasinorm with $h^{-1}(0) = \{0\}$. If one of the following conditions is satisfied*:

(a) *G is taut*;

(b) $G \Subset \mathbf{C}^n$;

(c) *G is $\tilde{k}$-hyperbolic and u is bounded from above,*

then $\Omega = \Omega_{u,h}(G)$ is $\tilde{k}$-hyperbolic.

Proof. Under our hypotheses the base G is always $\tilde{k}$-hyperbolic. So, it is enough to verify that the condition $(\star\star)$ is satisfied. In the cases (a) and (b), we can use so-called diagonal process to extract the desired subsequence for the condition $(\star)$ (in case (b), use Montel's theorem), so $(\star\star)$ holds. In case (c), we can take $N > 0$ so large that $\sup_{|\lambda|<r_j} u(f_j(\lambda)) \leq \log N$ for any $j \geq 1$, where $(f_j)_{j\geq 1}$ and $(r_j)_{j\geq 1}$ are as in the assumption of $(\star)$. Therefore, we can get the desired condition $(\star\star)$ by carrying out the same argument as in the proof Lemma 3.7. □

Remark 3.9. Other examples of a $\tilde{k}$-hyperbolic domain $\Omega_{u,h}(G)$ can be found in Example 5.3 below. Those examples do not satisfy any of the conditions (a), (b), (c) in Theorem 3.8. Nevertheless they satisfy the condition $(\star)$ in Lemma 3.7.

On the other hand, we can easily show the following

Proposition 3.10. *If G, D_h are Brody hyperbolic and $u > -\infty$ on G, then also $\Omega_{u,h}(G)$ is Brody hyperbolic. Conversely, if $\Omega_{u,h}(G)$ is Brody hyperbolic, so is G.*

Now we can give some concrete examples of domains which belong to $\mathfrak{G}_L \setminus \mathfrak{G}_K$.

Example 3.11. (1) For $\nu \geq 1$, choose $\alpha_\nu, \beta_\nu \in (0,1)$, $\lambda_\nu \in E \setminus \{0\}$, such that $\lim_{\nu\to\infty} \lambda_\nu = 0$, $\lim_{\nu\to\infty} \beta_\nu^{\alpha_\nu} = 0$, and $\sum_{\nu=1}^{\infty} \alpha_\nu \log|\lambda_\nu| > -\infty$. We define a function $u : E \to [-\infty, \infty)$ by $u(\lambda) := \sum_{\nu=1}^{\infty} \alpha_\nu \log(\beta_\nu^2 + |\lambda - \lambda_\nu|^2)$ for $\lambda \in E$. The Hartogs domain $G_{TT} := \Omega_{u,|\cdot|}(E) \subset \mathbf{C}^2$ was constructed by Thai and Thomas [**17**]. Obviously, $u \in \mathcal{SH}(E, \mathbf{R}) \cap \mathcal{C}^\infty(E \setminus \{0\})$ and $\liminf_{\lambda\to 0, \lambda\neq 0} u(\lambda) = -\infty$. Hence, by Proposition 3.2, $G_{TT} \in \mathfrak{G}_B \setminus \mathfrak{G}_K$; moreover, according to Theorem 3.8, G_{TT} belongs to $\mathfrak{G}_L$.

(2) The following was constructed by Diederich and Sibony [**2**]. Define $u(\lambda) := \sum_{\nu=2}^{\infty} \nu^{-2} \max\{-\nu^3, \log|\lambda - 1/\nu| - \log 2\}$ for $\lambda \in E$. Put $G_{DS} := \{z = (z_1, z_2) \in E \times \mathbf{C} : |z_2| e^{\|z\|^2 + u(z_1)} < 1\}$. Then $u \in \mathcal{SH}(E, \mathbf{R})$ and $G_{DS} \in \mathfrak{G}_B \setminus \mathfrak{G}_K$. In particular, $k_{G_{DS}}((0,0),(0,w)) = 0$, $(0,w) \in G$. But since $|z_2| e^{u(z_1)} < e^{-\|z\|^2} \leq 1$ for any $z \in G_{DS}$, Theorem 3.8 implies that G_{DS} belongs to $\mathfrak{G}_L$, but is not of Hartogs type.

4. Hyperbolicity of $\Sigma_{u,v}(G)$. Obviously, the d-hyperbolicity of Σ implies that $\max\{u, v\} > -\infty$ on G. If G is k-hyperbolic and u (or v) is locally bounded on G, it follows directly from Proposition 3.2 that Σ is also k-hyperbolic. Moreover,

Lemma 4.1. *If Σ is k-hyperbolic, then* $\max\{u, v\}$ *is locally bounded on G.*

Note that we do not know whether its converse also holds.

Proof. Suppose the contrary. Since $u, v \in \mathcal{C}^\uparrow(G)$, there exist a point $z_0 \in G$ and a sequence $(z_j)_{j\geq 1} \subset G$ converging to z_0 such that $0 > \max\{u(z_j), v(z_j)\} \to -\infty$ as $j \to \infty$. For $j \geq 1$, we put $\alpha_j := 1$ if $v(z_j) \leq u(z_j)$; $\alpha_j := v(z_j)/(2u(z_j))$ if $v(z_j) > u(z_j)$. Then $(\alpha_j)_{j\geq 1} \subset (0,1]$, $v(z_j) \leq \alpha_j u(z_j) < 0$, $j \geq 1$, and $\lim_{j\to\infty} \alpha_j u(z_j) = -\infty$. Take $\lambda_0 > 0$ such that $(z_0, \lambda_0) \in \Sigma$ and $\lambda_0 = e^{\zeta_0}$ for some $\zeta_0 \in \mathbf{R}$.

(i) *The case* $\lambda_0 \leq 1$. Without loss of generality, we may assume that $\alpha_j u(z_j) - \zeta_0 < 0$, $j \geq 1$. Let $j \geq 1$. Define a holomorphic mapping $\varphi_j : E \to G \times \mathbf{C}$ by $\varphi_j(\lambda) := (z_j, \lambda_0 e^{(\alpha_j u(z_j) - \zeta_0)\lambda})$, $\lambda \in E$. Then $\varphi_j(0) = (z_j, \lambda_0)$ and

$$e^{v(z_j)} \leq e^{\alpha_j u(z_j)} = \lambda_0 e^{\alpha_j u(z_j) - \zeta_0} < \left| e^{\alpha_j u(z_j)\lambda} \right| < e^{-\alpha_j u(z_j)} < e^{-u(z_j)}, \quad \lambda \in E,$$

so $\varphi_j \in \mathcal{O}(E, \Sigma)$. Take $w_0 \in \mathbf{C} \setminus \{0\}$ with $v(z_0) - \zeta_0 < \operatorname{Re} w_0 < -u(z_0) - \zeta_0$. Then $(z_0, \lambda_0 e^{w_0}) \in \Sigma$. Since $\zeta_j := \alpha_j u(z_j) - \zeta_0 \to -\infty$ as $j \to \infty$, it is clear that $w_0/\zeta_j \in E$ and $\varphi_j(w_0/\zeta_j) = (z_j, \lambda_0 e^{w_0})$ for $j \gg 1$. Therefore the continuity of k_Σ and the decreasing property of $\underline{k}$ give that

$$\begin{aligned} 0 &\leq k_\Sigma((z_0, \lambda_0), (z_0, \lambda_0 e^{w_0})) \\ &= \lim_{j \to \infty} k_\Sigma\left(\varphi_j(0), \varphi\left(\frac{w_0}{\zeta_j} \right) \right) \\ &\leq \lim_{j \to \infty} p\left(0, \frac{w_0}{\zeta_j} \right) = 0, \end{aligned}$$

which is a contradiction to the k-hyperbolicity of Σ.

(ii) *The case* $\lambda_0 > 1$. Let $\Sigma' := \{(z, \lambda) \in G \times \mathbf{C} : (z, \lambda) \in \Sigma\}$. Then the function $\Phi = (\Phi_1, \Phi_2) : \Sigma \to \Sigma'$ defined by $\Phi(z, \lambda) := (z, 1/\lambda)$ for $(z, \lambda) \in G \times \mathbf{C}$ is biholomorphic. Put $\Phi_2(z_0, \lambda_0) = 1/\lambda_0 =: \lambda_0' \in E \setminus \{0\}$. By applying the case $\lambda_0' \leq 1$ to (i) we obtain that Σ' is not k-hyperbolic; a contradiction. □

Lemma 4.2. *If* $\lim_{z \to z_0,\, z \neq z_0} \max\{u(z), v(z)\} = -\infty$ *for some* $z_0 \in G$*, then* $\tilde{k}_\Sigma = 0$ *on* $\mathcal{M} \times \mathcal{M}$*, where* $\mathcal{M} := (\{z_0\} \times \mathbf{C}) \cap \Sigma$*.*

Proof. For this it suffices that $\tilde{k}_\Sigma((z_0, w'), (z_0, w'')) = 0$ for (z_0, w'), $(z_0, w'') \in \Sigma$ with $w' \in \mathbf{R}$. To show this, fix two points $(z_0, w'), (z_0, w'') \in \Sigma$, $w' := e^\alpha$, $w'' := e^{\beta + i\theta}$, $\alpha, \beta \in \mathbf{R}$, $0 \leq \theta < 2\pi$, where $i^2 = -1$. For $j \geq 1$ put $r_j := \alpha - j(|\alpha - \beta| + 2\pi)$ and $R_j := |\alpha| + j(|\alpha| + |\beta| + 2\pi)$. By the hypothesis, we may take $j_0 \geq 1$ so large that, for any $j \geq j_0$, there exists $\delta_j > 0$ such that: $0 < \|z_0 - z\| < \delta_j \Rightarrow z \in G$, $\max\{u(z), v(z)\} < \min\{r_j, -R_j\}$. Fix $j \geq j_0$, and choose $C_j > 0$

so small that $C_j(j^2+j)\sqrt{n} < \delta_j$. Set $\mathbf{I} := (1, \dots, 1) \in \mathbf{C}^n$ and define two analytic disks $f_j : E \to \mathbf{C}^n$ and $g_j : E \to \mathbf{C}$ by

$$f_j(\lambda) := z_0 + C_j(1-j\lambda)j\lambda\mathbf{I}, \quad g_j(\lambda) := e^{(1-j\lambda)\alpha + j\lambda(\beta+i\theta)}, \quad \lambda \in E.$$

Then $\Psi_j := (f_j, g_j) \in \mathcal{O}(E, \Sigma)$ with $\Psi_j(0) = (z_0, w')$, $\Psi_j(1/j) = (z_0, w'')$; moreover,

$$0 \le \tilde{k}_\Sigma((z_0, w'), (z_0, w'')) = \tilde{k}_\Sigma(\Psi_j(0), \Psi_j(1/j)) \le p(0, 1/j) \overset{j\to\infty}{\longrightarrow} 0. \quad \square$$

The following statement follows immediately from Theorem 3.8.

Proposition 4.3. *Suppose that* $u \in \mathcal{PSH}(G, \mathbf{R})$, *respectively* $v \in \mathcal{PSH}(G, \mathbf{R})$. *If one of the following conditions is satisfied*:

(a) G *is taut*;

(b) $G \Subset \mathbf{C}^n$;

(c) G *is* $\tilde{k}$*-hyperbolic and* u*, respectively* v*, is bounded from above on* G*,*

then Σ *is* $\tilde{k}$*-hyperbolic.*

Using the argument of the proof of Lemma 3.7, we can also get:

Proposition 4.4. *If one of the following conditions is satisfied*:

(a) G *is taut and* $u, v \in \mathcal{PSH}(G)$ *with* $\max\{u, v\} > -\infty$ *on* G;

(b) $G \Subset \mathbf{C}^n$, $u, v \in \mathcal{PSH}(G)$ *with* $\max\{u, v\} > -\infty$ *on* G*,*

then Σ *is* $\tilde{k}$*-hyperbolic.*

The next property follows directly from the little Picard theorem.

Proposition 4.5. *If* G *is Brody hyperbolic and* $\max\{u, v\} > -\infty$ *on* G*, then* Σ *is Brody hyperbolic.*

Example 4.6. There exists a Hartogs-Laurent domain $\Sigma \subset \mathbf{C}^3$ which belongs to $\mathfrak{G}_B \setminus \mathfrak{G}_L$. For this, define $u(z) = v(z) := \log(1 + |z_1^2 +$

$z_1z_2|) - \log 3$ for $z = (z_1, z_2) \in G_{ET}$. Then $\max\{u, v\} > -\infty$ on G_{ET} and $G_{ET} \times \{1\} \subset \Sigma_{u,v}(G) =: \Sigma$. So

$$0 \le \tilde{k}_\Sigma(((0, \lambda), 1), ((0, 0), 1)) \le \tilde{k}_{G_{ET}}((0, \lambda), (0, 0)), \quad \lambda \in E.$$

Thus, by Example 3.6, the domain Σ is not $\tilde{k}$-hyperbolic.

Next, we are going to study the differences between the hyperbolicities of $\Sigma_{u,v}(G)$ and $\Omega_{u,|\cdot|}(G)$.

To give a negative answer to (2), we need the following auxiliary lemma:

Lemma 4.7. *Let $G \subset \mathbf{C}^n$ be a domain, and let $u \in \mathcal{PSH}(G, \mathbf{R})$ be nonconstant and bounded from below on G. Suppose that the domain G is not Brody hyperbolic and that $u \circ \varphi$ is not a constant for any nonconstant $\varphi \in \mathcal{O}(\mathbf{C}, G)$. Then the domain $\Sigma = \Sigma_{u,-\infty}(G)$ is Brody hyperbolic.*

Proof. Suppose that there exists a nonconstant mapping $\psi := (\psi_1, \psi_2) \in \mathcal{O}(\mathbf{C}, \Sigma)$, where $\psi_1 \in \mathcal{O}(\mathbf{C}, G)$ and $\psi_2 \in \mathcal{O}(\mathbf{C}, \mathbf{C})$. By our assumption, we can choose a constant $M > 0$ so large that $u > -\log M$ on G, which implies that $|\psi_2(\lambda)| < M$ for any $\lambda \in \mathbf{C}$. Then Liouville's theorem implies that $\psi_2 \equiv \text{constant} =: A \in \mathbf{C}\backslash\{0\}$. On the other hand, our assumption gives us that $u \circ \psi_1$ is not a constant on $\mathbf{C}$. Hence, it follows from the Liouville type theorem for subharmonic functions that there exists a sequence $(\lambda_\nu)_{\nu \ge 1} \subset \mathbf{C}$ such that $u(\psi_1(\lambda_\nu)) \to \infty$ as $\nu \to \infty$. Thus, we can take a $\nu_0 \in \mathbf{N}$ such that $0 < e^{-u(\psi_1(\lambda_\nu))} < |A|$ for any $\nu \ge \nu_0$, which is a contradiction to the fact that $\psi(\mathbf{C}) \subset \Sigma$. □

We give an example of a hyperbolic pseudoconvex Reinhardt domain Σ such that its base G is not hyperbolic.

Example 4.8. Let $n \ge 2$, and let $G := \{z \in \mathbf{C}^n : |z_1 \cdots z_n| < 1\}$. Define $u(z) := \max_{1 \le j \le n} |z_j|$ for $z \in G$. It is easy to check that $u \in \mathcal{PSH}(G)$ and G is not Brody hyperbolic. In view of the little Picard theorem, $u \circ \psi = \max_{1 \le j \le n} |\psi_j|$ is not a constant for any nonconstant mapping $\psi := (\psi_1, \dots, \psi_n) \in \mathcal{O}(\mathbf{C}, G)$, where $\psi_j \in \mathcal{O}(\mathbf{C})$, $j = 1, \dots, n$. Thus Lemma 4.7 implies that the pseudoconvex Reinhardt domain $\Sigma = \Sigma_{u,-\infty}(G)$ is Brody hyperbolic.

In spite of this example, the following result gives us a positive answer to (2).

Theorem 4.9. *If $\Sigma = \Sigma_{u,v}(G)$ is a pseudoconvex Reinhardt domain with $u \not\equiv -\infty$ and $v \not\equiv -\infty$, then Σ is hyperbolic if and only if G is hyperbolic and* $\max\{u, v\} > -\infty$ *on G.*

To prove this we need the following lemma.

Lemma 4.10. *If $\Sigma = \Sigma_{u,v}(G)$ is taut, then u and v are continuous on G.*

Proof. Let us suppose the contrary. Without loss of generality, we may assume that $u \notin \mathcal{C}(G)$. Choose a constant $A \in \mathbf{R}$ and a sequence $(z_j)_{j\geq 0} \subset G$ such that $z_j \to z_0$ as $j \in \mathbf{N}$ and $-u(z_0) < -A < -u(z_j)$ for any $j \in \mathbf{N}$. Note that $u(z_0) \neq -\infty$. Since $u(z_0) + v(z_0) < 0$, we may take an $\tilde{\alpha} \in \mathbf{R}$ such that $v(z_0) < -\tilde{\alpha} < -u(z_0)$. Because of $v \in \mathcal{C}^{\uparrow}(G)$, we may assume that $v(z_j) < -\tilde{\alpha}$, $j \geq 1$. Now, put $C := (1/2)\min\{-u(z_0) + \tilde{\alpha}, -A + u(z_0)\} > 0$ and $\breve{\Sigma} := \Sigma_{\breve{u},\breve{v}}(G)$, where $\breve{v} := v + u(z_0) + C/2$ and $\breve{u} := u - u(z_0) - C/2$. Clearly, the mapping $\Sigma \ni (z, w) \mapsto (z, w\exp(u(z_0) + C/2)) \in \breve{\Sigma}$ is well-defined and biholomorphic, so $\breve{\Sigma}$ is a taut domain. Moreover, if we put $\breve{A} := -u(z_0) + A - C/2$ and $\breve{\alpha} := -u(z_0) + \tilde{\alpha} - C/2$, then $\breve{v}(z_j) < -\breve{\alpha}$ for any $j \geq 1$. Hence, for any $j \geq 1$

(4.10.1)

$$\max\{\breve{v}(z_0), \breve{v}(z_j)\} < -\breve{\alpha} < -C < 0 < -\breve{u}(z_0) < C < -\breve{A} < -\breve{u}(z_j).$$

For $j \geq 1$ we define $f_j(\lambda) := (z_j, e^{C\lambda})$ for any $\lambda \in E$. Then

$$e^{\breve{v}(z_j)} < e^{-C} < |e^{C\lambda}| < e^{C} < e^{-\breve{u}(z_j)}, \quad j \geq 1, \quad \lambda \in E,$$

so $(f_j)_{j\in\mathbf{N}} \subset \mathcal{O}(E, \breve{\Sigma})$. Moreover, $f_j(0) = (z_j, e^0) = (z_j, 1) \overset{j\to\infty}{\to} (z_0, 1) \in \breve{\Sigma}$, because $e^{\breve{v}(z_0)} < e^{-C} < e^0 < e^{-\breve{u}(z_0)}$. The tautness of

$\breve{\Sigma}$ gives that $f_j \overset{\mathrm{K}}{\Rightarrow} (z_0, e^{C\lambda}) \in \mathcal{O}(E, \breve{\Sigma})$ as $j \to \infty$, which implies that $e^{\breve{v}(z_0)} < e^{C \operatorname{Re} \lambda} < e^{-\breve{u}(z_0)}$ for any $\lambda \in E$. Consequently, we obtain a contradiction to (4.10.1) by setting $E \ni \lambda \to 1$. □

Remark 4.11. In general, the tautness of $\Sigma_{u,v}(G)$ does not imply the tautness of G, cf. Example 4.8. However, if G is taut and $u, v \in (\mathcal{C} \cap \mathcal{PSH})(G, \mathbf{R})$, then $\Sigma_{u,v}(G)$ is taut, see Corollary 5.4.

Proof of Theorem 4.9. In view of Theorem 2.2 and Proposition 4.5, it is enough to verify the necessity. Assume that Σ is hyperbolic. By Lemma 4.1, the function $\max\{u, v\}$ is locally bounded on G. Seeking for a contradiction, suppose that G is not Brody hyperbolic. Then there is a nonconstant $\varphi \in \mathcal{O}(\mathbf{C}, G)$. Note that $(u+v) \circ \varphi < 0$ on $\mathbf{C}$. By the Liouville type theorem for subharmonic function, one has $u \circ \varphi + v \circ \varphi = \text{constant} =: \alpha \in [-\infty, 0)$.

(i) *The case* $-\infty < \alpha < 0$. Note that $u \circ \varphi = -v \circ \varphi + \alpha$. Since $u \circ \varphi, v \circ \varphi \in \mathcal{SH}(\mathbf{C})$, the function $v \circ \varphi$ is harmonic on $\mathbf{C}$, and so $v \circ \varphi = \operatorname{Re} F$ for some $F \in \mathcal{O}(\mathbf{C})$. Take a number $\beta \in \mathbf{R}$ such that $1 < \beta < e^{-\alpha}$ and define $\Psi = \Psi_{\varphi, F, \beta} : \mathbf{C} \to \mathbf{C}^{n+1}$ by $\Psi_{\varphi,F,\beta}(\lambda) := (\varphi(\lambda), \beta e^{F(\lambda)})$ for $\lambda \in \mathbf{C}$. Observe that

$$e^{v(\varphi(\lambda))} = e^{\operatorname{Re} F(\lambda)} < \beta |e^{F(\lambda)}| < e^{-\alpha} |e^{F(\lambda)}| = e^{-\alpha + v(\varphi(\lambda))} = e^{-u(\varphi(\lambda))}, \quad \lambda \in \mathbf{C},$$

which implies that Ψ is nonconstant holomorphic with $\Psi(\mathbf{C}) \subset \Sigma$, a contradiction.

(ii) *The case* $\alpha = -\infty$. Since the hyperbolicity is an invariant property under biholomorphic mappings, without loss of generality we may assume that $u(\varphi(\lambda_0)) > -\infty$, $v(\varphi(\lambda_0)) = -\infty$ for some $\lambda_0 \in \mathbf{C}$. Since $u \circ \varphi \in \mathcal{C}(\mathbf{C})$ by Lemma 4.10, one may take an open neighborhood $W = W(\lambda_0) \subset \mathbf{C}$ such that $u \circ \varphi > -\infty$ on W. Thus, it follows from the integrability theorem for subharmonic functions and the open mapping theorem that $v = -\infty$ on G; a contradiction. □

In the next example, we shall give a negative answer to the question (2) in the class of non-Reinhardt Hartogs-Laurent domains.

Example 4.12. Let $G := \{z \in \mathbf{C}^2 : |z_1 z_2| < 1\}$ and $u(z) := \max\{|z_1|, |z_2|\}$, $z \in G$. Put $\Sigma := \Sigma_{u,-\infty}(G)$ and $\tilde{\Sigma} := \Sigma_{u,-\infty}(G_{ET})$. Since $G_{ET} \subset G$, one has $\tilde{k}_{\tilde{\Sigma}} \geq \tilde{k}_{\Sigma}$. Because Σ is hyperbolic by Example 4.8, so is $\tilde{\Sigma}$.

5. Tautness of $\Omega_{u,h}(G)$ and $\Sigma_{u,v}(G)$. The following statement is an immediate consequence of Royden's criterion for taut domains (Proposition 2.1).

Lemma 5.1. *Let $G \subset \mathbf{C}^n$ be a domain. If G is not taut, then there exist an $R > 0$, sequences $(z_j)_{j\geq 0} \subset G$, $(f_j)_{j\geq 1}$, $(g_j)_{j\geq 1} \in \mathcal{O}(E, G)$, and $(\alpha_j)_{j\geq 0}$, $(\beta_j)_{j\geq 0} \in [0, 1)$, such that for any $j \geq 1$:*

$$k_G^{(2)}(z_0, z_j) < R, \tag{†1}$$

$$f_j(0) = z_0 \in G, \tag{†2}$$

$$f_j(\alpha_j) = g_j(0), \tag{†3}$$

$$g_j(\beta_j) = z_j, \quad z_j \overset{j\to\infty}{\longrightarrow} {}^{\exists}\hat{z}_0 \in \partial G \quad \text{or} \quad \|z_j\| \overset{j\to\infty}{\longrightarrow} \infty, \tag{†4}$$

$$\alpha_j \overset{j\to\infty}{\longrightarrow} \alpha_0, \quad \beta_j \overset{j\to\infty}{\longrightarrow} \beta_0. \tag{†5}$$

Using it, we now give a full characterization of a taut Hartogs domain (without the assumption of boundedness) with m-dimensional balanced fibers.

Theorem 5.2. *$\Omega = \Omega_{u,h}(G)$ is taut if and only if $G, D = D_h$ are taut, $u \in (\mathcal{C} \cap \mathcal{PSH})(G, \mathbf{R})$.*

Proof. To prove the necessity, we can use the same argument as in [**8**, Proposition 3.8]. For the sufficiency, suppose that Ω is not taut. By Lemma 5.1, one can choose $R > 0$, sequences $(z_j)_{j\geq 0} \subset \Omega$, $(f_j)_{j\geq 1}$, $(g_j)_{j\geq 1} \in \mathcal{O}(E, \Omega)$, and $(\alpha_j)_{j\geq 0}$, $(\beta_j)_{j\geq 0} \in [0, 1)$ having the properties (†1) $\sim$ (†5). Observe that $k_\Omega^{(2)}(z_0, z_j) \geq k_G^{(2)}(z_0^1, z_j^1)$, where $z_j =: (z_j^1, z_j^2) \in G \times \mathbf{C}^m$, $j \geq 0$. So the property (†1) implies that $(z_j^1)_{j\geq 1} \subset \mathbf{B}_{k_G^{(2)}}(z_0^1, R)$. But since G is taut, we may assume, in view of Royden's criterion, that $z_j^1 \overset{j\to\infty}{\to} {}^{\exists}a_0^1 \in G$. For any $j \geq 1$, denote

$f_j =: (f_j^1, f_j^2)$, $g_j =: (g_j^1, g_j^2) \in \mathcal{O}(E,G) \times \mathcal{O}(E,\mathbf{C}^m)$. Because of the tautness of G and the property (†2), we may extract a sequence $(f_{1j}^1)_{j\geq 1} \subset (f_j^1)_{j\geq 1}$ such that $f_{1j}^1 \overset{\mathrm{K}}{\rightrightarrows} {}^{\exists} f_0^1 \in \mathcal{O}(E,G)$ as $j \to \infty$. Hence, the properties (†3) and (†5) yield that

$$\lim_{j\to\infty} g_{1j}^1(0) = \lim_{j\to\infty} f_{1j}^1(\alpha_{1j}) = f_0^1(\alpha_0) \in G.$$

So, the tautness of G implies that there is a sequence $(g_{2j}^1)_{j\geq 1} \subset (g_{1j}^1)_{j\geq 1}$ such that $g_{2j}^1 \overset{\mathrm{K}}{\rightrightarrows} {}^{\exists} g_0^1 \in \mathcal{O}(E,G)$ as $j \to \infty$. In particular,

$$g_0^1(\beta_0) = \lim_{j\to\infty} g_{2j}^1(\beta_{2j}) = \lim_{j\to\infty} z_{2j}^1 = a_0^1.$$

On the other hand, since D is taut, it is clear that $D \Subset \mathbf{C}^m$, i.e., there is a constant $C > 0$ such that $h(w) \geq C\|w\|$, $w \in \mathbf{C}^m$. Since u is real-valued, one has $h(z_j^2) \leq \exp(-u(z_j^1))$ for $j \geq 1$. The continuity of u gives that

$$\limsup_{j\to\infty} \|z_j^2\| \leq \frac{1}{C} \limsup_{j\to\infty} h(z_j^2) \leq \frac{1}{C} \exp(-u(a_0^1)) < \infty,$$

which implies that $z_j^2 \not\to \infty$ as $j \to \infty$. Thus, in view of (†4), we may take a point $a_0^2 \in \mathbf{C}^m$ so that $\lim_{j\to\infty} z_j = (a_0^1, a_0^2) = \hat{z}_0 \in \partial\Omega$.

Step I. Choose $c_2 \in (0,1)$ so that $\beta_j \in c_2 E$ for $j \geq 0$. For any $j \geq 1$, we define a map $\tilde{g}_j : c_2^{-1}E \to \mathbf{C}^n \times \mathbf{C}^m$ by $\tilde{g}_j(\lambda) = (\tilde{g}_j^1(\lambda), \tilde{g}_j^2(\lambda)) := g_j(\beta_j\lambda)$, $\lambda \in c_2^{-1}E =: E_2$. Clearly, it is well-defined and $(\tilde{g}_j)_{j\geq 1} \subset \mathcal{O}(E_2, \Omega)$. Now we shall show that $(\tilde{g}_{2j}^2)_{j\geq 1}$ is bounded on E_2. Let $F_2 := \cup_{j\geq 0}(\beta_j E_2)$. Using (†5), it is easy to check that $F_2 \Subset E$. Let $L := g_0^1(\overline{F}_2)$. Obviously, $L \Subset G$ and so $\delta := \operatorname{dist}(L, \partial G)/3 > 0$. Since g_{2j}^1 converges uniformly on $\overline{F}_2$ to g_0^1 as $j \to \infty$, one can take $j_0 \in \mathbf{N}$ such that $\|g_{2j}^1(\lambda) - g_0^1(\lambda)\| < \delta$, $\lambda \in \overline{F}_2$, $j \geq j_0$. Hence,

$$\|g_{2j}^1(\lambda) - \hat{v}_0\| \geq \|g_0^1(\lambda) - \hat{v}_0\| - \|g_{2j}^1(\lambda) - g_0^1(\lambda)\| \geq \operatorname{dist}(L, \partial G) - \delta \geq 2\delta$$

for $j \geq j_0$, $\lambda \in \overline{F}_2$, $\hat{v}_0 \in \partial G$. That is, $\operatorname{dist}(g_{2j}^1(\overline{F}_2), \partial G) \geq 2\delta > 0$ for $j \geq j_0$, which implies that $K := g_0^1(\bar{F}_2) \cup (\cup_{j\geq j_0} g_{2j}^1(\overline{F}_2)) \Subset G$. In particular,

$$K' := \left\{ g_{2j}^1(\beta_{2j}\lambda),\, g_0^1(\beta_0\lambda) : \lambda \in E_2,\ j \geq j_0 \right\} \subset K.$$

Since u is uniformly continuous on $\overline{K}$, one can take a constant $C' > 0$ so that $|u(x) - u(y)| < C'$ for $x, y \in K$. Therefore, for any $j \geq j_0$ and $\lambda \in E_2$ one has

$$\begin{aligned} C\|\tilde{g}^2_{2j}(\lambda)\| &\leq h(\tilde{g}^2_{2j}(\lambda)) < e^{-u(\tilde{g}^1_{2j}(\lambda))} \\ &\leq e^{-u(g^1_0(\beta_0\lambda))+C'} \\ &\leq e^{-\inf_{x\in\bar{K}} u(x)+C'} < \infty. \end{aligned}$$

Here, in the third inequality, we used the fact that $K' \subset K$. So, the family $(\tilde{g}^2_{2j})_{j\geq 1}$ is uniformly bounded on E_2. In view of Montel's theorem, we can choose a sequence $(\tilde{g}^2_{3j})_{j\geq 1} \subset (\tilde{g}^2_{2j})_{j\geq 1}$ such that $\tilde{g}^2_{3j} \overset{\mathrm{K}}{\Rightarrow} {}^{\exists}\tilde{g}^2_0 \in \mathcal{O}(E_2, \mathbf{C}^m)$ as $j \to \infty$. In particular,

$$\tilde{g}^2_0(1) = \lim_{j\to\infty} \tilde{g}^2_{3j}(1) = \lim_{j\to\infty} g^2_{3j}(\beta_{3j}) = \lim_{j\to\infty} z^2_{3j} = a^2_0.$$

Put $H(z, w) := h(w)e^{u(z)}$ for $(z, w) \in G \times \mathbf{C}^m$. For $j \geq 1$ we put $\varphi_{3j} := H \circ \tilde{g}_{3j}$ on E_2. Since $\varphi_{3j} < 1$ on E_2 for any $j \geq 1$, one has $\varphi_0 := H \circ \tilde{g}_0 \leq 1$ on E_2, where $\tilde{g}_0 := (\tilde{g}^1_0, \tilde{g}^2_0)$, $\tilde{g}^1_0(\lambda) := g^1_0(\beta_0\lambda)$, $\lambda \in E_2$. In particular, $\varphi_0(1) = H(\hat{z}_0) = 1$. Hence, the maximum principle for subharmonic functions implies that $\varphi_0 \equiv 1$ on E_2, and also $\tilde{g}_0(0) = (\tilde{g}^1_0(0), \tilde{g}^2_0(0)) \in \partial\Omega$.

Step II. We are going to apply the same argument as in Step I to $(f_j)_{j\geq 1}$ and $(\alpha_j)_{j\geq 0}$. Choose $c_1 \in (0, 1)$ so that $\alpha_j \in c_1E$ for $j \geq 0$. Define a holomorphic mapping $\tilde{f}_j : c_1^{-1}E \to \Omega$ by $\tilde{f}_j(\lambda) = (\tilde{f}^1_j(\lambda), \tilde{f}^2_j(\lambda)) := f_j(\alpha_j\lambda)$, $\lambda \in c_1^{-1}E := E_1$. Then we may verify, as in step I, that $(\tilde{f}^2_{3j})_{j\geq 1}$ is bounded on E_1. Again, applying Montel's theorem, we can choose a sequence $(\tilde{f}^2_{4j})_{j\geq 1} \subset (\tilde{f}^2_{3j})_{j\geq 1}$ such that $\tilde{f}^2_{4j} \overset{\mathrm{K}}{\Rightarrow} {}^{\exists}\tilde{f}^2_0 \in \mathcal{O}(E_1, \mathbf{C}^m)$ as $j \to \infty$, from which and (†3), we obtain that

$$\tilde{g}_0(0) = \lim_{j\to\infty} \tilde{g}_j(0) = \lim_{j\to\infty} g_{4j}(0) = \lim_{j\to\infty} f_{4j}(\alpha_{4j}) = \lim_{j\to\infty} \tilde{f}_{4j}(1) = \tilde{f}_0(1),$$

where $\tilde{f}_0 := (\tilde{f}^1_0, \tilde{f}^2_0)$, $\tilde{f}^1_0(\lambda) := f^1_0(\alpha_0\lambda)$, $\lambda \in E_1$. Observe that $\psi_0(1) = H(\tilde{f}_0(1)) = 1$. But since $\psi_0 := H \circ \tilde{f}^2_0 \leq 1$ on $\lambda \in E_1$ as above, it follows from the maximum principle for $\psi_0 \in \mathcal{SH}(E_1)$ that $\psi_0 \equiv 1$ on E_2, which implies that

$$\partial\Omega \ni \tilde{f}_0(0) = \lim_{j\to\infty} \tilde{f}_{4j}(0) = \lim_{j\to\infty} f_{4j}(0) = z_0.$$

This is a contradiction to (†2) and we are done. □

We are now in a position to give some examples of domains which were mentioned in Remark 3.9.

Example 5.3. (1) There exists a non-taut pseudoconvex domain G such that for any $\tilde{\varphi} \in \mathcal{PSH}(G)$ the Hartogs domain $\tilde{\Omega} := \Omega_{\tilde{\varphi},|\cdot|}(G)$ is $\tilde{k}$-hyperbolic, pseudoconvex, but not taut. Here, it is possible that $\tilde{\varphi}$ is not bounded from above.

More explicitly, let $G_{TT} = \Omega_{u,|\cdot|}(E)$ and $(\alpha_\nu)_{\nu\ge 1}$ be as in (1) of Example 3.11. Put $A := (\log 5)\sum_{\nu=1}^{\infty}\alpha_\nu < \infty$ and take an $\alpha \in (0,1)$ so that $(\log\alpha) + A < 0$. Define $\varphi(z_1,z_2) := u(z_1)$ and $\psi(z_1,z_2) := \max\{\log\alpha, \log|z_2|\}$ for $(z_1,z_2)\in G_{TT}$. Clearly, $\varphi+\psi<0$ on G_{TT}, so the domain $\Sigma = \Sigma_{\varphi,\psi}(G_{TT})$ is well-defined. Moreover, it follows directly from Example 3.11 (1) and Proposition 4.3 (c) that Σ is $\tilde{k}$-hyperbolic. But since φ is not continuous at $(0,z_2)\in G_{TT}$, Σ is not taut by Lemma 4.10. Therefore, for any $\tilde{\varphi}\in\mathcal{PSH}(\Sigma)$, the unbounded domain $\tilde{\Omega} = \Omega_{\tilde{\varphi},|\cdot|}(\Sigma)$ is pseudoconvex, but not taut by Theorem 5.2; moreover, it is $\tilde{k}$-hyperbolic. For this, it suffices to show, as Σ replaces G in Lemma 3.7, that $(\star)$ for Σ is true, because Σ is $\tilde{k}$-hyperbolic. Let $a=(a_1,a_2,a_3)\in\Sigma$. Take a sequence $f_\nu\in\mathcal{O}(E_\nu,\Sigma)$, $\nu\ge 1$, with $f_\nu(0)=f_\nu(1)=a$, where $E_\nu := r_\nu E$ and $1<r_\nu \nearrow \infty$ as $\nu\to\infty$. For $j=1,2,3$ we define $\pi_j:\mathbf{C}^3\to\mathbf{C}$ by $\pi_j(z_1,z_2,z_3) := z_j$, and put $f_\nu := (f_\nu^1, f_\nu^2, f_\nu^3)$, where $f_\nu^j\in\mathcal{O}(E_\nu,\mathbf{C})$. Since $\pi_1(\Sigma)\subset\pi_1(G_{TT})\subset E$, the Montel theorem gives that there is a sequence $(f_{1\nu}^1)_{\nu\ge1}\subset(f_\nu^1)_{\nu\ge1}$ such that $f_{1\nu}^1 \overset{\mathrm{K}}{\Rightarrow} a_1$ as $\nu\to\infty$. Since $\psi\big|_\Omega \ge \log\alpha$, one has $\cup_{\nu=1}^{\infty} f_\nu^3(E_\nu)\subset\pi_3(G_{TT})\subset\mathbf{C}\setminus(\alpha E)$. By the tautness of $\mathbf{C}\setminus(\alpha E)$, we can take a sequence $(f_{2\nu}^3)_{\nu\ge1}\subset(f_{1\nu}^3)_{\nu\ge1}$ such that $f_{2\nu}^3 \overset{\mathrm{K}}{\Rightarrow} a_3$ as $\nu\to\infty$. On the other hand, since $f_{2\nu}(E_{2\nu})\subset\Sigma$ for $\nu\ge1$, one has

$$|f_{2\nu}^2(\lambda)| = e^{\log|f_{2\nu}^2(\lambda)|} \le e^{\psi(f_{2\nu}^1(\lambda), f_{2\nu}^2(\lambda))} < |f_{2\nu}^3(\lambda)|, \quad \forall\,\lambda\in E_{2\nu}.$$

Thus, there is $\nu_0\in\mathbf{N}$ such that $|f_{2\nu}^2(\lambda)|\le|a_3|+1$ for any $\lambda\in E_{2\nu}$ and $\nu\ge\nu_0$. In view of Montel's theorem, we can choose a sequence

$(f^2_{3\nu})_{\nu\geq 1} \subset (f^2_{2\nu})_{\nu\geq 1}$ such that $f^2_{3\nu} \overset{\mathrm{K}}{\Rightarrow} a_2$ as $\nu \to \infty$. Consequently, $f_{3\nu} \overset{\mathrm{K}}{\Rightarrow} a$ as $\nu \to \infty$.

(2) There is a non-taut k-hyperbolic domain G such that for any $u \in \mathcal{PSH}(G, \mathbf{R})$, which is not locally bounded, the Hartogs domain $\Omega := \Omega_{u,|\cdot|}(G)$ is $\tilde{k}$-hyperbolic but not k-hyperbolic; our construction shows that Ω could be chosen to be non-pseudoconvex.

More explicitly: Let $u_1 \in \mathcal{C}^{\uparrow}(E, \mathbf{R})$ and put $D_1 := \Omega_{u_1,|\cdot|}(E)$. For $j = 2, 3$, define $D_j := \Omega_{u_j,|\cdot|}(D_{j-1})$ where $u_j \in \mathcal{C}^{\uparrow}(D_{j-1}, \mathbf{R})$. If u_j, $j = 1, 2, 3$, are not continuous, u_j, $j = 1, 2$, are locally bounded, and u_3 is not locally bounded, then D_1, D_2 are k-hyperbolic by Proposition 3.2, but not taut by Theorem 5.2. On the other hand, D_3 is not k-hyperbolic by Proposition 3.2. If, moreover, $u_3 \in \mathcal{PSH}(D_2)$, then D_3 is $\tilde{k}$-hyperbolic. For this, as in (1), it suffices to show that $(\star)$ for D_2 holds. Let $b = (b_1, b_2, b_3) \in D_2$, and let $n \geq 1$. Take a sequence $g_\nu := (g^1_\nu, g^2_\nu) \in \mathcal{O}(E_\nu, D_2)$ such that $g_\nu(0) = g_\nu(1) = b$, where $E_\nu := r_\nu E$ and $1 < r_\nu \nearrow \infty$ as $\nu \to \infty$, let $g^1_\nu := (\varphi^1_\nu, \varphi^2_\nu) \in \mathcal{O}(E_\nu, D_1)$, where $\varphi^1_\nu \in \mathcal{O}(E_\nu, E)$. In view of Montel's theorem, we can extract a sequence $(\varphi^1_{1\nu})_{\nu\geq 1} \subset (\varphi^1_\nu)_{\nu\geq 1}$ such that $\varphi^1_{1\nu} \overset{\mathrm{K}}{\Rightarrow} {}^\exists\varphi^1 \in \mathcal{O}(\mathbf{C}, \overline{E})$ as $\nu \to \infty$, and it follows from the Liouville theorem that $\varphi^1 \equiv$ constant $= \varphi^1_{11}(0) = b_1 \in E$. Now, put $\varepsilon := \operatorname{dist}(b_1, \partial E)/2 > 0$ and fix $0 < s < 1$. Then we can choose $\nu_s \geq 1$ such that $\varphi^1_{1\nu}(\lambda) \in \mathbf{B}_1(b_1, \varepsilon)$, $\nu \geq \nu_s$, $\lambda \in s\overline{E}$. Because u_1 is locally bounded on E and $\mathbf{B}_1(b_1, \varepsilon) \Subset E$, one has

$$\left|\varphi^2_{1\nu}(\lambda)\right| \leq \exp\Big(\max_{|\zeta - b_1| \leq \varepsilon} u_1(\zeta)\Big) =: \alpha < \infty, \quad \lambda \in s\overline{E}, \quad \nu \geq \nu_s.$$

But since s is arbitrary, the family $(\varphi^2_{1\nu})_{\nu\gg 1}$ is locally bounded. So by Montel's theorem, we can choose a sequence $(\varphi^2_{2\nu})_{\nu\geq 1} \subset (\varphi^2_{1\nu})_{\nu\geq 1}$ such that $\varphi^2_{2\nu} \overset{\mathrm{K}}{\Rightarrow} {}^\exists\varphi^2 \in \mathcal{O}(\mathbf{C}, \alpha\overline{E})$ as $\nu \to \infty$. By applying Liouville's theorem to the entire function φ^2, we then get that $\varphi^2 \equiv$ constant $= \varphi^2_{21}(0) = b_2$. Hence, $g^1_{2\nu} \overset{\mathrm{K}}{\Rightarrow} (b_1, b_2) \in D_1$ as $\nu \to \infty$. Applying the same method to the family $(g_{2\nu})_{\nu\geq 1}$, we can obtain a sequence $(g_{3\nu})_{\nu\geq 1} \subset (g_{2\nu})_{\nu\geq 1}$ such that $g_{3\nu} \overset{\mathrm{K}}{\Rightarrow} b \in D_2$ as $\nu \to \infty$.

As a consequence of Theorem 5.2, we have

Corollary 5.4. *If G is taut and $u, v \in (\mathcal{C} \cap \mathcal{PSH})(G, \mathbf{R})$, then $\Sigma_{u,v}(G)$ is taut.*

However, its converse does not hold in general, see Example 4.8.

Proof. Let $(\varphi_j)_{j\geq 1} \subset \mathcal{O}(E, \Sigma_{u,v}(G))$ be a sequence. Observe that $\Sigma := \Sigma_{u,v}(G) \subset \Omega_{u,|\cdot|}(G) =: \Omega$. Since Ω is taut by Theorem 5.2, $(\varphi_j)_{j\geq 1}$ is a normal subfamily of $\mathcal{O}(E, \Omega)$, i.e., there is a sequence $(\varphi_{1j})_{j\geq 1} \subset (\varphi_j)_{j\geq 1}$ which is either normally convergent in $\mathcal{O}(E, \Omega)$ or compactly divergent. In the latter case, the sequence $(\varphi_{1j})_{j\geq 1}$, as a subfamily of $\mathcal{O}(E, \Sigma)$, diverges compactly.

For $j \geq 1$ we put $\varphi_j := (f_j, g_j)$, where $(f_j)_{j\geq 1} \subset \mathcal{O}(E, G)$ and $(g_j)_{j\geq 1} \subset \mathcal{O}(E)$.

From now on, we only suppose that $(\varphi_{1j})_{j\geq 1}$ is normally convergent in $\mathcal{O}(E, \Omega)$. Take a function $\varphi := (f, g) \in \mathcal{O}(E, \Omega)$, where $f \in \mathcal{O}(E, G)$, $g \in \mathcal{O}(E)$, such that $f_{1j} \overset{\mathrm{K}}{\rightrightarrows} f$ and $g_{1j} \overset{\mathrm{K}}{\rightrightarrows} g$ as $j \to \infty$. Note that

$$e^{(v\circ f_j)(\lambda)} < |g_j(\lambda)| < e^{-(u\circ f_j)(\lambda)}, \quad \lambda \in E, \quad j \geq 1,$$
$$|g(\lambda)| < e^{-(u\circ f)(\lambda)}, \quad \lambda \in E.$$

Since $g_j^{-1}(0) = \varnothing$ for any $j \geq 1$, it follows from Hurwitz's theorem that either $g \equiv 0$ or g never vanishes. In the former case, it is clear that $\varphi(E) \subset \partial\Sigma$, which implies that $(\varphi_{1j})_{j\geq 1}$, as a subfamily of $\mathcal{O}(E, \Sigma)$, is compactly divergent. Now we assume that $g \not\equiv 0$ and define

$$\tilde{u}(\lambda) := |g(\lambda)| e^{(u\circ f)(\lambda)}, \quad \tilde{v}(\lambda) := \frac{1}{|g(\lambda)|} e^{(v\circ f)(\lambda)}, \quad \lambda \in E.$$

Observe that $\tilde{u}, \tilde{v} \in \mathcal{SH}(E)$ and $\max\{\tilde{u}, \tilde{v}\} \leq 1$ on E. Then the maximum principle for subharmonic functions implies that either $\tilde{u}\big|_E = 1$ or $\tilde{u}\big|_E < 1$ and either $\tilde{v}\big|_E = 1$ or $\tilde{v}\big|_E < 1$. These properties yield that either $\varphi(E) \subset \partial\Sigma$ or $\varphi(E) \subset \Sigma$. Consequently, the sequence $(\varphi_{1j})_{j\geq 1}$ is either normally convergent in $\mathcal{O}(E, \Sigma)$ or compactly divergent. ◻

Next, we shall state and prove a version of Theorem 3.3 for the tautness

Theorem 5.5. *Let $G \subset \mathbf{C}^n$, $S \subset \mathbf{C}^m$ be domains, and let $\pi : G \to S$ be a holomorphic mapping. Suppose that for any $p \in S$ there exists an* *open neighborhood $U = U(p)$ of p in S such that $\pi^{-1}(U)$ is taut. If S is taut, then G is also taut.*

Proof. Suppose the contrary. Then, by Lemma 5.1, we can take sequences $(z_j)_{j\geq 0} \subset G$, $f_j, g_j \in \mathcal{O}(E, G)$, and $(\alpha_j)_{j\geq 0}, (\beta_j)_{j\geq 0} \subset [0, 1)$ satisfying (†2) $\sim$ (†5). Because S is taut and the family $(\pi \circ f_j)_{j\geq 1} \subset \mathcal{O}(E, S)$ satisfies

$$\lim_{j\to\infty} (\pi \circ f_j)(0) = \lim_{j\to\infty} \pi(f_j(0)) = \pi(z_0) \in S,$$

there exists a sequence $(f_{1j})_{j\geq 1} \subset (f_j)_{j\geq 1}$ such that

$$\pi \circ f_{1j} \overset{\mathrm{K}}{\Longrightarrow} {}^{\exists}\varphi_1 \in \mathcal{O}(E, S) \quad \text{as} \quad j \to \infty. \tag{5.5.1}$$

In particular, by (†3)

$$\lim_{j\to\infty} (\pi \circ g_{1j})(0) = \lim_{j\to\infty} (\pi \circ f_{1j})(\alpha_{1j}) = \varphi_1(\alpha_0) \in S.$$

Hence, $(\pi \circ g_{1j})_{j\geq 1} \subset \mathcal{O}(E, S)$ does not diverge compactly on G, and because of the tautness of S, we can extract a sequence $(g_{2j})_{j\geq 1} \subset (g_{1j})_{j\geq 1}$ such that

$$\pi \circ g_{2j} \overset{\mathrm{K}}{\Longrightarrow} {}^{\exists}\varphi_2 \in \mathcal{O}(E, S) \quad \text{as} \quad j \to \infty. \tag{5.5.2}$$

Step I. For any $\lambda \in E$, there exist open neighborhoods $V_\lambda \Subset E$ of λ, $U_{\varphi_2(\lambda)} \subset S$ of $\varphi_2(\lambda)$, and $j_\lambda \in \mathbf{N}$, such that $\pi^{-1}(U_{\varphi_2(\lambda)})$ is taut and $g_{2j}(V_\lambda) \Subset \pi^{-1}(U_{\varphi_2(\lambda)}) \subset G$ for any $j \geq j_\lambda$.

Subproof. Fix $\lambda \in E$. Clearly, $\varphi_2(\lambda) \in S$ and by our assumption, one can take an open neighborhood $U_{\varphi_2(\lambda)} \subset S$ of $\varphi_2(\lambda)$ such that $\pi^{-1}(U_{\varphi_2(\lambda)})$ is taut. Take $r_\lambda := r(\lambda) > 0$ so that $\mathbf{B}(\varphi_2(\lambda), 3r_\lambda) \subset U_{\varphi_2(\lambda)}$. Because of the continuity of φ_2, $B_\lambda := \varphi_2^{-1}(\mathbf{B}(\varphi_2(\lambda), r_\lambda)) \subset E$ is an open set containing the point λ, and also, one can take an open neighborhood $V_\lambda = V(\lambda) \Subset B_\lambda$ of λ so small that $\|\varphi_2(\zeta) - \varphi_2(\lambda)\| < r_\lambda$ for any $\zeta \in \bar{V}_\lambda$. Now, in view of (5.5.2), we may choose $j_\lambda \in \mathbf{N}$ so large that $\|(\pi \circ g_{2j})(\zeta) - \varphi_2(\zeta)\| < r_\lambda$ for any $\zeta \in \overline{V}_\lambda$ and $j \geq j_\lambda$. Hence

$$\|(\pi \circ g_{2j})(\zeta) - \varphi_2(\lambda)\| \leq \|(\pi \circ g_{2j})(\zeta) - \varphi_2(\zeta)\| + \|\varphi_2(\zeta) - \varphi_2(\lambda)\| < 2r_\lambda$$

for any $\zeta \in \overline{V}_\lambda$ and $j \geq j_\lambda$. Thus we get that

$$(\pi \circ g_{2j})(V_\lambda) \subset (\pi \circ g_{2j})(\overline{V}_\lambda) \subset \mathbf{B}(\varphi_2(\lambda), 3r_\lambda) \subset U_{\varphi_2(\lambda)}, \quad j \geq j_\lambda. \quad \square$$

Step II. Taking a point $0 < s < 1$ so that $[-s, \beta_0]$ is compact in E, we can choose a finite set $\{x_\mu : \mu = 1, \dots, q\} \subset [-s, \beta_0]$ such that $[-s, \beta_0] \subset \cup_{\mu=1}^q V_{x_\mu}$ and

$$\forall_{\mu \in \{1,\dots,q\}}, \quad \exists_{\nu \in \{1,\dots,q\} \setminus \{\mu\}} : V_{x_\nu} \cap V_{x_\mu} \neq \varnothing,$$

and moreover, after a rearrangement, we may assume that $\beta_0 \in V_q$ and $V_{x_\mu} \cap V_{x_{\mu+1}} \neq \varnothing$, $\mu \in \{1, \dots, q-1\}$. Now, we will consider the case $\lambda = \beta_0$. Suppose that there exists a subsequence $(g_{3j})_{j \geq 1} \subset (g_{2j})_{j \geq 1}$ such that $g_{3j} \overset{\mathrm{K}}{\Rightarrow} {}^\exists g_{\beta_0} \in \mathcal{O}(V_{\beta_0}, \pi^{-1}(U_{\varphi_2(\beta_0)}))$ as $j \to \infty$. By the first property of (†4), one has

$$\lim_{j \to \infty} z_{3j} = \lim_{j \to \infty} g_{3j}(\beta_{3j}) = g_{\beta_0}(\beta_0) \in G,$$

which is a contradiction to the divergence of the sequence $(z_j)_j$ in (†4). Hence, in view of Step I, the sequence $(g_{2j})_j$ diverges compactly on V_{β_0}. But since $\beta_0 \in V_{x_q} \cap V_{\beta_0}$, in view of Step I, we may extract a sequence $(g_{4j})_{j \geq 1} \subset (g_{2j})_{j \geq 1}$ such that $(g_{4j})_{j \geq 1}$ diverges compactly on V_{x_q}. Because $V_{x_{q-1}} \cap V_{x_q} \neq \varnothing$, in view of Step I, we may also extract a sequence $(g_{5j})_{j \geq 1} \subset (g_{4j})_{j \geq 1}$ such that $(g_{5j})_{j \geq 1}$ diverges compactly on $V_{x_{q-1}}$. Of course, we can proceed to $q-2$ and so on. In this manner, we may get $\mu_0 \in \{1, \dots, m\}$ with $0 \in V_{x_{\mu_0}}$ and a sequence $(g_{6j})_{j \geq 1} \subset (g_{5j})_{j \geq 1}$ such that $(g_{6j})_{j \geq 1}$ diverges compactly on $V_{x_{\mu_0}}$.

Thus, the result of Step II gives, in view of (†3), that

$$\text{(5.5.3)} \qquad \text{either } f_{6j}(\alpha_{6j}) \overset{j \to \infty}{\longrightarrow} {}^\exists \hat{a}_0 \in \partial G, \quad \text{or} \quad \|f_{6j}(\alpha_{6j})\| \overset{j \to \infty}{\longrightarrow} \infty.$$

This is a similar situation as in (†4) for the sequence $(g_j(\beta_j))_j$. Hence we can repeatedly carry out the procedures of Step I, using (5.5.1) and our assumption, and Step II, using the condition (5.5.3), to the sequence $(f_{6j}(\alpha_{6j}))_j$, so we may obtain a subsequence $(f_{7j})_{j \in \mathbf{N}}$ of $(f_{6j})_{j \in \mathbf{N}}$ such that $(f_{7j}(0))_{j \in \mathbf{N}}$ does not converge to a point in G; a contradiction to (†2). $\square$

As a consequence of Theorem 5.5, we get the following result, due to Thai and Huong [**16**]: *If* $\pi : G \to S$ *denotes a holomorphic covering between domains in* $\mathbf{C}^n$*, then* G *is taut if and only if so is* S.

We shall finish this section by studying the hyperconvexity of $\Omega_H(G)$ and $\Sigma_{u,v}(G)$.

Note that $\Omega_H(G) \Subset \mathbf{C}^{n+m}$ if and only if $G \Subset \mathbf{C}^n$ and $\exists_{C>0} : H(z,w) \geq C\|w\|$, $(z,w) \in G \times \mathbf{C}^m$; $\Sigma_{u,v}(G) \Subset \mathbf{C}^{n+1}$ if and only if $G \Subset \mathbf{C}^n$ and u is bounded from below on G.

The following result can be found in [**8**].

Proposition 5.6. *A bounded Hartogs domain* $\Omega_H(G)$ *is hyperconvex if and only if* G *is hyperconvex and* $H \in (\mathcal{C} \cap \mathcal{PSH})(G \times \mathbf{C}^m, \mathbf{R})$.

On the other hand, we have

Proposition 5.7. *Suppose that* $\Sigma = \Sigma_{u,v}(G)$ *is bounded. If* G *is hyperconvex and* $u, v \in (\mathcal{C} \cap \mathcal{PSH})(G, \mathbf{R})$*, then* Σ *is hyperconvex.*

Proof. Because of the hyperconvexity of G, there exists an exhaustion $\varphi \in (\mathcal{C} \cap \mathcal{PSH})(G, (-\infty, 0))$ of G. Define a function $\Phi : G \times \mathbf{C} \setminus \{0\} \to [-\infty, \infty)$ by $\Phi(z, \lambda) := \max\{\varphi(z), \psi(z, \lambda)\}$ for $(z, \lambda) \in G \times \mathbf{C} \setminus \{0\}$, where

$$\psi(z, \lambda) := \max\{u(z) + \log|\lambda|, v(z) - \log|\lambda|\}, \quad (z, \lambda) \in G \times \mathbf{C} \setminus \{0\}.$$

Since $u, v \in (\mathcal{C} \cap \mathcal{PSH})(G)$, one has $\psi \in (\mathcal{C} \cap \mathcal{PSH})(G \times \mathbf{C} \setminus \{0\})$. Therefore, $\Phi \in (\mathcal{C} \cap \mathcal{PSH})(G \times \mathbf{C} \setminus \{0\})$; moreover, $\Phi \in (\mathcal{C} \cap \mathcal{PSH})(\Sigma, (-\infty, 0))$, and it is an exhaustion of Σ. Thus Σ has a bounded continuous plurisubharmonic exhaustion function Φ, so it is hyperconvex. □

Remark 5.8. If a pseudoconvex Reinhardt Hartogs-Laurent domain $\Sigma_{u,v}(G)$ is hyperconvex, then G is hyperconvex and u is bounded from below on G. However, its converse, in general, does not hold. For example, consider the Hartogs triangle $\Sigma_{0,\log|\cdot|}(E)$.

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