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ON SOME TOPOLOGICAL PROPERTIES OF VECTOR-VALUED FUNCTION SPACES

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ABSTRACT. Let E be an ideal of L^0 over a σ-finite measure space (Ω, Σ, μ) with a Hausdorff locally convex-solid topology ξ, and let $(X, \|\cdot\|_X)$ be a real Banach space. Let $E(X)$ be a subspace of the space $L^0(X)$ of μ-equivalence classes of all strongly Σ-measurable functions $f : \Omega \to X$ and consisting of all those $f \in L^0(X)$ for which the scalar function $\|f(\cdot)\|_X$ belongs to E. In this paper we show that a number of topological properties of the spaces X and (E, ξ) can be lifted to the space $(E(X), \bar{\xi})$, where $\bar{\xi}$ stands for the topology on $E(X)$ associated with ξ. We characterize some important topological properties of the space $(E(X), \bar{\xi})$ (weak compactness of order intervals, almost reflexivity, weak sequential completeness, semi-reflexivity, relative weak compactness of solid hulls) in terms of the corresponding properties of X and (E, ξ).

1. Introduction and preliminaries. Let E be an ideal of L^0 (over a σ-finite measure space) with a Hausdorff locally convex-solid topology ξ, and let X be a real Banach space. The aim of this paper is to extend some important topological properties of the space (E, ξ) to the vector-valued function space $(E(X), \bar{\xi})$, where $\bar{\xi}$ stands for the topology on $E(X)$ associated with ξ. We characterize the following topological properties of the space $(E(X), \bar{\xi})$: weak compactness of order intervals: Section 2, almost reflexivity; Section 3, weak sequential completeness; Section 4, semi-reflexivity; Section 5, relative weak compactness of solid hull; Section 6, in terms of the corresponding properties of X and (E, ξ).

In the particular case of E being a Banach function space, over a finite measure space, the problem of characterizing the topological properties of the Köthe-Bochner space $E(X)$ in terms of the properties of both Banach spaces E and X has been considered by Pisier [**28**], Bombal

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[**3**], Geuiler and Chubarova [**16**], Bombal and Hernando [**4**], Talagrand [**30**], Bukhvalov and Lozanowskii [**6, 7**].

For terminology concerning Riesz spaces and function spaces we refer to [**1, 18, 33**]. Given a topological vector space (L,τ) by $(L,\tau)^*$ or L_τ^*, we will denote its topological dual. We denote by $\sigma(L,K)$ and $\beta(L,K)$ and $\tau(L,K)$ the weak topology, the strong topology and the Mackey topology on L with respect to a dual system $\langle L,K\rangle$.

Throughout the paper we assume that (Ω,Σ,μ) is a complete σ-finite measure space and let $\Sigma_f = \{A\in\Sigma : \mu(A)<\infty\}$. Let L^0 denote the corresponding space of μ-equivalence classes of all Σ-measurable real valued functions. Then L^0 is a super Dedekind complete Riesz space under the ordering $u\le v$ whenever $u(\omega)\le v(\omega)$, μ almost everywhere on Ω. Let χ_A stand for the characteristic function of a set A. By $\mathbf{N}$ and $\mathbf{R}$ we denote the sets of natural and real numbers.

Let E be an ideal of L^0 with $\operatorname{supp} E=\Omega$, and let E' stand for the Köthe dual of E, i.e.,

$$E' = \left\{ v\in L^0 : \int_\Omega |u(\omega)v(\omega)|\,\mathrm{d}\,\mu < \infty \text{ for all } u\in E \right\}.$$

Throughout the paper we assume that $\operatorname{supp} E'=\Omega$. Let $E^\sim$, $E_n^\sim$ and $E_s^\sim$ stand for the order dual, the order continuous dual and the singular dual of E, respectively. Then $E_n^\sim$ separates points of E and it can be identified with E' through the mapping: $E'\ni v\to\varphi_v\in E_n^\sim$, where

$$\varphi_v(u) = \int_\Omega u(\omega)v(\omega)\,\mathrm{d}\,\mu \quad \text{for all} \quad u\in E.$$

Then $E^\sim = E_n^\sim \oplus E_s^\sim$ and $E_s^\sim = (E_n^\sim)^d$ (= the disjoint complement of $E_n^\sim$ in $E^\sim$).

By a locally solid, respectively locally convex-solid, function space (E,ξ) we mean an ideal E provided with a locally solid, respectively locally convex-solid, topology ξ.

Note that in view of the super Dedekind completeness of E, both types of order convergence in E for sequences and for nets coincide, so $E_n^\sim = E_c^\sim$ (= the σ-order continuous dual of E). Recall that a Hausdorff locally convex-solid topology ξ on E is a Lebesgue, respectively σ-Lebesgue, topology if and only if $E_\xi^*\subset E_n^\sim$, respectively $E_\xi^*\subset E_c^\sim$, see

[**1**, Theorem 9.1, Theorem 9.2]. This shows that for ξ the σ-Lebesgue property and the Lebesgue property coincide. Moreover, one can show that for ξ the σ-Levy and the Levy property coincide, see [**13**, Proposition 3.2].

For terminology and basic concepts from the theory of vector-valued function spaces $E(X)$, in particular Lebesgue-Bochner spaces $L^p(X)$, we refer to the three main monographs: Diestel and Uhl's "vector measures" [**12**], Cembranos and Mendoza's "Banach spaces of vector valued functions" [**10**] and Pei-Kee Lin's "Köthe-Bochner function spaces" [**19**].

Now we recall terminology and some basic results concerning the topological properties and the duality theory of vector-valued function spaces $E(X)$ as set out in [**5, 7, 10, 12, 14, 19, 21**–**23**].

Let $(X, \|\cdot\|_X)$ be a real Banach space and let X^* stand for the Banach dual of X. Let S_X, B_X stand for the unit sphere and the unit ball of X. By $L^0(X)$ we denote the set of μ-equivalence classes of all strongly Σ-measurable functions $f : \Omega \to X$. For $f \in L^0(X)$, let us set $\tilde{f}(\omega) := \|f(\omega)\|_X$ for $\omega \in \Omega$. Let

$$E(X) = \{\, f \in L^0(X) : \tilde{f} \in E \,\}.$$

Recall that the algebraic tensor product $E \otimes X$ is the subspace of $E(X)$ spanned by the functions of the form $u \otimes x$, $(u \otimes x)(\omega) = u(\omega)x$, where $u \in E$, $x \in X$.

A subset H of $E(X)$ is said to be *solid* whenever $\tilde{f}_1 \le \tilde{f}_2$ and $f_1 \in E(X)$, $f_2 \in H$ imply $f_1 \in H$. A linear topology τ on $E(X)$ is said to be *locally solid* if it has a local base at zero consisting of solid sets. A linear topology τ on $E(X)$ that is as the same time locally solid and locally convex will be called a *locally convex-solid* topology on $E(X)$. A semi-norm ϱ on $E(X)$ is called *solid* if $\varrho(f_1) \le \varrho(f_2)$ whenever $f_1, f_2 \in E(X)$ and $\tilde{f}_1 \le \tilde{f}_2$. It is known that a locally convex topology τ on $E(X)$ is locally convex-solid if and only if it is generated by some family of solid semi-norms defined on $E(X)$, see [**14**]. A locally solid topology τ on $E(X)$ is said to be a *Lebesgue topology* whenever for a net (f_α) in $E(X)$, $\tilde{f}_\alpha \xrightarrow{(0)} 0$ in E implies $f_\alpha \xrightarrow{\tau} 0$, see [**23**, Definition 2.2].

Let (E, ξ) be a Hausdorff locally convex-solid function space. Then one can topologize the space $E(X)$ as follows, see [**14**]. Let $\{p_t : t \in T\}$

be a family of Riesz semi-norms on E that generates ξ. By putting

$$\bar{p}_t(f) := p_t(\tilde{f}) \quad \text{for} \quad f \in E(X), \quad t \in T,$$

we obtain a family $\{\bar{p}_t : t \in T\}$ of solid semi-norms on $E(X)$ that defines a Hausdorff locally convex-solid topology $\bar{\xi}$ on $E(X)$, (called the *topology associated with* ξ). Then $\bar{\xi}$ is a Lebesgue topology whenever ξ is a Lebesgue topology, see [**14**].

Conversely, let τ be a Hausdorff locally convex-solid topology on $E(X)$, and let $\{\varrho_t : t \in T\}$ be a family of solid semi-norms on $E(X)$ that generates τ. By putting, for a fixed $x_0 \in S_X$

$$\tilde{\varrho}_t(u) := \varrho_t(u \otimes x_0) \quad \text{for} \quad u \in E, \quad t \in T,$$

we obtain a family $\{\tilde{\varrho}_t : t \in T\}$ of Riesz semi-norms on E that defines a Hausdorff locally convex-solid topology $\tilde{\tau}$ on E.

One can show that $\tilde{\bar{\xi}} = \xi$ and $\bar{\tilde{\tau}} = \tau$, see [**14**]. Thus every Hausdorff locally convex-solid topology τ on $E(X)$ can be represented as the topology associated with some Hausdorff locally convex-solid topology ξ $(= \tilde{\tau})$ on E.

In particular, for a Banach function space $(E, \|\cdot\|_E)$ the space $E(X)$ provided with the norm $\|f\|_{E(X)} := \|\tilde{f}\|_E$ is usually called a *Köthe-Bochner space*, see [**19**].

For a linear functional F on $E(X)$, let us put

$$|F|(f) = \sup\{\, |F(h)| : h \in E(X),\ \tilde{h} \le \tilde{f}\,\} \quad \text{for} \quad f \in E(X).$$

The set

$$E(X)^{\sim} = \{\, F \in E(X)^{\#} : |F|(f) < \infty \text{ for all } f \in E(X)\,\}$$

will be called the *order dual* of $E(X)$ (here $E(X)^{\#}$ denotes the algebraic dual of $E(X)$).

For $F_1, F_2 \in E(X)^{\sim}$ we will write $|F_1| \le |F_2|$ whenever $|F_1|(f) \le |F_2|(f)$ for all $f \in E(X)$. A subset A of $E(X)^{\sim}$ is said to be *solid* whenever $|F_1| \le |F_2|$ with $F_1 \in E(X)^{\sim}$ and $F_2 \in A$ imply $F_1 \in A$. A linear subspace I of $E(X)^{\sim}$ will be called an *ideal* of $E(X)^{\sim}$ whenever

I is solid. It is known that if τ is a locally solid topology on $E(X)$, then $(E(X),\tau)^*$ is an ideal of $E(X)^\sim$, see [**21**, Theorem 3.2].

A linear functional F on $E(X)$ is said to be *order continuous* whenever, for a net (f_α) in $E(X)$, $\tilde{f}_\alpha \stackrel{(0)}{\to} 0$ in E implies $F(f_\alpha) \to 0$. The set consisting of all order continuous linear functionals on $E(X)$ will be denoted by $E(X)_n^\sim$ and called the *order continuous dual* of $E(X)$, see [**5, 21**]. Since we assume that $\operatorname{supp} E' = \Omega$, $E(X)_n^\sim$ separates points of $E(X)$. A Hausdorff locally convex-solid topology τ on $E(X)$ has the Lebesgue property if and only if $E(X)_\tau^* \subset E(X)_n^\sim$, see [**23**, Theorem 2.4].

We now recall terminology concerning the spaces of w^*-measurable functions, see [**5, 7, 10**].

For a given function $g : \Omega \to X^*$ and $x \in X$, we denote by g_x the real function on Ω defined by $g_x(\omega) = g(\omega)(x)$ for $\omega \in \Omega$. A function $g : \Omega \to X^*$ is said to be w^*-measurable if the functions g_x are measurable for each $x \in X$. We shall say the two w^*-measurable functions g_1, g_2 are w^*-equivalent whenever $g_1(\omega)(x) = g_2(\omega)(x)$, μ almost everywhere for each $x \in X$.

Let $L^0(X^*,X)$ be the set of weak*-equivalence classes of all weak*-measurable functions $g : \Omega \to X^*$. Following [**5, 7**] one can define the so-called *abstract norm* $\vartheta : L^0(X^*,X) \to L^0$ by $\vartheta(g) := \sup\{|g_x| : x \in B_X\}$.

Then for $f \in L^0(X)$ and $g \in L^0(X^*,X)$ the function $\langle f, g\rangle : \Omega \to \mathbf{R}$ defined by $\langle f, g\rangle(\omega) := \langle f(\omega), g(\omega)\rangle$ is measurable and $|\langle f, g\rangle| \le \widetilde{f}\vartheta(g)$. Moreover, $\vartheta(g) = \widetilde{g}$ for $g \in L^0(X^*)$.

For an ideal M of E', let

$$M(X^*,X) = \{\, g \in L^0(X^*,X) : \vartheta(g) \in M \,\}.$$

Then $M(X^*,X)$ is an ideal of $E'(X^*,X)$, i.e., if $\vartheta(g_1) \le \vartheta(g_2)$ with $g_1 \in E'(X^*,X)$ and $g_2 \in M(X^*,X)$, then $g_1 \in M(X^*,X)$, see [**21**, Definition 1.2]. Clearly $M(X^*) \subset M(X^*,X)$.

Due to Bukhvalov, see [**5**, Theorem 4.1], $E(X)_n^\sim$ can be identified with $E'(X^*,X)$ through the mapping: $E'(X^*,X) \ni g \mapsto F_g \in E(X)_n^\sim$, where

$$F_g(f) = \int_\Omega \langle f(\omega), g(\omega)\rangle \,\mathrm{d}\,\mu \quad \text{for all} \quad f \in E(X),$$

and moreover

$$|F_g|(f) = \int_\Omega \tilde{f}(\omega)\vartheta(g)(\omega)\,\mathrm{d}\,\mu \quad \text{for all} \quad f \in E(X).$$

It is well known that if X is reflexive, then $E'(X^*, X) = E'(X^*)$.

Let $F \in E(X)^\sim$ and $x_0 \in S_X$ be fixed. For $u \in E^+$, let us set:

$$\varphi_F(u) := |F|(u \otimes x_0) = \sup\{\, |F(h)| : h \in E(X),\ \widetilde{h} \le u \,\}.$$

Then $\varphi_F : E^+ \to \mathbf{R}^+$ is an additive mapping and φ_F has a unique positive extension to a linear mapping from E to $\mathbf{R}$, denoted by φ_F again, and given by

$$\varphi_F(u) := \varphi_F(u^+) - \varphi_F(u^-) \quad \text{for all} \quad u \in E,$$

see [**7**, Lemma 7]. We shall need the following two technical results.

Proposition 1.1. *Let (E, ξ) be a Hausdorff locally convex-solid function space. Then for $F \in E(X)^\sim$, the following statements are equivalent*:

(i) *F is $\bar{\xi}$-continuous.*

(ii) *φ_F is ξ-continuous.*

Proof. (i)⇒(ii). Let ξ be generated by a family $\{p_t : t \in T\}$ of Riesz semi-norms on E, and let F be $\bar{\xi}$-continuous. Then there exist $t_i \in T$, $i = 1, 2, \dots, n$, and $a > 0$ such that $|F(h)| \le a \max_{1 \le i \le n} p_{t_i}(\tilde{h})$ for all $h \in E(X)$. Then for $u \in E^+$,

$$\varphi_F(u) = \sup\{\, |F(h)| : h \in E(X),\ \tilde{h} \le u \,\} \le a \max_{1 \le i \le n} p_{t_i}(u).$$

It easily follows that $\varphi_F(u) \le 2a \max_{1 \le i \le n} p_{t_i}(u)$ for all $u \in E$, so φ_F is ξ-continuous.

(ii)⇒(i). Assume that $\varphi_F \in E^*_\xi$. Then, there exist $t_i \in T$, $i = 1, \dots, n$, and $a > 0$ such that for $f \in E(X)$ we have

$$|F(f)| \le |F|(f) = \varphi_F(\tilde{f}) \le a \max_{1 \le i \le n} p_{t_i}(\tilde{f}) = a \max_{1 \le i \le n} \bar{p}_{t_i}(f),$$

and this means that F is $\bar{\xi}$-continuous. □

Proposition 1.2. *Let* (E, ξ) *be a Hausdorff locally convex-solid function space with the Lebesgue property. Then there exists an ideal* M_ξ *of* E' *with* $\operatorname{supp} M_\xi = \Omega$ *and such that*

$$E^*_\xi = \{\varphi_v : v \in M_\xi\} \quad \text{and} \quad E(X)^*_\xi = \{F_g : g \in M_\xi(X^*, X)\}.$$

Proof. Since $E^*_\xi \subset E^\sim_n$, there exists an ideal M_ξ of E' with $\operatorname{supp} M_\xi = \Omega$ and such that $E^*_\xi = \{\varphi_v : v \in M_\xi\}$. Now we shall show that $E(X)^*_\xi = \{F_g : g \in M_\xi(X^*, X)\}$.

Indeed, let $F \in E(X)^*_\xi$. Then by Proposition 1.1, $\varphi_F \in E^*_\xi$, so $\varphi_F = \varphi_{v_0}$ for some $v_0 \in M^+_\xi$. On the other hand, since $F \in E(X)^\sim_n$, we have $F = F_g$ for some $g \in E'(X^*, X)$. It easily follows that $\varphi_{F_g} = \varphi_{\vartheta(g)}$, so $\vartheta(g) = v_0 \in M_\xi$. This means that $g \in M_\xi(X^*, X)$.

Now, assume that $F = F_g$, where $g \in M_\xi(X^*, X)$. Then $\varphi_F = \varphi_{F_g} = \varphi_{\vartheta(g)}$, where $\vartheta(g) \in M_\xi$. Hence $\varphi_F \in E^*_\xi$, and in view of Proposition 1.1, $F \in E(X)^*_\xi$. □

2. Order intervals in vector-valued function spaces. We start by recalling a characterization of weak compactness of order intervals in locally convex-solid function spaces (E, ξ), see [**9**, Proposition 5.1], [**1**, Theorem 22.1].

Theorem 2.1. *Let* (E, ξ) *be a Hausdorff locally convex-solid function space. Then the following statements are equivalent*:

(i) ξ *is a Lebesgue topology.*

(ii) *Each order interval in* E *is* $\sigma(E, E^*_\xi)$*-compact.*

(iii) E*, embedded in a natural way, is an ideal of the bidual of* (E, ξ)*.*

The aim of this section is to extend this result to the vector-valued setting. For this purpose, we first recall terminology and some results concerning the duality theory of the spaces $E(X)$ as set out in [**22**].

Let I be an ideal of $E(X)^\sim$ separating points of $E(X)$. For a linear functional V on I let us set

$$|V|(F) = \sup\{\, |V(G)| : G \in I,\ |G| \le |F|\,\} \quad \text{for} \quad F \in I.$$

Then the set

$$I^{\sim} = \{ V \in I^{\#} : |V|(F) < \infty \text{ for all } F \in I \}$$

will be called the *order dual* of I (here $I^{\#}$ denotes the algebraic dual of I).

For $V_1, V_2 \in I$ we will write $|V_1| \le |V_2|$ whenever $|V_1|(F) \le |V_2|(F)$ for all $F \in I$. A subset K of $I^{\sim}$ is said to be *solid* whenever $|V_1| \le |V_2|$ with $V_1 \in I^{\sim}$, $V_2 \in K$ imply $V_1 \in K$. A linear subspace L of $I^{\sim}$ is called an *ideal* of $I^{\sim}$ if L is solid.

Let τ be a Hausdorff locally convex-solid topology on $E(X)$. Then $E(X)^*_\tau$ is an ideal of $E(X)^{\sim}$. The strong topology $\beta(E(X)^*_\tau, E(X))$ is a Hausdorff locally convex-solid topology on $E(X)^*_\tau$, see [**22**, Section 4], and the topological dual $(E(X)^*_\tau)^*_\beta$ $(= (E(X)^*_\tau, \beta(E(X)^*_\tau, E(X)))^*)$, is an ideal of $(E(X)^*_\tau)^{\sim}$, see [**22**, Theorem 2.1]. The space $(E(X)^*_\tau)^*_\beta$ is called the *bidual* of $(E(X), \tau)$.

For $f \in E(X)$, let us put

$$\pi_f(F) = F(f) \quad \text{for} \quad F \in E(X)^*_\tau.$$

Then $|\pi_f|(F) = |F|(f)$ for $F \in E(X)^*_\tau$ and $\pi_f \in (E(X)^*_\tau)^{\sim}$, see [**22**, Section 1]. Hence $|\pi_{f_1}| \le |\pi_{f_2}|$ whenever $f_1, f_2 \in E(X)$ with $\tilde{f}_1 \le \tilde{f}_2$. Moreover, $\pi_f \in (E(X)^*_\xi)^*_\beta$, so we have a *natural embedding* $\pi : E(X) \ni f \mapsto \pi_f \in (E(X)^*_\tau)^*_\beta$.

Denote by $(E(X)^*_\tau)_{E(X)}$ the ideal of $(E(X)^*_\tau)^*_\beta$ generated by the set $\pi(E(X))$, that is, $(E(X)^*_\tau)_{E(X)}$ is the smallest ideal of $(E(X)^*_\tau)^*_\beta$ containing $\pi(E(X))$. Then

$$(E(X)^*_\tau)_{E(X)} = \{ V \in (E(X)^*_\tau)^*_\beta : |V| \le |\pi_f| \text{ for some } f \in E(X) \}.$$

For each $f \in E(X)$, let $\varrho_f(F) = |F|(f)$ for $F \in I$. We define the *absolute weak* topology* $|\sigma|(I, E(X))$ on I as the locally convex-solid topology generated by the family $\{\varrho_f : f \in E(X)\}$ of solid semi-norms on I, see [**22**].

Theorem 2.2 (see [**22**, Theorem 3.2]). *Let τ be a Hausdorff locally convex-solid topology on $E(X)$. Then $(E(X)^*_\tau, |\sigma|(E(X)^*_\tau, E(X)))^* = (E(X)^*_\tau)_{E(X)}$.*

For $f \in E(X)$, *let us put*

$$I_f = \{ V \in (E(X)^*_\tau)^*_\beta : |V| \le |\pi_f| \}.$$

Theorem 2.3 (see [**22**, Theorem 4.1]). *Let* τ *be a Hausdorff locally convex-solid topology on* $E(X)$. *Then for* $f \in E(X)$, *the set* I_f *is* $\sigma((E(X)^*_\tau)_{E(X)}, E(X)^*_\tau)$*-compact in* $(E(X)^*_\tau)_{E(X)}$.

For each $u \in E^+$ the set $D_u = \{f \in E(X) : \tilde{f} \le u\}$ will be called an *order interval* in $E(X)$.

Now we are in a position to state the main result of this section.

Theorem 2.4. *Let* (E, ξ) *be a Hausdorff locally convex-solid function space, and let* X *be a Banach space. Then the following statements are equivalent*:

(i) *Each order interval in* E *is* $\sigma(E, E^*_\xi)$*-compact and* X *is reflexive.*

(ii) ξ *is a Lebesgue topology and* X *is reflexive.*

(iii) *Each order interval in* $E(X)$ *is* $\sigma(E(X), E(X)^*_\xi)$*-compact.*

(iv) $\pi(E(X))$ *is an ideal of* $(E(X)^*_\xi)^*_\beta$, *i.e.*, $\pi(E(X)) = (E(X)^*_\xi)_{E(X)}$.

(v) $(E(X)^*_\xi, |\sigma|(E(X)^*_\xi, E(X)))^* = \pi(E(X))$.

Proof. (i)⇔(ii). See Theorem 2.1.

(ii)⇒(iii). Assume that ξ is a Lebesgue topology and X is reflexive. Then $E(X)^*_\xi \subset E(X)^\sim_n$, and by [**7**, Section 4, Corollary 1] each order interval in $E(X)$ is $\sigma(E(X), E(X)^*_\xi)$-compact.

(iii)⇒(ii). Assume that each order interval in $E(X)$ is $\sigma(E(X), E(X)^*_\xi)$-compact. First we show that ξ is a Lebesgue topology, that is, $E^*_\xi \subset E^\sim_n$, see [bf1, Theorem 9.1]. Indeed, assume on the contrary that there exists $\varphi_0 \in E^*_\xi$ such that $\varphi \notin E^\sim_n$. Hence there exist $\varepsilon_0 > 0$ and a net (u_α) in E such that $u_\alpha \downarrow 0$ in E and $|\varphi_0(u_\alpha)| \ge \varepsilon_0$ for all α. We can assume that $u_\alpha \le u$ for some $u \in E^+$ and all α. Let $f_\alpha = u_\alpha \otimes x_0$ for each α and a fixed $x_0 \in S_X$. Then $f_\alpha \in D_u$ for all α, so one can choose a subnet (f_β) of (f_α) and $f_0 \in D_u$ such that $f_\beta \to f_0$

for $\sigma(E(X), E(X)^*_{\bar{\xi}})$. Choose $x_0^* \in S_{X^*}$ such that $x_0^*(x_0) = 1$, and for each $\varphi \in E^*_\xi$, let us put

$$F_\varphi(f) = \varphi(x_0^* \circ f) \quad \text{for all} \quad f \in E(X).$$

We shall show that $F_\varphi \in E(X)^*_{\bar{\xi}}$. Indeed, let $\{p_t : t \in T\}$ be a family of Riesz semi-norms that generates ξ. Since $\varphi \in E^*_\xi$, there exist $a > 0$ and $t_i \in T$, $i = 1, \dots, n$, such that for each $f \in E(X)$ we have

$$\begin{aligned} |F_\varphi(f)| = |\varphi(x_0^* \circ f)| &\le 1 \max_{a \le i \le n} p_{t_i}(x_0^* \circ f) \le a \max_{1 \le i \le n} p_{t_i}(f) \\ &= a \max_{a \le i \le n} \bar{p}_{t_i}(f), \end{aligned}$$

that is, $F_\varphi \in E(X)^*_{\bar{\xi}}$. Hence

$$\varphi(u_\beta) = F_\varphi(u_\beta \otimes x_0) \longrightarrow F_\varphi(f_0) = \varphi(x_0^* \circ f_0).$$

This means that $u_\beta \to x_0^* \circ f_0 \in E$ for $\sigma(E, E^*_\xi)$, as desired.

On the other hand, since $u_\beta \downarrow 0$ in E, we conclude that $u_\beta \to 0$ for $\sigma(E, E^*_\xi)$, see [**18**, Corollary 10.2.2]. But $\varphi_0 \in E^*_\xi = (E, \sigma(E, E^*_\xi))^*$, so $\varphi_0(u_\beta) \to 0$, which is in contradiction with $|\varphi_0(u_\beta)| \ge \varepsilon_0 > 0$. This contradiction establishes that ξ is a Lebesgue topology.

Now we shall show that X is reflexive, i.e., the unit ball B_X is weakly compact. Indeed, let (x_α) be a net in B_X. Given a fixed $u \in E^+$ let us put $h_\alpha = u \otimes x_\alpha$ for each α. Then $h_\alpha \in D_u$ for each α, so there exist a subnet (h_β) of (h_α) and $h_0 \in D_u$ such that $u \otimes x_\beta = h_\beta \to h_0$ for $\sigma(E(X), E(X)^*_{\bar{\xi}})$. Let M_ξ be an ideal of E' determined by ξ, see Proposition 1.2. Choose $v_0 \in M_\xi^+$ such that $\int_\Omega u(\omega) v_0(\omega)\, \mathrm{d}\,\mu = 1$. Hence $v_0 \otimes x^* \in M_\xi(X^*) \subset M_\xi(X^*, X)$ for each $x^* \in X^*$, so

$$\begin{aligned} x^*(x_\beta) &= \int_\Omega u(\omega) v_0(\omega) x^*(x_\beta)\, \mathrm{d}\,\mu \\ &= F_{v_0 \otimes x^*}(u \otimes x_\beta) \to F_{v_0 \otimes x^*}(h_0) \\ &= \int_\Omega \langle h_0(\omega), v_0(\omega) x^* \rangle\, \mathrm{d}\,\mu \\ &= \int_\Omega x^*(v_0(\omega) h_0(\omega))\, \mathrm{d}\,\mu \\ &= x^*\Big(\int_\Omega v_0(\omega) h_0(\omega)\, \mathrm{d}\,\mu \Big). \end{aligned}$$

Hence $x_\beta \to x_0 = \int_\Omega v_0(\omega) h_0(\omega)\,\mathrm{d}\,\mu \in B_X$ for $\sigma(X, X^*)$.

(iv)$\Rightarrow$(ii). Assume that $\pi(E(X)) = (E(X)^*_\xi)_{E(X)}$, and let $u_0 \in E^+$. Let $f_0 = u_0 \otimes x_0$ for a fixed $x_0 \in S_X$. Then in view of [**22**, Theorem 1.3] for $f \in E(X)$, we have that $|\pi_f| \le |\pi_{f_0}|$ if and only if $\widetilde{f} \le \widetilde{f}_0 = u_0$. Hence

$$I_{f_0} = \{\pi_f : f \in E(X),\ |\pi_f| \le |\pi_{f_0}|\} = \{\pi_f : f \in E(X),\ \widetilde{f} \le u_0\}.$$

In view of Theorem 2.3, I_{f_0} is a $\sigma(\pi(E(X)), E(X)^*_\xi)$-compact subset of $\pi(E(X))$. Since $F(f) = \pi_f(F)$ for $f \in E(X)$ and $F \in E(X)^*_\xi$, the mapping

$$\pi^{-1} : (\pi(E(X)), \sigma(\pi(E(X)), E(X)^*_\xi)\,) \longrightarrow (E(X), \sigma(E(X), E(X)^*_\xi))$$

is continuous. Hence, the set $\pi^{-1}(I_{f_0})$ $(= D_{u_0})$ is $\sigma(E(X), E(X)^*_\xi)$-compact.

(ii)$\Rightarrow$(iv). Assume that (ii) holds. We shall show that $(E(X)^*_\xi)_{E(X)} \subset \pi(E(X))$. Indeed, let $V \in (E(X)^*_\xi)_{E(X)}$, i.e., $V \in (E(X)^*_\xi)^*_\beta$ and $|V| \le |\pi_{f_0}|$ for some $f_0 \in E(X)$. Let M_ξ be an ideal of E' determined by ξ, see Proposition 1.2. In view of [**22**, Theorem 1.1], the mapping $\psi : M_\xi(X^*) \ni g \mapsto F_g \in E(X)^*_\xi$ is an order continuous bijection, i.e., for a net (g_α) in $M_\xi(X^*)$, $\widetilde{g}_\alpha \xrightarrow{(0)} 0$ in M_ξ implies $F_{g_\alpha} \xrightarrow{(0)} 0$ in $E(X)^*_\xi$, see [**22**, Definition 1.2]. Hence one can easily show that $V \circ \psi \in M_\xi(X^*)^\sim_n$. Since X^* is reflexive, there is $h_0 \in M'_\xi(X^{**})$, $(= M'_\xi(X^{**}, X^*))$ such that

$$V(F_g) = V(\psi(g)) = \int_\Omega \langle g(\omega), h_0(\omega)\rangle\,\mathrm{d}\,\mu \quad \text{for all} \quad g \in M_\xi(X^*).$$

Let $j : X \to X^{**}$ stand for the canonical isometry. Define $k_0(\omega) = j^{-1}(h_0(\omega))$ for $\omega \in \Omega$. One can easily show that the function $k_0 : \Omega \to X$ is strongly Σ-measurable and $\|k_0(\omega)\|_X = \|h_0(\omega)\|_{X^{**}}$ for all $\omega \in \Omega$, i.e., $k_0 \in M'_\xi(X)$. We have

$$M'_\xi(X)^\sim_n = \{\bar{F}_g : g \in M''_\xi(X^*)\},$$

where

$$\overline{F}_g(k) = \int_\Omega \langle k(\omega), g(\omega)\rangle\,\mathrm{d}\,\mu \quad \text{for} \quad k \in M'_\xi(X).$$

Hence, for each $g \in M''_\xi(X^*)$, we get

$$(+)\qquad \begin{aligned}\pi_{k_0}(\bar{F}_g) = \overline{F}_g(k_0) &= \int_\Omega \langle k_0(\omega), g(\omega)\rangle \,\mathrm{d}\,\mu \\ &= \int_\Omega \langle j^{-1}(h_0(\omega)), g(\omega)\rangle \,\mathrm{d}\,\mu \\ &= \int_\Omega \langle g(\omega), h_0(\omega)\rangle \,\mathrm{d}\,\mu = V(F_g).\end{aligned}$$

We shall now show that $k_0 \in E(X)$, i.e., $\tilde{k}_0 \in E$. Indeed, let $g_0 \in M''_\xi(X^*)$. Then by [**21**, Corollary 2.5] for $g \in M''_\xi(X^*)$, $|F_g| \le |F_{g_0}|$ if and only if $\tilde{g} \le \tilde{g}_0$. Hence by making use of (+) and [**5**, Theorem 4.1] we get

$$\begin{aligned}|V|(F_{g_0}) &= \sup\{|V(F_g)| : g \in M''_\xi(X^*),\ \tilde{g} \le \tilde{g}_0\} \\ &= \sup\{|\pi_{k_0}(\bar{F}_g)| : g \in M''_\xi(X^*),\ \tilde{g} \le \tilde{g}_0\} \\ &= \sup\{|\bar{F}_g(k_0)| : g \in M''_\xi(X^*),\ \tilde{g} \le \tilde{g}_0\} \\ &= \sup\left\{ \left| \int_\Omega \langle k_0(\omega), g(\omega)\rangle \,\mathrm{d}\,\mu \right| : g \in M''_\xi(X^*),\ \tilde{g} \le \tilde{g}_0 \right\} \\ &= \int_\Omega \tilde{g}_0(\omega)\tilde{k}_0(\omega)\,\mathrm{d}\,\mu.\end{aligned}$$

Since $|V| \le |\pi_{f_0}|$ and for each $g \in M_\xi(X^*) \subset M''_\xi(X^*)$, $|\pi_{f_0}|(F_g) = |F_g|(f_0)$, we get

$$\int_\Omega \tilde{g}(\omega)\tilde{k}_0(\omega)\,\mathrm{d}\,\mu = |V|(F_g) \le |\pi_{f_0}|(F_g) = |F_g|(f_0) = \int_\Omega \tilde{f}_0(\omega)\tilde{g}(\omega)\,\mathrm{d}\,\mu.$$

It follows that $\tilde{k}_0 \le \tilde{f}_0$, where $\tilde{k}_0 \in M'_\xi$ and $\tilde{f}_0 \in E \subset E'' \subset M'_\xi$. Hence $\tilde{k}_0 \in E$, i.e., $k_0 \in E(X)$. Hence, in view of (+) for each $g \in M_\xi(X^*)$, we get

$$V(F_g) = \int_\Omega \langle k_0(\omega), g(\omega)\rangle \,\mathrm{d}\,\mu = F_g(k_0) = \pi_{k_0}(F_g).$$

Thus $V = \pi_{k_0}$, where $k_0 \in E(X)$, i.e., $V \in \pi(E(X))$, as desired.

(iv)⇔(v). It follows from Theorem 2.2. □

Remark. In [**6**, Proposition 2] it is shown that every order interval in $E(X)$ is $\sigma(E(X), E(X)_n^\sim)$-compact if and only if X is reflexive.

3. Almost reflexivity of vector-valued function spaces. First we recall some definitions. Let $\langle L, K\rangle$ be a dual pair. A subset A of L is said to be *conditionally* $\sigma(L,K)$-*compact* whenever each sequence in A contains a $\sigma(L,K)$-Cauchy subsequence. Recall that a normed space X is said to be *almost reflexive* if every norm-bounded subset of X is conditionally weakly compact, see [**11, 17**]. The fundamental ℓ^1-Rosenthal theorem [**29**] says that a Banach space X is almost reflexive if and only if it contains no isomorpic copy of ℓ^1.

Due to Geuiler and Chubarova [**16**], see also [**4**, Proposition 3.2], the Köthe-Bochner space $E(X)$ is almost reflexive if and only if both Banach spaces X and E are almost reflexive. This result is a broad generalization of the corresponding theorems of Pisier [**28**] and Bombal [**3**], who proved it in the special cases of $L^p(X)$, $1 < p < \infty$, and Orlicz-Bochner spaces respectively.

Now we extend the concept of almost reflexivity to the class of locally convex spaces.

Definition 3.1. A Hausdorff locally convex space (L, ξ) is said to be *almost reflexive* whenever every $\sigma(L, L_\xi^*)$-bounded subset of L is conditionally $\sigma(L, L_\xi^*)$-compact.

In this section we characterize almost reflexivity of $(E(X), \bar{\xi})$ whenever (E, ξ) is a Hausdorff locally convex-solid function space with the Lebesgue property and X is a Banach space.

Let M be an ideal of E' with $\operatorname{supp} M = \Omega$. Assume that B is a $\sigma(E, M)$-bounded subset of E. Then B is also $|\sigma|(E, M)$-bounded, see [**1**, Theorem 6.6], so one can define a Riesz semi-norm p_B on M by

$$p_B(v) = \sup\Big\{ \int_\Omega |u(\omega)v(\omega)| \, \mathrm{d}\,\mu : u \in B\Big\}.$$

Let (E, ξ) be a Hausdorff locally convex-solid function space with the Lebesgue property and let M_ξ be an ideal of E' with $\operatorname{supp} M_\xi = \Omega$ such that $E_\xi^* = \{\varphi_v : v \in M_\xi\}$. The following characterization of conditionally $\sigma(E, M_\xi)$-compact sets in E will be needed, see [**26**, Theorem 3.2].

Proposition 3.1. *Let (E, ξ) be a Hausdorff locally convex-solid function space with the Lebesgue property. Then for a subset B of E, the following statements are equivalent*:

(i) *B is conditionally $\sigma(E, M_\xi)$-compact.*

(ii) *B is $\sigma(E, M_\xi)$-bounded and p_B is order continuous on M_ξ.*

The *strong topology* $\beta(M_\xi, E)$ is a locally convex-solid topology on M_ξ that is generated by a family $\{p_B : B \in \mathcal{B}_s\}$, where $\mathcal{B}_s$ is the collection of all $\sigma(E, M_\xi)$-bounded solid subsets of E, see [**1**, Section 9].

As a simple consequence of Proposition 3.1, we get the following:

Proposition 3.2. *Let (E, ξ) be a Hausdorff locally convex-solid function space with the Lebesgue property. Then the following statements are equivalent*:

(i) *Every $\sigma(E, M_\xi)$-bounded set in E is conditionally $\sigma(E, M_\xi)$-compact.*

(ii) *$\beta(M_\xi, E)$ is a Lebesgue topology.*

Proof. (i)$\Rightarrow$(ii). It follows from Proposition 3.1.

(ii)$\Rightarrow$(i). Assume that $\beta(M_\xi, E)$ is a Lebesgue topology and, let B be a $\sigma(E, M_\xi)$-bounded subset of E. Then its solid hull $S(B)$ in E is also $\sigma(E, M_\xi)$-bounded and the semi-norm $p_{S(B)}$ on M_ξ is order continuous. Hence by Proposition 3.1 B is conditionally $\sigma(E, M_\xi)$-compact. □

As an application of Proposition 3.2 we have the following characterization of almost reflexivity of (E, ξ).

Corollary 3.3. *Let (E, ξ) be a Hausdorff locally convex-solid function space with the Lebesgue property. Then the following statements are equivalent*:

(i) *(E, ξ) is almost reflexive.*

(ii) *$\beta(E^*_\xi, E)$ is a Lebesgue topology on E^*_ξ.*

Now we are ready to characterize almost reflexivity of $(E(X), \bar{\xi})$.

Theorem 3.4. *Let (E, ξ) be a Hausdorff locally convex-solid function space with the Lebesgue property, and let X be a Banach space. Then the following statements are equivalent*:

(i) *X is almost reflexive and (E, ξ) is almost reflexive.*

(ii) *$(E(X), \bar{\xi})$ is almost reflexive.*

Proof. (i)⇒(ii). Assume that (i) holds, and let H be a $\sigma(E(X), M_\xi(X^*, X))$-bounded subset of $E(X)$. Then by [**25**, Proposition 1.3], the set $\{\tilde{f} : f \in H\}$ is $\sigma(E, M_\xi)$-bounded, so it is conditionally $\sigma(E, M_\xi)$-compact. Hence by [**25**, Proposition 2.3], H is conditionally $\sigma(E(X), M_\xi(X^*, X))$-compact, so $(E(X), \bar{\xi})$ is almost reflexive.

(ii)⇒(i). Assume that $(E(X), \bar{\xi})$ is almost reflexive. To show that B_X is conditionally weakly compact, let (x_n) be a sequence in B_X. Given $u \in E^+$ let $h_n = u \otimes x_n$ for $n \in \mathbf{N}$. We shall show that the set $\{h_n : n \in \mathbf{N}\}$ is $\sigma(E(X), M_\xi(X^*, X))$-bounded. Indeed, for $g \in M_\xi(X^*, X)$, we have

$$\sup_n \Big| \int_\Omega \langle u(\omega) x_n, g(\omega) \rangle \, \mathrm{d}\, \mu \Big| \le \int_\Omega u(\omega) \vartheta(g)(\omega) \, \mathrm{d}\, \mu < \infty.$$

Hence the set $\{h_n : n \in \mathbf{N}\}$ is conditionally $\sigma(E(X),\ M_\xi(X^*, X))$-compact, so there exists a $\sigma(E(X), M_\xi(X^*, X))$-Cauchy subsequence (h_{k_n}) of (h_n). Arguing as in the proof of implication (iii)⇒(ii) in Theorem 2.4, we see that (x_{k_n}) is weakly Cauchy.

Now assume that Z is a $\sigma(E, M_\xi)$-bounded subset of E. Then Z is also $|\sigma|(E, M_\xi)$-bounded, so for each $g \in M_\xi(X^*, X)$ and a fixed $x_0 \in S_X$, we get

$$\operatorname{supp}_{u \in Z} \Big| \int_\Omega \langle u(\omega) x_0, g(\omega) \rangle \, \mathrm{d}\, \mu \Big| \le \sup_{u \in Z} \int_\Omega |u(\omega)| \vartheta(g)(\omega) \, \mathrm{d}\, \mu < \infty.$$

Hence the set $\{u \otimes x_0 : u \in Z\}$ is $\sigma(E(X), M_\xi(X^*, X))$-bounded, so it is conditionally $\sigma(E(X), M_\xi(X^*, X))$-compact. By [**25**, Theorem 2.2], the set $\{|u| : u \in Z\}$ is conditionally $\sigma(E, M_\xi)$-compact, so Z is conditionally $\sigma(E, M_\xi)$-compact, see [**8**, Theorem 3.4, Proposition 2.2]. This means that (E, ξ) is almost reflexive. ◻

4. Weak sequential completeness of vector-valued function spaces. In his celebrated paper Talagrand [**30**] showed that a Köthe-Bochner space $E(X)$ is weakly sequentially complete if and only if both Banach spaces E and X are weakly sequentially complete. In particular, it is well known that a Banach function space $(E, \|\cdot\|_E)$ is weakly sequentially complete if and only if E is KB-space, i.e., $\|\cdot\|_E$ has both the σ-Lebesgue property and the σ-Levy property, see [**20**, Theorem 1.c.4], [**18**, Theorem 10.4.9]. In this section we study weak sequential completeness of $(E(X), \bar{\xi})$ whenever (E, ξ) is a Hausdorff locally convex-solid function space and X is a Banach space.

We start by recalling a characterization of weak sequential completeness of locally convex-solid function spaces. For this purpose, we now establish notation and some results as set out in [**33**, Exercise 102.24, pp. 331–332].

Let (E, ξ) be a Hausdorff locally convex-solid function space. Then putting

$$E^*_{\xi,n} := E^*_\xi \cap E^\sim_n \quad \text{and} \quad E^*_{\xi,s} := E^*_\xi \cap E^\sim_s$$

we get $E^*_\xi = E^*_{\xi,n} \oplus E^*_{\xi,s}$. We have

$$\begin{aligned} E^a(\xi) &:= \big\{u \in E : |u| \geq u_n \downarrow 0 \text{ in } E \text{ implies } \varphi(u_n) \to 0 \text{ for all } \varphi \in E^*_\xi\big\} \\ &= \big\{u \in E : |u| \geq u_n \downarrow 0 \text{ in } E \text{ implies } u_n \to 0 \text{ for all } \xi\big\}. \end{aligned}$$

Observe that ξ is a σ-Lebesgue topology (= Lebesgue topology) if and only if $E^a(\xi) = E$. Moreover, if $\operatorname{supp} E^a(\xi) = \Omega$, i.e., $E^a(\xi)$ is order dense in L^0, then we have

$$E^*_{\xi,s} = E^a(\xi)^\perp = \{\varphi \in E^*_\xi : \varphi(u) = 0 \text{ for all } u \in E^a(\xi)\}$$

and $E^*_{\xi,n}$ separates points of E.

Now we are in position to state our desired result, see [**26**, Theorem 2.2].

Theorem 4.1. *Let (E, ξ) be a Hausdorff locally convex-solid function space such that* $\operatorname{supp} E^a(\xi) = \Omega$. *Then the following statements are equivalent*:

(i) *E is $\sigma(E, E^*_\xi)$-sequentially complete.*

(ii) ξ *has both the* σ*-Lebesgue property and the* σ*-Levy property.*

(iii) ξ *has both the Lebesgue property and the Levy property.*

As a simple consequence of Theorem 4.1, we have the following well-known result, see [**8**, Corollary 4.2], [**1**, Theorem 20.26]).

Theorem 4.2. *Let* (E, ξ) *be a Hausdorff locally convex-solid function space with the Lebesgue property. Then the following statements are equivalent*:

(i) E *is* $\sigma(E, E_\xi^*)$*-sequentially complete.*

(ii) ξ *is a* σ*-Levy topology.*

Now we pass on to the vector-valued setting. Recall that a functional $F \in E(X)^\sim$ is said to be *singular* if there is an ideal B of E with $\operatorname{supp} B = \Omega$ and such that $F(f) = 0$ for all $f \in E(X)$ with $\tilde{f} \in B$. The set consisting of all singular functionals on $E(X)$ will be denoted by $E(X)_s^\sim$ and called the *singular dual* of $E(X)$, see [**7, 21**].

Due to Bukhvalov and Lozanowski, see [**7**, Section 3, Theorem 2], the following Yosida-Hewitt type decomposition holds:

$$E(X)^\sim = E(X)_n^\sim \oplus E(X)_s^\sim \tag{4.1}$$

and moreover, if $F = F_g + F_s$, where $g \in E'(X^*, X)$ and $F_s \in E(X)_s^\sim$, then $\varphi_F = \varphi_{F_g} + \varphi_{F_s}$, where $\varphi_{F_g}(u) = \int_\Omega u(\omega)\vartheta(g)(\omega)\,\mathrm{d}\,\mu$ for $u \in E$ and $\varphi_{F_s} \in E_s^\sim$.

Let us put

$$E(X)_{\xi,n}^* := E(X)_\xi^* \cap E(X)_n^\sim$$

and

$$E(X)_{\xi,s}^* := E(X)_\xi^* \cap E(X)_s^\sim.$$

Then

$$E(X)_\xi^* = E(X)_{\xi,n}^* \oplus E(X)_{\xi,s}^* \tag{4.2}$$

and there is an ideal M_ξ of E' with $\operatorname{supp} M_\xi = \Omega$ such that

$$E^*_{\xi,n} = \{\varphi_v : v \in M_\xi\}.$$

We shall now show that

$$E(X)^*_{\tilde{\xi},n} = \{F_g : g \in M_\xi(X^*, X)\}. \tag{4.3}$$

Indeed, let $F \in E(X)^*_{\tilde{\xi},n}$, i.e., $F = F_g$ for some $g \in E'(X^*, X)$ and $F \in E(X)^*_{\tilde{\xi}}$. Hence $\varphi_F = \varphi_{F_g} = \varphi_{\vartheta(g)}$, where $\vartheta(g) \in E'$ and $\varphi_F \in E^*_\xi$, see Proposition 1.1. Hence $\varphi_F \in E^*_{\xi,n}$, so $\vartheta(g) \in M_\xi$. Thus $g \in M_\xi(X^*, X)$.

Now, let $F = F_g$, where $g \in M_\xi(X^*, X)$. Then $F \in E(X)^\sim_n$ and $\varphi_F = \varphi_{F_g} = \varphi_{\vartheta(g)}$, where $\vartheta(g) \in M_\xi$. Hence $\varphi_F \in E^*_\xi$, so $F \in E(X)^*_{\tilde{\xi}}$, see Proposition 1.1. Thus $F \in E(X)^*_{\tilde{\xi},n}$.

The following "topological versions" of [**25**, Theorem 2.2 and Theorem 3.3] will be of importance in the proof of Theorem 4.5.

Theorem 4.3. *Let (E, ξ) be a Hausdorff locally convex-solid function space with the Lebesgue property, and let X be a Banach space. Then for a subset H of $E(X)$ the following statements are equivalent*:

(i) *H is conditionally $\sigma(E(X), E(X)^*_{\tilde{\xi}})$-compact.*

(ii) (a) *The set $\{\tilde{f} : f \in H\}$ is conditionally $\sigma(E, E^*_\xi)$-compact.* (b) *For each subset $A \in \Sigma_f$ with $\chi_A \in M_\xi$ and each sequence (f_n) in H there exists a sequence (h^A_n) with $h^A_n \in \operatorname{conv}\{\chi_A f_k : k \geq n\}$ such that $(h^A_n(\omega))$ is weakly Cauchy in X for almost every $\omega \in A$.*

Theorem 4.4. *Let (E, ξ) be a Hausdorff locally convex-solid function space with the Lebesgue property, and the σ-Levy property, and let X be a Banach space. Then for a subset H of $E(X)$, the following statements are equivalent*:

(i) *H is relatively $\sigma(E(X), E(X)^*_{\tilde{\xi}})$-sequentially compact.*

(ii) (a) *The set $\{\tilde{f} : f \in H\}$ is relatively $\sigma(E, E^*_\xi)$-sequentially compact.* (b) *For each $A \in \Sigma_f$ with $\chi_A \in M_\xi$ and each sequence*

(f_n) in H, there is a sequence (h_n^A) with $h_n^A \in \operatorname{conv}\{f_k : k \geq n\}$ such that $(h_n^A(\omega))$ is weakly convergent in X for almost every $\omega \in A$.

Now we are in position to state our main result.

Theorem 4.5. *Let (E, ξ) be a Hausdorff locally convex-solid function space with $\operatorname{supp} E^a(\xi) = \Omega$, and let X be a Banach space. Then the following statements are equivalent*:

(i) *$E(X)$ is $\sigma(E(X), E(X)_\xi^*)$-sequentially complete.*

(ii) *E is $\sigma(E, E_\xi^*)$-sequentially complete and X is weakly sequentially complete.*

Proof. (i)$\Rightarrow$(ii). Assume that $E(X)$ is $\sigma(E(X), E(X)_\xi^*)$-sequentially complete.

First, we show that E is $\sigma(E, E_\xi^*)$-sequentially complete. Indeed, let (u_n) be a $\sigma(E, E_\xi^*)$-Cauchy sequence in E. For a fixed $x_0 \in S_X$ let $h_n = u_n \otimes x_0$ for $n \in \mathbf{N}$. We shall show that (h_n) is a $\sigma(E(X), E(X)_\xi^*)$-Cauchy sequence in $E(X)$. Indeed let $F \in E(X)_\xi^*$. Then $F = F_g + F_s$, where $g \in M_\xi(X^*, X)$ and $F_s \in E(X)_{\xi,s}^*$, see (4.2) and (4.3). We have $\varphi_{F_g} = \varphi_{\vartheta(g)}$, where $|g_{x_0}| \leq \vartheta(g) \in M_\xi$. Hence $g_{x_0} \in M_\xi$, so

$$F_g(h_n) = \int_\Omega u_n(\omega) g(\omega)(x_0)\, \mathrm{d}\,\mu = \int_\Omega u_n(\omega) g_{x_0}(\omega)\, \mathrm{d}\,\mu \longrightarrow a_g \in \mathbf{R}.$$

Now let us set

$$\varphi_s(u) := F_s(u \otimes x_0) \quad \text{for} \quad u \in E.$$

In view of (4.1) and Proposition 1.1, we have that $\varphi_{F_s} \in E_{\xi,s}^*$. Moreover, for $u \in E^+$, we have

$$\begin{aligned} |\varphi_s|(u) &= \sup\{|F_s(w \otimes x_0)| : w \in E,\ |w| \leq u\} \\ &\leq \sup\{|F_s(h)| : h \in E(X),\ \tilde{h} \leq u\} \\ &= |F_s|(u \otimes x_0) = \varphi_{F_s}(u). \end{aligned}$$

It follows that $\varphi_s \in E_{\xi,s}^*$, because $\varphi_{F_s} \in E_{\xi,s}^*$ and $E_{\xi,s}^*$ is an ideal of $E^\sim$. Hence

$$F_s(h_n) = F_s(u_n \otimes x_0) = \varphi_s(u_n) \longrightarrow a_s \in \mathbf{R}.$$

Thus

$$F(h_n) = F_g(h_n) + F_s(h_n) \longrightarrow a_g + a_s \in \mathbf{R},$$

and this means that (h_n) is a $\sigma(E(X), E(X)^*_{\bar{\xi}})$-Cauchy sequence. Hence there exists $h_0 \in E(X)$ such that $h_n \to h_0$ for $\sigma(E(X), E(X)^*_{\bar{\xi}})$. Choose $x_0^* \in S_{X^*}$ such that $x_0^*(x_0) = 1$. Then $u_0 = x_0^* \circ h_0 \in E$.

We shall show that $u_n \longrightarrow u_0$ for $\sigma(E, E^*_\xi)$. Indeed, let $\varphi \in E^*_\xi$, i.e., $\varphi = \varphi_v + \varphi_s$, where $v \in M_\xi$ and $\varphi_s \in E^*_{\xi,s}$. Then $g = v \otimes x_0^* \in M_\xi(X^*, X)$, so $F_g \in E(X)^*_{\bar{\xi},n}$, see (4.3). Hence

$$\varphi_v(u_n) = F_g(h_n) \longrightarrow F_g(h_0) = \int_\Omega (x_0^* \circ h_0)(\omega) v(\omega) \, \mathrm{d}\,\mu = \varphi_v(u_0).$$

Let us put

$$G_s(f) := \varphi_s(x_0^* \circ f) \quad \text{for} \quad f \in E(X).$$

We shall first show that $G_s \in E(X)^*_{\bar{\xi},s}$. Indeed, for $u \in E^+$ we have

$$\begin{aligned} \sup\{|G_s(f)| : f \in E(X),\ \tilde{f} \le u\} &\le \sup\{|\varphi_s(w)| : w \in E,\ |w| \le u\} \\ &= |\varphi_s|(u). \end{aligned}$$

Hence $G_s \in E(X)^\sim$. Since $\varphi_s \in E^\sim_s$, there exists an ideal B of E with $\operatorname{supp} B = \Omega$ and such that $\varphi_s(u) = 0$ for all $u \in B$. Assume now that $f \in E(X)$ with $\tilde{f} \in B$. Then $|x_0^* \circ f| \le \tilde{f}$, so $x_0^* \circ f \in B$. Hence $G_s(f) = \varphi_s(x_0^* \circ f) = 0$, so $G_s \in E(X)^\sim_s$. Let ξ be generated by a family $\{p_t : t \in T\}$ of Riesz semi-norms on E. Since $\varphi_s \in E^*_\xi$ there exist $a > 0$ and $t_i \in T$, $i = 1, \ldots, n$, such that for $f \in E(X)$,

$$\begin{aligned} |G_s(f)| = |\varphi_s(x_0^* \circ f)| &\le a \max_{1 \le i \le n} p_{t_i}(x_0^* \circ f) \le a \max_{1 \le i \le n} p_{t_i}(\tilde{f}) \\ &= a \max_{1 \le i \le n} \bar{p}_{t_i}(f). \end{aligned}$$

Hence $G_s \in E(X)^*_{\bar{\xi}}$, so $G_s \in E(X)^*_{\bar{\xi},s}$ and

$$\varphi_s(u_n) = \varphi_s(x_0^* \circ (u_n \otimes x_0)) = G_s(h_n) \longrightarrow G_s(h_0) = \varphi_s(x_0^* \circ h_0) = \varphi_s(u_0).$$

It follows that

$$\varphi(u_n) = \varphi_v(u_n) + \varphi_s(u_n) \longrightarrow \varphi_v(u_0) + \varphi_s(u_0) = \varphi(u_0),$$

i.e., $u_n \longrightarrow u_0$ for $\sigma(E, E_\xi^*)$. Thus E is $\sigma(E, E_\xi^*)$-sequentially complete.

Making use of Theorem 4.1 we obtain that ξ is a Lebesgue topology, so in view of (4.3), $E(X)_{\bar\xi}^* = E(X)_{\bar\xi,n}^* = \{F_g : g \in M_\xi(X^*, X)\}$.

We shall now show that X is weakly sequentially complete. Indeed, let (x_n) be a weakly Cauchy sequence in X. Then $\sup_n \|x_n\|_X = a < \infty$. Given a fixed $u \in E^+$, let us put $h_n = u \otimes x_n$ for $n \in \mathbf{N}$. We shall now show that (h_n) is a $\sigma(E(X), M_\xi(X^*, X))$-Cauchy sequence in $E(X)$. In fact, let $g \in M_\xi(X^*, X)$. Let $x_n' = x_n/a$ for $n \in \mathbf{N}$. Then $|g_{x_n'}| \le \vartheta(g) \in M_\xi \subset E'$ for $n \in \mathbf{N}$, and since $g_{x_n'}(\omega) = g(\omega)(x_n')$, $g_{x_n'}(\omega) \to v(\omega)$ for some $v \in M_\xi$ and all $\omega \in \Omega$. It follows that $(g_{x_n'} - v) \xrightarrow{(0)} 0$ in E'. Since $u \in E \subset E''$, we get $\varphi_u(g_{x_n'} - v) = \int_\Omega u(\omega)(g_{x_n'}(\omega) - v(\omega))\,\mathrm{d}\,\mu \to 0$. Hence

$$\int_\Omega \langle u_n(\omega)x_n, g(\omega)\rangle\,\mathrm{d}\,\mu = a\int_\Omega u(\omega)g_{x_n'}(\omega)\,\mathrm{d}\,\mu \longrightarrow a\int_\Omega u(\omega)v(\omega)\,\mathrm{d}\,\mu \in \mathbf{R}.$$

This means that (h_n) is $\sigma(E(X), M_\xi(X^*, X))$-Cauchy, so there exists $h_0 \in E(X)$ such that $h_n \to h_0$ for $\sigma(E(X), M_\xi(X^*, X))$. Choose $v_0 \in M_\xi^+$ such that $\int_\Omega u(\omega)v_0(\omega)\,\mathrm{d}\,\mu = 1$. Then $v_0 \otimes x^* \in M_\xi(X^*)$ for $x^* \in X$, so

$$\begin{aligned} x^*(x_n) &= \int_\Omega u(\omega)v_0(\omega)x^*(x_n)\,\mathrm{d}\,\mu \\ &= F_{v_0\otimes x^*}(u \otimes x_n) \longrightarrow F_{v_0\otimes x^*}(h_0) \\ &= \int_\Omega \langle h_0(\omega), v_0(\omega)x^*\rangle\,\mathrm{d}\,\mu \\ &= \int_\Omega x^*(v_0(\omega)h_0(\omega))\,\mathrm{d}\,\mu \\ &= x^*\Big(\int_\Omega v_0(\omega)h_0(\omega)\,\mathrm{d}\,\mu\Big). \end{aligned}$$

Hence $x_n \to x_0 = \int_\Omega v_0(\omega)h_0(\omega)\,\mathrm{d}\,\mu \in X$ for $\sigma(X, X^*)$, as desired.

(ii)⇒(i). Assume that (ii) holds. In view of Theorem 4.1 ξ is a Lebesgue topology, so $E(X)_{\bar\xi}^* = E(X)_{\bar\xi,n}^* = \{F_g : g \in M_\xi(X^*, X)\}$, see (4.3). Let (f_n) be a $\sigma(E(X), M_\xi(X^*, X))$-Cauchy sequence in $E(X)$. Then the set $\{f_n : n \in \mathbf{N}\}$ is conditionally $\sigma(E(X), M_\xi(X^*, X))$-compact, so in view of Theorem 4.3 and Theorem 4.4 it is also relatively

$\sigma(E(X), M_\xi(X^*, X))$-sequentially compact. Hence, one can choose a subsequence (f_{k_n}) of (f_n) and $f_0 \in E(X)$ such that $f_{k_n} \to f_0$ for $\sigma(E(X), M_\xi(X^*, X))$. It follows that $f_n \to f_0$ for $\sigma(E(X), M_\xi(X^*, X))$, as desired. □

We know that $E^a(\xi) = E$ whenever ξ has the Lebesgue property, so as a consequence of Theorem 4.5, we get the following vector-valued analogue of Theorem 4.2.

Corollary 4.6. *Let (E, ξ) be a Hausdorff locally convex-solid function space with the Lebesgue property, and let X be a Banach space. Then the following statements are equivalent*:

(i) $E(X)$ *is* $\sigma(E(X), E(X)^*_\xi)$*-sequentially complete.*

(ii) E *is* $\sigma(E, E^*_\xi)$*-sequentially complete and* X *is weakly sequentially complete.*

Now we apply Theorem 4.5 and Corollary 4.6 to two particular cases: $\xi = \tau(E, E^\sim)$ and $\xi = \tau(E, E^\sim_n)$.

Recall that an ideal E is said to be *perfect* whenever $E = E''$. Note that E is $\sigma(E, E^\sim_n)$-sequentially complete if and only if E is perfect, see Theorem 4.1 and [**1**, Theorem 9.4].

It is well known that the Mackey topology $\tau(E, E^\sim)$ is locally solid, see [**2**]. Let

$$E^a := \{u \in E : |u| \geq u_n \downarrow 0 \text{ in } E \text{ implies } \varphi(u_n) \longrightarrow 0 \text{ for all } \varphi \in E^\sim\}.$$

Since the Mackey topology $\tau(E(X), E(X)^\sim)$ is locally solid and $\tau(E(X), E(X)^\sim) = \overline{\tau(E, E^\sim)}$, see [**22**, Theorem 3.7], [**24**, proof of Theorem 3.3], by making use of Theorem 4.1 and Theorem 4.5, we get:

Corollary 4.7. *Let E be an ideal of L^0 with* $\operatorname{supp} E^a = \Omega$, *and let X be a Banach space. Then the following statements are equivalent*:

(i) *$E(X)$ is $\sigma(E(X), E(X)^\sim)$-sequentially complete.*

(ii) *$E^\sim = E^\sim_n$, E is perfect and X is weakly sequentially complete.*

It is well known that the Mackey topology $\tau(E, E^\sim_n)$ is the finest Hausdorff locally convex-solid topology on E with the Lebesgue property, see [**2, 15**]. Since $(E, \tau(E, E^\sim_n))^* = \{\varphi_v : v \in E'\}$, by Proposition 1.2 we get $E(X)^*_{\tau(E,E^\sim_n)} = \{F_g : g \in E'(X^*, X)\} = E(X)^\sim_n$. Hence, in view of Corollary 4.6, we obtain the following:

Corollary 4.8. *Let E be an ideal of L^0, and let X be a Banach space. Then the following statements are equivalent*:

(i) *$E(X)$ is $\sigma(E(X), E(X)^\sim_n)$-sequentially complete.*

(ii) *E is perfect and X is weakly sequentially complete.*

5. Semi-reflexivity of vector-valued function spaces. Recall that a Hausdorff locally convex space (L, ξ) is said to be *semi-reflexive* if the natural embedding of L into its bidual is onto. It is well known that (L, ξ) is semi-reflexive if and only if every $\sigma(L, L^*_\xi)$-bounded subset of L is relatively $\sigma(L, L^*_\xi)$-compact, see [**31**, Chapter 10.2]. In particular, a Banach space X is reflexive (= semi-reflexive) if and only if it is almost reflexive and weakly sequentially complete.

The following characterization of semi-reflexivity of function spaces will be of importance, see [**9**, Proposition 5.4], [**1**, Theorem 22.4], [**32**].

Theorem 5.1. *Let (E, ξ) be a Hausdorff locally convex-solid function space. Then the following statements are equivalent*:

(i) *(E, ξ) is semi-reflexive.*

(ii) *ξ is a Lebesgue, Levy topology and $\beta(E^*_\xi, E)$ is a Lebesgue topology.*

In this section we extend this characterization to the vector-valued setting. In particular, it is known that if E is a Banach function space, (over a finite measure space) with an order continuous norm, then the Köthe-Bochner space $E(X)$ is reflexive if and only if both Banach spaces E and X are reflexive, [**4**, Proposition 3.2].

We will need the following version of the Eberlein-Šmulian theorem for the locally convex space $(E(X), \sigma(E(X), E(X)^*_{\bar{\xi}}))$.

Theorem 5.2 (see [**25**, Theorem 3.2]). *Let (E, ξ) be a Hausdorff locally convex-solid function space with the Lebesgue property, and let X be a Banach space. Assume that the absolute weak topology $|\sigma|(E, E^*_{\xi})$ has the σ-Levy property. Then for a subset H of $E(X)$, the following statements are equivalent*:

(i) *H is relatively $\sigma(E(X), E(X)^*_{\bar{\xi}})$-compact.*

(ii) *H is relatively $\sigma(E(X), E(X)^*_{\bar{\xi}})$-sequentially compact.*

Now we are in position to state our main result.

Theorem 5.3. *Let (E, ξ) be a Hausdorff locally convex-solid function space, and let X be a Banach space. Then the following statements are equivalent*:

(i) *$(E(X), \bar{\xi})$ is semi-reflexive.*

(ii) *X is reflexive and (E, ξ) is semi-reflexive.*

Proof. (i)⇒(ii). Assume that $(E(X), \bar{\xi})$ is semi-reflexive, i.e., $\pi(E(X)) = (E(X)^*_{\bar{\xi}})^*_{\beta}$. Then, in view of Theorem 2.4, X is reflexive and ξ is a Lebesgue topology. Let M_ξ be an ideal of E' determined by ξ, see Proposition 1.2.

To show that (E, ξ) is semi-reflexive, let Z be a $\sigma(E, M_\xi)$-bounded subset of E. It is enough to show that Z is relatively $\sigma(E, M_\xi)$-compact. Indeed, let (u_α) be a net in Z and $x_0 \in S_X$. Then $\{u \otimes x_0 : u \in Z\}$ is a $\sigma(E(X), M_\xi(X^*, X))$-bounded subset of $E(X)$, see the proof of (ii)⇒(i) in Theorem 3.4, so it is also relatively $\sigma(E(X), M_\xi(X^*, X))$-compact. Hence there exist a subnet (u_β) of (u_α) and $h_0 \in E(X)$ such that $u_\beta \otimes x_0 \to h_0$ for $\sigma(E(X), M_\xi(X^*, X))$. Choose $x_0^* \in S_{X^*}$ such that $x_0^*(x_0) = 1$. Then $v \otimes x_0^* \in M_\xi(X^*, X)$ for every $v \in M_\xi$, so

$$\begin{aligned}\varphi_v(u_\beta) &= F_{v \otimes x_0^*}(u_\beta \otimes x_0) \longrightarrow F_{v \otimes x_0^*}(h_0) \\ &= \int_\Omega \langle h_0(\omega), v(\omega) x_0^* \rangle \, \mathrm{d}\,\mu\end{aligned}$$

$$= \int_\Omega (x_0^* \circ h_0)(\omega) v(\omega) \, \mathrm{d}\, \mu$$
$$= \varphi_v(x_0^* \circ h_0),$$

i.e., $u_\beta \to x_0^* \circ h_0 \in E$ for $\sigma(E, M_\xi)$, as desired.

(ii)$\Rightarrow$(i). Assume that X is reflexive and (E,ξ) is semi-reflexive, i.e., ξ is a Lebesgue, Levy topology and $\beta(E_\xi^*, E)$ has the Lebesgue property, see Theorem 5.1. By making use of Theorem 3.4, we obtain that the space $(E(X),\bar{\xi})$ is almost reflexive. Moreover, by Theorem 4.2 and Corollary 4.6, $E(X)$ is $\sigma(E(X), E(X)_{\bar{\xi}}^*)$-sequentially complete. It follows that every $\sigma(E(X), E(X)_{\bar{\xi}}^*)$-bounded subset H of $E(X)$ is relatively $\sigma(E(X), E(X)_{\bar{\xi}}^*)$-sequentially compact. Since $|\sigma|(E, E_\xi^*)$ is a Levy topology, by Theorem 5.2, H is relatively $\sigma(E(X), E(X)_{\bar{\xi}}^*)$-compact. This means that $(E(X),\bar{\xi}\,)$ is semi-reflexive. □

6. Relative weak compactness of solid hulls in vector-valued function spaces. Bukhvalov [**6**, Proposition 5] has showed that if a Banach function space E is a KB-space and X is a reflexive Banach space, then the convex-solid hull of every relatively weakly compact subset of the Köthe-Bochner space $E(X)$ is again relatively weakly compact. In this section we extend this result to the general setting whenever (E,ξ) is a Hausdorff locally convex-solid function space and X is a Banach space.

By $S(H)$ we will denote the solid hull of a set H in $E(X)$, i.e., the smallest solid set in $E(X)$ containing H. Then $S(H) = \{f \in E(X) : \tilde{f} \le \tilde{h}$ for some $h \in H\}$. It is known that the convex hull of a solid subset H of $E(X)$ is again solid, see [**14**, Theorem 1.2].

The following result will be of importance.

Theorem 6.1. *Let (E,ξ) be a Hausdorff locally convex-solid function space with the Lebesgue property, and let X be a Banach space. Then the following statements are equivalent*:

(i) *X is almost reflexive.*

(ii) *The convex solid hull of every conditionally $\sigma(E(X), E(X)_{\bar{\xi}}^*)$-compact subset of $E(X)$ is conditionally $\sigma(E(X), E(X)_{\bar{\xi}}^*)$-compact.*

(iii) *The solid hull of every conditionally* $\sigma(E(X), E(X)^*_\xi)$*-compact subset of* $E(X)$ *is conditionally* $\sigma(E(X), E(X)^*_\xi)$*-compact.*

Proof. (i)⇒(ii). See [**6**, Corollary 1 of Proposition 4].

(ii)⇒(iii). It is obvious.

(iii)⇒(i). Assume that (iii) holds. It is enough to show that the unit ball B_X is conditionally weakly compact. Indeed, let (x_n) be a sequence in B_X, and let $u \in E^+$. Then the order interval D_u $(= S(\{u \otimes x_0\})$ for a fixed $x_0 \in S_X)$ is conditionally $\sigma(E(X), E(X)^*_\xi)$-compact. Then $h_n = u \otimes x_n \in D_u$ for $n \in \mathbf{N}$, so there exists a $\sigma(E(X), E(X)^*_\xi)$-Cauchy subsequence (h_{k_n}) of (h_n). Arguing similarly as in the proof of implication (iii)⇒(ii) in Theorem 2.4, we obtain that (x_{k_n}) is weakly Cauchy, as desired. □

Now we are in position to state our desired result.

Theorem 6.2. *Let* (E, ξ) *be a Hausdorff locally convex-solid function space with the Levy property and* X *a Banach space. Then the following statements are equivalent*:

(i) ξ *is a Lebesgue topology and* X *is reflexive.*

(ii) *The convex solid hull of every relatively* $\sigma(E(X), E(X)^*_\xi)$*-compact subset of* $E(X)$ *is relatively* $\sigma(E(X), E(X)^*_\xi)$*-compact.*

(iii) *The solid hull of every relatively* $\sigma(E(X), E(X)^*_\xi)$*-compact subset of* $E(X)$ *is relatively* $\sigma(E(X), E(X)^*_\xi)$*-compact.*

Proof. (i)⇒(ii). Assume that (i) holds, and let H be a relatively $\sigma(E(X), E(X)^*_\xi)$-compact subset of $E(X)$. In view of Theorem 5.2, H is $\sigma(E(X), E(X)^*_\xi)$-sequentially compact, so it is $\sigma(E(X), E(X)^*_\xi)$-conditionally compact. Hence, by Theorem 6.1, $\operatorname{conv}(S(H))$ is also conditionally $\sigma(E(X), E(X)^*_\xi)$-compact. Since the space $E(X)$ is $\sigma(E(X), E(X)^*_\xi)$-sequentially complete, see Corollary 4.6, $\operatorname{conv}(S(H))$ is relatively $\sigma(E(X), E(X)^*_\xi)$-sequentially compact. Making use of Theorem 5.2, we obtain that $\operatorname{conv}(S(H))$ is relatively $\sigma(E(X), E(X)^*_\xi)$-compact, as desired.

(ii)⇒(iii). It is obvious.

(iii)⇒(i). Assume that (iii) holds. Then for every $u \in E^+$ the order interval D_u $(=S(\{u \otimes x_0\})$ for a fixed $x_0 \in S_X)$ is $\sigma(E(X), E(X)^*_\xi)$-compact. In view of Theorem 2.4 ξ has the Lebesgue property and X is reflexive. □

Now we consider a particular case whenever $\xi = \tau(E, E^\sim_n)$. Since $(E, \tau(E, E^\sim_n))^* = (E, |\sigma|(E, E^\sim_n))^* = E^\sim_n$, see [**1**, Theorem 6.6], E is perfect, i.e., $E = E''$, if and only if $\tau(E, E^\sim_n)$ is a Levy topology, see [**1**, Theorem 9.4]. Moreover, $E(X)^*_{\tau(E,E^\sim_n)} = E(X)^\sim_n$ holds. Thus as an application of Theorem 6.2 we get:

Corollary 6.3. *Let E be a perfect ideal, and let X be a Banach space. Then the following statements are equivalent*:

(i) *X is reflexive.*

(ii) *The convex solid hull of every relatively $\sigma(E(X), E(X)^\sim_n)$-compact subset of $E(X)$ is relatively $\sigma(E(X), E(X)^\sim_n)$-compact.*

(iii) *The solid hull of every relatively $\sigma(E(X), E(X)^\sim_n)$-compact subset of $E(X)$ is relatively $\sigma(E(X), E(X)^\sim_n)$-compact.*

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