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SOME OBSERVATIONS ON GENERALIZED LIPSCHITZ FUNCTIONS

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ABSTRACT. In this paper we present a notion of generalized Lipschitz function, suggested by the Riemann-Stieltjes integral viewed as a generalization of the Riemann integral. We make some remarks on the connection of this notion with the Hellinger integral, we give a McShane's type result for generalized Lipschitz functions and a result on approximation of bounded generalized uniformly continuous function with generalized Lipschitz functions.

The classical notion of Lipschitz function is the following one:

Definition 1. Let (X, d) and (Y, d') be metric spaces. A function $f : X \to Y$ is called Lipschitz if there exists a constant $M \geq 0$ such that

$$d'(f(x), f(y)) \leq M \cdot d(x, y)$$

for all $x, y \in X$. The smallest number $M \geq 0$ satisfying the above relation is called the Lipschitz constant of f and is denoted by $\operatorname{lip} f$.

Remark 1. Intuitively speaking, a Lipschitz function is one that obeys speed limits.

From the point of view of real analysis, the condition of being Lipschitz should be viewed as a weakened version of differentiability, because of the followings result due to Rademacher, see [**6**]:

Theorem 1. *If U is an open set in $\mathbf{R}^n$ and $f : U \to \mathbf{R}^m$ is a Lipschitz function, then f is differentiable outside of a Lebesgue null subset of U.*

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The class of Lipschitz functions has been intensively studied. The paper of Luukkainen and Väisälä, see [**2**], is a very good introduction to the study of Lipschitz topology.

Several generalizations of the notion of Lipschitz function were given.

Mankiewicz, see [**3**], generalized the definition of a Lipschitz function for locally convex spaces, as follows:

Definition 2. A function f from a subset A of a locally convex vector space X into a locally convex space Y is said to be Lipschitz if for every continuous pseudonorm p on Y there exists a continuous pseudonorm q on X and a constant M such that

$$p(f(x) - f(y)) \leq Mq(x - y),$$

for every $x, y \in A$.

He has obtained some extensions of the classical theorem of Rademacher and he has given applications of this result to the problem of the topological classification of Fréchet spaces.

Recently, Romaguera and Sanchis, see [**7**], defined the notion of semi-Lipschitz function, namely:

Definition 3. Let X be set and d a quasi-metric on X. A function $f : X \to \mathbf{R}$ is called semi-Lipschitz if there exists a constant $M \in \mathbf{R}_+$ such that, for every $x, y \in X$, we have

$$f(x) - f(y) \leq Md(x, y).$$

They use this notion to characterize the points of best approximations and semi-Chebyshev sets in quasi-metric spaces.

Jouini [**1**] has considered a generalization of the notion of Lipschitz function, which in particular includes all nondecreasing functions. More precisely,

Definition 4. Given a cone Q with vertex 0 in $\mathbf{R}^n \times \mathbf{R}$, if K is a compact subset of $\mathbf{R}^n$, a function $f : K \to \mathbf{R}$ is said to be Q-Lipschitz if it is lower semi-continuous and

$$(\operatorname{epi}(f) + Q) \cap (K \times \mathbf{R}) \subseteq \operatorname{epi}(f).$$

Remark 2. f is k-Lipschitz, in the usual sense, if and only if it is Q_k-Lipschitz, where $Q_k = \{(x, u) \mid u \geq k \|x\|\}$.

He has obtained, based on this notion, an analog of Ascoli's theorem and has given some applications for certain differential equations.

In this paper we present a notion of generalized Lipschitz function which is suggested by the Riemann-Stieltjes integral viewed as a generalization of the Riemann integral.

To be more specific, we have:

Definition 5. Let (X, d) and (Y, d') be metric spaces. A function $f : X \to Y$ is called Lipschitz with respect to a function $g : X \to X$ if there exists a constant $M \geq 0$ such that

$$d'(f(x), f(y)) \leq M \cdot d(g(x), g(y))$$

for all $x, y \in X$.

The smallest number $M \geq 0$ satisfying the above relation is called the Lipschitz constant of f and is denoted by $\operatorname{lip}_g f$.

Remark 3. Obviously, for $g = \mathrm{Id}_X$, we obtain the classical notion of Lipschitz function.

Remark 4. Let $g : \mathbf{R} \to \mathbf{R}$ be a strictly increasing function and $f : \mathbf{R} \to \mathbf{R}$ a Lipschitz function with respect to g. Then, for each $a, b \in \mathbf{R}$, $a < b$, the Hellinger integral $\int_a^b (df)^2/dg$ exists.

Indeed, if $\{x_0, x_1, \dots, x_n\}$ is a subdivision of $[a,b]$, then

$$\sum_{k=1}^{n} \frac{(f(x_k)-f(x_{k-1}))^2}{g(x_k)-g(x_{k-1})} \le \sum_{k=1}^{n} (\mathrm{lip}_g f)^2 (g(x_k)-g(x_{k-1})) = (\mathrm{lip}_g f)^2 (g(b)-g(a)),$$

so $\int_a^b (df)^2/dg$ exists and, moreover,

$$\int_a^b \frac{(df)^2}{dg} \le (\mathrm{lip}_g f)^2 (g(b)-g(a)).$$

On the other hand, if for $a, b \in \mathbf{R}$, $a < b$, the Hellinger integral $\int_a^b (df)^2/dg$ exists and the function $H : [a,b] \to \mathbf{R}$, given, for each $x \in [a,b]$, by $H(x) = \int_a^b (df)^2/dg$, is Lipschitz with respect to g, then f is Lipschitz with respect to g.

Indeed, for each $x, y \in [a,b]$ with $x < y$, we have

$$\frac{|f(y)-f(x)|^2}{g(y)-g(x)} \le |H(y)-H(x)| \le \mathrm{lip}_g H (g(y)-g(x)),$$

so

$$|f(y)-f(x)| \le \sqrt{\mathrm{lip}_g H}\,(g(y)-g(x)).$$

Our first result is about the extension of generalized Lipschitz functions. It is just a reformulation, in our frame, of the classical McShane's result, see [**5**].

Theorem 2. *Let (X,d) be a metric space, $g : X \to X$ and $f : S \to \mathbf{R}$, where S is a subset of X, a Lipschitz function with respect to g. Then, there exists $F : X \to \mathbf{R}$ a Lipschitz function with respect to g such that*:

a)

$$F_{|S} = f$$

and

b)

$$\mathrm{lip}_g F = \mathrm{lip}_g f.$$

Proof. Let us define, for every $x \in X$,

$$F(x) = \sup\{f(y) - (\mathrm{lip}_g f) d(g(x), g(y)) \mid y \in S\}.$$

For a fixed $x \in S$, we have

$$f(y) - (\mathrm{lip}_g f) d(g(x), g(y)) \leq f(x),$$

for each $y \in S$.

Moreover, since in the above inequality for $x = y$, we have equality, we conclude that

$$F(x) = f(x),$$

for each $x \in S$, i.e., $F_{|S} = f$.

Now we show that $F(x) < \infty$ for all $x \in X$.

Indeed, let us choose an arbitrary $x \in X$ and fix $y_0 \in S$.

Then, for each $y \in S$, we have

$$\begin{aligned} f(y) - (\mathrm{lip}_g f) d(g(x), g(y)) \leq f(y) &- (\mathrm{lip}_g f) d(g(y_0), g(y)) \\ &+ (\mathrm{lip}_g f) d(g(y_0), g(x)). \end{aligned}$$

Hence

$$F(x) \leq f(y_0) + (\mathrm{lip}_g f) d(g(y_0), g(x)) < \infty.$$

For $x, x' \in X$ such that $F(x') \geq F(x)$, we have

$$\begin{aligned} 0 \leq F(x') - F(x) & \\ &= \sup\{f(y) - (\mathrm{lip}_g f) d(g(x'), g(y)) \mid y \in S\} \\ &\quad - \sup\{f(y) - (\mathrm{lip}_g f) d(g(x), g(y)) \mid y \in S\} \\ &\leq \sup\{(\mathrm{lip}_g f)(d(g(x), g(y) - d(g(x'), g(y)))) \mid y \in S\} \\ &\leq (\mathrm{lip}_g f) d(g(x'), g(x)). \end{aligned}$$

Consequently, we have

$$|F(x) - F(x')| \leq (\mathrm{lip}_g f) d(g(x'), g(x)),$$

for all $x, x' \in X$.

In conclusion F is Lipschitz with respect to g and

$$\text{lip}_g F \leq \text{lip}_g f.$$

If $\text{lip}_g F < \text{lip}_g f$, then, as $F_{|S} = f$, we would get a contradiction with the definition of $\text{lip}_g f$.

So

$$\text{lip}_g F = \text{lip}_g f. \qquad \square$$

Our second result is concerning the approximation of bounded uniformly continuous functions by generalized Lipschitz functions.

Definition 6. Let (X, d) and (Y, d') be metric spaces. Given a function $g : X \to X$, the function $f : X \to Y$ is called uniformly continuous with respect to g if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$d(g(x), d(y)) < \delta \quad \text{implies} \quad d'(f(x), f(y)) \leq \varepsilon.$$

Definition 7. Let (X, d) and (Y, d') be metric spaces and $g : X \to X$. The pair (X, d) and (Y, d') is said to have the g Lipschitz extension property if there is a constant C, which depends only on X, Y and g, such that for each subset B of X and each function $f : B \to Y$ which is Lipschitz with respect to g, there exists a function $F : X \to Y$ which is Lipschitz with respect to g, such that

$$F_{|B} = f$$

and

$$\text{lip}_g(F) \leq C \cdot \text{lip}_g(f).$$

Theorem 3. *Let (X, d) and (Y, d') be a pair of metric spaces which has the g Lipschitz extension property, where g is nonconstant. Then, for every bounded uniformly continuous function $f : X \to Y$ with respect to g and every $\varepsilon > 0$, there exists a function $F : X \to Y$ which is Lipschitz with respect to g, such that*

$$\sup_{x \in X} d'(f(x), F(x)) < \varepsilon.$$

Proof. If f is constant, the conclusion is trivial. So we can assume f to be nonconstant.

Let us fix an element a in X. Therefore,

$$\sup_{x\in X} d'(f(x), f(a)) \neq 0.$$

For $\varepsilon > 0$, let us consider ε', such that

$$0 < \varepsilon' < \frac{2\varepsilon}{C+1}.$$

Since f is uniformly continuous with respect to g, there exists $\delta > 0$ such that

$$d(g(x), d(y)) < \delta \quad \text{implies} \quad d'(f(x), f(y)) \leq \frac{\varepsilon'}{2}.$$

Let us consider $\eta > 0$ such that

$$\eta < \min\{\delta, (\varepsilon' \cdot \delta/4 \cdot \sup_{x\in X} d'(f(x), f(a)))\}.$$

Eventually working with a smaller η we can suppose that the set

$$\mathcal{M} = \{A \subseteq X \mid \text{ for each } x, y \in A,\ x \neq y,\ \text{ we have } d(g(x), g(y)) \geq \eta\}$$

is not empty.

Ordering $\mathcal{M}$ by inclusion, we obtain an inductive ordered set, hence, taking into account Zorn's lemma, there exists a maximal element S of $\mathcal{M}$.

If $x, y \in S$ and $d(g(x), g(y)) \geq \delta$, then

$$\begin{aligned} d'(f(x), f(y)) &\leq d'(f(x), f(a)) + d'(f(a), f(y)) \\ &\leq 2 \cdot \sup_{z\in X} d'(f(z), f(a)) = 2 \cdot \frac{\sup_{z\in X} d'(f(z), f(a))}{\delta} \cdot \delta \\ &\leq 2 \cdot \frac{\sup_{z\in X} d'(f(z), f(a))}{\delta} \cdot d(g(x), g(y)). \end{aligned}$$

If $x, y \in S$ and $0 < d(g(x), g(y)) < \delta$, then

$$d'(f(x), f(y)) \leq \frac{\varepsilon'}{2} = \frac{\varepsilon'}{2 \cdot \eta} \cdot \eta \leq \frac{\varepsilon'}{2 \cdot \eta} \cdot d(g(x), g(y)).$$

Therefore,

$$d'(f(x), f(y)) \leq \max\left\{2 \cdot \frac{\sup_{z \in X} d'(f(z), f(a))}{\delta}, \frac{\varepsilon'}{2 \cdot \eta}\right\} \cdot d(g(x), g(y)),$$

for all $x, y \in S$, so $f_{|S} : S \to Y$ is a Lipschitz function with respect to g.

We can consider, according to our hypothesis, $F : X \to Y$ a Lipschitz function with respect to g, such that

$$F_{|S} = f_{|S}$$

and

$$d'(F(x), F(y)) \leq C \max\left\{2 \cdot \frac{\sup_{z \in X} d'(f(z), f(a))}{\delta}, \frac{\varepsilon'}{2 \cdot \eta}\right\} \cdot d(g(x), g(y)),$$

for all $x, y \in X$.

For $x \in X - S$, there exists $x_0 \in S$, such that $d(g(x), g(x_0)) < \eta$ because otherwise $S \cup \{x\} \in \mathcal{M}$, which contradicts the fact that S is a maximal element of $\mathcal{M}$.

For $x \in S$, there exists $x_0 = x \in S$, such that $d(g(x), g(x_0)) = 0 < \eta$.

Hence, for each $x \in X$, there exists $x_0 \in S$, such that $d(g(x), g(x_0)) < \eta < \delta$.

Then, for all $x \in X$, we have

$$\begin{aligned} d'(f(x), F(x)) &\leq d'(f(x), F(x_0)) + d'(F(x_0), F(x)) \\ &= d'(f(x), f(x_0)) + d'(F(x_0), F(x)) \\ &\leq \frac{\varepsilon'}{2} + C \max\left\{2 \cdot \frac{\sup_{x \in X} d'(f(x), f(a))}{\delta}, \frac{\varepsilon'}{2 \cdot \eta}\right\} \cdot \eta \\ &\leq \frac{\varepsilon'(C+1)}{2} < \varepsilon. \end{aligned}$$

Hence,

$$d'(f(x), F(x)) \leq \varepsilon$$

for all $x \in X$, and $F : X \to Y$ is a Lipschitz function with respect to g. $\square$

Remark 5. Our first result provides a pair of metric spaces which has the g Lipschitz extension property, namely (X, d) and $\mathbf{R}$.

Remark 6. Theorem 3 is not valid if the condition on f to be bounded is not satisfied.

Indeed, let (Y, d') be $\mathbf{R}$ with the usual metric.

Let $(e_n)_{n \in \mathbf{N}^*}$ be the canonical basis of l_2, and let $\|.\|$ denote the Euclidian norm.

Let

$$X = \bigcup_{n=1}^{\infty} X_n,$$

where

$$X_n = \{e_n + t(e_{n+1} - e_n) \mid t \in [0, 1]\},$$

endowed with the metric d given by

$$d(x, y) = \|x - y\|,$$

$x, y \in X$.

Let $g = \mathrm{Id}_X$. We consider $f : X \to \mathbf{R}$, defined by

$$f(x) = n + t,$$

if

$$x = e_n + t(e_{n+1} - e_n),$$

$t \in [0, 1]$.

Then f is not bounded and it is uniformly continuous with respect to g on (X, d), see [**4**, p. 796].

Let us suppose that Theorem 3 is valid for f.

Then there exists a function $F : X \to \mathbf{R}$ which is Lipschitz with respect to g, such that

$$|f(x) - F(x)| < \frac{1}{2},$$

for each $x \in X$.

Hence

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - F(x)| + |F(x) - F(y)| + |F(y) - f(y)| \\ &< \frac{1}{2} + \operatorname{lip} F \cdot \|x - y\| + \frac{1}{2} = 1 + \operatorname{lip} F \cdot \|x - y\|, \end{aligned}$$

for all $x, y \in X$.

In particular, for

$$x = e_n + \frac{1}{2}(e_{n+1} - e_n) = \frac{e_n + e_{n+1}}{2}$$

and

$$y = e_m + \frac{1}{2}(e_{m+1} - e_m) = \frac{e_m + e_{m+1}}{2},$$

where $m, n \in \mathbf{N}^*$, we have

$$\left| \left(n + \frac{1}{2} \right) - \left(m + \frac{1}{2} \right) \right| < 1 + \frac{\operatorname{lip} F}{2} \cdot \| e_n + e_{n+1} - (e_m + e_{m+1}) \|,$$

so

$$|n - m| < 1 + \frac{\operatorname{lip} F}{2} \cdot (\|e_n\| + \|e_{n+1}\| + \|e_m\| + \|e_{m+1}\|),$$

for all $m, n \in \mathbf{N}^*$.

Therefore we obtain the following contradiction:

$$|n - m| < 1 + 2 \cdot \operatorname{lip} F,$$

for all $m, n \in \mathbf{N}^*$.

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