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# OPERATOR ALGEBRAS AND MAULDIN-WILLIAMS GRAPHS

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ABSTRACT. We describe a method for associating a $C^*$-correspondence to a Mauldin-Williams graph and show that the Cuntz-Pimsner algebra of this $C^*$-correspondence is isomorphic to the $C^*$-algebra of the underlying graph. In addition, we analyze certain ideals of these $C^*$-algebras.

We also investigate Mauldin-Williams graphs and fractal $C^*$-algebras in the context of the Rieffel metric. This generalizes the work of Pinzari, Watatani and Yonetani. Our main result here is a "no go" theorem showing that such algebras must come from the commutative setting.

**1. Introduction.** In recent years many classes of $C^*$-algebras have been shown to fit into the Pimsner construction of what are known now as Cuntz-Pimsner algebras, see [**20, 22**]. This construction is based on a so-called $C^*$-correspondence over a $C^*$-algebra. For example, a natural $C^*$-correspondence can be associated with a graph $G$, see [**10**], [**11**, Example 1.5]. The Cuntz-Pimsner algebra of this $C^*$-correspondence is isomorphic to the graph $C^*$-algebra $C^*(G)$ as defined in [**16**]. Another example is the $C^*$-correspondence associated with a local homeomorphism on a compact metric space studied by Deaconu in [**6**], and the $C^*$-correspondence associated with a local homeomorphism on a locally compact space studied by Deaconu, Kumjian, and Muhly in [**7**]. They showed that the Cuntz-Pimsner algebra is isomorphic to the groupoid $C^*$-algebra associated with a local homeomorphism in [**5, 7, 26**].

By a (directed) *graph* we mean a system $G = (V, E, r, s)$ where $V$ and $E$ are finite sets, called the sets of *vertices* and *edges*, respectively, of the graph, and where $r$ and $s$ are maps from $E$ to $V$, called the *range* and *source* maps, respectively. Thus, $s(e)$ is the source of an edge $e$ and $r(e)$ is its range. A *Mauldin-Williams graph* is a graph $G$ together with a collection of compact metric spaces, one for each

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vertex of the graph, and a collection of contraction maps, one for each edge of the graph which satisfy certain properties, see Definition 2.1 below. In this note we follow the notations from [**8**]. We associate with such a system a $C^*$-correspondence which reflects the dynamics of the Mauldin-Williams graph, and we analyze the Cuntz-Pimsner algebra of this $C^*$-correspondence. Our construction is related with topological generalizations of graph $C^*$-algebras of Katsura [**14**] and Muhly and Tomforde [**21**]. The study of the Cuntz-Pimsner algebra associated with graph dynamical systems was initiated in [**23**], where the authors consider the case when the graph $G$ consists of a single vertex $v$ and edges $e_1, e_2, \dots, e_n$. In this case, the $\phi_e$s constitute what is known as an *iterated function system* acting on the space $(T_v, \rho_v)$. They conclude that $\mathcal{O}(\mathcal{X})$ is isomorphic to the Cuntz algebra $\mathcal{O}_n$. The first of our two principal theorems in this note generalizes this result. Our proof is different from the proof in [**23**] and reveals extra structure.

**Theorem.** *Let* $(G, \{T_v, \rho_v\}_{v\in V}, \{\phi_e\}_{e\in E})$ *be a Mauldin-Williams graph such that the graph* $G$ *has no sinks and no sources. Let* $A$ *and* $\mathcal{X}$ *be the associated* $C^*$*-algebra and* $C^*$*-correspondence. Then the Cuntz-Pimsner algebra* $\mathcal{O}(\mathcal{X})$ *is isomorphic to* $C^*(G)$ *of* [**4**].

Thus the structure of $\mathcal{O}(\mathcal{X})$ is completely determined by the graph $G$. From one perspective, this result is somewhat disappointing. Given the richness of dynamical systems expressed as Mauldin-Williams graphs and given the fact that Cuntz-Pimsner algebras generalize crossed products, one might expect a lively interplay between the dynamics and the structure of $\mathcal{O}(\mathcal{X})$. However, the "rigidity" that this theorem expresses is quite remarkable, and it may inspire one to wonder about the natural limits of the result.

In particular, one might wonder if there are noncommutative versions of Mauldin-Williams graphs and whether these might prove to have a richer theory. This thought was taken up in [**23**, Section 4.3] where Pinzari, Watatani and Yonetani considered noncommutative iterated function systems based on Rieffel's notion of "noncommutative metric spaces" [**27, 28**]. The second objective of this note is to show that noncommutative iterated function systems of Pinzari, Watatani and Yonetani can be formulated in the setting of Mauldin-Williams-type graphs, but that the generality gained is illusory. Roughly, the Rieffel metric is a metric on the state space of a (not necessarily commutative) $C^*$-algebra $A$ defined by a certain subset of "Lipschitz elements" in $A$ (see Definition 3.1 below). When we associate a $C^*$-algebra $A_v$ to each vertex $v \in V$, for a prescribed graph $G = (V, E, r, s)$, and when we associate a $*$-homomorphism $\phi_e : A_{s(e)} \to A_{r(e)}$ to each edge $e \in E$ that is strictly contractive with respect to the Rieffel metrics on $A_{s(e)}$ and $A_{r(e)}$, we call the resulting system a noncommutative Mauldin-Williams graph. It also gives rise to a Cuntz-Pimsner algebra $\mathcal{O}(\mathcal{X})$. Our second objective in this note is to show that once more $\mathcal{O}(\mathcal{X})$ is isomorphic to the Cuntz-Krieger algebra associated to $G$. In fact, we shall show in Theorem 3.4 that, in such situations, the $C^*$-algebras $A_v$ are necessarily commutative. This implies, in particular, that the structures considered by Pinzari, Watatani and Yonetani are necessarily no more general than those arising from ordinary iterated function systems.

## 2. The Cuntz-Pimsner algebra associated to a Mauldin-Williams graph.

**Definition 2.1.** By a *Mauldin-Williams graph*, see [**8, 18**], we mean a system $(G, \{T_v, \rho_v\}_{v\in V}, \{\phi_e\}_{e\in E})$ where $G = (V, E, r, s)$ is a graph and where $\{T_v, \rho_v\}_{v\in V}$ and $\{\phi_e\}_{e\in E}$ are families such that:

(1) For each $v \in V$, $T_v$ is a compact metric space with a prescribed metric $\rho_v$.

(2) For $e \in E$, $\phi_e$ is a continuous map from $T_{r(e)}$ to $T_{s(e)}$ such that

$$\rho_{s(e)}(\phi_e(x), \phi_e(y)) \le c\rho_{r(e)}(x, y)$$

for some constant $c$ satisfying $0 < c < 1$, independent of $e$, and all $x, y \in T_{r(e)}$.

We shall assume, too, that the functions $s$ and $r$ are surjective. Thus, we assume that there are no sinks and no sources in the graph $G$. An *invariant list* associated with a Mauldin-Williams graph is a family $(K_v)_{v\in V}$ of compact sets, such that $K_v \subset T_v$ for all $v \in V$ and

$$K_v = \bigcup_{\substack{e\in E\\ s(e)=v}} \phi_e(K_{r(e)}).$$

Since each $\phi_e$ is a contraction, a Mauldin-Williams graph

$$(G, \{T_v, \rho_v\}_{v\in V}, \{\phi_e\}_{e\in E})$$

has a unique invariant list, see [**18**, Theorem 1]. We set $T := \cup_{v\in V} T_v$ and $K := \cup_{v\in V} K_v$, and we call $K$ the *invariant set* of the Mauldin-Williams graph.

In the particular case when we have one vertex $v$ and $n$ edges, i.e., in the setting of an *iterated function system*, the invariant set is the unique compact subset $K := K_v$ of $T = T_v$ such that

$$K = \phi_1(K) \cup \cdots \cup \phi_n(K).$$

**Definition 2.2.** Given a Mauldin-Williams graph

$$(G, \{T_v, \rho_v\}_{v\in V}, \{\phi_e\}_{e\in E}),$$

we construct a so called $C^*$-correspondence $\mathcal{X}$ over the $C^*$-algebra $A = C(T)$, where $T = \coprod_{v\in V} T_v$ is the disjoint union of the spaces $T_v$, as follows. Let $E \times_G T = \{(e, x) \colon | \colon x \in T_{r(e)}\}$. Then, by our finiteness assumptions, $E \times_G T$ is a compact space. We set $\mathcal{X} = C(E \times_G T)$ and view $\mathcal{X}$ as a bimodule over $C(T)$ via the formulae:

$$\xi \cdot a(e, x) := \xi(e, x)a(x)$$

and

$$a \cdot \xi(e, x) := a \circ \phi_e(x)\xi(e, x),$$

where $a \in C(T)$ and $\xi \in C(E \times_G T)$. Further, $\mathcal{X}$ comes equipped with the structure of a Hilbert $C^*$-module over $C(T)$ via the formula

$$\langle \xi, \eta\rangle_A(x) := \sum_{\substack{e\in E\\ x\in T_{r(e)}}} \overline{\xi(e, x)}\, \eta(e, x)$$

for all $\xi, \eta \in \mathcal{X}$ so that, in the language of [**20**], $\mathcal{X}$ may be viewed as a *$C^*$-correspondence over* $C(T)$. Since there are no sources in the graph

$G$, the $A$-valued inner product is well defined. Let $n = |E|$, and let $C^n(A)$ be the column space over $A$, i.e., $C^n(A) = \{(\xi_e)_{e\in E} : \xi_e \in A, \text{ for all } e \in E\}$. Then we view $\mathcal{X}$ as a subset of $C^n(A)$.

We note that the left action is given by the $*$-homomorphism $\Phi : A \to \mathcal{L}(\mathcal{X})$, $(\Phi(a)\xi)(e,x) = a \circ \phi_e(x)\xi(e,x)$. Then $\Phi$ is faithful if and only if $K = T$.

In the case of an iterated function system the $C^*$-correspondence will be the full column space over the $C^*$-algebra $A = C(T)$, that is $\mathcal{X} = C^n(A)$, with the structure:

• The right action is the untwisted right multiplication, i.e., $\xi \cdot a(i,x) = \xi(i,x)a(x)$ for all $i \in \{1, \ldots, n\}$ and $x \in T$.

• The left action is given by the $*$-homomorphism $\Phi : A \to \mathcal{L}(\mathcal{X})$ defined by the formula $\Phi(a)(\xi)(i,x) = a \circ \varphi_i(x)\xi_i(x)$.

• The $A$-valued inner product given by the formula:

$$\langle \xi, \eta \rangle_A(x) = \sum_{i=1}^{n} \xi^*(i,x)\, \eta(i,x).$$

Given a $C^*$-correspondence $\mathcal{X}$ over a $C^*$-algebra $A$ a *Toeplitz representation* of $\mathcal{X}$ in a $C^*$-algebra $B$ consists of a pair $(\psi, \pi)$, where $\psi : \mathcal{X} \to B$ is a linear map and $\pi : A \to B$ is a $*$-homomorphism such that

$$\psi(x \cdot a) = \psi(x)\pi(a)\ ,\ \psi(a \cdot x) = \pi(a)\psi(x),$$

i.e., the pair $(\psi, \pi)$ is a bimodule map and

$$\psi(x)^*\psi(y) = \pi(\langle x, y \rangle_A).$$

That is, the map $\psi$ preserves inner product, see [**11**, Section 1]. Given such a Toeplitz representation, there is an $*$-homomorphism $\pi^{(1)}$ from $\mathcal{K}(\mathcal{X})$ into $B$ which satisfies

$$\pi^{(1)}(\Theta_{x,y}) = \psi(x)\psi(y)^* \quad \text{for all} \quad x, y \in \mathcal{X},$$

where $\Theta_{x,y} = x \otimes \tilde{y}$ is the rank one operator defined by $\Theta_{x,y}(z) = x \cdot \langle y, z \rangle_A$.

We define then

$$J(\mathcal{X}) := \Phi^{-1}(\mathcal{K}(\mathcal{X})),$$

which is a closed two sided-ideal in $A$, see [**11**, Definition 1.1]. Let $K$ be an ideal in $J(\mathcal{X})$. We say that a Toeplitz representation $(\psi, \pi)$ of $\mathcal{X}$ is *coisometric* on $K$ if

$$\pi^{(1)}(\Phi(a)) = \pi(a) \quad \text{for all} \quad a \in K.$$

When $(\psi, \pi)$ is coisometric on all of $J(\mathcal{X})$, we say that it is *Cuntz-Pimsner covariant.*

It is shown in [**11**, Proposition 1.3] that, for an ideal $K$ in $J(\mathcal{X})$, there is a $C^*$-algebra $\mathcal{O}(K, \mathcal{X})$ and a Toeplitz representation $(k_{\mathcal{X}}, k_A)$ of $\mathcal{X}$ into $\mathcal{O}(K, \mathcal{X})$ which is coisometric on $K$ and satisfies:

(1) for every Toeplitz representation $(\psi, \pi)$ of $\mathcal{X}$ which is coisometric on $K$, there is an $*$-homomorphism $\psi \times_K \pi$ of $\mathcal{O}(K, \mathcal{X})$ such that $(\psi \times_K \pi) \circ k_{\mathcal{X}} = \psi$ and $(\psi \times_K \pi) \circ k_A = \pi$; and

(2) $\mathcal{O}(K, \mathcal{X})$ is generated as a $C^*$-algebra by $k_{\mathcal{X}}(\mathcal{X}) \cup k_A(A)$.

The algebra $\mathcal{O}(\{0\}, \mathcal{X})$ is the *Toeplitz algebra* $\mathcal{T}_{\mathcal{X}}$, and $\mathcal{O}(J(\mathcal{X}), \mathcal{X})$ is the *Cuntz-Pimsner algebra* $\mathcal{O}_{\mathcal{X}}$.

For a finite graph $G = (E, V, r, s)$, a *Cuntz-Krieger $G$-family* consists of a family $\{P_v \; : v \in V\}$ of mutually orthogonal projections and a family of partial isometries $\{S_e\}_{e \in E}$ such that

$$S_e^* S_e = P_{r(e)} \quad \text{for} \quad e \in E, \quad \text{and} \quad P_v = \sum_{s(f)=v} S_f S_f^* \quad \text{for} \quad v \in s(E).$$

The edge matrix of $G$ is the $E \times E$ matrix defined by

$$A_G(e, f) = \begin{cases} 1 & \text{if } r(e) = s(f) \\ 0 & \text{otherwise.} \end{cases}$$

Then, a Cuntz-Krieger $G$-family satisfies:

$$S_e^* S_e = \sum_{f \in E} A_G(e, f) \, S_f S_f^*$$

for every $e \in E$ such that $A_G(e, \cdot)$ has nonzero entries. It is shown in [**16**, Theorem 1.2] that there exists a $C^*$-algebra $C^*(G)$ generated by a Cuntz-Krieger $G$-family $\{S_e, P_v\}$ of nonzero elements such that, for every Cuntz-Krieger $G$-family $\{W_e, T_v\}$ of partial isometries on $H$, there is a representation $\pi$ of $C^*(G)$ on $H$ such that $\pi(S_e) = W_e$ and $\pi(P_v) = T_v$ for all $e \in E$ and $v \in V$. The triple $(C^*(G), S_e, P_v)$ is unique up to isomorphism. Since we are assuming that $G$ has no sinks, $\{S_e\}_{e\in E}$ is a Cuntz-Krieger family for the edge matrix $A_G$ in the sense of [**4**], see [**16**, Section 1], and the projections $P_v$ are redundant. If the matrix $A_G$ satisfies Condition (I) from [**4**] (or, equivalently, since $G$ is finite, if $G$ satisfies Condition (L) from [**16**], which asserts that every loop has an exit), then $C^*(G)$ is unique and is isomorphic to the Cuntz-Krieger algebra from [**4**], see [**16**, Theorem 3.7].

**Theorem 2.3.** *Let $(G, \{T_v, \rho_v\}_{v\in V}, \{\phi_e\}_{e\in E})$ be a Mauldin-Williams graph such that the graph $G$ has no sinks and no sources. Let $A$ and $\mathcal{X}$ be defined as above. Then the Cuntz-Pimsner algebra $\mathcal{O}_{\mathcal{X}}$ is isomorphic to $C^*(G)$.*

Before proving the theorem, we introduce some notation. For $k \geq 2$, set

$$E^k := \{\alpha = (\alpha_1, \dots, \alpha_k) : \alpha_i \in E \text{ and } r(\alpha_i) = s(\alpha_{i+1}),\ i = 1, \dots, k-1\},$$

the set of *paths of length* $k$ in the graph $G$. Let $E^* = \cup_{k\in N} E^k$ be the space of *finite paths* in the graph $G$. Also the *infinite path space* $E^\infty$ is defined to be

$$E^\infty := \{(\alpha_i)_{i\in\mathbf{N}} : \alpha_i \in E \text{ and } r(\alpha_i) = s(\alpha_{i+1})\ \forall\, i \in \mathbf{N}\}.$$

For $v \in V$, we also define $E^k(v) := \{\alpha \in E^k : s(\alpha) = v\}$, and we define $E^*(v)$ and $E^\infty(v)$ in a similar way. We consider $E^\infty(v)$ endowed with the metric $\delta_v(\alpha, \beta) = c^{|\alpha\wedge\beta|}$ if $\alpha \neq \beta$ and 0 otherwise, where $\alpha \wedge \beta$ is the longest common prefix of $\alpha$ and $\beta$, and $|w|$ is the length of the word $w \in E^*$, see [**8**, p. 116]. Then $E^\infty(v)$ is a compact metric space and, since $E^\infty$ equals the disjoint union of the spaces $E^\infty(v)$, $E^\infty$ becomes a compact metric space in a natural way.

For $\alpha \in E^k$, we write $\phi_\alpha = \phi_{\alpha_1} \circ \cdots \circ \phi_{\alpha_k}$ and $S_\alpha = S_{\alpha_1} \cdots S_{\alpha_k}$. Let $\mathcal{S}_v$ be the state space of $A_v = C(T_v)$ and $\mathcal{S} = \prod_{v\in V} \mathcal{S}_v$. We consider the metrics $L_v$ defined on $\mathcal{S}_v$ by the formula

$$L_v(\mu,\nu) = \sup\{|\mu(f)-\nu(f)| : f \in \mathrm{Lip}\,(T_v),\ c_f \le 1\}, \tag{2.1}$$

where $\mathrm{Lip}\,(T_v)$ is the space of Lipschitz functions on $T_v$ and $c_f$ is the Lipschitz constant of the Lipschitz function $f$. For $f \in \mathrm{Lip}\,(T_v)$, $v \in V$ and $\mu, \nu \in \mathcal{S}_v$,

$$|\mu(f)-\nu(f)| \le c_f L_v(\mu,\nu). \tag{2.2}$$

Further, if $\alpha \in E^k$, $k \ge 1$, and $\mu,\nu \in \mathcal{S}_{r(\alpha)}$, then $\mu \circ \phi_{\alpha_k}^{-1} \circ \cdots \circ \phi_{\alpha_1}^{-1}, \nu \circ \phi_{\alpha_k}^{-1} \circ \cdots \circ \phi_{\alpha_1}^{-1} \in \mathcal{S}_{s(\alpha)}$ and

$$L_{s(\alpha_1)}(\mu \circ \phi_{\alpha_k}^{-1} \circ \cdots \circ \phi_{\alpha_1}^{-1}, \nu \circ \phi_{\alpha_k}^{-1} \circ \cdots \circ \phi_{\alpha_1}^{-1}) \le c^k \mathrm{diam}_L(\mathcal{S}), \tag{2.3}$$

where $\mathrm{diam}_L(\mathcal{S}) = \max_{v\in V} \mathrm{diam}_{L_v}(\mathcal{S}_v)$.

For $\alpha \in E^\infty$, the sequence $\big(\phi_{\alpha_1\cdots\alpha_k}(T_{r(\alpha_k)})\big)_{k\in\mathbf{N}} \subset T_{s(\alpha)}$ is a decreasing sequence of compact sets. Moreover, $\mathrm{diam}\,\big(\phi_{\alpha_1\cdots\alpha_k}(T_{r(\alpha_k)})\big) \le c^k D$, where $D := \max_{v\in V} \mathrm{diam}\,(T_v)$. Therefore

$$\lim_{k\to\infty} \mathrm{diam}\,\big(\phi_{\alpha_1\cdots\alpha_k}(T_{r(\alpha_k)})\big) = 0,$$

so the intersection $\cap_{k\in\mathbf{N}}\phi_{\alpha_1\cdots\alpha_k}(T_{r(\alpha_k)})$ consists of a single point, $x_\alpha \in T_{s(\alpha)}$. Hence, we can define a map $\Pi : E^\infty \to T$ by the formula

$$\Pi(\alpha) = x_\alpha.$$

Then $\Pi$ is a continuous map and its image is the invariant set $K$ of the Mauldin-Williams graph.

*Proof of Theorem* 2.3. Let $\mu_0 = (\mu_v^0)_{v\in V} \in \mathcal{S}$ be fixed and $a = \sum_{v\in V}^{\oplus} a_v \in \mathrm{Lip}\,(T)$. We define

$$i_A(a) = \lim_{k\to\infty} \sum_{\alpha\in E^k} \mu_{r(\alpha)}^0(a_{s(\alpha)} \circ \phi_\alpha)\, S_\alpha S_\alpha^*.$$

We prove that $i_A$ is a norm decreasing $*$-homomorphism from the $*$-algebra $\mathrm{Lip}\,(T)$ into $C^*(G)$. Then, since $\mathrm{Lip}\,(T)$ is a dense $*$-subalgebra of $A = C(T)$, we can extend $i_A$ to $A$.

We show first that the limit from the definition of $i_A(a)$ exists. Let $a \in \mathrm{Lip}\,(T)$, and let $\varepsilon > 0$. Choose $k_0 \in \mathbf{N}$ such that $c^k \mathrm{diam}_L(\mathcal{S}) c_a < \varepsilon$ for all $k \geq k_0$. Set $a_k := \sum_{\alpha \in E^k} \mu^0_{r(\alpha)}(a_{s(\alpha)} \circ \phi_\alpha) S_\alpha S_\alpha^*$. Let $k, m \geq k_0$, and suppose that $k > m$. Then

$$\begin{aligned} a_m - a_k &= \sum_{\alpha \in E^m} \sum_{\substack{\beta \in E^{k-m} \\ s(\beta)=r(\alpha)}} \mu^0_{r(\alpha)}(a_{s(\alpha)} \circ \phi_\alpha)\, S_\alpha S_\beta S_\beta^* S_\alpha^* \\ &\quad - \sum_{\alpha \in E^m} \sum_{\substack{\beta \in E^{k-m} \\ s(\beta)=r(\alpha)}} \mu^0_{r(\beta)}(a_{s(\alpha)} \circ \phi_{\alpha\beta})\, S_\alpha S_\beta S_\beta^* S_\alpha^* \\ &= \sum_{\alpha \in E^m} \sum_{\substack{\beta \in E^{k-m} \\ s(\beta)=r(\alpha)}} (\mu^0_{r(\alpha)}(a_{s(\alpha)} \circ \phi_\alpha) \\ &\qquad\qquad - \mu^0_{r(\beta)}(a_{s(\alpha)} \circ \phi_{\alpha\beta}))\, S_\alpha S_\beta S_\beta^* S_\alpha^* \\ &= \sum_{\alpha \in E^m} \sum_{\substack{\beta \in E^{k-m} \\ s(\beta)=r(\alpha)}} (\mu^0_{r(\alpha)}(a_{s(\alpha)} \circ \phi_\alpha) \\ &\qquad\qquad - \mu^0_{r(\beta)} \circ \phi_\beta^{-1}(a_{s(\alpha)} \circ \phi_\alpha))\, S_\alpha S_\beta S_\beta^* S_\alpha^*. \end{aligned}$$

Since $|\mu^0_{r(\alpha)}(a_{s(\alpha)} \circ \phi_\alpha) - \mu^0_{r(\beta)} \circ \phi_\beta^{-1}(a_{s(\alpha)} \circ \phi_\alpha)| < \varepsilon$, by equations (2.2) and (2.3), for all $\alpha \in E^m, \beta \in E^{k-m}$ such that $s(\beta) = r(\alpha)$, and since $S_\alpha S_\beta S_\beta^* S_\alpha^*$ are orthogonal projections, $\|a_m - a_k\| < \varepsilon$, for all $m, k \geq k_0$. So $(a_k)_{k \in \mathbf{N}}$ is a Cauchy sequence, hence convergent. Since $\|a_m\| = \max_{\alpha \in E^m} |\mu^0_{r(\alpha)}(a_{s(\alpha)} \circ \phi_\alpha)| \leq \|a\|$ for all $m \in \mathbf{N}$, $\|i_A(a)\| \leq \|a\|$ for all $a \in A$.

Next we prove that $i_A$ is a homomorphism. Let $a, b \in \mathrm{Lip}\,(T)$. Then for each $\alpha \in E^\infty$ there is a point $x_\alpha \in K$ such that $\cap_{k \in \mathbf{N}} \phi_{\alpha_1 \cdots \alpha_k}(T_{r(\alpha_k)}) = \{x_\alpha\}$. Then $\lim_{k \to \infty} \mu^0_{r(\alpha_k)}(a_{s(\alpha)} \circ \phi_{\alpha_1 \cdots \alpha_k}) = a(x_\alpha)$, $\lim_{k \to \infty} \mu^0_{r(\alpha_k)} (b_{s(\alpha)} \circ \phi_{\alpha_1 \cdots \alpha_k}) = b(x_\alpha)$ and $\lim_{k \to \infty} \mu^0_{r(\alpha_k)}((ab)_{s(\alpha)} \circ \phi_{\alpha_1 \cdots \alpha_k}) = a(x_\alpha) b(x_\alpha)$. Let $\varepsilon > 0$. Since $\mathrm{diam}\,(\phi_{\alpha_1 \cdots \alpha_k}(T_{r(\alpha_k)})) \leq c^k D$ for all $\alpha \in E^\infty$ and $k \in \mathbf{N}$, there exists some $N \in \mathbf{N}$ such that $|\mu^0_{r(\alpha)}(a_{s(\alpha)} \circ \phi_{\alpha_1 \cdots \alpha_k}) - a(x_\alpha)| < \varepsilon$, $|\mu^0_{r(\alpha)}(b_{s(\alpha)} \circ \phi_{\alpha_1 \cdots \alpha_k}) - b(x_\alpha)| < \varepsilon$ and $|\mu^0_{r(\alpha)}(a_{s(\alpha)} \circ \phi_{\alpha_1 \cdots \alpha_k} b_{s(\alpha)} \circ \phi_{\alpha_1 \cdots \alpha_k}) - a(x_\alpha) b(x_\alpha)| < \varepsilon$ for all $k \geq N$

and *for all* $\alpha \in E^\infty$. Then

$$\Bigg\| \sum_{\alpha\in E^k} \mu^0_{r(\alpha)}(a_{s(\alpha)}\circ\phi_\alpha b_{s(\alpha)}\circ\phi_\alpha)\, S_\alpha S_\alpha^* - \sum_{\alpha\in E^k} \mu^0_{r(\alpha)}(a_{s(\alpha)}\circ\phi_\alpha)\mu^0_{r(\alpha)}(b_{s(\alpha)}\circ\phi_\alpha)\, S_\alpha S_\alpha^* \Bigg\|$$
$$\leq \max_{\alpha\in E^k} |\mu^0_{r(\alpha)}(a_{s(\alpha)}\circ\phi_\alpha b_{s(\alpha)}\circ\phi_\alpha) - \mu^0_{r(\alpha)}(a_{s(\alpha)}\circ\phi_\alpha)\mu^0_{r(\alpha)}(b_{s(\alpha)}\circ\phi_\alpha)|$$
$$< \varepsilon(1+\|a\|+\|b\|)$$

for all $k \geq N$. Thus, $i_A(ab) = i_A(a)i_A(b)$. Hence, $i_A$ is a homomorphism and one can easily see that it is an $*$-homomorphism.

Further, $i_A$ satisfies the equation

$$i_A(a)S_e = S_e i_A(a_{s(e)}\circ\phi_e) \quad \text{for all} \quad a\in A, \quad \text{and} \quad e\in E, \tag{2.4}$$

where we extend the map $a_{s(e)}\circ\phi_e$ to all $T$ by setting it to be 0 when $x \notin T_{r(e)}$, since, for $a \in \mathrm{Lip}\,(T)$, we have:

$$\begin{aligned} i_A(a)S_e &= \left(\lim_{k\to\infty}\sum_{\alpha\in E^k}\mu^0_{r(\alpha)}(a_{s(\alpha)}\circ\phi_\alpha)\, S_\alpha S_\alpha^*\right) S_e \\ &= \lim_{k\to\infty}\sum_{\alpha\in E^k,\alpha_1=e}\mu^0_{r(\alpha)}(a_{s(\alpha)}\circ\phi_\alpha)\, S_e S_{\alpha_2}\cdots S_{\alpha_k}S^*_{\alpha_k}\cdots S^*_{\alpha_2} \\ &= S_e \lim_{k\to\infty}\sum_{\alpha'\in E^{k-1}(r(e))}\mu^0_{r(\alpha')}(a_{s(e)}\circ\phi_e\circ\phi_{\alpha'})\, S_{\alpha'}S^*_{\alpha'} \\ &= S_e i_A(a_{s(e)}\circ\phi_e). \end{aligned}$$

We also define the linear map $i_{\mathcal{X}} : \mathcal{X} \to C^*(G)$ by the formula

$$i_{\mathcal{X}}(\xi) = \sum_{e\in E} S_e i_A(\xi_e),$$

where $\xi_e \in C(T)$ is defined by $\xi_e(x) = \xi(e,x)$ if $x \in T_{r(e)}$ and 0 otherwise. We have

$$i_{\mathcal{X}}(\xi\cdot a) = \sum_{e\in E} S_e i_A(\xi_e a) = \sum_{e\in E} S_e i_A(\xi_e) i_A(a) = i_{\mathcal{X}}(\xi) i_A(a),$$

$$\begin{aligned} i_{\mathcal{X}}(a\cdot\xi) &= \sum_{e\in E} S_e i_A(a_{s(e)}\circ\phi_e\xi_e) = \sum_{e\in E} S_e i_A(a_{s(e)}\circ\phi_e) i_A(\xi_e) \\ &= \sum_{e\in E} i_A(a) S_e i_A(\xi_e) = i_A(a) i_{\mathcal{X}}(\xi) \end{aligned}$$

and

$$\begin{aligned} i_{\mathcal{X}}(\xi)^* i_{\mathcal{X}}(\eta) &= \Big(\sum_{e\in E} S_e i_A(\xi_e)\Big)^* \Big(\sum_{f\in E} S_f i_A(\eta_f)\Big) \\ &= \sum_{e\in E} i_A(\xi_e)^* i_A(\eta_e) = i_A\Big(\sum_{e\in E}\xi_e^*\eta_e\Big) = i_A(\langle\xi,\eta\rangle_A). \end{aligned}$$

Hence $(i_A, i_{\mathcal{X}})$ is a Toeplitz representation.

Let $J(\mathcal{X}) := \Phi^{-1}(\mathcal{K}(\mathcal{X}))$. Note that $J(\mathcal{X}) = A$ since for $a\in A$ we have

$$\Phi(a)\xi = \sum_{e\in E}\Theta_{x^e,\delta^e}(\xi),$$

where $x^e\in\mathcal{X}$ is defined by $x^e_f = a_{s(e)}\circ\phi_e\delta^e_f$,

$$\delta^e(f,x) = \begin{cases} 1 & \text{if } f=e \text{ and } x\in T_{r(e)} \\ 0 & \text{otherwise.}\end{cases}$$

Then, for $a\in A$, we have

$$\begin{aligned} i_A^{(1)}(\Phi(a)) &= i_A^{(1)}\Big(\sum_{e\in E}\Theta_{x^e,\delta^e}\Big) = \sum_{e\in E} i_A^{(1)}(\Theta_{x^e,\delta^e}) \\ &= \sum_{e\in E} i_{\mathcal{X}}(x^e) i_{\mathcal{X}}(\delta^e)^* \\ &= \sum_{e\in E}\Big(\sum_{f\in E} S_f i_A(x^e_f)\Big)\Big(\sum_{g\in E} S_g i_A(\delta^e_g)\Big)^* \\ &= \sum_{e\in E}(S_e i_A(a_{s(e)}\circ\phi_e))(i_A(1_{T_{r(e)}})\, S_e^*) \\ &= i_A(a)\sum_{e\in E} S_e S_e^* = i_A(a). \end{aligned}$$

Therefore, $(i_A, i_{\mathcal{X}})$ is a Cuntz-Pimsner covariant representation.

For $\delta^e$ defined as above, we notice that

$$i_{\mathcal{X}}(\delta^e) = \sum_{f\in E} S_f i_A(\delta^e_f) = S_e i_A(1_{T_{r(e)}}) = S_e.$$

Then $i_{\mathcal{X}}(\mathcal{X}) \cup i_A(A)$ generates $C^*(G)$.

Since $(i_A, i_{\mathcal{X}})$ is a Cuntz-Pimsner covariant representation, there exists a homomorphism $i_{\mathcal{X}} \times i_A$ of $\mathcal{O}_{\mathcal{X}}$ onto $C^*(G)$ such that $(i_{\mathcal{X}} \times i_A) \circ k_{\mathcal{X}} = i_{\mathcal{X}}$ and $(i_{\mathcal{X}} \times i_A) \circ k_A = i_A$. We prove that $i_{\mathcal{X}} \times i_A$ is also injective. Let $\gamma : \mathbf{T} \to \mathrm{Aut}\,(\mathcal{O}_{\mathcal{X}})$ defined by $\gamma_z(k_{\mathcal{X}}(\xi)) = zk_{\mathcal{X}}(\xi)$, and let $\gamma_z(k_A(a)) = k_A(a)$ be the gauge action on $\mathcal{O}_{\mathcal{X}}$. Let $\beta : \mathbf{T} \to \mathrm{Aut}\,(C^*(G))$ defined by $\beta_z(S_e) = zS_e$ for all $e \in E$ be the gauge action on $C^*(G)$. Therefore, by the definition of $i_A$ and $i_{\mathcal{X}}$, $\beta_z(i_{\mathcal{X}}(\xi)) = zi_{\mathcal{X}}(\xi)$ and $\beta_z(i_A(a)) = i_A(a)$ for all $\xi \in \mathcal{X}$ and $a \in A$. Hence $\beta_z \circ (i_{\mathcal{X}} \times i_A) = (i_{\mathcal{X}} \times i_A) \circ \gamma_z$ for all $z \in \mathbf{T}$. Then the gauge-invariant uniqueness theorem [**11**, Theorem 4.1] implies that $i_{\mathcal{X}} \times i_A$ is injective. Thus, $C^*(G)$ is isomorphic to the Cuntz-Pimsner algebra associated to the $C^*$-correspondence $\mathcal{X}$. □

**Corollary 2.4.** *The Cuntz-Pimsner algebra $\mathcal{O}_{\mathcal{X}} = \mathcal{O}(J(\mathcal{X}), \mathcal{X})$ of the $C^*$-correspondence associated with an iterated function system $(\phi_1, \phi_2, \dots, \phi_n)$ is isomorphic to the Cuntz algebra $\mathcal{O}_n$.*

If $K$ (the invariant set of the Mauldin-Williams graph) is a proper subset of $T$, then $U := T \setminus K$ is a nonempty open set of $T$. Let $I_U := C_0(U)$ be the corresponding ideal in $A$. Then

$$\mathcal{X}_{I_U} := \{\xi \in \mathcal{X} : \langle \xi, \eta\rangle_A \in I_U \text{ for all } \eta \in \mathcal{X}\}$$

is a right Hilbert $I_U$-module, and we know that $\mathcal{X}_{I_U} = \mathcal{X} I_U := \{\xi \cdot i : \xi \in \mathcal{X}, i \in I_U\}$, see [**11**, Section 2]. It follows that $\mathcal{X}_{I_U} = \{\xi \in \mathcal{X} : \xi_e \in C_0(U)\}$ ($\xi_e \in C_0(U)$ means that $\xi(e, x) = 0$ if $x \in K$). We claim that $I_U$ is an $\mathcal{X}$-invariant ideal in $A$, i.e., $\Phi(I_U)\mathcal{X} \subset \mathcal{X} I_U$. For $i \in I_U$ and $\xi \in \mathcal{X}$, we have $(\Phi(i)\xi)_e = i \circ \phi_e \xi_e$, and, since $i \in I_U$ and $\phi_e(K_{r(e)}) \subset K_{s(e)}$, $i \circ \phi_e \in I_U$. Hence, $(\Phi(i)\xi)_e \in I_U$. Therefore, $I_U$ is an $\mathcal{X}$-invariant ideal in $A$ and $\mathcal{X}/\mathcal{X} I_U$ is a $C^*$-correspondence over $A/I_U \simeq C(K)$, see [**11**, Lemma 2.3]. Moreover $\mathcal{X}/\mathcal{X} I_U \simeq \mathcal{X}(K)$, where $\mathcal{X}(K) = C(E \times_G K)$ is the $C^*$-correspondence defined as in Definition 2.2 for the $C^*$-algebra

$C(K)$. Then the ideal $\mathcal{I}(I_U)$ of $\mathcal{O}_{\mathcal{X}}$ generated by $i_A(I_U)$ is Morita equivalent to $\mathcal{O}_{\mathcal{X}I_U}$, and since $\Phi(A) \subset \mathcal{K}(\mathcal{X})$, $\mathcal{O}_{\mathcal{X}}/\mathcal{I}(I_U) \cong \mathcal{O}_{\mathcal{X}/\mathcal{X}I_U}$, see [**11**, Corollary 3.3].

**Proposition 2.5.** *The ideal $\mathcal{I}(I_U)$ generated by $i_A(I_U)$ is equal to* 0.

*Proof.* Let $a \in I_U$ be a Lipschitz function. As in the proof of Theorem 2.3, we have that $i_A(a) = \lim_{k\to\infty} \sum_{\alpha\in E^k} \mu^0_{r(\alpha)}(a_{s(\alpha)} \circ \phi_\alpha)S_\alpha S_\alpha^*$. Let $\varepsilon > 0$. Since $\cap_{k\in\mathbf{N}}\phi_{\alpha_1\cdots\alpha_k}(T_{r(\alpha_k)}) = \{x_\alpha\}$ with $x_\alpha \in K_{s(\alpha)}$ and $\operatorname{diam}(\phi_{\alpha_1\cdots\alpha_k})(T_{r(\alpha_k)}) < c^k D$ for all $\alpha \in E^\infty$, there exists $N \in \mathbf{N}$ such that $|\mu^0_{r(\alpha_k)}(a \circ \phi_{\alpha_1\cdots\alpha_k}) - a(x_\alpha)| < \varepsilon$ for all $\alpha \in E^\infty$ and $k \geq N$. Since $a(x) = 0$ for all $x \in K$, $\|\sum_{\alpha\in E^k} \mu^0_{r(\alpha)}(a_{s(\alpha)} \circ \phi_\alpha)S_\alpha S_\alpha^*\| < \varepsilon$ for all $k \geq N$. Hence, $i_A(a) = 0$. □

**Corollary 2.6.** *The Cuntz-Pimsner algebra associated to the $C^*$-correspondence $C(E \times_G K)$ over $C(K)$ with the actions defined as in Definition* 2.2 *is isomorphic to $C^*(G)$.*

One can interpret the previous results in the particular case of the iterated function system and obtain the result from [**23**, Remark 4.6].

**3. On noncommutative Mauldin-Williams graphs.** We give a generalization of the work of Pinzari, Watatani and Yonetani from [**23**, Section 4.3] on noncommutative iterated function systems in the context of "noncommutative" Mauldin-Williams graphs and the Rieffel metric. We show that, in fact, these situations are no more general than those just discussed.

We begin by reviewing the Rieffel metric.

**Definition 3.1.** Let $A$ be a unital $C^*$-algebra, let $\mathcal{L}(A) \subset A$ be a dense subspace of $A$ (the Lipschitz elements), and let $L$ be a semi-norm (the Lipschitz semi-norm) on $\mathcal{L}(A)$ such that $\mathcal{K} := \{a \in \mathcal{L}(A) : L(a) = 0\}$ equals the scalar multiples of the identity. The *Rieffel metric* $\rho$ on the state space $\mathcal{S}$ of $A$ is defined by the equation

$$\rho(\mu,\nu) = \sup\{|\mu(a) - \nu(a)| : a \in \mathcal{L}(A), L(a) \leq 1\}$$

for all $\mu,\nu \in \mathcal{S}$. We will suppose that the metric $\rho$ is bounded on $S$ and that the corresponding topology coincides with the weak-$*$ topology on $\mathcal{S}$.

For a compact metric space $(X,\rho)$, let

$$\mathcal{C}(X) := \{E : E \text{ is a non-empty compact subset of } X\}.$$

The *Hausdorff metric* on $\mathcal{C}(X)$ is defined by the formula

$$\delta_\rho(E,F) = \inf\{r > 0 : U_r(E) \supseteq F \text{ and } U_r(F) \supseteq E\}$$

for all $E,F \in \mathcal{C}(X)$, where $U_r(E) = \{x \in X : \rho(x,y) < r \text{ for some } y \in E\}$, see [**8**, Theorem 2.4.1] or [**15**, Proposition 1.1.5]. Then $(\mathcal{C}(X),\delta_\rho)$ is a compact metric space.

For a $C^*$-algebra $A$, Rieffel defines the *quantum closed subsets* of $A$ in [**29**, p. 14] to be the closed convex subsets of the state space $\mathcal{S}(A)$ of $A$. If $L$ is a Lipschitz semi-norm on $A$ and $\rho_L$ is the corresponding Rieffel metric, the space $\mathcal{Q}(A)$ of quantum closed subsets of $A$ is a compact metric space for the associated Hausdorff metric, see [**29**, p. 14]).

Following the definition of the classical Mauldin-Williams graphs, we define a noncommutative variant.

**Definition 3.2.** A *noncommutative Mauldin-Williams graph* is a system $(G,\{A_v,\mathcal{L}_v,L_v,\rho_v\}_{v\in V},\{\phi_e\}_{e\in E})$ where $G=(V,E,s,r)$ is a graph and where $\{A_v,\mathcal{L}_v,L_v\}_{v\in V}$ and $\{\phi_e\}_{e\in E}$ are families such that

(1) For each $v \in V$, $A_v$ is a unital $C^*$-algebra with a prescribed Lipschitz semi-norm $L_v$ on a prescribed subspace $\mathcal{L}_v$ of Lipschitz elements in $A_v$ and $\rho_v$ is the corresponding Rieffel metric.

(2) For $e \in E$, $\phi_e$ is a unital $*$-homomorphism from $A_{s(e)}$ to $A_{r(e)}$ such that

$$\rho_{s(e)}(\phi_e^*(\mu),\phi_e^*(\nu)) \leq c\rho_{r(e)}(\mu,\nu)$$

for some constant $c$ satisfying $0 < c < 1$ and all $\mu,\nu \in \mathcal{S}_{r(e)}$ (where $\mathcal{S}_v$ is the state space of the $C^*$-algebra $A_v$).

We shall assume, too, that there are no sinks in the graph $G$. We also let $\mathcal{S} := \prod_{v\in V} \mathcal{S}_v$.

When we have one vertex and $n$ edges we call the system a *noncommutative iterated function system*.

Let $\mathcal{C}(\mathcal{S}_v)$ be the space of compact subsets of $\mathcal{S}_v$ endowed with the Hausdorff metric $\delta_{\rho_{L_v}}$, for each $v \in V$. Let $\mathcal{C} = \prod_{v\in V} \mathcal{C}(\mathcal{S}_v)$. Then $\mathcal{C}$ is a compact metric space. Moreover, the map $F : \mathcal{C} \to \mathcal{C}$ defined by the formula

$$F\left((K_v)_{v\in V}\right) = \left( \bigcup_{\substack{e\in E\\ s(e)=v}} \phi_e^*(K_{r(e)}) \right)_{v\in V}$$

is a contraction, since each $\phi_e^*$ is a contraction with respect to the Rieffel metric. Thus, there exists a *unique* element $(K_v)_{v\in V} \in \mathcal{C}$ such that

$$K_v = \bigcup_{\substack{e\in E\\ s(e)=v}} \phi_e^*(K_{r(e)}) \tag{3.1}$$

for all $v \in V$. Let $T_v$ be the closed convex hull of $K_v$, for $v \in V$. That is $T_v \in \mathcal{Q}(A_v)$ for all $v \in V$. Since by [**29**, Proposition 3.6] there is a bijection between isomorphism classes of quotients of $A_v$ and closed convex subsets of $\mathcal{S}_v$, we will assume that

$$\mathcal{S}_v = T_v \quad \text{for all} \quad v \in V, \tag{3.2}$$

by taking a quotient of the original $C^*$-algebra $A_v$, if necessary. In particular, if $(M_v)_{v\in V} \in \mathcal{S}$ is any family which satisfies equation (3.1), then $M_v = K_v$ and the closed convex hull of $M_v$ equals $\mathcal{S}_v$ for all $v \in V$.

**Lemma 3.3.** *In the above situation, if $\mathcal{I}$ is an ideal in $\sum_{v\in V}^{\oplus} A_v$ of the form $\mathcal{I} = (\mathcal{I}_v)_{v\in V}$, with $\mathcal{I}_v$ a proper ideal of $A_v$, then*

$$\mathcal{I}_v = \bigcap_{\substack{e\in E\\ s(e)=v}} \phi_e^{-1}(\mathcal{I}_{r(e)}) \quad \textit{if and only if} \quad \mathcal{I}_v = (0_v) \quad \textit{for all} \quad v \in V. \tag{3.3}$$

*Proof.* Let $\mathcal{I} = (\mathcal{I}_v)_{v\in V}$ be such that $\mathcal{I}_v = \cap_{e\in E, s(e)=v}\phi_e^{-1}(\mathcal{I}_{r(e)})$ for all $v \in V$. Let $M_v := \{\mu \in \mathcal{S}_v : \mu(a) = 0 \text{ for all } a \in \mathcal{I}_v\}$. We show that

$(M_v)_{v\in V}$ is a family which satisfies equation (3.1). Let $v \in V$, and let $\mu \in \cup_{e\in E, s(e)=v}\phi_e^*(M_{r(e)})$. Then there exists some $e \in E$ and $\nu \in M_{r(e)}$ such that $\mu = \phi_e^*(\nu)$. Let $a \in \mathcal{I}_v$. Then $\phi_e(a)$ belongs to $\mathcal{I}_{r(e)}$. Hence $\mu(a) = \nu(\phi_e(a)) = 0$. Therefore $\mu \in M_v$.

Now suppose that there is some $\mu \in M_v$ such that

$$\mu \notin \bigcup_{\substack{e\in E\\ s(e)=v}} \phi_e^*(M_{r(e)}).$$

Hence, there is some $a \in A_v$ such that $\mu(A) \neq 0$ and $\phi_e^*(\nu)(a) = 0$ for all $\nu \in M_{r(e)}$ and for all $e \in E$ such that $s(e) = v$. Then $\phi_e(a_v) \in \mathcal{I}_{r(e)}$ for all $e \in E$, therefore $a_v \in \mathcal{I}_v$. Thus $\mu(a) = 0$, which is a contradiction. Then the family $(M_v)_{v\in V}$ satisfies equation (3.1). Therefore $\mu(a) = 0$ for all $\mu \in \mathcal{S}_v$ and $a \in \mathcal{I}_v$, hence $\mathcal{I}_v = 0$ for all $v \in V$.

Suppose that there exists some $a \in A_v$ which is not zero, but $a \in \cap_{e\in E, s(e)=v} \operatorname{Ker}\phi_e$. Then there is some $\mu \in \mathcal{S}_v$ such that $\mu(a) = 0$. Since $K_v = \cup_{e\in E, s(e)=v}\phi_e^*(K_{r(e)})$, there is some $e \in E$ with $s(e) = v$ and some $\nu \in S_{r(e)}$ such that $\mu = \phi_e^*(\nu)$. Since $\phi_e(a) = 0$, we obtain that $\mu(a) = \nu(\phi_e(a)) = 0$, which is a contradiction. Hence $(0_v) = \cap_{e\in E, s(e)=v} \operatorname{Ker}\phi_e$. □

Recall that $E^k$ denotes the set of paths of length $k$, $E^\infty$ denotes the set of infinite paths in the graph $G$, $E^k(v)$ denotes the set of paths of length $k$ starting at the vertex $v$, and $E^\infty(v)$ denotes the set of infinite paths starting at the vertex $v$. For $k \in \mathbf{N}$ and $\alpha \in E^k$, we write $\phi^*_{\alpha_1\cdots\alpha_k}$ for the map $\phi^*_{\alpha_1} \circ \cdots \circ \phi^*_{\alpha_k} : \mathcal{S}_{r(\alpha_k)} \to \mathcal{S}_{s(\alpha_1)}$ and $\phi_{\alpha_k\cdots\alpha_1}$ for the map $\phi_{\alpha_k} \circ \cdots \circ \phi_{\alpha_1} : A_{s(\alpha_1)} \to A_{r(\alpha_k)}$. We will use the following results (which are similar to the commutative case): if $v \in V$, $a \in \mathcal{L}_v(A_v)$ and $\mu, \nu \in \mathcal{S}_v$ then $|\mu(a) - \nu(a)| \le \rho_v(\mu,\nu)\cdot L_v(a)$; if $\alpha \in E^k$ and $\mu,\nu \in \mathcal{S}_{r(\alpha)}$, then

$$\rho_{s(\alpha)}(\phi^*_\alpha(\mu), \phi^*_\alpha(\nu)) \le c^k \rho_{r(\alpha)}(\mu,\nu) \le c^k D, \tag{3.4}$$

where $s(\alpha) = s(\alpha_1)$, $r(\alpha) = r(\alpha_k)$ and $D = \max_{v\in V} \operatorname{diam}_{L_v}(\mathcal{S}_v)$.

Since $(G, \{\mathcal{S}_v, \rho_v\}_{v\in V}, \{\phi_e^*\}_{e\in E})$ is a (classical) Mauldin-Williams graph, for each $\alpha \in E^\infty$ there is a unique state $\mu_\alpha \in \mathcal{S}_{s(\alpha)}$ such that $\{\mu_\alpha\} = \cap_{k\in\mathbf{N}}\phi^*_{\alpha_1\cdots\alpha_k}(\mathcal{S}_{r(\alpha_k)})$. In particular $\lim_{k\to\infty}\phi^*_{\alpha_1\cdots\alpha_k}(\mu_{r(\alpha_k)}) = \mu_\alpha$ for all $\mu = (\mu_v)_{v\in V} \in \mathcal{S}$.

**Theorem 3.4.** *Let* $(G, \{A_v, \mathcal{L}_v, L_v, \rho_v\}_{v\in V}, \{\phi_e\}_{e\in E})$ *be a (noncommutative) Mauldin-Williams graph. Suppose that the graph* $G$ *has no sinks. Then there is an injective* $*$*-homomorphism from* $A$ *into* $C(E^\infty)$.

*Proof.* Fix $v_0 \in V$. Define $\pi_{v_0} : A_{v_0} \to C(E^\infty(v_0))$ by the formula

$$\pi_{v_0}(a)(\alpha) = \mu_\alpha(a)$$

for all $a \in A_{v_0}$. Thus, if $a \in \mathcal{L}_{v_0}$,

$$\pi_{v_0}(a)(\alpha) = \lim_{k\to\infty} \mu_{r(\alpha_k)}(\phi_{\alpha_k\cdots\alpha_1}(a))$$

for all $\mu = (\mu_v)_{v\in V} \in \mathcal{S}$. By the comments preceding the theorem the map $\pi_{v_0}$ is well defined. We prove that it is a homomorphism.

Let $\mu_0 = (\mu_v^0)_{v\in V} \in \mathcal{S}$ be *fixed.* Let $a \in \mathcal{L}_{v_0}$, $\alpha \in E^\infty(v_0)$ and let $\varepsilon > 0$. Let $k \in \mathbf{N}$ be such that $c^k D L_{v_0}(a) < \varepsilon$. For *any* $\mu = (\mu_v)_{v\in V} \in \mathcal{S}$, we have

$$\begin{aligned}&\left|\mu_{r(\alpha_k)}\left(\mu^0_{r(\alpha_k)}(\phi_{\alpha_k\cdots\alpha_1}(a))1_{A_{r(\alpha_k)}} - \phi_{\alpha_k\cdots\alpha_1}(a)\right)\right|\\ &\qquad = \left|\mu^0_{r(\alpha_k)}(\phi_{\alpha_k\cdots\alpha_1}(a)) - \mu_{r(\alpha_k)}(\phi_{\alpha_k\cdots\alpha_1}(a))\right| < c^k D L_{v_0}(a) < \varepsilon.\end{aligned}$$

Hence,

$$\|\mu^0_{r(\alpha_k)}(\phi_{\alpha_k\cdots\alpha_1}(a))1_{A_{r(\alpha_k)}} - \phi_{\alpha_k\cdots\alpha_1}(a)\| < 4\varepsilon. \tag{3.5}$$

Let $a, b \in A_{v_0}$. We have

$$\begin{aligned}&|\mu^0_{r(\alpha_k)}(\phi_{\alpha_k\cdots\alpha_1}(ab)) - \mu^0_{r(\alpha_k)}(\phi_{\alpha_k\cdots\alpha_1}(a))\mu^0_{r(\alpha_k)}(\phi_{\alpha_k\cdots\alpha_1}(b))|\\ &\qquad \le \|\mu^0_{r(\alpha_k)}(\phi_{\alpha_k\cdots\alpha_1}(ab)) - \phi_{\alpha_k\cdots\alpha_1}(a)\phi_{\alpha_k\cdots\alpha_1}(b)\|\\ &\qquad\quad + \|\phi_{\alpha_k\cdots\alpha_1}(a)\phi_{\alpha_k\cdots\alpha_1}(b) - \phi_{\alpha_k\cdots\alpha_1}(a)\mu^0_{r(\alpha_k)}(\phi_{\alpha_k\cdots\alpha_1}(b))\|\\ &\qquad\quad + \|\phi_{\alpha_k\cdots\alpha_1}(a)\mu^0_{r(\alpha_k)}(\phi_{\alpha_k\cdots\alpha_1}(b))\\ &\qquad\qquad - \mu^0_{r(\alpha_k)}(\phi_{\alpha_k\cdots\alpha_1}(a))\mu^0_{r(\alpha_k)}(\phi_{\alpha_k\cdots\alpha_1}(b))\|\\ &\qquad < 4\varepsilon + 4\|a\|\varepsilon + 4\varepsilon = (8 + 4\|a\|)\varepsilon,\end{aligned}$$

by inequality (3.5). Since $\lim_{k\to\infty} \mu^0_{r(\alpha_k)}(\phi_{\alpha_k\cdots\alpha_1}(ab)) = \pi_{v_0}(ab)(\alpha)$ and $\lim_{k\to\infty} \mu^0_{r(\alpha_k)}(\phi_{\alpha_k\cdots\alpha_1}(a))\mu^0_{r(\alpha_k)}(\phi_{\alpha_k\cdots\alpha_1}(b)) = \pi_{v_0}(a)(\alpha)\pi_{v_0}(b)(\alpha)$, we see that $\pi_{v_0}$ is a homomorphism.

Hence, for each $v \in V$, we have defined an $*$-homomorphism $\pi_v : A_v \to C(E^\infty(v))$. We prove that $\pi_v$ is injective for all $v \in V$. Let $v \in V$. Let $a \in A_v$. Then

$$\begin{aligned} a \in \operatorname{Ker} \pi_v &\iff \pi_v(a)(\alpha) = 0 \ \forall\ \alpha = (\alpha_n)_{n\in\mathbf{N}} \in E^\infty(v) \\ &\iff \pi_v(a)(e\beta) = 0 \ \forall\ \beta \in E^\infty(r(e)) \quad \text{and} \quad e \in E(v) \\ &\iff \pi_{r(e)}(\phi_e(a))(\beta) = 0 \ \forall\ \beta \in E^\infty(r(e)) \quad \text{and} \quad e \in E(v) \\ &\iff a \in \bigcap_{s(e)=v} \operatorname{Ker} \pi_{r(e)} \circ \phi_e. \end{aligned}$$

Lemma 3.3 implies that $\operatorname{Ker} \pi_v = 0$ for all $v \in V$, hence the $*$-homomorphism $\pi : A \to C(E^\infty)$ defined by the formula

$$\pi\left((a_v)_{v\in V}\right) = \sum_{v\in V}{}^{\oplus} \pi_v(a_v)$$

is an injective $*$-homomorphism. □

**Corollary 3.5.** *Under the hypothesis of Theorem* 3.4*, we conclude that $A$ must be a commutative $C^*$-algebra.*

Even in the setting of a "noncommutative" iterated function system studied in [**23**, Section 4.2], if we have defined a Rieffel metric such that the underlying topology and the weak-$*$ topology coincide, and if the duals of the endomorphisms restricted to the state space of $A$ are contractions with respect to the Rieffel metric, then (under the hypothesis that $A$ satisfies equation (3.2)) $A$ is forced to be commutative and the endomorphisms $\phi_i$ must come from an ordinary iterated function system, i.e., $A = C(K)$ for some compact metric space and there are contractions $\{\varphi_i\}_{i=1,\dots,n}$ defined on $K$ such that $\phi_i(a) = a \circ \varphi_i$. This seems not to have been noticed by the authors of [**23**].

The assumption that the graph $G$ has no sinks is essential in the proofs of Theorems 2.3 and 3.4, since it forces the presence of infinite paths in the graph. Also, the assumption that the graph $G$ has no sources was needed to define the $C^*$-correspondence associated with a Mauldin-Williams graph. It is not needed, though, in the proof of Theorem 3.4.

We would like to call attention to a recent preprint of Kajiwara and Watatani [**13**] in which they considered a somewhat different $C^*$-correspondences associated with an iterated function system and arrive at a $C^*$-algebra that is sometimes different from $\mathcal{O}_n$. It appears that their construction can be modified to cover the setting of Mauldin-Williams graphs, leading to $C^*$-algebras different from the Cuntz-Krieger algebras of the underlying graphs. We intend to pursue the ramifications of this in a future note.

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