ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 37, Number 3, 2007

SUBGROUPS OF PURE BRAID GROUPS GENERATED BY POWERS OF DEHN TWISTS

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ABSTRACT. Let B_n be the group of braids on n strings, and let P_n be the corresponding pure braid group. In this paper we consider subgroups of B_n generated by powers of Dehn twists. For example, let $A_{12}, A_{13}, \dots, A_{n-1,n}$ be the standard Dehn twist generators for P_n and consider subgroups of the form $\langle A_{ij}^{\varepsilon_{ij}} \rangle$; we give conditions guaranteeing that such a subgroup has finite index in P_n. We then consider subgroups obtained by adding in powers of other Dehn twists. In the cases considered the finite index property is characterized in terms of certain inequalities.

1. Introduction. The braid group B_n has the presentation

$$\left\langle \sigma_1, \dots, \sigma_{n-1} \;\middle|\; \begin{array}{c} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},\ 1 \le i < n-1; \\ \sigma_i\sigma_j = \sigma_j\sigma_i,\ |i-j| > 1 \end{array} \right\rangle.$$

This makes it clear that there is an epimorphism $B_n \to S_n, \sigma_i \mapsto (i, i+1)$. The kernel of this map is P_n, the *pure braid group* of index $n!$. It is well known [**1**] that P_n is generated by elements $A_{ij},\ 1 \le i < j \le n$, where

$$A_{ij} = \sigma_i^{-1} \cdots \sigma_{j-2}^{-1}\, \sigma_{j-1}^{-1}\, \sigma_j^2\, \sigma_{j-1}\, \sigma_{j-2} \cdots \sigma_i.$$

A presentation for P_n with these generators is indicated in [**1, 5, 7**]. It thus seems natural to investigate subgroups of the form

$$H = \langle A_{ij}^{\varepsilon_{ij}} \mid 1 \le i < j \le n \rangle, \tag{1.1}$$

which we call A_{ij} *subgroups*. Other relevant results on properties of Dehn twists and groups generated by Dehn twists can be found in [**4, 8**].

For H as in (1.1) the criterion for $[P_n : H]$ to be finite is given in

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AMS *Mathematics Subject Classification.* Primary 20F36.
Key words and phrases. Braid group, pure braid group, Dehn twist, finite index subgroup, free group.
Received by the editors on November 23, 2004.

Theorem 1. *Let $\varepsilon_{ij} = \varepsilon_{ji} \in \mathbf{Z}^{\geq 0}$. Then the following are equivalent*:

(i) *the A_{ij}-subgroup $H = \langle A_{ij}^{\varepsilon_{ij}} \rangle$ has finite index in P_n*;

(ii) *we have*

(1) *$\varepsilon_{ij} \neq 0$ for all $1 \leq i < j \leq n$; and*

(2) *for all distinct $1 \leq i, j, k \leq n$ we have* $\min\{\varepsilon_{ij}, \varepsilon_{jk}\} = 1$.

(iii) *We have*

(1) *$\varepsilon_{ij} \neq 0$ for all $1 \leq i < j \leq n$; and*

(2) *for all distinct $1 \leq i, j, k \leq n$ we have*

$$\frac{1}{\varepsilon_{ij}} + \frac{1}{\varepsilon_{jk}} + \frac{1}{\varepsilon_{ik}} > 2.$$

If $H = \langle A_{ij}^{\varepsilon_{ij}} \rangle$ is of finite index in P_n, then H is normal in P_n, with H containing P_n', and the index is

$$[P_n : H] = \prod_{1 \leq i < j \leq n} \varepsilon_{ij}.$$

The action of P_n on the cosets of H gives the group $P_n/H \cong \prod_{i<j} \mathbf{Z}_{\varepsilon_{ij}}$.

Recall that the braid group B_n can be interpreted as the mapping class group of the n-punctured disc $D_n \subset \mathbf{R}^2$ [**1**], where the punctures $p_1, \dots, p_n$ are on the x-axis. In this situation each σ_i is a positive half twist [**1**] relative to a simple closed curve $a_{i,i+1}$ containing only the puncture points p_i, p_{i+1} in its interior. The curve $a_{i,i+1}$ is the boundary of a tubular neighborhood of the horizontal line $c_{i,i+1}$ joining p_i, p_{i+1}. Each A_{ij} is a Dehn twist [**1**] about a simple closed curve a_{ij} containing the puncture points p_i, p_j; here a_{ij}, for $|i - j| > 1$, is the boundary of a tubular neighborhood of a semi-circular arc $c_{i,j}$ joining p_i, p_j under the x-axis.

Next we introduce some more elements of P_n. These are

$$\bar{A}_{ij} = (\sigma_i \sigma_{i+1} \cdots \sigma_{j-1})\, \sigma_j^2 (\sigma_i \sigma_{i+1} \cdots \sigma_{j-1})^{-1}.$$

Then the $\bar{A}_{ij}$ are Dehn twists about curves $\bar{a}_{ij}$ which are the reflections of a_{ij} in the x-axis. Note that $\bar{A}_{i,i+1} = A_{i,i+1}$.

We now investigate subgroups of the form $\langle A_{ij}^{\varepsilon_{ij}}, \bar{A}_{ij}^{\delta_{ij}}\rangle$. For example, for $n = 3$ we have subgroups of the form

$$H = H(a,\, b,\, c,\, d) = \langle A_{12}^a,\, A_{23}^b,\, A_{13}^c,\, \bar{A}_{13}^d\rangle,$$

where $a, b, c, d \in \mathbf{Z}^{\geq 0}$ and $\bar{A}_{13} = A_{12}A_{13}A_{12}^{-1} = A_{23}^{-1}A_{13}A_{23}$.

The conditions $a \leq b$, $c \leq d$ can always be assumed since

$$H(a,\, b,\, c,\, d) \cong H(b,\, a,\, c,\, d) \cong H(a,\, b,\, d,\, c).$$

Conditions for $[P_3 : H(a, b, c, d)]$ to be finite are given in

Theorem 2. *For $a \leq b$, $c \leq d$, the subgroup $H(a, b, c, d)$ has finite index in P_3 if and only if we have one of the following four distinct cases*:

(1) $a = b = 1$, $c + d \neq 0$;

(2) $a = 1$, $b > 1$, $\gcd(c, d) = 1$;

(3) $a = 2$, $b = 2$, $c = 1$, $d > 0$;

(4) $a = 2$, $b > 2$, $c = 1$, $d = 1$.

Equivalently, in the cases where $abcd \neq 0$, the index $[P_3 : H]$ is finite if and only if

$$\frac{4}{a^2} + \frac{4}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} + \frac{4}{\gcd(c,d)^2} > 7. \tag{1.2}$$

Each subgroup $H(a, b, c, d)$ in case (3)*, except for $H(2, 2, 1, 1)$, is non-normal*; *all of* (1)*,* (2) *and* (4) *give normal subgroups.*

Lastly the indices in the cases (1)–(4) *above are, respectively*,

(1) $\gcd(c, d)$; (2) b; (3) $4d$; (4) $2b$.

If we have a subgroup of the form $H = \langle A_{ij}^{\varepsilon_{ij}}, \bar{A}_{ij}^{\delta_{ij}}\rangle$, then Theorem 2 gives necessary conditions for H to have finite index in P_n: for every triple $1 \leq i < j < k \leq n$ there is an epimorphism

$$\psi_{ijk} : P_n \to P_{i,j,k} = \langle A_{ij},\, A_{jk},\, A_{ik}\rangle,$$

where $\psi_{ijk}A_{rs} = id$ unless $r, s \in \{i, j, k\}$ and $\psi_{ijk}A_{rs} = A_{rs}$ for $r, s \in \{i, j, k\}$. Note that $P_{i,j,k} \cong P_3$. Then, for this choice of i, j, k the necessary condition is that $[P_{i,j,k} : \psi_{ijk}(H)]$ is finite, this being determined by the numbers $a = \varepsilon_{ij}, b = \varepsilon_{jk}, c = \varepsilon_{ik}, d = \delta_{ik}$ satisfying (1.2). It would be nice if the collection of all $\binom{n}{3}$ such necessary conditions was also sufficient; this we now show not to be the case:

Theorem 3. *Let*

$$H = \langle A_{12}^2,\, A_{23}^2,\, A_{34}^2,\, A_{13},\, A_{24},\, A_{14},\, \bar{A}_{13}^2,\, \bar{A}_{24}^2,\, \bar{A}_{14}^2 \rangle \subset P_4.$$

Then for all $1 \le i < j < k \le 4$ the index $[P_{ijk} : \psi_{ijk}(H)]$ is finite, however H has infinite index in P_4.

All of the finite index subgroups H generated by Dehn twist powers that we have considered thus far have had the property that the action of P_n on the cosets of H has given a finite solvable group. We show by example that this is not always the case:

Example 4. Define the following elements of P_3 and note that each of them is a Dehn twist, since they are all conjugates of σ_1^2 or σ_2^2.

$$\begin{array}{lll}
t_1 = \sigma_1^2\sigma_2^2\sigma_1^{-2}; & t_2 = \sigma_1^{-2}\sigma_2^2\sigma_1^2; & t_3 = \sigma_2^3\sigma_1^2\sigma_2^{-3}; \\
t_4 = \sigma_1^{-1}\sigma_2^2\sigma_1^2\sigma_2^{-2}\sigma_1 & t_5 = \sigma_2^2\sigma_1\sigma_2^2\sigma_1^{-1}\sigma_2^{-2}; & t_6 = \sigma_1^2; \\
t_7 = \sigma_1\sigma_2^2\sigma_1^{-1}; & t_8 = \sigma_2\sigma_1^{-1}\sigma_2^2\sigma_1\sigma_2^2\sigma_1^{-1}\sigma_2^{-2}\sigma_1\sigma_2^{-1}; & t_9 = \sigma_2^2\sigma_1^2\sigma_2^{-2}.
\end{array}$$

For $k = 1, \ldots, 100$ the P_3 subgroups

$$\langle t_1^2,\, t_2^2,\, t_3^3,\, t_4^2,\, t_5^3,\, t_6^2,\, t_7^2,\, t_8^3,\, t_9^k \rangle$$

have index $25k$ and for each such subgroup H the action of P_3 on the cosets of H gives a nonsolvable group (having A_5 as one of its composition factors). The proof is a computer calculation that we made using MAGMA [**2**].

2. Necessary conditions for Theorem 1. A presentation of P_n is given in [**4**, Lemma 4.1] or [**7**]. Note that the relations are all commutators in the generators A_{ij}. Thus, the abelianization P_n/P_n' is a free abelian group of rank $\binom{n}{2}$. Let

$$Ab : P_n \to P_n/P_n' \cong \mathbf{Z}^{\binom{n}{2}},$$

be the abelianization map. If $[P_n : H] < \infty$, then $[Ab\,(P_n) : Ab\,(H)] < \infty$ and so for all $1 \leq r < s \leq n$ there is $m_{rs} > 0$ with $Ab\,(A_{rs}^{m_{rs}}) \in Ab\,(H)$. Now we have the direct product

$$Ab\,(P_n) = \prod_{1 \leq i < j \leq n} Ab\,(A_{ij}),$$

and since $H = \langle A_{ij}^{\varepsilon_{ij}} \rangle$ we see that $Ab\,(H) = \prod_{1 \leq i < j \leq n} Ab\,(A_{ij}^{\varepsilon_{ij}})$. It follows that $\varepsilon_{rs} \neq 0$ for all r, s. This gives the first necessary condition from Theorem 1.

Now suppose that there are distinct $i, j, k \leq n$ with $\varepsilon_{ij}, \varepsilon_{jk} \geq 2$. We will show that in this case $[P_n : H]$ is infinite. This will give the second necessary condition from Theorem 1.

For any subset $S \subset \{1, 2, \ldots, n\}$ there is a punctured disc $D_S \subset D_n$, unique up to isotopy, which contains only the punctures p_i, $i \in S$, and only the a_{ij} for $i, j \in S$. There is a corresponding braid group $B(S) = B(D_S)$ which we can think of as a subgroup of $B_n = B(\{1, 2, \ldots, n\})$. Note that $B(S) \cong B_{|S|}$. Let $P(S)$ denote the pure braid subgroup of $B(S)$. Then there is a projection $\pi_S : P_n \to P(S)$ which can be easily described by saying that we fill in all the punctures p_i where $i \notin S$. More formally: $\pi_S(A_{ij}) = A_{ij}$ if $i, j \in S$ and otherwise $\pi_S(A_{ij}) = 1$. In particular, we have $\psi_{ijk} = \pi_{\{ijk\}}$.

Returning to the situation above (where we have $\varepsilon_{ij}, \varepsilon_{jk} \geq 2$) we let $S = \{i, j, k\}$ and notice that

$$\pi_S(H) = \langle A_{ij}^{\varepsilon_{ij}}, A_{ik}^{\varepsilon_{ik}}, A_{jk}^{\varepsilon_{jk}} \rangle.$$

Since $P(S) \cong P(\{1, 2, 3\}) = P_3$ the second necessary condition will follow from:

Proposition 2.1. *If $a, b, c \in \mathbf{N}$ with at most one of a, b, c equal to* 1*, then*

$$H = \langle A_{12}^a,\, A_{13}^b,\, A_{23}^c \rangle \subset P_3$$

is a free group of rank 3 *and has infinite index in P_3.*

Proof. First note that conjugating by $\sigma_1\sigma_2$ permutes A_{12}, A_{23}, A_{13} cyclically and so we may assume, without loss of generality, that $b, c > 1$. We may also assume that $a = 1$, since the result for $a = 1$ implies the result for general a (since $\langle A_{12}^a, A_{13}^b, A_{23}^c \rangle$ is a subgroup of $\langle A_{12}, A_{13}^b, A_{23}^c \rangle$ and subgroups of free groups are free [**7**]).

It is well known that P_3 has infinite cyclic center [**3**]. In fact from the presentation for P_3 one easily sees that $P_3 \cong F_2 \times \mathbf{Z}$ where $F_2 = \langle A_{13}, A_{23} \rangle$ is a free group of rank 2 [**1, 5, 7**]. Thus, the second assertion of the above result is a consequence of the first, since any subgroup of P_3 of finite index would have nontrivial center.

Now a special case of the epimorphism $\pi_S : P_n \to P(S)$ is when $S = \{1, 2, \ldots, n-1\}$. In this situation we have the split exact sequence

$$1 \longrightarrow F_{n-1} \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow 1, \tag{2.1}$$

where $F_{n-1} = \langle A_{1,n}, A_{2,n}, \ldots, A_{n-1,n} \rangle$ is a free group of rank $n-1$ and P_{n-1} is naturally a subgroup of P_n (this gives the splitting) [**1**]. Thus, there is an action (by conjugation) of P_{n-1} on F_{n-1}. We apply this in the situation where $n = 3$, so that $P_{n-1} = P_2 = \langle A_{12} \rangle$ acts on $F_2 = \langle x = A_{13}, y = A_{23} \rangle$. The action is:

$$A_{12}(x) = (xy)^{-1}x(xy), \qquad A_{12}(y) = (xy)^{-1}y(xy), \tag{2.2}$$

and since xy is fixed by this action we see that the action of $\langle A_{12} \rangle$ is just conjugation by powers of xy.

The proof of Proposition 2.1 will follow using a more general result:

Lemma 2.2. *For $i, j = 1, \ldots, n$, $i < j$, let $C_{ij} = \{\alpha_{ijk} A_{ij}^{\varepsilon_{ijk}} \alpha_{ijk}^{-1} \mid \alpha_{ijk} \in P_n\}$ be a set of P_n-conjugates of powers of A_{ij}, and let $d_{ij} = \gcd\{\varepsilon_{ijk}\}_k$. Let $H < P_n$ be the subgroup*

$$H = \langle C_{ij} \mid 1 \le i < j \le n \rangle.$$

If there are $1 \le u < v < n$ such that $d_{un}, d_{vn} > 1$, then $[P_n : H] = \infty$.

Proof. In many of the results that we now prove, including Lemma 2.2, we will need the following result of elementary group theory:

Lemma 2.3. *Let*

$$1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1$$

be a split exact sequence of groups, and let H be a subgroup of G. Then $[G:H] < \infty$ if and only if

(i) $[N : N \cap H] < \infty$; *and*

(ii) $[Q : \pi(H)] < \infty$.

Proof. Elementary. □

The idea for the proof of Lemma 2.2 will be to apply Lemma 2.3 to the split short exact sequence (2.1). We will show that $F_{n-1} \cap H$ has infinite index in F_n.

First note [**1, 5**] that if $\phi \in P_{n-1}$, $i < n$, then ϕ acts on A_{in} by conjugation; we write this as $\phi(A_{in}) = \phi A_{in} \phi^{-1}$.

Let us partition the C_{ij} as follows:

$$C_1 = \bigcup_{1 \leq i < n} C_{in}, \qquad C_2 = \bigcup_{1 \leq i < j < n} C_{ij}.$$

Thus, $C_1 \subset F_{n-1}$, $C_2 \subset P_{n-1}$.

Now if $w \in F_{n-1} \cap H$, then we can write

$$w = \phi_1 A_1 \, \phi_2 A_2 \cdots \phi_r A_r \, \phi_{r+1},$$

where $\phi_i \in \langle C_2 \rangle \subset P_{n-1}$, $A_i \in C_1 \subset F_{n-1}$ for all i. Since $w \in F_{n-1}$ we see that $\phi_1 \phi_2 \cdots \phi_{r+1} = \text{id}$ so that

$$\begin{aligned} w &= \phi_1 A_1 \, \phi_2 A_2 \cdots \phi_r A_r \, \phi_r^{-1} \cdots \phi_2^{-1} \, \phi_1^{-1} \\ &= \phi_1 (A_1) \, (\phi_1 \phi_2) (A_2) \cdots (\phi_1 \phi_2 \cdots \phi_r) (A_r) \, . \end{aligned} \tag{2.3}$$

It follows that $F_{n-1} \cap H$ is generated by $\phi(A)$ where $\phi \in \langle C_2 \rangle$, $A \in C_1$.

For u, v as in Lemma 2.2 we let N denote the normal closure in F_{n-1} of

$$\left\{A_{un}^{d_{un}}, A_{vn}^{d_{vn}}\right\} \cup \{A_{in} \,|\, 1 \le i < n,\ i \ne u, v\}\,.$$

From the above discussion it follows that $\phi(A) \in N$ for all $\phi \in \langle C_2 \rangle$, $A \in C_1$. In particular we see that $F_{n-1} \cap H \subset N$.

However, since $d_{un}, d_{vn} > 1$ the subgroup N has infinite index in F_{n-1} since the quotient $F_{n-1}/N \cong \mathbf{Z}_{d_{un}} * \mathbf{Z}_{d_{vn}}$ is an infinite group in these circumstances. Thus N, and so H, has infinite index in F_{n-1}. This concludes the proof of Lemma 2.2. □

Returning to the situation $H = \langle A_{12}, A_{13}^b, A_{23}^c \rangle$, where $b, c > 1$, we may apply Lemma 2.2 where $u = 1$, $v = 2$ and so conclude that H has infinite index in P_3. □

3. Sufficiency for Theorem 1. Suppose that $H = \langle A_{ij}^{\varepsilon_{ij}} \rangle$ satisfies (1) and (2) of Theorem 1 (ii). We show that $[P_n : H] < \infty$. Let

$$J = \{\{i, j\} \,|\, \varepsilon_{ij} \ne 1\}\,.$$

Let $A_J = \{A_{ij} \,|\, \{i, j\} \in J\}$. Note that by (2) if $x, y \in J$, $x \ne y$, then $x \cap y = \varnothing$. For $x = \{i, j\} \in J$ we will let A_x also denote A_{ij}.

Proposition 3.1. *The set*

$$C\left(\varepsilon_{uv}\right) = \left\{\prod_{x \in J} A_x^{\delta_x}\right\},$$

where $0 \le \delta_x < \varepsilon_x$, is a set of coset representatives for H in P_n. The subgroup H is normal and

$$P_n/H \cong \oplus_{i<j} \mathbf{Z}_{\varepsilon_{ij}}\,.$$

Proof. Since we have the abelianization map $Ab : P_n \to \mathbf{Z}^{\binom{n}{2}}$ it is easy to see that the elements in $C(\varepsilon_{uv})$ determine different cosets of H. In fact we have the following result:

Lemma 3.2. *If H is as in the above, then H is normal in P_n and H contains P_n'.*

Proof. We first show a) that each simple commutator $A_{ij}^{\varepsilon} A_{rs}^{\delta} A_{ij}^{-\varepsilon} A_{rs}^{-\delta}$ is in H for some choice of $\varepsilon, \delta \in \{\pm 1\}$. We then show b) that H is normal in P_n. The result will follow upon showing that a) and b) imply that H contains P_n'.

Let $x = \{i, j\}$, $y = \{r, s\}$ and for any two simple closed curves c, d on D_n let $\iota(c, d)$ denote the geometric intersection number of c and d. Then there are three cases to be considered:

(i) $x, y \notin J$.

(ii) $x, y \in J$.

(iii) $x \in J$, $y \notin J$.

(i) If $x, y \notin J$, then (1) and (2) imply that $A_x, A_y \in H$ and so $A_x A_y A_x^{-1} A_y^{-1} \in H$.

(ii) If $x, y \in J$ and $\iota(a_{ij}, a_{rs}) = 0$, then we have $A_x A_y A_x^{-1} A_y^{-1} = 1 \in H$ and this does this case. If $\iota(a_{ij}, a_{rs}) \neq 0$, then (2) implies that $\iota(a_{ij}, a_{rs}) = 4$ and in this case we can assume that $i < r < j < s$ so that $A_{ir}, A_{rj}, A_{js}, A_{is} \in H$, by (2). In this situation the result now follows from Lemma 3.3 (since it is sufficient to do the case $i = 1$, $r = 2$, $j = 3$, $s = 4$).

Lemma 3.3. *Let $x = A_{24}^{-1} A_{13} A_{24} A_{13}^{-1}$. Then*

$$x \in K = \langle A_{12},\, A_{23},\, A_{34},\, A_{14} \rangle.$$

Proof. It will suffice to show that $xK = K$. From [**5**, Lemma 4.2], [**1, 7**] we find the following relation in P_4:

$$A_{13}\, A_{24}\, A_{13}^{-1} = \left(A_{34}^{-1}\, A_{14}^{-1}\, A_{34}\, A_{14}\right) A_{24} \left(A_{34}^{-1}\, A_{14}^{-1}\, A_{34}\, A_{14}\right)^{-1}. \tag{3.1}$$

Then, using the fact that $A_{23}A_{24}A_{34}$ commutes with A_{23}, A_{24}, A_{34} and that $A_{14}A_{24}A_{12}$ commutes with A_{14}, A_{24}, A_{12} we have:

$$\begin{aligned} xK &= A_{24}^{-1}\, A_{13}\, A_{24}\, A_{13}^{-1}\, K \\ &= A_{24}^{-1}\, A_{34}^{-1}\, A_{14}^{-1}\, A_{34}\, A_{14}\, A_{24} \left(A_{34}^{-1}\, A_{14}^{-1}\, A_{34}\, A_{14}\right)^{-1} K \end{aligned}$$

$$
\begin{aligned}
&= A_{24}^{-1}\, A_{34}^{-1}\, A_{14}^{-1}\, A_{34}\, A_{14}\, A_{24}\, K \\
&= A_{24}^{-1}\, A_{34}^{-1}\, A_{14}^{-1}\, A_{34}\, A_{14}\, (A_{24}\, A_{12}\, A_{14})\; K \\
&= A_{24}^{-1}\, A_{34}^{-1}\, A_{14}^{-1}\, A_{34}\, (A_{24}\, A_{12}\, A_{14})\, A_{14}\, K \\
&= A_{24}^{-1}\, A_{34}^{-1}\, A_{14}^{-1}\, A_{34}\, (A_{24}\, A_{34}\, A_{23})\, K \\
&= A_{24}^{-1}\, A_{34}^{-1}\, A_{14}^{-1}\, (A_{24}\, A_{34}\, A_{23})\, A_{34}\, K \\
&= A_{24}^{-1}\, A_{34}^{-1}\, A_{14}^{-1}\, A_{24} K \\
&= A_{24}^{-1}\, A_{34}^{-1}\, A_{14}^{-1}\, (A_{24}\, A_{12}\, A_{14})\, K \\
&= A_{24}^{-1}\, A_{34}^{-1}\, (A_{24}\, A_{12}\, A_{14})\, A_{14}^{-1}\, K \\
&= A_{24}^{-1}\, A_{34}^{-1}\, A_{24}\, K \\
&= A_{24}^{-1}\, A_{34}^{-1}\, (A_{24}\, A_{34}\, A_{23})\, K \\
&= A_{24}^{-1}\, (A_{24}\, A_{34}\, A_{23})\, A_{34}^{-1}\, K \\
&= K. \qquad \square
\end{aligned}
$$

Remark 3.4. Although the above is the most convenient proof of the lemma, one can use the same calculation to obtain the following expression for $x = a_{24}^{-1} a_{13} a_{24} a_{13}^{-1}$ as an element of K:

$$
\begin{gathered}
a_{34}\, a_{23}\, a_{34}^{-1}\, a_{23}^{-1}\, a_{34}^{-1}\, a_{12}\, a_{14}^{-1}\, a_{12}^{-1}\, a_{34}\, a_{23}\, a_{34} \\
a_{23}^{-1}\, a_{34}^{-1}\, a_{12}\, a_{14}\, a_{12}^{-1}\, a_{14}^{-1}\, a_{34}^{-1}\, a_{14}\, a_{34}.
\end{gathered}
$$

Checking that this element is equal to x is a second, but messier proof of Lemma 3.3.

This concludes the proof of case (ii).

(iii) Here we have $\iota(a_{ij}, a_{rs}) \in \{0, 2, 4\}$. Assume first that $x \in J$, $y \notin J$ and that $\iota(a_{ij}, a_{rs}) = 0$. Then $A_x A_y A_x^{-1} A_y^{-1} = 1$, showing that this case follows.

The next possibility for (iii) is that $\iota(a_{ij}, a_{rs}) = 2$. Then we may put $x = \{i, j\}$, $y = \{r, s\}$, where $i < j = r < s$ (any other case is similar). It thus suffices to deal with the case $i = 1$, $j = 2$, $s = 3$. So by (2) we see that $\{2, 3\}, \{1, 3\} \notin J$ and so $A_{23}, A_{13} \in H$.

Recall [**1, 3**] that the center of B_n is the infinite cyclic group generated by

$$Z_n = (A_{12}\, A_{13} \cdots A_{1n})\, (A_{23}\, A_{24} \cdots A_{2n}) \cdots (A_{n-2,n-1}\, A_{n-2,n})\, A_{n-1,n}.$$

In the case $n = 3$ we have $Z_3 = A_{12}A_{13}A_{23} = A_{23}A_{12}A_{13}$ and so we have:

$$\begin{aligned} A_{12}\, A_{23}\, A_{12}^{-1}\, A_{23}^{-1} &= \left(A_{23}^{-1}\, A_{13}^{-1}\, A_{12}^{-1}\right) A_{12}\, A_{23}\, A_{12}^{-1}\, (A_{12}\, A_{13}\, A_{23})\, A_{23}^{-1} \\ &= A_{23}^{-1}\, A_{13}^{-1}\, A_{23}\, A_{13} \in H, \end{aligned}$$

as required.

Now assume that in (iii) we have $\iota(a_{ij}, a_{rs}) = 4$. Here we may assume that $i < r < j < s$. As in the above we may apply $\pi_{\{i,j,r,s\}}$ and so simplify to the situation where $i = 1$, $r = 2$, $j = 3$, $s = 4$. Now $\varepsilon_{ij} > 1$ together with (1) and (2) imply that we have $\varepsilon_{12} = \varepsilon_{23} = \varepsilon_{34} = \varepsilon_{14} = 1$. Then $K = \langle A_{12}, A_{23}, A_{34}, A_{14}\rangle \subset H$. The result now follows from Lemma 3.3. This concludes the proof of cases (i), (ii), (iii).

We continue the proof of Lemma 3.2 by showing that H is normal in P_n. This is again accomplished by cases. Let $x = \{i, j\}$, $y = \{r, s\}$, our goal being to show that $A_x A_y^{\varepsilon_y} A_x^{-1},\ A_x^{-1} A_y^{\varepsilon_y} A_x \in H$. The cases are:

(1) $x, y \notin J$;

(2) $x \notin J,\ y \in J$;

(3) $x \in J,\ y \notin J$.

(4) $x, y \in J$.

For (1) we have $A_x, A_y \in H$ and so $A_x A_y^{\varepsilon_y} A_x^{-1},\ A_x^{-1} A_y^{\varepsilon_y} A_x \in H$.

For (2) we have $A_x \in H,\ A_y^{\varepsilon_y} \in H$, so $A_x A_y^{\varepsilon_y} A_x^{-1},\ A_x^{-1} A_y^{\varepsilon_y} A_x \in H$.

For (3) we have $A_x^{\varepsilon_x},\ A_y \in H$. If $x = y$ or $a_{ij} \cap a_{rs} = \varnothing$, then we have $A_x A_y^{\varepsilon_y} A_x^{-1} = A_y^{\varepsilon_y},\ A_x^{-1} A_y^{\varepsilon_y} A_x = A_y^{\varepsilon_y} \in H$, as required.

If we have (3) and $\iota(a_x, a_y) = 2$, then $x \cap y \in \{i, j\}$ and again there are subcases to check depending on the relative sizes of i, j, r, s. We may clearly assume that $i < j$, $r < s$. We also have $A_y \in H$.

One subcase is where $i = r < j < s$. Then $\{i, s\} \notin J$ so that $A_{js} \in H$ and we have

$$\begin{aligned} A_x\, A_y\, A_x^{-1} &= A_{ij}\, A_{is}\, A_{ij}^{-1} = A_{js}^{-1}\, A_{is}\, A_{js} \in H; \\ A_x^{-1}\, A_y\, A_x &= A_{ij}^{-1}\, A_{is}\, A_{ij} = A_{is}\, A_{js}\, A_{is}\, A_{js}^{-1}\, A_{is}^{-1} \in H. \end{aligned}$$

If $i < j = r < s$, then $A_{is} \in H$ and we have

$$\begin{aligned} A_x\, A_y\, A_x^{-1} &= A_{ij}\, A_{js}\, A_{ij}^{-1} = A_{js}^{-1}\, A_{is}^{-1}\, A_{js}\, A_{is}\, A_{js} \in H; \\ A_x^{-1}\, A_y\, A_x &= A_{ij}^{-1}\, A_{js}\, A_{ij} = A_{is}\, A_{js}\, A_{is}^{-1} \in H. \end{aligned}$$

All other cases are similar; this does case (3).

If we have (4), then by hypothesis (2) of Theorem 1 (ii) we see that i, j, r, s are distinct and that, by perhaps interchanging x and y, we may assume $i < r < j < s$. (Here we are ignoring the trivial cases where A_x and A_y commute.) In fact we simplify notation so as to assume $i = 1,\ r = 2,\ j = 3,\ s = 4$. Now we have $A_y^{\varepsilon_y} \in H$ and hypothesis (2) indicates that $A_{12}, A_{23}, A_{34}, A_{14} \in H$. Thus, using (3.1) we have

$$\begin{aligned} A_x\, A_y^{\varepsilon_y}\, A_x^{-1} &= A_{13}\, A_{24}^{\varepsilon_y}\, A_{13}^{-1} \\ &= \left(A_{34}^{-1}\, A_{14}^{-1}\, A_{34}\, A_{14}\right) A_{24}^{\varepsilon_y} \left(A_{34}^{-1}\, A_{14}^{-1}\, A_{34}\, A_{14}\right)^{-1} \in H. \end{aligned}$$

For $A_x^{-1} A_y^{\varepsilon_y} A_x$ a similar argument works where we replace (3.1) by (3.2) below. That (3.2) is true can be checked using any of the solutions to the word problem for B_n [**1, 2**].

$$(3.2) \quad A_{13}^{-1}\, A_{24}\, A_{13} = \left(A_{14}\, A_{34}\, A_{14}^{-1}\, A_{34}^{-1}\right) A_{24} \left(A_{14}\, A_{34}\, A_{14}^{-1}\, A_{34}^{-1}\right)^{-1}.$$

This completes the proof of the fact that H is normal in P_n. To conclude the proof of Lemma 3.2. we need:

Lemma 3.4. *Let G be a group generated by a set X, and let N be a normal subgroup such that for all $x, y \in X$ there are $\varepsilon, \delta \in \{1, -1\}$ such that $x^\varepsilon y^\delta\, x^{-\varepsilon} y^{-\delta} \in N$. Then N contains the derived subgroup G'.*

Proof. The proof will follow if we can show that for all $u, v \in G$ we have $uvu^{-1}v^{-1} \in N$. We do this by induction on $n = |u| + |v|$, where $|u|$ is the length of u as a word in the generators X.

The cases $n = 0, 1$ are trivial, while the case $n = 2$ follows easily from the hypothesis. So assume that $n > 3$, so that we have $|u| > 1$ or $|v| > 1$.

We first however need to do the case $|u| = 1$ (so $u = x \in X$), where we induct on $m = |v|$. The case $m = 1$ is easy so assume that $v = yw,\ y \in X$ where $|w| < |u|$. Then $xwx^{-1}w^{-1} \in N$ and we have

$$uvu^{-1}\, v^{-1} = xywx^{-1}\, w^{-1}\, y^{-1} = \left(xyx^{-1}\, y^{-1}\right) \left(yxwx^{-1}\, w^{-1}y^{-1}\right),$$

and one sees that the two terms are each in N, since N is normal in G. This does the case $|u| = 1$.

So now we do the induction on $n = |u| + |v|$, where we can assume that $|u| > 1$ so that $u = xw$ where $x \in X$ and $|w| < |u|$. Then by induction $wvw^{-1}v^{-1}$, $xvx^{-1}v^{-1} \in N$ and

$$uvu^{-1}\,v^{-1} = xwvw^{-1}\,x^{-1}\,v^{-1} = \left(xwvw^{-1}\,v^{-1}\,x^{-1}\right)\left(xvx^{-1}\,v^{-1}\right),$$

which also belongs to N. □

This concludes the proof of Lemma 3.2. □

From Lemma 3.2 we see that

$$[P_n : H] = \left[Ab(P_n) \cong \mathbf{Z}^{\binom{n}{2}} : Ab(H)\right] = \prod_{x \in J} \varepsilon_x.$$

Proposition 3.1, the sufficiency of the first statement of Theorem 1, together with the rest of Theorem 1 now follow. □

4. The $\langle A_{12}^a, A_{23}^b, A_{13}^c, \bar{A}_{13}^d\rangle$ cases: Infinite index subgroups. In this section we investigate the subgroup

$$H = H(a,b,c,d) = \langle A_{12}^a,\ A_{23}^b,\ A_{13}^c,\ \bar{A}_{13}^d\rangle,$$

where $a,b,c,d \in \mathbf{Z}^{\geq 0}$ and $\bar{A}_{13} = ab^2a^{-1} = A_{12}A_{13}A_{12}^{-1} = A_{23}^{-1}A_{13}A_{23}$. We consider under what conditions $[P_3 : H]$ is infinite. First we have:

Lemma 4.1. *For H as above the index $[P_3 : H]$ is infinite if either* (i) $a = 0$ *or* (ii) $a,b,c,d > 1$.

Proof. To prove Lemma 4.1 we apply Lemma 2.3 to (2.1) with $n = 3$, so that $N = F_2 = \langle A_{13}, A_{23}\rangle$ and $Q = \langle A_{12}\rangle$. First we note that if $a = 0$, then $\pi(H) = \{\mathrm{id}\}$ which does not have finite index in $\langle A_{12}\rangle$. This finishes (i) of Lemma 4.1.

For (ii) assume that $a,b,c,d > 1$. Let $x = A_{13},\ y = A_{23}$, and let

$$K = \Big\{(xy)^{ak}x^{\pm c}\,(xy)^{-ak},\,(xy)^{ak}\,y^{\pm b}\,(xy)^{-ak},\\ (xy)^{ak}\,y^{-1}x^{\pm d}y\,(xy)^{-ak}\ \mid k \in \mathbf{Z}\Big\}.$$

As in (2.3) we note that any element of $F_2 \cap H$ has the form

$$A_{12}^{n_1}\, w_1\, A_{12}^{n_2}\, w_2 \cdots A_{12}^{n_r}\, w_r\, A_{12}^{n_{r+1}},$$

where $n_i \neq 0$, $i \leq r$, $n_1 + n_2 + \cdots + n_{r+1} = 0$ and $w_i \in \langle x^c, y^b, y^{-1}x^d y\rangle$. Then using (2.2) it follows that K is a generating set for $F_2 \cap H$.

For any element u in a free group F we let $\#u$ denote the freely reduced length of u.

Recall [**6**] that a subset $U = \{u_1, u_2, \dots\}$ of a free group F is *Nielsen reduced* if we have:

(NR1) $u_i \neq 1$ for all i;

(NR2) if $u_i^\varepsilon\, u_j^\delta \neq 1$, then

$$\#\left(u_i^\varepsilon\, u_j^\delta\right) \geq \max\left(\#u^\varepsilon,\, \#v^\delta\right),$$

for $\varepsilon, \delta \in \{\pm 1\}$;

(NR3) for all u_i, u_j, u_k such that $u_i^\varepsilon\, u_j^\delta \neq 1$ and $u_j^\delta\, u_k^\gamma \neq 1$ we have

$$\#\left(u_i^\varepsilon\, u_j^\delta\, u_k^\gamma\right) > \#u_i^\varepsilon - \#u_j^\delta + \#u_k^\gamma,$$

for $\varepsilon,\ \delta,\ \gamma \in \{\pm 1\}$.

The key property of a Nielsen reduced set is indicated in

Lemma 4.2 [**6**, Proposition 2.5]. *Any Nielsen reduced subset of a free group F freely generates the subgroup that it generates.*

Now one has:

Lemma 4.3. *The set K is Nielsen reduced.*

Proof. One checks the conditions (NR1), (NR2), (NR3) in the definition. The condition (NR1) is clear. From the definition of K we see that there are three types of elements which we denote

$$E = E(k) = (xy)^{ak}\, x^{\pm c}\, (xy)^{-ak}, \qquad F = F(k) = (xy)^{ak}\, y^{\pm b}\, (xy)^{-ak},$$
$$G = G(k) = (xy)^{ak}\, y^{-1} x^{\pm d} y\, (xy)^{-ak}.$$

To check (NR2) there are now 9 cases. We indicate how to check some of these, leaving the rest to the reader. Let $u, v \in K$. If u and v have the same type, then (NR2) is clear. If u and v do not have the same type, then neither do v^{-1} and u^{-1} and checking the pair u, v (with product uv) also checks the pair v^{-1}, u^{-1} (with product $v^{-1}u^{-1} = (uv)^{-1}$). Thus there are only three cases left to check.

If $u = E(k)$, $v = F(m)$ and $k \neq m$, then (NR2) is clear since $a > 1$. If $k = m$, then (NR2) follows easily from the fact that $b, c > 1$.

If $u = E(k)$, $v = G(m)$ and $k \neq m$, then (NR2) is clear since $a > 1$. If $k = m$, then (NR2) follows since $c, d > 1$.

The last case, $u = F(k)$, $v = G(m)$, is similar. This checks (NR2).

For (NR3) there are 27 cases for $u, v, w \in K$. If u, v, w all have the same type, then it is easy to see that (NR3) follows. Again the rest of the cases may be partitioned as triples $\{(u, v, w), (w^{-1}, v^{-1}, u^{-1})\}$, leaving 12 cases to check. One now checks these cases. □

Since K is an infinite set we see that $F_2 \cap H$ is a free group of infinite rank and so cannot have finite index in F_2. From Lemma 2.3 we see that H has infinite index in P_3. This completes the proof of Lemma 4.1. □

As indicated in Section 1, by symmetries of D_3 we only have to consider the cases where $a \leq b$ and $c \leq d$. Then from Lemma 4.1 we now need only consider the situation where either $a = 1$ or $c = 1$. By Theorem 1 we see that H has finite index if either (i) $a = b = 1$; or (ii) $1 \in \{a, b\} \cap \{c, d\}$.

Lemma 4.4. *In the following cases H has infinite index*:

(1) $a, b > 2$;

(2) $a = 2$, $b > 2$, $c = 1$, $d > 1$.

Proof. (1) If $a, b > 2$ and $cd = 0$, then the result follows from Theorem 1; so we may assume that $c, d > 0$. Then without loss we may assume that $c = d = 1$, as all other cases follow from this case (these being subgroups of this case). The proof now follows the same pattern as for Lemma 4.1, namely, writing down the set K:

$$K = \Big\{ (xy)^{ak}\, x\, (xy)^{-ak}, (xy)^{ak}\, y^{\pm b}\, (xy)^{-ak}, \\ (xy)^{ak}\, y^{-1}\, x^{\pm 1} y\, (xy)^{-ak} \mid k \in \mathbf{Z} \Big\}.$$

Unfortunately, K is not Nielsen reduced. Thus, we need to show that K can be reduced to an infinite set which is Nielsen reduced. Thus, we will produce from K an infinite sequence of elements $U = (u_1, u_2, \dots ,)$ which will be Nielsen reduced and which generate the same subgroup as does K (namely $F_2 \cap H$).

We start by putting

$$\begin{aligned} &u_1 = x, \quad u_2 = x^{-1}, \quad u_3 = y^b, \quad u_4 = y^{-b}, \\ &u_5 = y^{-1} x^1 y, \quad u_6 = y^{-1} x^{-1} y, \\ &u_7 = y^b \left(y^{-1} xy\right) \left[(xy)^{-a} x (xy)^a\right] \left(y^{-1} x^{-1} y\right) y^{-b} \\ &\quad = y^{b-2} x^{-1} (xy)^{3-a}\, y^{-1}\, (xy)^{a-2}\, y^{2-b}, \\ &u_8 = x^{-1} \left[(xy)^a y^{-1} x^{-1} y (xy)^{-a}\right] x = y\, (xy)^{a-2}\, x^{-1}\, (xy)^{2-a}\, y^{-1}. \end{aligned}$$

Now using the above we define elements u_i corresponding to other elements of K. For example, consider the elements $Y_k = (xy)^{-ka}\, x (xy)^{ka} \in K$ for $k \geq 2$ (the case $k = 1$ is given by u_7 above). Then the element in U corresponding to Y_k will be

$$\begin{aligned} &u_7\, u_3\, u_5\, Y_k\, u_6\, u_4 \\ &\qquad = y^b \left(y^{-1} xy\right) (xy)^{-a}\, x\, (xy)^a \left[(xy)^{-ka} x (xy)^{ka}\right] \left(y^{-1} x^{-1} y\right) y^{-b} \\ &\qquad = y^{b-1}\, (xy)^{1-a}\, x\, (xy)^{a(1-k)}\, x\, (xy)^{ka-1}\, y^{1-b}. \end{aligned}$$

Here we have written the element a second way so as to indicate its freely reduced form; we will also do this in each of the cases below.

For $Y_k = (xy)^{ka}\, x (xy)^{-ka} \in K$ for $k \geq 1$ the element in U corresponding to Y_k will be

$$\begin{aligned} u_8\, u_2\, Y_k\, u_1 &= x^{-1}\, (xy)^a \left(y^{-1} x^{-1} y\right) (xy)^{-a} \left[(xy)^{ka} x (xy)^{-ka}\right] x \\ &= y\, (xy)^{a-2}\, y\, (xy)^{a(k-1)}\, x\, (xy)^{1-ka}\, y^{-1}. \end{aligned}$$

For $Y_k = (xy)^{ka}\, y^b (xy)^{-ka} \in K$ for $k \geq 1$ the element in U corresponding to Y_k will be

$$\begin{aligned} u_8\, u_2\, Y_k\, u_1 &= x^{-1}\, (xy)^a\, y^{-1} x^{-1} y\, (xy)^{-a} \left[(xy)^{ka} y^b (xy)^{-ka}\right] x \\ &= y\, (xy)^{a-2}\, x^{-1}\, (xy)^{ka+1-k}\, y^b\, (xy)^{1-ka}\, y^{-1}. \end{aligned}$$

For $Y_k = (xy)^{-ka}\, y^b (xy)^{ka} \in K$ for $k \geq 1$ the element in U corresponding to Y_k will be

$$\begin{aligned} u_7\, u_3\, u_5\, Y_k\, u_6\, u_4 &= y^b y^{-1} xy\, (xy)^{-a}\, x\, (xy)^a \left[(xy)^{-ka} y^b (xy)^{ka}\right] y^{-1} x^{-1} x y^{-b} \\ &= y^{b-2} x^{-1}\, (xy)^{2-a}\, x\, (xy)^{a-ak}\, y^b\, (xy)^{ka-1}\, y^{1-b}. \end{aligned}$$

For $Y_k = (xy)^{ka} (y^{-1}xy)\, (xy)^{-ka} \in K$ for $k \geq 1$ the element in U corresponding to Y_k will be

$$\begin{aligned} u_8\, u_2\, Y_k\, u_1 &= x^{-1}\, (xy)^a\, y^{-1} x^{-1} y\, (xy)^{-a} \left[(xy)^{ka} y^{-1} xy (xy)^{-ka}\right] x \\ &= y\, (xy)^{a-2}\, y\, (xy)^{ka-a}\, y^{-1}\, (xy)^{2-ka}\, y^{-1}. \end{aligned}$$

For $Y_k = (xy)^{-ka} (y^{-1}xy) (xy)^{ka} \in K$ for $k \geq 1$ the element in U corresponding to Y_k will be

$$\begin{aligned} &u_7\, u_3\, u_5\, Y_k\, u_6\, u_4 \\ &\quad = y^b y^{-1} xy\, (xy)^{-1}\, x\, (xy)^a \left[(xy)^{-ka} y^{-1} xy (xy)^{ka}\right] y^{-1} x^{-1} y y^{-b} \\ &\quad = y^{b-2} x^{-1}\, (xy)^{2-a}\, x\, (xy)^{a-ka}\, y^{-1}\, (xy)^{ka}\, y^{1-b}. \end{aligned}$$

It is clear from the above construction of the elements of U that the elements in U determine the same subgroup of F_2 as does K. It is also clear that the infinitely many elements of the sequence U are distinct.

It remains to show that the elements of U are Nielsen reduced and this is a routine checking of a number of cases; as in the above the number of cases to be considered can be somewhat reduced. This proves Lemma 4.4 (1).

For Lemma 4.4 (2) we assume that $a = 2$, $b > 2$, $c = 1$, $d > 1$. Again we have the set

$$\begin{aligned} K = \Big\{ &(xy)^{2k}\, x^{\pm 1}\, (xy)^{-2k}, (xy)^{2k}\, y^{\pm b}\, (xy)^{-2k}, \\ &\qquad (xy)^{2k}\, y^{-1} x^{\pm d} y\, (xy)^{-2k} \mid k \in \mathbf{Z} \Big\}, \end{aligned}$$

of generators of $F_2 \cap H$. Again K is not Nielsen reduced and as in case (1) we now define an infinite sequence $U = (u_1, u_2, \dots)$ such that $\langle K \rangle = \langle U \rangle$ and U is Nielsen reduced. We let

$$\begin{aligned} &u_1 = x, && u_2 = y^{-1}x^d y, \\ &u_3 = y^b, && u_4 = (xy)^{-2}\, x\, (xy)^2 = (xy)^{-1}\, y^{-1}\, (xy)^2\,; \\ &u_5 = (xy)^2\, x\, (xy)^{-2}\,; && u_6 = (xy)^{-2}\, y^b\, (xy)^2\,; \\ &u_7 = (xy)^2\, y^b\, (xy)^{-2} = xyxy^b x^{-1}\, (xy)^{-1}\,. \end{aligned}$$

For $Y_k = (xy)^{2k} x\, (xy)^{-2k} \in K$ for $k \geq 2$ the element in U corresponding to Y_k will be

$$\begin{aligned} u_1^{-1}\, Y_k\, u_5\, u_7\, u_1 &= x^{-1} \left[(xy)^{2k} x (xy)^{-2k} \right] (xy)^2\, x\, (xy)^{-2}\, (xy)^2\, y^b\, (xy)^{-2}\, x \\ &= y\, (xy)^{2k-1}\, x\, (xy)^{3-2k}\, y^{b-2} x^{-1} y^{-1}. \end{aligned}$$

For $Y_k = (xy)^{-2k} x\, (xy)^{2k} \in K$ for $k \geq 2$ the element in U corresponding to Y_k will be

$$\begin{aligned} &u_4\, u_6\, Y_k\, u_4^{-1} \\ &\quad = (xy)^{-2}\, x\, (xy)^2\, (xy)^{-2}\, y^b\, (xy)^2 \left[(xy)^{-2k} x (xy)^{2k} \right] (xy)^{-2}\, x^{-1}\, (xy)^2 \\ &\quad = (xy)^{-1}\, y^{b-2} x^{-1}\, (xy)^{3-2k}\, x\, (xy)^{2k-2}\, yxy. \end{aligned}$$

For $Y_k = (xy)^{2k} y^b (xy)^{-2k} \in K$ for $k \geq 2$ the element in U corresponding to Y_k will be

$$\begin{aligned} u_1^{-1}\, Y_k\, u_5\, u_7\, u_1 &= x^{-1} [(xy)^{2k} y^b (xy)^{-2k}]\, (xy)^2\, x\, (xy)^{-2}\, (xy)^2\, y^b\, (xy)^{-2}\, x \\ &= y\, (xy)^{2k-1}\, y^{b-1} x^{-1}\, (xy)^{4-2k}\, y^{b-1}\, (xy)^{-1}\, y^{-1}. \end{aligned}$$

For $Y_k = (xy)^{-2k} y^b (xy)^{2k} \in K$ for $k \geq 2$ the element in U corresponding to Y_k will be

$$\begin{aligned} &u_4\, u_6\, Y_k\, u_4^{-1} \\ &\quad = (xy)^{-2}\, x\, (xy)^2\, (xy)^{-2}\, y^b\, (xy)^2 \left[(xy)^{-2k} y^b (xy)^{2k} \right] (xy)^{-2}\, x^{-1}\, (xy)^2 \\ &\quad = y^{-1} x^{-1} y^{b-1}\, (xy)^{2-2k}\, y^b\, (xy)^{2k-2}\, yxy. \end{aligned}$$

For $Y_k = (xy)^{2k}y^{-1}x^d y(xy)^{-2k} \in K$ for $k \geq 2$ the element in U corresponding to Y_k will be

$$\begin{aligned} u_1^{-1}Y_k u_5 u_7 u_1 &= x^{-1}[(xy)^{2k}y^{-1}x^d y(xy)^{-2k}](xy)^2x(xy)^{-2}(xy)^2y^b(xy)^{-2}x \\ &= y\,(xy)^{2k-2}\,x^d\,(xy)^{3-2k}\,xy^{b-1}x^{-1}y^{-1}. \end{aligned}$$

For $Y_k = (xy)^{-2k}y^{-1}x^d y(xy)^{2k} \in K$ for $k \geq 2$ the element in U corresponding to Y_k will be

$$\begin{aligned} &u_4\,u_6\,Y_k\,u_4^{-1} \\ &= (xy)^{-2}\,x\,(xy)^2\,(xy)^{-2}y^b\,(xy)^2\left[(xy)^{-2k}y^{-1}xy(xy)^{2k}\right](xy)^{-2}x^{-1}(xy)^2 \\ &= (xy)^{-1}\,y^{b-2}x^{-1}\,(xy)^{3-2k}\,y^{-1}\,(xy)^{2k-1}\,yxy. \end{aligned}$$

Again it is easy to see that $\langle K\rangle = \langle U\rangle$ and that the elements in U are distinct, there thus being infinitely many of them. It remains to show that they are Nielsen reduced and this again consists of checking various cases. Doing this completes the proof of Lemma 4.4. □

Lemma 4.5. *Suppose that $b > 1$ and that $e = \gcd(c, d) > 1$. Then $H(1, b, c, d)$ has infinite index in P_3.*

Proof. Note that $A_{13}^d = A_{12}^{-1}\,\bar{A}_{13}^d\,A_{12}$ and so

$$H(1, b, c, d) = H(1, b, \gcd(c, d), 0).$$

Thus, by Theorem 1, $H(1, b, \gcd(c, d), 0)$ has infinite index in P_3 if $b, \gcd(c, d) > 1$. □

We gather together the results of this section in

Lemma 4.6. *Let $a, b, c, d \in \mathbf{Z}^{\geq 0}$ where $a \leq b$ and $c \leq d$. If $H(a, b, c, d)$ has finite index in P_3, then we have one of the following situations*:

(1) $a = b = 1,\ c + d \neq 0$;

(2) $a = 1,\ b > 1,\ \gcd(c, d) = 1$;

(3) $a = 2$, $b = 2$, $c = 1$, $d > 0$;

(4) $a = 2$, $b > 2$, $c = 1$, $d = 1$.

Proof. Note that $a = 0$ is precluded by Lemma 4.1 (i), and so $b = 0$ is also not allowed. If $c = 0$, then we have the situation of Theorem 1 and so must have $a = 1$, $b = 1$ or $a = 1$, $d = 1$, both of which are included in Lemma 4.6. Thus, all other cases must have $a, b, c, d \neq 0$.

The case $a = 1$, $b = 1$, $c + d \neq 0$ is included in the list, while $a = 1$, $b = 1$, $c + d = 0$ is not allowed by Theorem 1. If $a = 1$, $b > 1$, then Lemma 4.5 shows that we must have $\gcd(c, d) = 1$, one of the included cases.

All further cases must have $a, b > 1$ and so we have $c = 1$ by Lemma 4.1. Now $a, b > 2$ is not allowed by Lemma 4.1 and so we must have $a = 2$, (with $c = 1$). By Lemma 4.4 if $b > 2$, then $d = 1$, a possibility in Lemma 4.6. So, lastly, we now consider $a = 2$, $b = 2, c = 1$, $d \geq c = 1$ and this case is included. This concludes the proof of Lemma 4.6. □

5. Finite index cases for $H(a, b, c, d)$**.** In this section we show that all of the cases listed in Lemma 4.6 have finite index.

We start with case (1): $a = b = 1, c + d > 0, c \leq d$. Recall that $\bar{A}_{13} = A_{12}A_{13}A_{12}^{-1}$. Thus, $H(1, 1, c, d)$ contains $A_{13}^{\gcd(c,d)}$. If $c \neq 0$, then the finite index result follows from Theorem 1. So assume that $c = 0$, in which case $d \neq 0$ and Theorem 1 again gives the result. Theorem 1 also tells us that the index is $\gcd(c, d)$ and that the action on cosets gives an abelian group and that H is normal in P_3.

(2) Here we have $H = \langle A_{12}, A_{23}^b, A_{13}^c, \bar{A}_{13}^d = A_{12}A_{13}A_{12}^{-1}\rangle = \langle A_{12}, A_{23}^b, A_{13}^1\rangle$, since $1 = \gcd(c, d)$. Thus, H has index b in P_3 by Theorem 1. This case also gives an abelian action on cosets and H is normal in P_3.

(3) We will need:

Lemma 5.1. *Let* $H = \langle A_{12}^2, A_{23}^2, A_{13}, \bar{A}_{13}^d \rangle$ *where* $d \geq 1$. *Then the following elements include a set of coset representatives for* H:

$$\begin{aligned}
&\bar{A}_{13}^k,\ k = 0, \dots, d-1;\\
&\bar{A}_{13}^k\, A_{12},\ k = 0, \dots, d-1;\\
&\bar{A}_{13}^k\, A_{23},\ k = 0, \dots, d-1;\\
&\bar{A}_{13}^k\, A_{12}\, A_{23},\ k = 0, \dots, d-1.
\end{aligned}$$

Proof. Let $\mathfrak{C}$ denote the above set of elements, so that $|\mathfrak{C}| = 4d$. Note that $\langle A_{12}, A_{23}, \bar{A}_{13} \rangle = P_3$. The idea will be to show that for $x \in \{A_{12}, A_{23}, \bar{A}_{13}\}$ and for $y \in \mathfrak{C}$ there is $z \in \mathfrak{C}$ such that $Hyx = Hz$. This will then conclude the proof of case (3).

We will need:

Lemma 5.2. *Let* $H' = \langle A_{12}^2, A_{23}^2, A_{13} \rangle \subset H$. *Then we have*

(i) *Let* $X \in H'$. *Then for all* $k \geq 0$ *we have* $\bar{A}_{13}^k\, X \bar{A}_{13}^{-k} \in H'$. *If* $X \in H$, *then for all* $k \geq 0$ *we have* $\bar{A}_{13}^k\, X \bar{A}_{13}^{-k} \in H$.

(ii) *For all* $k \geq 0$ *we have* $\bar{A}_{13}^k\, A_{12}^2\, \bar{A}_{13}^{-k}$, $\bar{A}_{13}^k\, A_{13}\, \bar{A}_{13}^{-k} \in H'$.

(iii) *For all* $k \geq 1$ *we have* $\bar{A}_{13}^k\, A_{12}\, A_{23}\, A_{12}\, A_{23}^{-1}\, \bar{A}_{13}^{1-k} \in H'$.

(iv) *For all* $k \geq 0$ *we have* $\bar{A}_{13}^k\, A_{12}\, A_{23}^2\, A_{12}^{-1}\, \bar{A}_{13}^{-k} \in H'$.

(v) *For all* $k \geq 0$ *we have* $\bar{A}_{13}^k\, A_{12}\, \bar{A}_{13}\, A_{12}^{-1}\, \bar{A}_{13}^{-k} \in H'$.

(vi) *For all* $k \geq 0$ *we have* $\bar{A}_{13}^k\, A_{23}\, \bar{A}_{13}\, A_{23}^{-1}\, \bar{A}_{13}^{-k} \in H'$.

(vii) *For all* $k \geq 0$ *we have* $\bar{A}_{13}^k\, A_{12}\, A_{23}\, \bar{A}_{13}\, A_{23}^{-1}\, A_{12}^{-1}\, \bar{A}_{13}^{-i-1} \in H'$.

Proof. (i) Clearly the second statement of (i) follows from the first. Now note that if, in a group, z centralizes $\langle w, x, y \rangle$, then $xywy^{-1}x^{-1} = xzywy^{-1}z^{-1}x^{-1}$. Using this we have:

$$\begin{aligned}
\bar{A}_{13}^k X \bar{A}_{13}^{-k} &= \left(A_{12} A_{13} A_{12}^{-1}\right)^k X \left(A_{12} A_{13} A_{12}^{-1}\right)^{-k}\\
&= \left[A_{12} \left(A_{12}^{-1} A_{23}^{-1} A_{13}^{-1}\right) A_{13} \left(A_{23}^{-1} A_{13}^{-1} A_{12}^{-1}\right) A_{12}^{-1}\right]^k X\\
&\quad \times \left[A_{12} \left(A_{12}^{-1} A_{23}^{-1} A_{13}^{-1}\right) A_{13} \left(A_{23}^{-1} A_{13}^{-1} A_{12}^{-1}\right) A_{12}^{-1}\right]^{-k}\\
&= \left(A_{23}^{-2} A_{13}^{-1} A_{12}^{-2}\right)^k X \left(A_{23}^{-2} A_{13}^{-1} A_{12}^{-2}\right)^{-k} \in H
\end{aligned}$$

as required. This proves (i).

Now each case of (ii) follows from (i).

For (iii) we note that

$$\bar{A}_{13}\, A_{12}\, A_{23}\, A_{12}\, A_{23}^{-1} = A_{13} \in H.$$

This is (iii) with $k = 1$. The result for $k > 1$ now follows from (i). This concludes the proof of (iii).

For each of (iv), (v) , (vi), (vii), we only need to check the case $k = 1$ (using a solution to the word problem in B_n [**1, 2**]) and then the general result follows from (i). □

Now we consider xA_{12} for $x \in \mathfrak{C}$. If $x = \bar{A}_{13}^k$, $k = 0, \ldots, d-1$, then $xA_{12} \in \mathfrak{C}$.

If $x = \bar{A}_{13}^k A_{12}$, then from Lemma 5.2 (ii) we see that

$$H\, xA_{12} = H\, \bar{A}_{13}^k\, A_{12}^2 = H\bar{A}_{13}^k \in H\mathfrak{C},$$

as required.

If $x = \bar{A}_{13}^k A_{23}$, then from Lemma 5.2 (iii) we see that

$$H\, xA_{12} = H\, \bar{A}_{13}^k\, A_{23}\, A_{12} = H\, \bar{A}_{13}^{k-1}\, A_{12}\, A_{23} \in H\mathfrak{C},$$

as required.

Lastly, if $x = \bar{A}_{13}^k A_{12} A_{23}$, then from Lemma 5.2 (iv) we see that

$$H\, xA_{12} = H\, \bar{A}_{13}^k\, A_{12}\, A_{23}\, A_{12} = H\, \bar{A}_{13}^{k-1}\, A_{23} \in H\mathfrak{C},$$

as required.

This concludes the case of multiplication by A_{12}.

We now consider HxA_{23} for $x \in \mathfrak{C}$. If $x = \bar{A}_{13}^k$ or $x = \bar{A}_{13}^k A_{12}$, then $xA_{23} \in \mathfrak{C}$, and this concludes consideration of these cases.

If $x = \bar{A}_{13}^k A_{23}$, then from Lemma 5.2 (i) we see that $\bar{A}_{13}^k A_{23}^2 \bar{A}_{13}^{-k} \in H$ and so $xA_{23}^2 = H\bar{A}_{13}^k$, which does this case.

If $x = \bar{A}_{13}^k A_{12} A_{23}$, then from Lemma 5.2 (iv) we have

$$H\, xA_{23} = H\, \bar{A}_{13}^k\, A_{12}\, A_{23}^2 = H\, \bar{A}_{13}^k\, A_{12} \in H\mathfrak{C};$$

this finishes the case $x = \bar{A}_{13}^k A_{12} A_{23}$ and concludes the case of multiplication by A_{23}.

We now consider $Hx\bar{A}_{13}$ for $x \in \mathfrak{C}$. If $x = \bar{A}_{13}^k$, then $Hx\bar{A}_{13} = H\bar{A}_{13}^{k+1}$ and using the fact that $\bar{A}_{13}^d \in H$ this case follows.

The rest of the cases follow as above using Lemma 5.2 (v), (vi), (vii).

This shows that the action of multiplication by $A_{12}, A_{23}, \bar{A}_{13}$ on the right preserves the finite set of cosets $\{Hx, x \in \mathfrak{C}\}$ and so H has finite index at most $4d$ in case (3) of Lemma 4.6. The fact that the index is $4d$ follows since the above coset representatives are distinct under mapping to the abelianization of P_3. This concludes discussion of (3) of Lemma 4.6.

We now consider case (4) from Lemma 4.6. Here $H = \langle A_{12}^2, A_{23}^b, A_{13}, \bar{A}_{13}\rangle$, and we show that H is normal in P_3 and H contains P_3'. We follow the same strategy as in the proof of Lemma 3.2, namely, we show that H contains commutators $[A_{ij}^{\pm 1}, A_{rs}^{\pm 1}]$ and then that it is normal.

First note that H contains

$$\bar{A}_{13}\, A_{13}^{-1} = A_{12}\, A_{13}\, A_{12}^{-1}\, A_{13}^{-1} = A_{23}^{-1}\, A_{13}\, A_{23}\, A_{13}^{-1};$$

this finishes two of the commutators, and lastly one can show that $\bar{A}_{13}^{-1} A_{13} = A_{12} A_{23} A_{12}^{-1} A_{23}^{-1}$.

Now we show that H is normal in P_3 by showing that for all $x \in \{A_{12}^2, A_{23}^b, A_{13}, \bar{A}_{13}\}$ and all $y \in \{A_{12}^{\pm 1}, A_{23}^{\pm 1}, A_{13}^{\pm 1}\}$ we have $yxy^{-1} \in H$. Of course, for $y = A_{13}^{\pm 1}$, then we clearly have $yxy^{-1} \in H$. Thus, we let $y \in \{A_{12}^{\pm 1}, A_{23}^{\pm 1}\}$.

The nontrivial cases where $y = A_{12}^{\pm 1}$ are indicated in:

$$\begin{aligned}
A_{12}^{-1}\, A_{23}^b\, A_{12} &= A_{13}\, A_{23}^b\, A_{13}^{-1} \in H;\\
A_{12}\, A_{23}^b\, A_{12}^{-1} &= \left(A_{12}\left(A_{13}^{-1}\, A_{12}^{-1}\, A_{23}^{-1}\right)\right) A_{23}^b \left(A_{12}\left(A_{13}^{-1}\, A_{12}^{-1}\, A_{23}^{-1}\right)\right)^{-1}\\
&= \bar{A}_{13}^{-1}\, A_{23}^b\, \bar{A}_{13} \in H;\\
A_{12}\, A_{13}\, A_{12}^{-1} &= \bar{A}_{13} \in H;\\
A_{12}^{-1}\, A_{13}\, A_{12} &= A_{12}^{-2}\, A_{12}\, A_{13}\, A_{12}^{-1}\, A_{12}^2 = A_{12}^{-2}\, \bar{A}_{13}\, A_{12}^2 \in H;\\
A_{12}^{-1}\, \bar{A}_{13}\, A_{12} &= A_{13} \in H;\\
A_{12}\, \bar{A}_{13}\, A_{12}^{-1} &= A_{12}^2\, A_{13}\, A_{12}^{-2} \in H.
\end{aligned}$$

For $y = A_{23}$ we have:

$$
\begin{aligned}
&A_{23}A_{12}^2A_{23}^{-1} = A_{13}^{-1}A_{12}^2A_{13} \in H;\\
&A_{23}^{-1}A_{12}^2A_{23} = A_{12}A_{13}A_{12}^2A_{13}^{-1}A_{12}^{-1} = \bar{A}_{13}A_{12}^2\bar{A}_{13}^{-1} \in H;\\
&A_{23}A_{13}A_{23}^{-1} = A_{13}^{-1}A_{12}^{-2}(A_{12}A_{13}A_{12}^{-1})A_{12}^2A_{13} = A_{13}^{-1}A_{12}^{-2}\bar{A}_{13}A_{12}^2A_{13} \in H;\\
&A_{23}^{-1}A_{13}A_{23} = A_{12}A_{13}A_{12}^{-1} = \bar{A}_{13} \in H;\\
&A_{23}\bar{A}_{13}A_{23}^{-1} = A_{23}A_{12}A_{13}A_{12}^{-1}A_{23}^{-1} = A_{13};\\
&A_{23}^{-1}\bar{A}_{13}A_{23} = A_{23}^{-1}A_{12}A_{13}A_{12}^{-1}A_{23}\\
&\qquad = \left(A_{23}^{-1}A_{12}A_{23}A_{23}^{-1}\right)A_{13}\left(A_{23}^{-1}A_{12}A_{23}A_{23}^{-1}\right)^{-1}\\
&\qquad = \left(A_{12}A_{13}A_{12}A_{13}^{-1}A_{12}^{-1}A_{23}^{-1}\right)A_{13}(A_{12}A_{13}A_{12}A_{13}^{-1}A_{12}^{-1}A_{23}^{-1})^{-1}\\
&\qquad = \bar{A}_{13}A_{12}^2A_{13}A_{12}^{-2}\bar{A}_{13} \in H.
\end{aligned}
$$

This concludes considerations of all cases and so proves Lemma 5.1. □

It is now easy to show that in cases (1)–(4) of Lemma 4.6 with $abcd \neq 0$ we have

$$\frac{4}{a^2} + \frac{4}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} + \frac{4}{\gcd(c,d)^2} > 7.$$

We now show that

$$\frac{4}{a^2} + \frac{4}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} + \frac{4}{\gcd(c,d)^2} \leq 7$$

in all cases where $[P_3 : H] = \infty$. From the proof of Lemma 4.6 it suffices to show that

$$\frac{4}{a^2} + \frac{4}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} + \frac{4}{\gcd(c,d)^2} \leq 7$$

in the situations described in Lemma 4.1 (ii), Lemma 4.4 (1), (2) and Lemma 4.5. Doing this will conclude the proof of Theorem 2.

For Lemma 4.1 (ii) we have $a, b, c, d > 1$ and so

$$\frac{4}{a^2} + \frac{4}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} + \frac{4}{\gcd(c,d)^2} \leq 1 + 1 + 1/4 + 1/4 + 4 < 7.$$

For Lemma 4.4 (1) we have $a, b > 2$ and so

$$\frac{4}{a^2} + \frac{4}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} + \frac{4}{\gcd(c,d)^2} \leq 4/9 + 4/9 + 1 + 1 + 4 < 7.$$

For Lemma 4.4 (2) we have $a = 2, b > 2, c = 1, d > 1$ and so

$$\frac{4}{a^2} + \frac{4}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} + \frac{4}{\gcd(c,d)^2} \leq 1 + 4/9 + 1 + 1/4 + 4 < 7.$$

For Lemma 4.5 we have $a = 1,\ b > 1,\ \gcd(c,d) > 1$ so that $c, d > 1$, and so

$$\frac{4}{a^2} + \frac{4}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} + \frac{4}{\gcd(c,d)^2} \leq 4 + 1 + 1/4 + 1/4 + 1 < 7.$$

This concludes the proof of Theorem 2. □

6. $H = \langle A_{12}^2, A_{23}^2, A_{34}^2, A_{13}, A_{24}, A_{14}, \bar{A}_{13}^2, \bar{A}_{24}^2, \bar{A}_{14}^2\rangle$ **has infinite index in** P_4**.** In this section we prove Theorem 3.

If $1 \leq i < j < k < 4$, then $[P_{ijk} : \psi_{ijk}(H)]$ is finite by Theorem 2. We now show that H has infinite index in P_4.

Let $F = \langle x_1 = A_{14}, x_2 = A_{24}, x_3 = A_{34}\rangle \cong F_3$ be the free group of rank 3. The method of proof will be to show that $H \cap F$ has infinite index in F. Then Theorem 3 follows by Lemma 2.3.

Now $\psi_{123}(H) = \langle A_{12}^2, A_{23}^2, A_{13}, A_{12}A_{13}^2A_{12}^{-1}\rangle$ has finite index in P_3 by Theorem 2. Also $H \cap F$ contains

$$G = \{A_{34}^2,\ A_{24},\ A_{14},\ \bar{A}_{24}^2 = A_{34}^{-1}\, A_{24}^2\, A_{34},\ \bar{A}_{14}^2 = A_{34}^{-1}\, A_{24}^{-1}\, A_{14}^2\, A_{24}\, A_{34}\}.$$

The expressions for $\bar{A}_{24}^2, \bar{A}_{14}^2$ can be checked [**1, 3, 5**]. Further, as we saw (2.3), $H \cap F$ is generated by elements of the kind $\alpha(g)$, where $g \in G$ and $\alpha \in \psi_{123}(H)$, this latter being an infinite set.

To show that $H \cap F$ has infinite index in F we do the following: let N be the normal subgroup of F generated by the finite set of elements

$$\left\{\alpha(g)g^{-1} \,|\, g \in G,\ \alpha \in \{A_{12}^2,\ A_{23}^2,\ A_{13},\ A_{12}\, A_{13}^2\, A_{12}^{-1}\}\right\}. \tag{6.1}$$

Let $Q = F/N$, let $\pi_Q : F \to Q$ be the quotient map, and let $H' = \pi_Q(H \cap F)$. It will suffice to show that $H' = \pi_Q(H \cap F)$ has infinite index in $Q = \pi_Q(F)$. We will let $Q_1 = \pi_Q(A_{14})$, $Q_2 = \pi_Q(A_{24})$, $Q_3 = \pi_Q(A_{34})$.

To do this we construct an infinite family of permutation representations

$$\rho_n : Q \to S_{8n},$$

which have the following properties:

(1) $\rho_n(Q)$ is transitive.

(2) $\rho_n(H')$ fixes 1.

(3) $[\rho_n(Q) : \rho_n(H')] \geq 8n$.

We define ρ_n by

$$\begin{aligned}
\rho_n(Q_1) &= \prod_{i=0}^{n-1} (3+8i, 4+8i)\,(5+8i, 7+8i)\,;\\
\rho_n(Q_2) &= (8n-3, 8n-11, 8n-19, \dots, 21, 13, 5)\\
&\quad \times (8n, 8n-8, 8n-16, \dots, 24, 16, 8)\\
&\quad \times \prod_{i=0}^{n-1} (2+8i, 3+8i)\,(4+8i, 6+8i)\,;\\
\rho_n(Q_3) &= \prod_{i=0}^{n-1} (1+8i, 2+8i)\,(3+8i, 7+8i)\,(4+8i, 5+8i)\,(6+8i, 8+8i)\,.
\end{aligned}$$

We first need to show that this is a representation of Q, i.e., we need to show that each element listed in (6.1) acts trivially. If we let $r_i = \rho_n(Q_i)$, then this amounts to checking that the r_i satisfy a small number of relations.

For example, if $\alpha = A_{12}^2$, $g = A_{14}$, then

$$\alpha(g)g^{-1} = A_{24}^{-1}\,A_{14}^{-1}\,A_{24}^{-1}\,A_{14}\,A_{a24}\,A_{14}\,A_{24}\,A_{14}^{-1}$$

and so we must show that r_1 commutes with $r_2 r_1 r_2$. In fact the relations that must hold are the following commutators:

$$\begin{aligned}
&(r_1,\, r_2\, r_1\, r_2) = 1,\\
&(r_2,\, r_3\, r_2\, r_3) = 1,\\
&\bigl(r_1^2,\, r_2\, r_3\, r_2^{-1}\, r_1\, r_2\, r_3\, r_2^{-1}\bigr) = 1,\\
&\bigl(r_3,\, r_2^{-1}\, r_1\, r_2\, r_3\, r_2^{-1}\, r_1\, r_2\bigr) = 1,\\
&\bigl(r_2^2,\, r_3^{-1}\, r_2^{-1}\, r_1^{-1}\, r_2\, r_3^{-1}\, r_2^{-1}\, r_1^{-1}\, r_2\, r_3\, r_2^{-1}\, r_1\, r_2\, r_3\, r_2^{-1}\, r_1\bigr) = 1.
\end{aligned}$$

These one now checks.

The point of quotienting F by N to get Q is that this makes $\pi_Q(H \cap F)$ finitely generated, namely, generated by the finite set $\pi_Q(G)$.

We now have a representation, and it is easy to see that it acts transitively. It is also easy to see that $\rho_n(H')$ fixes 1. Thus $\rho_n(H')$ is contained in the stabilizer $\mathrm{St}_n = \mathrm{Stab}_{\rho_n(Q)}(1)$. Since $\rho_n(Q)$ acts transitively we have $[\rho_n(Q) : \mathrm{St}_n] = 8n$ and so

$$[\rho_n(Q) : \rho_n(H')] = [\rho_n(Q) : \mathrm{St}_n]\,[\mathrm{St}_n : \rho_n(H')] \geq 8n,$$

as required.

Having shown (1), (2) and (3) for all n it easily follows that $[Q : H']$ is infinite. This proves Theorem 3. □

Acknowledgments. This paper was written while the author enjoyed the hospitality of the Mathematics Institute at the University of Warwick. All calculations made in the preparation of this manuscript were accomplished using MAGMA [**2**].

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